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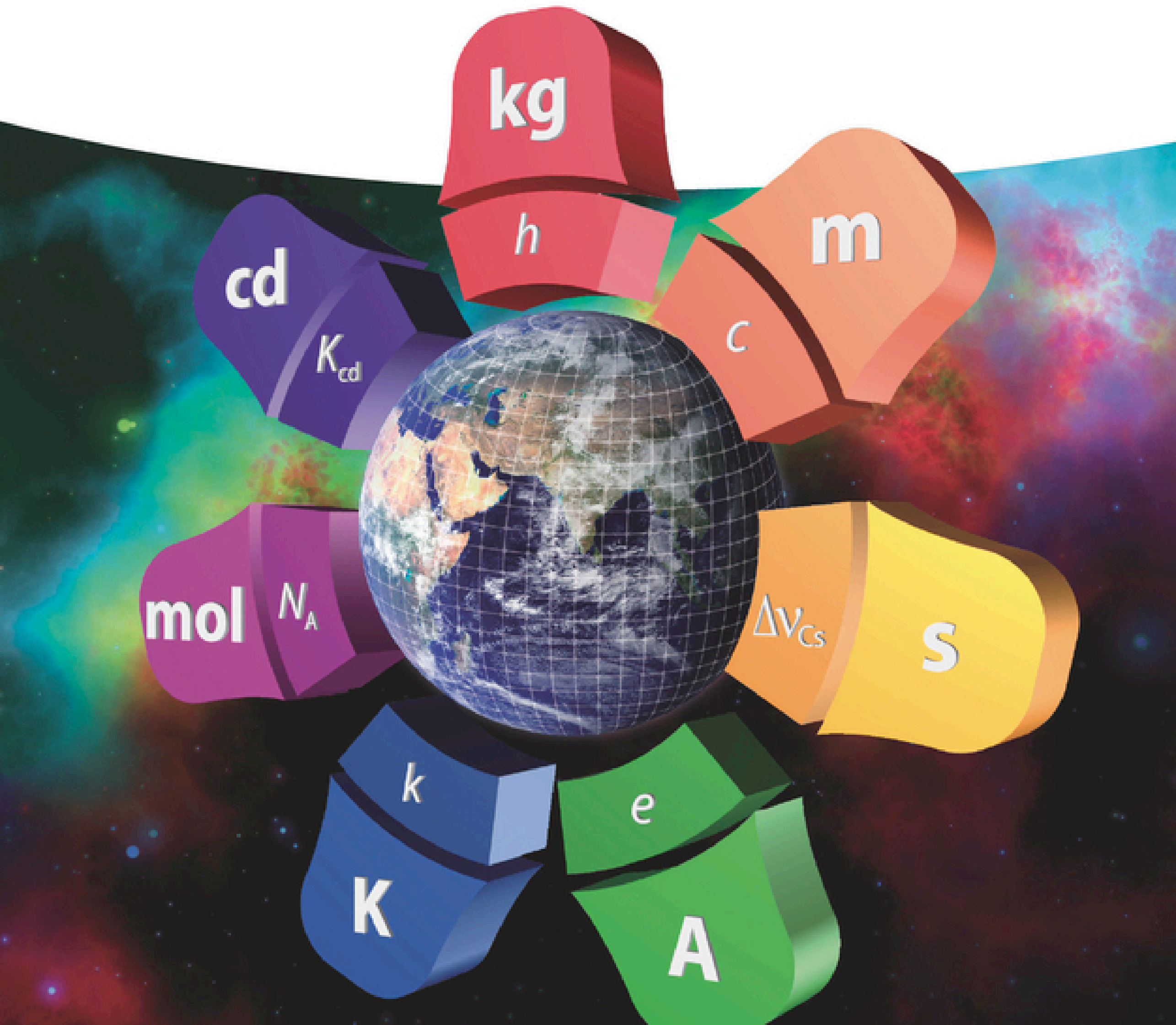
Quantum Metrology and Quantum Standards



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Library of Congress Card No.:
applied for

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available on the Internet at <http://dnb.d-nb.de>.

