AN INTRODUCTION TO PROOF THROUGH REAL ANALYSIS

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Preface

Many mathematics departments in universities in the United States now offer courses intended to introduce students to mathematical proof and transition students to the study of advanced Mathematics. Such courses typically focus on proof techniques, mathematical content foundational to the study of advanced Mathematics, and some explicit attention to the conventions and best practices of mathematical writing.

Across such courses, there seems to be general agreement about the important proof techniques students should learn, and similarly, there is little substantial disagreement regarding the principles of good mathematical writing. However, these transition courses do vary widely in regard to the mathematical content taught. Some courses focus almost entirely on proof techniques and introduce almost no new mathematical content. Some focus first on elementary logic and set theory and then move on to other content, such as discrete mathematics, geometry, or analysis.

As hinted by the title, this book is intended to be an introduction to *proof* through *analysis*. It is a development of notes Daniel Madden has created over many years of teaching the proofs course at the University of Arizona, and the approach taken in this text is different in a number of ways.

First, although this is not an analysis book, the content is heavily focused on analysis. And second, foundational material such as logic, sets, relations, functions are not explicitly studied until the middle of the book, after we have had a go at developing the real numbers. We have found that this approach, while challenging, rewards the effort. Students come away with a solid understanding of mathematical proof techniques and ample experience using those techniques in a robust mathematical context. In addition, students leave the course very well prepared for their advanced mathematics courses and with particularly strong readiness for analysis.

This study has three parts. First, there is a careful review of the basic ideas of numbers, not entirely rigorous, but distinctly careful. This first part will cover select results about natural numbers, integers, rational numbers. We will look at the things we learned in grade school very carefully. Our goal is to reset the stage so that we can examine all our basic notions about numbers. This will end in a definition of the real numbers based on "the completeness axiom." This is the key to truly understanding the real numbers as most people know them, decimals. As we learn more mathematical analysis in this study and any that follow, we will learn how to correctly understand and apply all sorts of infinite processes that describe real numbers.

In the second part, we will shore up our intuitive understanding of logic and set theory by formalizing both subjects. We will go over basic logic and simple set theory. Here we begin the mathematical practice of giving precise definitions for even the simplest of mathematical terms. This is not surprising at all, but it is abstract. We will talk about true and false statements without regard to what those statements are. We will see how to interpret (parse) a complicated sentence to extract its logical meaning. We will use logic to redefine our terminology for numbers so that it can be used in more general mathematical context. We will take the ideas about the various systems of numbers in part 1 to set up a mathematical language that can be used for other mathematical systems. None of this is difficult, but it will be a challenge to keep up with a large number of abstract (but very familiar) definitions.

The third part begins with a repeat of most of part 1. With the terminology and logic of part 2, many things that seemed difficult or unnecessarily long in the first pass at numbers will be much clearer. The second pass will go by quicker, but it should go much easier. A lot of results and proofs will be repeated almost as new. By this time, the basic structure of all proofs will be much more familiar and setup time greatly reduced. The ideas behind a proof will be much more apparent now that the logic and structure of the exposition are more familiar. Finally, in this third part, the new Mathematics begins with the introduction of topology on the real line. The mathematical goal of the course is to prove that the real numbers are all that is needed to measure all distances. This goal is achieved with a proof of "the intermediate value theorem." The educational goal of the course, however, is to learn how to use logic to understand, explain, and prove Mathematics in a careful and rigorous manner.

Introduction

Why proof?

For most people, Mathematics is about using mathematical facts to solve practical problems. Users of Mathematics are rarely concerned about why the methods work and care only that they do work. To too many people, Mathematics is a collection of arcane techniques known only to a select few with "math brains." It is troublesome when those arcane techniques that confuse people are differentiation, integration, or matrix manipulation. It is downright frightening when the confusing problems are adding fractions or computing a restaurant tip. The worst way to view Mathematics is as a long collection of hard-to-remember techniques for solving specific problems. A much better way is to think of Mathematics as an organization of basic ideas that can solve all sorts of problems as needed. When you understand what Mathematics actually means, you can use that understanding to produce your own problem-solving techniques. The key to understanding any piece of Mathematics (or anything else for that matter) is to understand why it works the way it does.

Since the ancient Greeks first studied Mathematics in a careful way, the subject has been built on deductive proof. Mathematical results are accepted as facts only after they have been logically proved from a few basic facts. Once mathematical facts are established, they can be used to solve practical and theoretical mathematical problems. Mathematicians have two reasons for proving a mathematical statement rigorously: first, to be sure that the result is true, and second, to understand when and how it works.

Following the ancient Greek process, mathematicians want a proof for everything - whether it is on the cutting edge of mathematics and science or it is an apparently obvious fact about grade school arithmetic. The idea is to understand why a mathematical result is true and to move on to what you know because it is true. Most of the Mathematics we see in school is about the "moving on" variety. Once school children understand the connection between combining small groups of objects and adding numbers, they can move on to the arithmetic algorithm of adding larger numbers. Thus,

$$\begin{array}{r} 278 \\ +\quad 394 \\ \hline 672 \end{array}$$

is just the theoretical way to combining 278 objects and 394 objects and counting the combination. Once school children understand the connection between groups of groups and multiplication, they can learn the algorithm for multiplication. Then

$$\begin{array}{cccc} & 2 & 5 & 7 \\ \times & & 3 & 5 \\ \hline 1 & 2 & 8 & 5 \\ 7 & 7 & 1 & \\ \hline 8 & 9 & 9 & 5 \end{array}$$

is just the theoretical way of counting 35 rows of 257 objects.

At the very beginning, every child is given some simple justifications for the validity of these algorithms. The strong belief among math educators and education researchers is that students who understand those justifications best are the students that will learn the algorithms best. Granted in the long run, it is a child's ability with the algorithm that is considered most important. In time, greater facility with the algorithms supplants a person's need for the logic behind those algorithms. But the complete understanding of the operation behind the algorithm is always essential for its proper use in odd situations.

There is a popular notion that the logic behind the techniques of Mathematics can be ignored once the procedures of Mathematics are learned. This notion seems to work well for the basic arithmetic of whole numbers. There is a lot of evidence, however, that this is why so many people stumble over problems involving fractions. Too many people "move on" to memorizing the algorithms of fractional arithmetic before they understand the meaning of that arithmetic or why the things they are memorizing work. It is hard to memorize anything and harder still to hold that memory without knowing the context of what you are learning. "To add fractions, find a common denominator." "To divide fractions, invert and multiply." Everyone knows this, but how many can correctly add $3\frac{3}{4}$ to $5\frac{7}{8}$ or divide 21 by $\frac{2}{3}$?

As perplexing as fractions are to the general population, decimal numbers are even worse. Thanks to calculators, everyone knows $\pi = 3.14159\ldots$ where the dots tell us a better calculator would give more digits. Everyone also seems to know that $\frac{1}{3} = 0.33333\ldots$where here the dots mean that the 3s go on forever, or at least they would if it were actually possible for written digits to go on forever. Most people understand decimal numbers well enough that they can move on to using them very well and very effectively without error. But even the most highly trained person can be tripped up by an unexpected decimal question that involves infinitely many decimals. In the next section, we consider some surprisingly confusing questions about simple numbers.

Before we get to these confusing examples, let us set up a plan for curing any resulting mathematical confusion. Early school mathematical training generally concentrates on the problem-solving problems using Mathematics. Some theoretical or intuitive explanations of the ideas and techniques are given, but the level of logical rigor in these justifications varies greatly depending on the topic under discussion. If we are interested in a more advanced education in Mathematics, we must revisit these past justifications of the mathematical ideas we now hold so dear. The time must come when we understand and appreciate a rigorous justification of every mathematical result we will use. This turns out to be a rather difficult step to make. We will work on it in stages.

Why analysis?

Our main objective in this study is to develop a precise description of the real numbers for use as a foundation for the ideas and methods of calculus. There are two ingredients in this development: algebra and analysis. "Algebra" generally refers to the arithmetic of the numbers: addition, subtraction, multiplication, and division. The ways in which these operations interact form the "algebraic structure" of the number systems that we will consider. "Analysis" refers to the study of the distinctions between exact numbers and their approximations. It is simply a fact that certain real numbers cannot be expressed exactly using only finitely many whole numbers. Analysis allows us to say precise things about real numbers that cannot be precisely described with a finite expression.

Problems in analysis typically occur when we use numbers to measure things. Given an isosceles right triangle, two squares drawn with sides the length of the short sides of the triangle will have a combined area equal to a square with a side whose length is the same as the hypotenuse. If we measure the sides as n units, the hypotenuse will measure $n\sqrt{2}$ units. Thus, to measure the hypotenuse, there must be a number we write as $\sqrt{2}$, which when multiplied by itself is 2. A good calculator will approximate $\sqrt{2}$ as 1.41421. A better calculator will approximate it as 1.41421356237, and a sensational one as

$$1.4142135623730950488016887242096980785696718753769.$$

But, as the Greeks discovered, the only way to write an exact representation of the number is by saying that it is a number that when squared is 2 and then to make up a symbol for it, such as $\sqrt{2}$.

Since our goal is to develop a rigorous description of the real numbers, we must be able to use it to work with numbers we can describe exactly but cannot calculate exactly. We will use algebra and analysis to allow us to do arithmetic with numbers such as this. Suppose, for example, that we need a number x so that $x^3 + x = 7$. Once we are sure that it exists, we can assign it a symbol. For now, let us say $\natural$. As it turns out, $\natural$ is like $\sqrt{2}$. We can approximate it as accurately as we like, but it may be that the only way to write it exactly is $\natural$. We can use algebra to do some exact calculations with $\natural$. For example, $\natural^4 = 7\natural - \natural^2$, but it is a matter of opinion whether $7\natural - \natural^2$ is a better name for $\natural^4$ or if it is the other way around.

For a more famous example, suppose that we need a number that is the ratio between the circumference of a circle and the diameter of the circle. First, we need to know that it exists, but we can thank the ancient Greeks for that. We can assign it a symbol π. We can approximate it as accurately as we like, but the only way to write it exactly is π. The situation is even worse than $\sqrt{2}$ or $\natural$; mathematicians have proved that there is no polynomial $P(x)$ of any degree with rational coefficients so that $P(\pi) = 0$. This means that the only possible way to write π^4 exactly is π^4.

The way most people know π is "3.14159.... where the digits continue forever without a pattern." So the question is, "Does anyone know π exactly?" If there is no pattern to the digits and they go on forever, then no one can know them all. These digits may look random after a while, but because we believe π is a real number, we believe that all the digits are exactly described even if they may never be all known. Most educated people have a working knowledge of the real numbers, but mostly because they have a reasonable understanding of decimal approximation. Thus, they are not bothered by questions about exact values of π.

On the other hand, consider 2^π. With a calculator, almost anyone can find that $2^\pi = 8.8249778$, and many will guess that this is simply an approximation of the exact value. But scratch the surface of this general understanding of real numbers and you discover a problem: what have we approximated? That is, "What is the meaning of 2^π?" Now $2^{\frac{22}{7}} = \sqrt[7]{2^{22}}$, but π is not a rational fraction. So this is of little help describing what the number 2^π means. The only reason most people have to believe that it has a meaning at all is that their calculator will calculate it.

Next consider a problem with infinite decimal arithmetic that most people avoid by using approximations. Consider the numbers: $\alpha = 0.91260\,91260\,91260\ldots$ and $0.142857\,142857\,142857\ldots$, where the ellipsis (...) means that the pattern of digits repeats forever. Now if we believe that we can make π a number by saying "π is 3.14159.... where the digits continue forever without a pattern," then knowing all the digits of α and β should make them even better known numbers. The question is, can we find an exact decimal expression for $\alpha - \beta$? Does it even have one? If we line them up to subtract using the familiar algorithm, it is hard to know where to start working on the digits. If we know enough about real and rational numbers, we may know a better approach that tells us that the answer will have its own repeating decimal form. But finding that exact answer means having the patience to calculate and recognize the 30 digit repeating pattern it turns out to have.