Print ISBN: 978-3-527-34459-8
ePDF ISBN: 978-3-527-81451-0
ePub ISBN: 978-3-527-81449-7
oBook ISBN: 978-3-527-81448-0

Cover Design SCHULZ Grafik-Design, Fußgönheim, Germany
Typesetting SPi Global, Chennai, India
Printing and Binding

Printed on acid-free paper

10 9 8 7 6 5 4 3 2 1

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Foreword

The International System of Units (*Système International d'Unités,* SI) provides the basis for internationally harmonized measurements that are indispensable for scientific, economic, and social progress. The SI was established in the Metre Convention, which was signed in 1875 and presently has 60 Signatory States as well as 42 Associate States and Economies, who together represent more than 97% of the world economy. It is thus the cornerstone of global trade and quality infrastructure. Since 1875, the SI has been continuously advanced by the organs of the Metre Convention: the General Conference on Weight and Measures (*Conférence Générale des Poids et Mesures*, CGPM) and the International Committee for Weights and Measures (*Comité International des Poids et Mesures*, CIPM), including its Consultative Committee for Units (CCU) and the International Bureau of Weights and Measures (*Bureau International des Poids et Mesures*, BIPM), a scientific institute in Sèvres near Paris.

In 2018, the evolution of the SI took a quantum leap forward: in a landmark decision in November 2018, the 26th CGPM voted to fundamentally revise the SI by abandoning all physical artifacts, material properties, and measurement descriptions used to date to define the kilogram/mole, the kelvin, and the ampere, respectively. On 20 May 2019, the revised SI, which is defined by fixing the numerical values of seven "defining constants," will come into force. Among these are fundamental constants such as the Planck constant, the speed of light in vacuum, and the elementary charge, which together form the fine-structure constant α. The units will thus be independent of space and time with a relative accuracy below 10^{-17} per year, according to the state-of-the-art experiments on the constancy of α. The revised SI guarantees long-term stability and realization of the units anywhere in the known universe with ever-increasing accuracy as technology develops, thus opening the door to innovation in science, industry, and technology.

This book provides a complete review of the revised SI. The definition of units based on the defining constants is examined alongside the realization of the units, which often incorporates the most recent progress in quantum technologies. The book explains and illustrates the physics and technology behind the definitions and their impact on measurements, emphasizing the decisive role quantum metrology has played in the revision. It also reviews what progress based on quantum metrology is anticipated. The book is thus indispensable and highly topical – indeed, it is urgently needed in order to communicate the background and consequences of the revised SI to the broad scientific community and to other interested readers, including lecturers and teachers.

The authors are well qualified for this undertaking. Both have extensive experience and an excellent track record in metrology: Ernst Göbel was president of PTB, the national metrology institute of Germany, for more than 16 years. He was also a member of the CIPM for more than 15 years and served as its president from 2004 to 2010. Uwe Siegner joined the PTB in 1999, working on metrological applications of femtosecond laser technology and on electrical quantum metrology. He has been the head of the electricity division of PTB since 2009. Both authors are experienced university lecturers; in fact, this book is based on lectures they have given at the Technische Universität Braunschweig.

I have studied the book with great interest and pleasure, and I wish the same to a broad readership.

Braunschweig
November 2018

Prof. Dr. Joachim Ullrich
President of PTB, Vice President of CIPM,
President of the Consultative Committee for Units (CCU)

Preface

The General Conference on Weights and Measures (*Conférence Générale des Poids et Mesures*, CGPM) is the governing body of the Metre Convention. The CGPM rules the International System of Units (*Système International d'Unités*), the SI, which provides the basis for all measurements worldwide. At its 26th meeting in November 2018, the CGPM decided that all SI units would be based on seven "defining constants," among them fundamental constants of nature, such as the Planck constant, the speed of light in vacuum, and the elementary charge. To a significant extent, quantum metrology has provided the scientific foundation for this revolutionary change of the system of measurement units. The essence of quantum metrology is to base measurements on counting of discrete quanta.

The concept of some indivisible discrete single particles that are the basic building blocks of all matter goes back to philosophers many centuries BCE. In particular, the Greek philosopher *Demokrit* and his students specified the idea of *atoms* (from the Greek *àtomos*) as the base elements of all matter.

These concepts found support in natural science beginning in the eighteenth century. This was particularly driven by chemistry (e.g. *A. Lavoisier*, *J. Dalton*, and *D. Mendeleev*), kinetic gas theory (e.g. *J. Loschmidt* and *A. Avogadro*), and statistical physics (e.g. *J. Stefan*, *L. Boltzmann*, and *A. Einstein*).

The discovery of the electron by *J.J. Thomson* (1897) and the results of the scattering experiments by *J. Rutherford* and his coworkers (1909) opened a new era in physics, based on their conclusions that atoms are not indivisible but instead composite species. In the atomic model developed by *N. Bohr* in 1913, the atom consists of electrons carrying a negative elementary charge ($-e$) and a tiny nucleus which carries almost all the mass of an atom composed of positively charged ($+e$) protons and electrically neutral neutrons. In Bohr's model, the electrons in an atom can only occupy discrete energy levels, consistent with the experimental findings of atomic spectroscopy.

In the standard model of modern particle physics, electrons are in fact elementary particles belonging to the group of leptons. Protons and neutrons are composite particles composed of fractionally charged elementary particles, named quarks, which are bound together by the strong force.

In the last 50 years or so, scientists have learned to handle single quantum objects, for example, atoms, ions, electrons, and Cooper pairs, not least due to the tremendous progress in laser physics and nanotechnology. This progress has also laid the base for "quantum metrology." The paradigm of quantum metrology is to base measurements on the counting of discrete quanta (e.g. charge or magnetic flux quanta). In contrast, in classical metrology, the values of continuous variables are determined. Proceeding from classical to quantum metrology, the measurement of real numbers is replaced by counting of integers.

The progress in quantum metrology stimulated the discussion about a revision of the SI more than 10 years ago. In particular, it was recognized early on that quantum metrology would allow a new definition of the base units of the SI in terms of constants of nature. This concept was implemented by the decision of the CGPM in November 2018 to revise the SI and to base it on seven defining constants. This book describes this new SI, which will be used from 20 May 2019, its definitions and the underlying physics and technology.

The discrete nature of a physical system is sometimes obvious, for example, by counting cycles when microwave or optical transitions between discrete energy states in atoms or ions are considered. The discrete quantum character of solid-state systems is less obvious because their single-particle energy spectra are quasi-continuous energy bands. Discrete quantum entities can then result from collective effects called macroscopic quantum effects.

The paradigm of quantum metrology becomes particularly obvious when the new definition of the electrical units (ampere, volt, and ohm) is considered. We, therefore, give a more comprehensive description of the underlying solid-state physics and the relevant macroscopic quantum effects. For example, we partly summarize the textbook knowledge and deduce results starting from general principles in Chapter 4 where we introduce superconductivity, the Josephson effect, and quantum interference phenomena in superconductors.

This book addresses advanced students, research workers, scientists, practitioners, and professionals in the field of modern metrology as well as a general readership interested in the foundations of the new SI definition. However, we consider this book as an overview that shall not cover all subjects in the same detail as it covers the electrical units. For further reading, we refer to the respective literature.

This book is based on the previous book by the same authors "Quantum Metrology: Foundation of Units and Measurements," however, reorganized and revised by including the final wording of the new SI definitions and the final values of the defining constants as decided by the 26th CGPM. The differences between the previous and the present SI are highlighted. Further, the individual chapters are updated by including latest results and progress.

This book would not have been possible without the support of many colleagues and friends. We would like to especially mention Stephen Cundiff (JILA, now University of Michigan), Wolfgang Elsäßer (University of Darmstadt), Peter Michler (University Stuttgart), and Alfred Leitenstorfer (University Konstanz) as well as our PTB colleagues Franz Ahlers, Peter Becker, Ralf Behr, Bernd Fellmuth, Joachim Fischer, Christian Hahn, Frank Hohls, Oliver Kieler, Johannes Kohlmann, Stefan Kück, Andre Müller, Ekkehard Peik, Klaus Pierz, Hansjörg Scherer, Piet Schmidt, Sibylle Sievers, Lutz Trahms, Stephan Weyers, and Robert Wynands. We are also grateful for the technical support provided by Alberto Parra del Riego and Jens Simon. We further acknowledge the support of the Wiley-VCH staff members.