The final example has been known to be good bait used by trolls on mathematical discussion boards since the invention of the internet. Consider two other numbers $\alpha = 0.5$ and $\beta = 0.499999\ldots$. The question is, "Is one of these numbers greater than the other, and if so which?" Now as we know, the number α has a better name. The decimal point in 0.5 is mathematical notation where the next digits give the number of parts where the previous unit is divided into 10 equal parts. Thus, $\alpha = \frac{5}{10} = \frac{1}{2}$. Comparing the first decimal digits, we know that, α is definitely greater than or equal to β. Its first digit is larger than the first digit of β, and some might say that that makes it greater. But it really only tells us that $\alpha \geq \beta$. We might try to subtract to see if the difference is 0. If we line them up

$$
\begin{array}{l}
0.50000000000000\ldots \\
0.49999999999999\ldots
\end{array}
$$

we run into the same problem we just saw; where to start? The fact that most of the digits in β are greater than the ones in α above them forces us to guess how that arithmetic will go. Still, we can certainly see that the result will start: 0.00000. We can guess that it will never give a digit other than 0 until it ends and that it will, in fact, never end. The result of the subtraction will be a decimal with infinitely many 0 digits. That must be 0, right? In the end, we can only use the finite versions of subtraction to approximate the infinite arithmetic. If we are lucky, we can identify a pattern and guess an answer. But can we be sure? It does look like $\beta - \alpha = 0$ and so $\alpha = \beta$, but can one real number really have two decimal expansions?

In Mathematics, we often describe a precise number that we can only approximate using decimal numbers. We then give the number a name or symbol and work with the number by working with the name. We did this earlier by setting $\alpha = 0.5$ and $\beta = 0.499999\ldots$. We then interpreted $\alpha = 0.5$ to mean 5 divided by 10. We then argued that there was reason to suspect $\alpha = \beta$. The most famous case of naming numbers we do not know exactly is π, but the base of the natural logarithms e is basically the same. From this point of view, for any positive real number a, we use the symbol $\sqrt[n]{a}$ as a name for the real solution to $x^n = a$. In addition, for any real number θ, we use geometry to precisely describe a number between 0 and 1 that we call $\sin(\theta)$. A lot of Mathematics is about finding precise relations between the different numbers we have named. If the real numbers work as we expect, it should come as no surprise that $0.5 = 0.499999\ldots$. We should be able to prove this from basic undeniable principles. We also should know that $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, and we expect someone is able to prove it. A bit more surprising is that $\natural$, the real solution of $x^3 + x = 7$ can also be given as

$$\natural = -\sqrt[3]{\frac{2}{189+\sqrt{35829}}} + \sqrt[3]{\frac{63+\sqrt{3981}}{18}}.$$

However, mathematicians were mostly shocked when Niels Abel proved that the real solution $\flat$ to $x^5 + 10x^2 = 40$ cannot be given precisely in terms of natural numbers and radical signs alone.

Numbers such as π and e have no pattern in their decimal expansions. We can, however, describe π and e using infinite representations where all the terms are known:

$$\pi = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} + \ldots$$
$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$
$$\pi = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots.$$

These are at least a bit better than the decimal approximations because the patterns they follow do give all the terms. If we prove that these infinite expressions actually give numbers, we can claim to know them exactly. We still cannot write them down exactly without alluding to infinitely many terms. We can use our names for them to do calculations with them using algebra. We can do approximate calculations with them by keeping just the first terms in their infinite expressions. However, knowing why the first and last infinite expressions can be given the same name is an issue for analysis. If we can find some argument that the difference between $\gamma = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} + \ldots$ and $\delta = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots$ is zero, we can at least say $\gamma = \delta$. But why either of these make $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ true requires analysis.

Our goal is to develop a precise description of the real numbers that allows us to deal with real numbers we can describe precisely but not write out precisely with finite terms. We will generally use analysis to determine when we have actually described one and only one real number, that is, to determine when a number exists and is unique. This will allow us to give it a name. We will then typically use algebra to use the name to study that number or other numbers we might be interested in.

We start by reviewing the most basic aspects of numbers. These are things that we may not have looked at closely since we learned about then in preschool, kindergarten, or elementary school. The object is to practice being very careful and precise with the most familiar of all Mathematics. But this time, we have algebra to help. As we have seen, some things about numbers can be confusing. We can learn to work past any confusion by starting with an extra careful look at things we know very well.

Part I

A First Pass at Defining $\mathbb{R}$

1

Beginnings

1.1 A naive approach to the natural numbers

1.1.1 Preschool: foundations of the natural numbers

One of the first things we learn in mathematics is the counting chant: one, two, three, four, five.... We quickly learn how to count to higher and higher numbers, and finally, the day comes when we realize that we can continue on counting forever. At that point, believe it or not, we have all the necessary assumptions we need to discover all of mathematics. The counting numbers are often called *whole numbers*, but mathematicians call them *natural numbers*. We can express our childhood discovery in four adult principles:

- There is a unique first natural number.
- Every natural number has a unique immediate successor.
- Every natural number except the first has a unique immediate predecessor.
- Every natural number is an eventual successor of the first.

Algebra begins when we introduce symbols to express these principles. Now there is a unique first natural number; we will write it as 1. Every natural number has a unique immediate successor. There are many choices for denoting the successor of a natural number. In a more rigorous course on the foundations of mathematics, we might write the successor of a natural number n as $s(n)$. We will choose a notation that anticipates later definitions. The successor of a natural number n will be written as $n + 1$. Notice that this is not addition (yet); $n + 1$ means "the successor of n," no more and no less. Every natural number except the first has a unique immediate predecessor. Again, we choose a notation with an eye on what is coming later. If $n \neq 1$, the predecessor of a natural number n will be written as $n - 1$. This is not subtraction; it is simply the symbol for the predecessor. The relationship between successors and predecessors can be described using this notation. Notice that $1 - 1$ is not defined because the first number does not have a predecessor.

An Introduction to Proof through Real Analysis, First Edition. Daniel J. Madden and Jason A. Aubrey.

Remark. If n is a natural number, then $(n + 1) - 1 = n$.

Remark. If n is a natural number and $n \neq 1$, then $(n - 1) + 1 = n$.

These are our first algebraic results. Note that they are nothing more than symbolic representations of the meanings of the words "successor" and "predecessor." Thus, $(n + 1) - 1 = n$ is just a symbolic statement that means "the predecessor of the successor of a natural number n is just the number n." Thus, $(n - 1) + 1 = n$ means "the successor of the predecessor of a natural number n other than the first number 1 is just the number n." That is all algebra really is: the encoding of ideas expressed in words into symbolic representations of those ideas.

The fourth principle is the hardest to precisely express in symbols. However, in this first chapter, we are just setting some groundwork to make later logically rigorous mathematics easier. We are willing to forgo some rigor to lay this groundwork. To say this more clearly, we are not going to restrict ourselves to completely logical proofs and definitions until the end of this chapter.