Braunschweig
December 2018

Ernst O. Göbel and Uwe Siegner

List of Abbreviations

2DEG	two-dimensional electron gas
AGT	acoustic gas thermometer/thermometry
AIST	National Institute of Advanced Industrial Science and Technology (National Metrology Institute of Japan)
APD	avalanche photo diode
BIPM	International Office for Weights and Measures (*Bureau International des Poids et Mesures*)
CBT	Coulomb blockade thermometer/thermometry
CCC	cryogenic current comparator
CCEM	Consultative Committee for Electricity and Magnetism
CCL	Consultative Committee for Length
CCM	Consultative Committee for Mass
CCT	Consultative Committee for Temperature
CCU	Consultative Committee for Units
CERN	European Organization for Nuclear Science
CGPM	General Conference on Weights and Measures (*Conférence Générale des Poids et Mesures*)
CIPM	International Committee for Weights and Measures (*Comité International des Poids et Mesures*)
CIPM MRA	CIPM mutual recognition arrangement
CODATA	International Council for Science: Committee on Data for Science and Technology
CVD	chemical vapor deposition
CVGT	constant volume gas thermometer/thermometry
DBT	Doppler broadening thermometer/thermometry
DCGT	dielectric constant gas thermometer/thermometry
ECG	electrocardiography
EEG	electroencephalography
EEP	Einstein's equivalence principle
FQHE	fractional quantum Hall effect
GUM	guide to the expression of uncertainties in measurements
HEMT	high electron mobility transistor
IAC	international Avogadro coordination
IDMS	isotope dilution mass spectroscopy
IERS	International Earth Rotation and Reference Systems Service

INRIM	National Institute of Metrology of Italy (*Istituto Nazionale di Ricerca Metrologia)*
ISO	International Organization for Standards
ITS	international temperature scale
JNT	Johnson noise thermometer/thermometry
KRISS	Korea Research Institute of Standards and Science
LED	light emitting diode
LNE	French Metrology Institute (*Laboratoire National de Métrologie et d'Essais*)
MBE	molecular beam epitaxy
MCG	magnetocardiography
MEG	magnetoencephalography
METAS	Federal Institute of Metrology, Switzerland
MOCVD	metalorganic chemical vapor deposition
MODFET	modulation-doped field-effect transistor
MOS	metal-oxide-semiconductor
MOSFET	metal-oxide-semiconductor field-effect transistor
MOT	magneto-optical trap
MOVPE	metalorganic vapor-phase epitaxy
MSL	Measurement Standards Laboratory of New Zealand
NBS	National Bureau of Standards
NEXAFS	near-edge absorption fine structure
NIM	National Institute of Metrology (National Metrology Institute of China)
NININ	normal metal/insulator/normal metal/insulator/normal metal
NIST	National Institute of Standards and Technology (National Metrology Institute of the United States)
NMIJ	National Metrology Institute of Japan
NMR	nuclear magnetic resonance
NPL	National Physics Laboratory (National Metrology Institute of the United Kingdom)
NRC	National Research Council, Canada
NV	nitrogen vacancy
OM	optical molasses
PLTS	provisional low temperature scale
PMT	photomultiplier tube
PTB	Physikalisch-Technische Bundesanstalt (National Metrology Institute of Germany)
PTR	Physikalisch-Technische Reichsanstalt (former National Metrology Institute of Germany)
QED	quantum electrodynamics
QHE	quantum Hall effect
QMT	quantum metrology triangle
QVNS	quantized voltage noise source
RCSJ	resistively and capacitively shunted junction
RHEED	reflection high-energy electron diffraction
RIGT	refractive index gas thermometer/thermometry

rms	root-mean-square
RT	radiation thermometry
SEM	scanning electron microscopy
SET	single-electron transport
SI	International System of Units (*Système International d'Unités*)
SINIS	superconductor/insulator/normal metal/insulator/ superconductor
SIS	superconductor/insulator/superconductor
SNS	superconductor/normal metal/superconductor
SNT	shot noise thermometer
SOI	silicon-on-insulator
SPAD	single-photon avalanche diode
SQUID	superconducting quantum interference device
TAI	international atomic time (*temps atomique international*)
TES	transition-edge sensor
TEM	transmission electron microscope
TPW	triple point of water
ULCA	ultrastable low-noise current amplifier
UME	TÜBITAK Ulusal Metroloji Enstitüsü
UV	ultraviolet
UTC	coordinated universal time
XRCD	X-ray crystal density
XRF	X-ray fluorescence
XPS	X-ray photoelectron spectroscopy
XXR	X-ray reflectometry
YBCO	yttrium barium copper oxide

1 Introduction

Metrology is the science of measurement including all theoretical and experimental aspects, in particular, the experimental and theoretical investigations of uncertainties in measurement results. According to Nobel Prize Winner *J. Hall*, "metrology truly is the mother of science" [1].