The fourth principle states: Every natural number is an eventual successor of the first. That is, every natural number is the successor of the successor of the successor of … the successor of 1. The loose notation for this is: if n is a natural number, then n can be written as

$$n = (((\dots((1 + 1) + 1) + \dots + 1) + 1) + 1. \tag{1.1}$$

The use of the ellipsis in this bit of algebra kills any hope of making an unambiguous statement. It should be clear what this means: n is made up of a series of $(+1)$s, each of which signals the successor of a previous number. This is not the best way to begin a course in rigorous mathematics, and soon we will need to replace it with something else.

There is one more bit of notation we set for dealing with these basic principles. We say m is an eventual successor of n if

$$m = (((\dots((n + 1) + 1) + \dots + 1) + 1) + 1. \tag{1.2}$$

Again, the use of ellipsis kills any rigor this idea might have. When m is an eventual successor of n, we say "m is greater than n"; and we write $m > n$. Actually, we might prefer to move smaller to larger and write $n < m$ and say "n is less than m." This leads to some algebra, and a careful name for an important algebraic property:

Remark. Let k, m and n be natural numbers. If $n < m$ and $m < k$, then $n < k$.

We can refer to this remark by saying, "The order of the natural numbers is *transitive*."

This remark is true because $n < m$ means

$$m = (((\ldots((n + 1) + 1) + \ldots + 1) + 1) + 1, \tag{1.3}$$

and $m < k$ means

$$k = (((\ldots((m + 1) + 1) + \ldots + 1) + 1) + 1. \tag{1.4}$$

Equality means that m is exactly the same as the expression that follows the equal sign. So we can "substitute" that expression for the m in the later equation.

$$k = (((\ldots((\ldots(n + 1)\ldots + 1) + \ldots + 1) + 1) + 1. \tag{1.5}$$

So, indeed, k is an eventual successor of n.

Finally, suppose that we have natural numbers n and m. Since we have not said otherwise, they could be the same. Thus, it might be that $n = m$. Both numbers are eventual successors of 1. If $n \neq m$, one of the two must be an eventual successor of 1 that appears before the first. Thus, either $n < m$ or $m < n$. This leads to our final observation about the order of the natural numbers and another mathematical term.

Remark. If n and m are natural numbers, then exactly one of the following must be true: $n < m$; $m < n$; or $n = m$.

We refer to this remark by saying, "The order on the natural numbers has *trichotomy*."

Thus, if $n < m$ is not true, then either $m < n$ or $n = m$. We have notation that allows us to abbreviate this further. We write $n \leq m$ to mean either $n < m$ or $n = m$. Similarly, we write $n \geq m$ to mean either $n > m$ or $n = m$. There is no notational shortcut for saying either $n > m$ or $n < m$ other than $n \neq m$.

1.1.2 Kindergarten: addition and subtraction

The first use we learn for numbers is for counting things. We learn names and symbols for all the eventual successors of 1.

$$\begin{aligned} 1 + 1 &= 2. \\ (1 + 1) + 1 &= 3. \\ ((1 + 1) + 1) + 1 &= 4. \\ (((1 + 1) + 1) + 1) + 1 &= 5. \\ \cdots\cdots &= \cdots \end{aligned} \tag{1.6}$$

In the early grades, we add the two numbers 2 and 5 by creating two sets (say, of marbles), one with 2 marbles and another set with 5 marbles. We combine the two sets into one and count to find a total of 7 marbles. We learn that the notation for this is $2 + 5 = 7$.

$$\begin{aligned} 2 &= 1 + 1; \\ 5 &= (((1 + 1) + 1) + 1) + 1; \\ 2 + 5 &= (1 + 1) + ((((1 + 1) + 1) + 1) + 1) \\ &= (((((1 + 1) + 1) + 1) + 1) + 1) + 1) \\ &= 7. \end{aligned} \tag{1.7}$$

While a main goal in elementary school arithmetic is learning the algorithm for adding natural numbers, this would be pointless without a few years of counting and combining so that we know what the addition algorithm does for us. This algorithm is a theoretical method that allows us to avoid long counts. We eventually learn how to find that $27 + 35 = 62$ without knowing what objects we are trying to count. The concrete problem of counting combined sets becomes the abstract problem of adding numbers. We learn what addition is mostly by repeated counting. Later, we learn a shortcut that uses an arithmetic procedure. But addition has never been taught by someone defining it for us, until now.

As adults we need to invent (or define) an operation on natural numbers where two natural numbers n and m are combined to produce a new natural number. We denote this new number as $n + m$. We define this new number by writing n and m as eventual successors of 1:

$$\begin{aligned} n &= (((\ldots((1 + 1) + 1) + \ldots + 1) + 1) + 1; \\ m &= (((\ldots((\mathbf{1} + \mathbf{1}) + \mathbf{1}) + \ldots + \mathbf{1}) + \mathbf{1}. \end{aligned} \tag{1.8}$$

Then

$$\begin{aligned} n + m &= [(((\ldots((1 + 1) + 1) + \ldots + 1) + 1) + 1] \\ &\quad + [(((\ldots((\mathbf{1} + \mathbf{1}) + \mathbf{1}) + \ldots + \mathbf{1}) + \mathbf{1}] \\ &= (((\ldots((1 + 1) + 1) + \ldots + 1) + 1) + 1) \\ &\quad + \cdots + \mathbf{1}) + \mathbf{1}) + \mathbf{1}) + \ldots \mathbf{1}) + \mathbf{1}. \end{aligned} \tag{1.9}$$

The imprecision of the ellipsis almost renders this definition useless, but the bold 1s help a bit. In a course on the rigorous foundations of mathematics, we would need to do much better than this. Luckily, years of combining sets of marbles allows us to realize what we are trying to say in this study with the aforementioned definition. This almost unintelligible definition does lead to one very important algebraic fact. It is clear that the definition of addition is just the rearrangement of the parenthesis around 1s and +s. Thus, we have an algebraic fact about the addition of counting numbers: parentheses do not matter.

Remark. If k, m, and n are natural numbers, then $(k + n) + m = k + (n + m)$.

We refer to this by saying, "Addition of natural numbers is *associative*."

A few other algebraic facts follow just as quickly.

Remark. If m and n are natural numbers, then $n < n + m$.

We refer to this by paraphrasing Euclid, "The whole is greater than the part."

Remark. If k, m, and n are natural numbers and $n < m$, then $n + k < m + k$.

We refer to this by saying, "Addition of natural numbers respects the order."

If we remember our lessons from counting blocks, we realize that it doesn't make a difference which set of blocks we start with when we combine the two sets – the total always comes out the same. We can turn this observation into another useful algebraic fact.

Remark. If m and n are natural numbers, then $m + n = n + m$.

We refer to this by saying, "Addition of natural numbers is *commutative*."

The first step after learning the arithmetic operation of addition is the introduction of a new operation, subtraction. At first we learned it as the solution to an addition puzzle, such as "What number added to 5 gives 7?" We all recall the problem: Fill in the box

$$5 + [\ ] = 7. \tag{1.10}$$

Only later, after we understood this type of question better, did we learn a procedure for subtracting. Soon we learned that there were two arithmetic operations: addition and subtraction. As mathematicians, we will not talk about subtraction as its own operation, but rather look at it in terms of addition. It is not that there is anything wrong with thinking of subtraction as its own operation, but just that it will help later algebraic ideas to try to keep the language focused on addition. Subtraction will still be a possibility, but we will not fully admit it, but rather refer to the following property of the natural numbers:

Remark. If n and m are natural numbers with $n < m$, then there exists a unique natural number k so that $m = n + k$.

We refer to this by saying, "There is a conditional subtraction on the natural numbers."

We say that this subtraction is conditional because we cannot subtract the natural number n from m unless $n < m$ (and get a natural number as a result). Of course, one of our first orders of business will be to create the integers as a larger collection of numbers that removes this condition on subtraction. As for notation, it is no surprise that we will eventually write k as $m - n$. Thus, the sign "−" for subtraction is still there. For at least a while, we will not take advantage of this notation because we are trying to avoid treating subtraction as an operation. The reason for this should be clearer when we start to discuss the integers where things work better algebraically.

There are two other "subtraction" properties that we will use frequently.

Remark. If k, n, and m are natural numbers with $n + k = m + k$, then $n = m$.

Remark. If k, n, and m are natural numbers with $n + k < m + k$, then $n < m$.

Rather than talking about these in terms of subtraction, we will refer to these as "cancellation properties of addition."

1.1.3 Grade school: multiplication and division

Once we know that we can add any two natural numbers, we can use that to invent a new operation, multiplication. Two natural numbers n and m are combined to produce a new natural number. We denote this new number as $n \cdot m$ or nm. We define this new number by writing n as eventual successor of 1:

$$n = (((\ldots((1 + 1) + 1) + \ldots + 1) + 1) + 1. \tag{1.11}$$

Then

$$n \cdot m = (((\ldots((m + m) + m) + \ldots + m) + m) + m. \tag{1.12}$$

Again, because of the ellipsis, the only reason this might be considered a definition is because we already know what it means: to find $n \cdot m$ add m to itself n times. For example,

$$3 \times 7 = (7 + 7) + 7. \tag{1.13}$$

As we move on to a discussion of the properties of multiplication, we lose any pretense of rigor. We need to refer to geometric intuition to justify our observations. Luckily, we spent endless hours playing with various objects in the elementary grades, developing this intuition just to understand the multiplication properties. A geometric representation of $n \cdot m$ is the number of objects arranged in a rectangle n blocks wide and m blocks long. A geometric representation of $(n \cdot m) \cdot k$ is the number of objects arranged in k rectangles each n blocks wide and m blocks long and stacked into a 3-D box. If we turn an n by m rectangle on its side, it turns into a rectangle that is m objects wide and n objects long. So we have our first algebraic property of multiplication.

Remark. If m and n are natural numbers, then $m \cdot n = n \cdot m$.

We refer to this by saying, "Multiplication of natural numbers is *commutative*."

If we pile k of these rectangles one on top of each other, we get a box n blocks wide, m blocks long, and k blocks high. The number of blocks in the box is $k \cdot (n \cdot m)$. But if we stack m walls of rectangles that are m blocks long and k blocks high, we get the same box. The number of blocks in the box is $m \cdot (n \cdot k)$. But by commutativity of multiplication, we can say

Remark. If k, m and n are natural numbers, then $(k \cdot n) \cdot m = k \cdot (n \cdot m)$.

We refer to this by saying, "Multiplication of natural numbers is *associative*."

The next observation follows directly from the definition of multiplication.

Remark. If n is a natural number, then $n \cdot 1 = 1 \cdot n = n$.

We refer to this by saying, "1 is a multiplicative identity."

If $n < m$, then m is an eventual successor of n, and we can write

$$\begin{aligned} m &= (((\ldots((n+1)+1)+\ldots+1)+1)+1 \\ &= (\ldots(.(\mathbf{1}+\mathbf{1})+\ldots\mathbf{1})+\ldots+1)+1)+1. \end{aligned} \tag{1.14}$$

So

$$\begin{aligned} m \cdot k &= (\ldots(.(\mathbf{k}+\mathbf{k})+\ldots\mathbf{k})+\ldots+k)+k)+k \\ &= (\ldots(kn+k)\ldots+k)+k)+k. \end{aligned} \tag{1.15}$$

So we know $k \cdot n < k \cdot m$. Thus,

Remark. If k, m, and n are natural numbers and $n < m$, then $n \cdot k < m \cdot k$.

We refer to this by saying, "Multiplication of natural numbers respects the order."

Notice that we have defined three things for the natural numbers: an order $<$, and two operations: addition $+$ and multiplication $\cdot$. We know how addition interacts with the order. Addition respects the order. We know how multiplication interacts with the order; multiplication respects the order. Next, we see how multiplication interacts with addition. We leave a geometric justification of this as an exercise.

Remark. If k, m, and n are natural numbers, then $k \cdot (n + m) = k \cdot n + k \cdot m$.

We refer to this by saying, "Multiplication of natural numbers distributes over addition."

If we were reluctant to talk about subtraction of natural numbers simply because to subtract n from m we must know $n < m$, we are definitely going to wait before we discuss division of natural numbers. Division of natural numbers is a much more complicated procedure involving remainders as well as quotients. We will get to it, but not just now.

Still we would like some division-like algebraic results to make things easier. We have two painfully obvious observations:

Remark. If k, n, and m are natural numbers with $n \cdot k = m \cdot k$, then $n = m$.

Remark. If k, n, and m are natural numbers with $n \cdot k < m \cdot k$, then $n < m$.

We refer to either of these as "cancellation properties of multiplication." Be warned, however, these are very dangerous. We are basically going to find safer replacements for them as soon as we can.