Metrology is almost as old as humankind. When people began to exchange goods, they had to agree on commonly accepted standards as a base for their trade. Indeed, many of the ancient cultures such as China, India, Egypt, Greece, and the Roman Empire had a highly developed measurement infrastructure. Examples are the Nippur cubit from the third millennium BCE found in the ruins of a temple in Mesopotamia and now exhibited in the archeology museum in Istanbul and the famous Egyptian royal cubit as the base length unit for the construction of pyramids. However, the culture of metrology faded during the Middle Ages when many different standards were in use. In Germany, for instance, at the end of the eighteenth century, 50 different standards for mass and more than 30 standards for length were used in different parts of the country. This, of course, had been a barrier to trade and led to abuse and fraud. It was then during the French Revolution that the *French Académie des Sciences* took the initiative to define standards independent of the measures taken from the limbs of royal representatives. Instead, their intent was to base the standards on stable quantities of nature available for everyone at all times. Consequently, in 1799, the standard for length was defined as the ten millionth part of the quadrant of the earth, and a platinum bar was fabricated to represent this standard (*Mètre des Archives*). Subsequently, the kilogram, the standard of mass, was defined as the mass of one cubic decimeter of pure water at the temperature of its highest density at 3.98 °C. This can be seen as the birth of the metric system, which, however, at that time was not generally accepted through Europe or even in France. It was only with the signature of the Metre Convention in 1875 by 17 signatory countries that the metric system based on the meter and the kilogram received wider acceptance [2]. At the time of this writing, the Metre Convention was signed by 60 states with another 42 states being associated with the General Conference on Weights and Measures *(Conférence Générale des Poids et Mesures, CGPM)* (as of November 2018). At the General Conferences, following the first one in 1889, the system of units was continuously extended. Finally, at the 11th CGPM in 1960, the previous SI (*Système International*

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d'Unités) (see Section 2.2) with the kilogram, second, meter, ampere, kelvin, and candela as base units was defined. The mole, unit of amount of substance, was added at the 14th CGPM in 1971. Within the SI, the definition of some units has been adopted according to progress in science and technology; for example, the meter was defined in 1960 based on the wavelength of a specific emission line of the noble gas krypton. But then, in 1983, it was replaced by the distance light travels in a given time and by assigning a fixed value to the speed of light in vacuum. Similarly, the second, originally defined as the ephemeris second, was changed by the 13th CGPM and defined via an electronic transition in the Cs isotope 133. Thus, in the previous SI, the meter and the second were defined by constants of nature. In the present revised SI, as accepted by the 26th CGPM in 2018, all units are based on constants of nature [3–7]. In fact, in this context, single quanta physics has a decisive role as will be outlined in this book.

We shall begin with introducing some basic principles of metrology in Chapter 2. We start in Section 2.1 by repeating some basic facts related to measurement and discuss the limitations for measurement uncertainty. The present SI is then presented in Section 2.2. The previous definitions of the respective units are also given for comparison.

Chapter 3 treats the realization of the present definition of the second employing atomic clocks based on the hyperfine transition in the ground state of ^{133}Cs applying thermal beams and laser-cooled atoms, respectively.

Chapter 4 is devoted to superconductivity and its utilization in metrology. Because of its prominent role for electrical metrology, we introduce superconductivity, the Josephson effect, magnetic flux quantization, and quantum interference. By means of the Josephson effect, the volt (the unit for the electrical potential difference) is traced back to the Planck constant and the elementary charge as realized in today's most precise voltage standards. We further discuss magnetic flux quantization and quantum interference allowing the realization of quantum magnetometers (superconducting quantum interference devices) with unprecedented resolution and precision.

The underlying solid-state physics and the metrological application of the quantum Hall effect are discussed in Chapter 5. In the present SI, the unit of electric resistance, ohm, is traced back to the Planck constant and the elementary charge by the quantum Hall effect.

In Chapter 6, we describe the physics of single-electron transport devices, which allow the realization of the unit of electric current, the ampere, according to its present definition based on the elementary charge and frequency. We further discuss the so-called metrological triangle experiment aimed to prove the consistency of the present realizations of the volt, ampere, and ohm.

Chapter 7 is then devoted to the present definition of the kilogram and the mole based on, respectively, the Planck constant and the Avogadro constant. We present the Kibble balance and the silicon single-crystal experiment, which have been seminal for the precise determination of the Planck constant and are now primary realizations of the kilogram replacing the International Kilogram Prototype (IKP).

Various experiments that have contributed to the precise determination of the value of the Boltzmann constant and that are potential realizations of the unit of thermodynamic temperature, kelvin, are described in Chapter 8.

In Chapter 9, we take an even further look into the future of the SI when we discuss optical clocks, which may in due time cause a change of the defining constant for the unit of time, the second, resulting in an improved realization. Further, we discuss the prospect of single-photon emitters for a possible new definition of radiometric and photometric quantities, for example, for (spectral) irradiance and luminous intensity.

In an outlook in Chapter 10, we finally discuss a few examples how the present definitions of the SI pave the way to bring quantum metrology and quantum technology to the "workbench," thereby considerably improving the quality of measurements for industry, science, and society.

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2

Some Basics

2.1 Measurement

Measurement is a physical process to determine the value or magnitude of a quantity. The quantity value can be calculated as follows:

$$Q = \{q\} \cdot [Q] \tag{2.1}$$

where $\{q\}$ is the numerical value and $[Q]$ the unit (see Section 2.2). The unit is thus simply a particular example of a quantity value. Equation (2.1) also applies for Q being a constant. If the numerical value of a constant is fixed, it defines the unit because their product must be equal to the quantity value, Q. This is the underlying concept of the present SI.

Repeated measurements of the same quantity, however, will generally result in slightly different results. In addition, systematic effects that impact the measurement result must be considered. Thus, any measurement result **must** be completed by an uncertainty statement. This measurement uncertainty quantifies the dispersion of quantity values being attributed to a measurand, based on the information used. Measurement uncertainty comprises, in general, many components. Some of the components may be evaluated by type A evaluation of measurement uncertainty from the statistical distribution of quantity values from a series of measurements and can be characterized by standard deviations. The other components, which may be evaluated by type B evaluation of measurement uncertainty, can also be characterized by standard deviations, evaluated from probability density functions based on experience or other information. For the evaluation of uncertainties in measurements, an international agreed guide has been published jointly by ISO and the *Bureau International des Poids et Mesures* (BIPM), the ***G**uide to the Expression of **U**ncertainty in **M**easurement* (GUM) [1–3]. Generally, precision measurements are those with smallest measurement uncertainty.

2.1.1 Limitations of Measurement Uncertainty

One might tend to believe that measurement uncertainty can be continuously decreased as more efforts are put in the respective experiment. However, this is not the case since there are fundamental as well as practical limitations for measurement precision. The fundamental limit is a consequence of the Heisenberg

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uncertainty principle of quantum mechanics, and the major practical limit is due to noise.

2.1.1.1 The Fundamental Quantum Limit

Note that throughout this book, we use the letter f to denote technical frequencies and the Greek letter ν to denote optical frequencies.

The Heisenberg uncertainty principle is a fundamental consequence of quantum mechanics stating that there is a minimum value for the physical quantity action, H:

$$H_{\min} \approx h \tag{2.2}$$

where h is the Planck constant. Action has the dimensions of energy multiplied by time and its unit is joule seconds. From the Heisenberg uncertainty principle, it follows that conjugated variables, such as position and momentum or time and energy, cannot be measured with ultimate precision at a time. For example, if Δx and Δp are the standard deviation for position, x, and momentum, p, respectively, the inequality relation holds ($\hbar = h/2\pi$):

$$\Delta x \Delta p \geq \frac{1}{2}\hbar \tag{2.3}$$

Applied to measurement, the argument is as follows: during a measurement, information is exchanged between the measurement system and the system under consideration. Related to this is an energy exchange. For a given measurement time, τ, or bandwidth of the measurement system, $\Delta f = 1/\tau$, the energy extracted from the system is limited according to Eq. (2.2) [4]:

$$E_{\min} \cdot \tau = \frac{E_{\min}}{\Delta f} \approx h \tag{2.4}$$

Let us now consider, for example, the relation between inductance, L, and, respectively, magnetic flux, Φ, and current, I (see Figure 2.1). The energy is given by $E = (1/2)LI^2 = (1/2)(\Phi^2/L)$, and consequently,

$$I_{\min} \approx \sqrt{\frac{2h}{\tau \cdot L}}; \quad \Phi_{\min} \approx \sqrt{\frac{2h \cdot L}{\tau}} \tag{2.5}$$