These are "painfully" obvious because while they are quite obvious after years of practicing arithmetic, the justifications that they are correct are rather painful to follow. There are a few ingredients in this justification: trichotomy, the results of multiplication are unique, multiplication respects order, and logical reasoning. Let us give a justification a try.

We know that the results of multiplication are unique; however we multiply two numbers m and k, the result will always be the same. Thus, we can state this algebraically as: if $n = m$, then for all natural numbers k, we have $n \cdot k = m \cdot k$. We really want to be clear about what this says.

$$\text{If it is true that } n = m, \text{ then it absolutely must be true that } n \cdot k = m \cdot k.$$

(We are just being resolute about our earlier statement.) But then, if we ever see that $n \cdot k = m \cdot k$ is false, then there is no way that $n = m$ could be true. This is to say:

$$\text{If } n \cdot k \neq m \cdot k, \text{ then } n \neq m.$$

Let us remember this for now.

Because multiplication respects order, we know that if k, m, and n are natural numbers and $n < m$, then $n \cdot k < m \cdot k$. So assuming that k, m, and n are natural numbers, if it is true that $n < m$, then it absolutely must be the case that $n \cdot k < m \cdot k$. So as before, if we ever see that $n \cdot k < m \cdot k$ is false, then there is no way that $n < m$ could be true. So

$$\text{If } n \cdot k < m \cdot k \text{ is not true, then } n < m \text{ is not true either.}$$

But by trichotomy, saying that $n \cdot k < m \cdot k$ is false is the same as saying $n \cdot k \geq m \cdot k$. By basically the same argument, we can also say:

$$\text{If } m \cdot k < n \cdot k \text{ is not true, then } m < n \text{ is not true either.}$$

Now we can justify our first statement that, if $n \cdot k = m \cdot k$, then $n = m$. Suppose it is true that $n \cdot k = m \cdot k$. Then by trichotomy, both $(n \cdot k < m \cdot k)$ and $(m \cdot k < n \cdot k)$ are not true. (Trichotomy says *exactly* one must be true.) By our two observations, we know ($n < m$) is not true, and ($m < n$) is not true. But trichotomy leaves only one possibility. It must be that $n = m$. Thus, as we said in our second remark: if k, n, and m are natural numbers with $n \cdot k < m \cdot k$, then $n < m$.

Next, we justify our second statement that, if $n \cdot k < m \cdot k$, then $n < m$. Suppose $n \cdot k < m \cdot k$. Then by trichotomy, both ($n \cdot k = m \cdot k$) and ($m \cdot k < n \cdot k$) are not true. By the first observation, we know that ($n \cdot k \neq m \cdot k$) implies $n \neq m$. The last observation says that ($m \cdot k < n \cdot k$) is not true implies that ($m < n$) is not true. But again, trichotomy leaves only one possibility. It must be that $n < m$.

It was a bit painful to follow these justifications of those simple remarks, but we do now see that they are simply consequences of trichotomy and a unique result from multiplication. One of our goals is to create an algebraic and logical language that makes arguments such as this easier to understand.

There is only one last remark we need to make about the natural numbers.

Remark. Let n and m be natural numbers with $n \leq m \leq n + 1$, then either $n = m$ or $m = n + 1$.

We refer to this by saying, "The natural numbers are discrete."

Again, the justification for this depends on the statements in the earlier remarks. Suppose $n < m < n + 1$. Then by subtraction (whoops), we know that there is a natural number k so that $m = n + k$. But then $n + k = m$ and $m < n + 1$. So by transitivity, $n + k < n + 1$. But we have a cancellation rule for addition; so $k < 1$. But since every natural number is an eventual successor of 1 and trichotomy holds, this cannot happen.

The purpose of algebra is to help make all these justifications easier to manage.

1.1.4 Natural numbers: basic properties and theorems

We have just reviewed several years of elementary school arithmetic so that we can identify and name various basic algebraic properties of the natural numbers. They are as follows:

- There is a first natural number, which we call 1.
- There is an order on the natural numbers.
- The order is transitive.
- The order has trichotomy.
- For any two natural numbers n and m, there is a unique natural number $n + m$.
- This addition is associative.
- This addition is commutative.
- If m and n are natural numbers, then $n < n + m$.
- If k, m, and n are natural numbers and $n < m$, then $n + k < m + k$.

- If n and m are natural numbers with $n < m$, then there exists a unique natural number k so that $m = n + k$.
- If k, n, and m are natural numbers with $n + k = m + k$, then $n = m$.
- If k, n, and m are natural numbers with $n + k < m + k$, then $n < m$.
- For any two natural numbers n and m, there is a unique natural number $n \cdot m$.
- This multiplication is associative.
- This multiplication is commutative.
- The natural number 1 is a multiplicative identity.
- If k, m, and n are natural numbers and $n < m$, then $n \cdot k < m \cdot k$.
- If k, n, and m are natural numbers with $n \cdot k = m \cdot k$, then $n = m$.
- If k, n, and m are natural numbers with $n \cdot k < m \cdot k$, then $n < m$.
- If m and n are natural numbers and $n \leq m \leq n + 1$, then either $m = n$ or $m = n + 1$.
- Multiplication distributes over addition.

1.2 First steps in proof

There are, of course, many more true facts about the natural numbers, but they all should follow from these basic properties. We will state many further facts about these numbers as theorems. We will prove these theorems by using the aforementioned basic properties. If our justifications for these properties are accepted and are correct, then the theorems we prove by using them must be perfectly true. Granted our justifications of these properties are a bit dicey, but we are going to have to start being rigorous somewhere, and it will be easier starting by assuming a list of basic properties such as those aforementioned.

Let us now use these properties to *prove* something.

1.2.1 A direct proof

The first proof we will give is called a *direct proof*. Suppose that we wish to prove a statement of the form "If P, then Q." In a direct proof of this statement, we begin by assuming P. Then we *deduce* Q using P and any other assumptions we have available. Let us now prove the statement

If n is a natural number, then $(n + 1)^2 = n^2 + 2n + 1$

using a direct proof. This is of the form "If P, then Q" where P is the statement "n is a natural number" and Q is the statement "$(n + 1)^2 = n^2 + 2n + 1$." We will begin the proof by assuming that n is a natural number. Knowing that, we can use all of the basic properties of the natural numbers listed earlier. So we will use those assumptions to deduce that $(n + 1)^2 = n^2 + 2n + 1$.

Theorem 1.2.1. *If n is a natural number, then $(n + 1)^2 = n^2 + 2n + 1$.*

Proof. Assume that n is a natural number. Then $n+1$ is a natural number because addition is always defined. Then

$$(n+1)^2 = (n+1)(n+1), \tag{1.16}$$

because that is what the exponent means.

$$(n+1)(n+1) = (n+1)n + (n+1)\cdot 1, \tag{1.17}$$

by the distributive property.

$$(n+1)n + (n+1)\cdot 1 = (n+1)n + (n+1), \tag{1.18}$$

because 1 is a $\cdot$ identity.

$$(n+1)n + (n+1) = n(n+1) + (n+1), \tag{1.19}$$

because $\cdot$ is commutative.

$$n(n+1) + (n+1) = (n\cdot n + n\cdot 1) + (n+1), \tag{1.20}$$

by the distributive property.

$$(n\cdot n + n\cdot 1) + (n+1) = (n\cdot n + n) + (n+1), \tag{1.21}$$

because 1 is a $\cdot$ identity.

$$(n\cdot n + n) + (n+1) = (n^2 + n) + (n+1), \tag{1.22}$$

because that is what the exponent means.

$$(n^2 + n) + (n+1) = n^2 + (n + (n+1)), \tag{1.23}$$

because $+$ is associative.

$$n^2 + (n + (n+1)) = n^2 + ((n+n) + 1), \tag{1.24}$$

because $+$ is associative.

$$n^2 + ((n+n)+1) = n^2 + ((n\cdot 1 + n\cdot 1) + 1), \tag{1.25}$$

because 1 is a $\cdot$ identity.

$$n^2 + ((n\cdot 1 + n\cdot 1) + 1) = n^2 + (n(1+1) + 1), \tag{1.26}$$

by the distributive property.

$$n^2 + (n(1+1) + 1) = n^2 + (n\cdot 2 + 1), \tag{1.27}$$

because that is what 2 means.

$$n^2 + (n\cdot 2 + 1) = n^2 + (2n + 1), \tag{1.28}$$

because $\cdot$ is commutative.

$$n^2 + (2n+1) = n^2 + 2n + 1, \tag{1.29}$$

because $+$ is associative, this is unambiguous.

Thus, we have

$$(n+1)^2 = n^2 + 2n + 1. \tag{1.30}$$

□

This is a completely algebraic proof; it is also a completely boring proof to anyone who knows algebra. This is the stuff of middle school algebra and is not the kind of proof that should give us any problems. While we should be able to justify any step in any algebraic part of any proof we give, there is rarely a reason to do so. In addition, we can take advantage of algebra's disregard for the rules of proper language composition. Notice that each step in the aforementioned proof is a full English sentence with a subject, a verb (always "equals"), and an object followed by a prepositional phrase. This is how a paragraph should be in any English composition.

But in an algebraic proof, we can violate one the major rules of good writing: no run-on sentences. The aforementioned proof is completely over the top for mathematical adults. In any work past a high school text, it would be written more like:

$$\begin{aligned}(n+1)^2 &= (n+1)(n+1) \\ &= (n+1)n + (n+1) \\ &= n^2 + n + n + 1 \\ &= n^2 + 2n + 1.\end{aligned} \tag{1.31}$$

Even this might be longer that necessary. Notice that this is a run-on English sentence. It has one subject, $(n+1)^2$, several objects, and one word "equals" used as a verb four times. This is unacceptable in an English composition, but perfectly acceptable in an algebraic proof. We need to remember that this proof is an abbreviation of the full proof written earlier as a composition. Each equal sign has two subjects: the object of the previous line, and by deduction, the original subject of the sentence. The conclusion drawn from the four intermediate sentences is that the original subject is equal to the final object.

In this study, we will not bother to do much more than outline an algebraic proof such as this. This does not, however, reduce at all our need for detailed algebraic proofs. As humans we will make algebra mistakes, and we need to be ready to find them before someone else does. Finding an algebraic mistake is often nothing more than giving a complete and thorough line-by-line step through the use of our basic properties until the error reveals itself.

1.2.2 Mathematical induction

Unfortunately, not all theorems about the natural numbers are easily proved by a direct proof or simple algebra. Consider

For all natural numbers n, $2 \cdot (1 + 2 + 3 + \dots (n-1) + n) = n(n+1)$.

The dreaded rigor killer, ellipsis, appears again. Mathematics has notation that allows us to write such a summation in a more precise mathematical way. However, in this case, it is pretty clear what this claim is: if we add all the numbers starting at 1 and stop when we get to n and then double the result, the answer would be the same as if we multiplied n by its successor. Unfortunately, the only direct proof of this involves using geometric intuition. This is a perfectly fine proof, but there is an alternate proof that uses a much more general method with many more applications.

We will prove this claim using a "proof by mathematical induction." Such a proof is a two-step process. Both steps must be completed successfully for the proof to be valid. The first step is to prove that the result is true for the first natural number. The second step takes advantage of a logical loophole. To prove a statement of the form "If something, then something else," one may assume that something is true. Once something is assumed true for a valid logical reason, we can use that assumption to draw additional conclusions. The second step in induction is to prove the following: "If the statement is true for a particular natural number, then it will be true for its successor."

If we can accomplish both these steps, we will know

- that the statement is true for 1;
- that anytime the statement is true for a particular number, it will be true for its successor.

So we know that the statement is true for 1, and 1 is certainly a particular number. Since the statement is true for 1, it is true for the successor of 1. But 2 is a particular number, and the statement is true for it; so because we have proved the second step of induction, the statement is true for the successor of 2. Because every natural number is an eventual successor of 1, we will eventually know that the statement is true for any number.

Here is the claim written as a theorem, and this is followed by its (mostly rigorous) proof. Notice that, as we write out exactly what we are proving, our statement about n reappears three times. It may look like we are proving or assuming the same thing over and over. But a more careful look reveals that in each statement, the meaning of the variable n changes. Thus, the statements are actually about different numbers.

Theorem 1.2.2. *For all natural numbers n,* $2 \cdot (1 + 2 + 3 + \ldots (n-1) + n) = n(n+1)$.