These relations are also depicted in Figure 2.1. The gray area corresponds to the regime that is accessible by measurement. Note that this is a heuristic approach that does not consider a specific experiment. Nevertheless, it may provide useful conclusions on how to optimize an experiment. For instance, if an ideal coil (without losses) is applied to measure a small current, inductance should be large (e.g. $L = 1$ H, $\tau = 1$ s, and then $I_{\min} = 3.5 \times 10^{-17}$ A). If instead the coil is applied to measure magnetic flux, L should be small (e.g. $L = 10^{-10}$ H, $\tau = 1$ s, and then $\Phi_{\min} = 4 \times 10^{-22}$ V s $= 2 \times 10^{-7} \times \Phi_0$, where $\Phi_0 = h/2e$ is the flux quantum $= 2.067 \times 10^{-15}$ V s).

Similarly, for a capacitor with capacitance, C, the energy is given by

$$E = \frac{1}{2}Q^2/C = \frac{1}{2}U^2 \cdot C \tag{2.6}$$

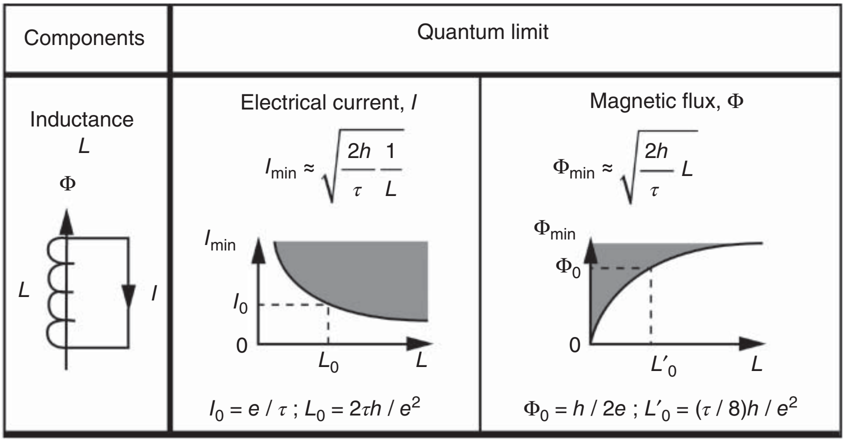

Figure 2.1 Components and quantities considered (left) and the minimum current, I_{min}, and the minimum magnetic flux, Φ_{min}, versus inductance, L, for an ideal coil. Source: Kose and Melchert 1991 [4]. Reproduced with permission of John Wiley and Sons.

and thus,

$$Q_{min} \approx \sqrt{\frac{2h \cdot C}{\tau}}; \quad U_{min} \approx \sqrt{\frac{2h}{\tau \cdot C}}. \tag{2.7}$$

Finally, for a resistor with resistance, R, the energy is given by

$$E = I^2 \cdot R \cdot \tau = \frac{U^2}{R} \cdot \tau \tag{2.8}$$

and thus, for the minimum current and voltage, respectively, we obtain

$$I_{min} \approx \frac{1}{\tau} \cdot \sqrt{\frac{h}{R}}; \quad U_{min} \approx \frac{1}{\tau} \cdot \sqrt{h \cdot R} \tag{2.9}$$

2.1.1.2 Noise

In this chapter, we briefly summarize some aspects of noise theory. For a more detailed treatment of this important and fundamental topic, the reader is referred to, for example, [5].

Noise limits the measurement precision in most practical cases. The noise power spectral density, $P(T,f)/\Delta f$, can be approximated by (Planck formula)

$$\frac{P(T,f)}{\Delta f} = h \cdot f + \frac{h \cdot f}{e^{hf/kT} - 1} \tag{2.10}$$

where f is the frequency, k the Boltzmann constant, and T the temperature. Two limiting cases can be considered as follows.

(i) *Thermal noise (Johnson noise) ($kT \gg hf$)*:

$$\frac{P_{th}(T)}{\Delta f} = k \cdot T \tag{2.11}$$

According to this "Nyquist relation," the thermal noise power spectral density is independent of frequency (white noise) and increases linearly with