Proof. The proof is by induction on n. Thus, we will actually prove two other mini theorems:

1. If $n = 1$, then $2 \cdot (1 + 2 + 3 + \ldots (n-1) + n) = n(n+1)$.

2. If for a particular $n = n_0$,

$$2 \cdot (1 + 2 + 3 + \dots (n - 1) + n) = (n + 1)n, \tag{1.32}$$

then for $n = n_0 + 1$,

$$2 \cdot (1 + 2 + 3 + \dots (n - 1) + n) = n(n + 1). \tag{1.33}$$

Proof of Step 1. Assume that $n = 1$. To prove that two expressions are the same, consider them one at a time. First, $(1 + 2 + 3 + \dots (n - 1) + n)$ means start at 1 and stop when you get to n. But we are working under the assumption that $n = 1$. So

$$(1 + 2 + 3 + \dots (n - 1) + n) = 1. \tag{1.34}$$

So

$$2 \cdot (1 + 2 + 3 + \dots (n - 1) + n) = 2 \cdot 1 = 2. \tag{1.35}$$

Now consider the other expression, $n(n + 1)$. We are still assuming $n = 1$.

$$(n + 1)n = (1 + 1) \cdot 1 = 2. \tag{1.36}$$

Since $2 = 2$, we have shown that if $n = 1$, then $2 \cdot (1 + \dots (n - 1) + n) = n(n + 1)$. ■

Proof of Step 2. Assume for a particular $n = n_0$, $2 \cdot (1 + \dots (n - 1) + n) = n(n + 1)$. Thus, we can say

$$2 \cdot (1 + \dots (n_0 - 1) + n_0) = n_0.(n_0 + 1). \tag{1.37}$$

Under this assumption, we want to prove, for $n = n_0 + 1$, that we also have $2 \cdot (1 + 2 + 3 + \dots (n - 1) + n) = n(n + 1)$. That is to say, we want to show

$$2 \cdot (1 + 2 + \dots ((n_0 + 1) - 1) + (n_0 + 1)) = (n_0 + 1)((n_0 + 1) + 1). \tag{1.38}$$

To prove that two expressions are equal, we consider each side. Consider $2 \cdot (1 + \dots ((n_0 + 1) - 1) + (n_0 + 1))$. We have

$$\begin{aligned} &2 \cdot (1 + \dots ((n_0 + 1) - 1) + (n_0 + 1)) \\ &= 2 \cdot [(1 + 2 + 3 + \dots n_0) + (n_0 + 1)] \\ &= 2 \cdot [1 + 2 + 3 + \dots n_0] + 2[n_0 + 1] \\ &= n_0(n_0 + 1) + 2(n_0 + 1) \end{aligned} \tag{1.39}$$

because that is the assumption we are working under in this step. Then

$$\begin{aligned} &2 \cdot (1 + \dots ((n_0 + 1) - 1) + (n_0 + 1)) \\ &= n_0(n_0 + 1) + 2(n_0 + 1) \\ &= (n_0 + 2)(n_0 + 1). \end{aligned} \tag{1.40}$$

Next, consider the other side, $(n_0 + 1)((n_0 + 1) + 1)$.

$$(n_0 + 1)((n_0 + 1) + 1) = (n_0 + 1)(n_0 + 2). \tag{1.41}$$

The two expressions are equal. So we have proved: if for a particular $n = n_0$, we have $2 \cdot (1 + 2 + 3 + \dots (n - 1) + n) = (n + 1)n$, then for $n = n_0 + 1$, we have $2 \cdot (1 + 2 + 3 + \dots (n - 1) + n) = n(n + 1)$. ■

These two steps complete the proof by induction. So we have proved: for all natural numbers n, $2 \cdot (1 + 2 + 3 + \dots (n - 1) + n) = n(n + 1)$. □

There are a few final comments on this write-up. Much of the exposition is a matter of taste, but no matter what, the proof must be an English essay. It may contain some headings, but everything in the content should be a full sentence. This includes the algebraic calculations. The logic is easier if all statements to be proved are written in the "If P, then Q" form. The proof of one of these statements should begin with "Assume P." After that assumption, the goal becomes to prove Q. The use of n_0 to stand for a particular value of n in the induction step is completely optional. With more experience in writing induction proofs, it becomes a distraction. However, even with experience, the second step of an induction step can get rather confusing when the statement being proved is long. Using the n_0 can be a valuable tool in fighting through that kind of confusion. For beginners, it is not a bad idea to take the time to use that extra notation so that it will always be available when needed.

1.3 Problems

1.1 (a) Use $n = 2$, $m = 3$, and $k = 4$ to provide an example of the distributive property $n(m + k) = nm + nk$ using either ellipsis arguments or a geometric construction.

(b) Provide a justification of the general distributive property $n(m + k) = nm + nk$ using either ellipsis arguments or a geometric construction.

1.2 Provide justifications for the cancellation properties of addition. (Hint: look at the justifications for multiplication.)

1.3 Prove that for all natural numbers n, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

1.4 Be careful while reading these formulas.

(a) Prove that for all natural numbers n, $\sum_{k=1}^{n} (2k - 1) = n^2$.

(b) Prove that for all natural numbers n, $\sum_{k=1}^{n} 2k - 1 = n^2 + n - 1$.

1.5 Prove that for all natural numbers n, $n^2 \geq n$.

1.6 Prove that for all natural numbers $n \geq 2$, $n^2 \geq n + 2$. (Hint: when trying to prove an inequality $a \leq b$, it can help to write the objective as $a \leq ? \leq b$. Then the idea is to find a value we can use in place of the question mark. If we can prove the two inequalities $a \leq ?$ and $? \leq b$, the result we want follows from transitivity. If we are lucky, one of these two inequalities is already known to be true.)

1.7 Prove that for all natural numbers n, $\prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) = n + 1$. (Hint: the symbol $\prod$ is similar to the symbol $\sum$ except it means multiply instead of add.)

1.8 Let n be any natural number greater than or equal to 7.

(a) Prove that if there is a natural number q so that $n = 7 \cdot q$, then $n + 1 = 7 \cdot q + 1$.
(b) Prove that if there are natural numbers q and r so that $n = 7 \cdot q + r$ and $r < 6$, then there is a natural number r' so that $n + 1 = 7 \cdot k + r'$ with $r' < 7$.
(c) Prove that if there are natural numbers q and r so that $n = 7 \cdot q + r$ and $r = 6$, then there is a natural number q' so that $n + 1 = 7 \cdot q'$.
(d) Prove the following statement using induction.

> For all natural numbers $n \geq 7$, either there exists a natural number q so that $n = 7q$ or there exists a pair of natural numbers q and r so that $n = 7q + r$ with $r < 7$.