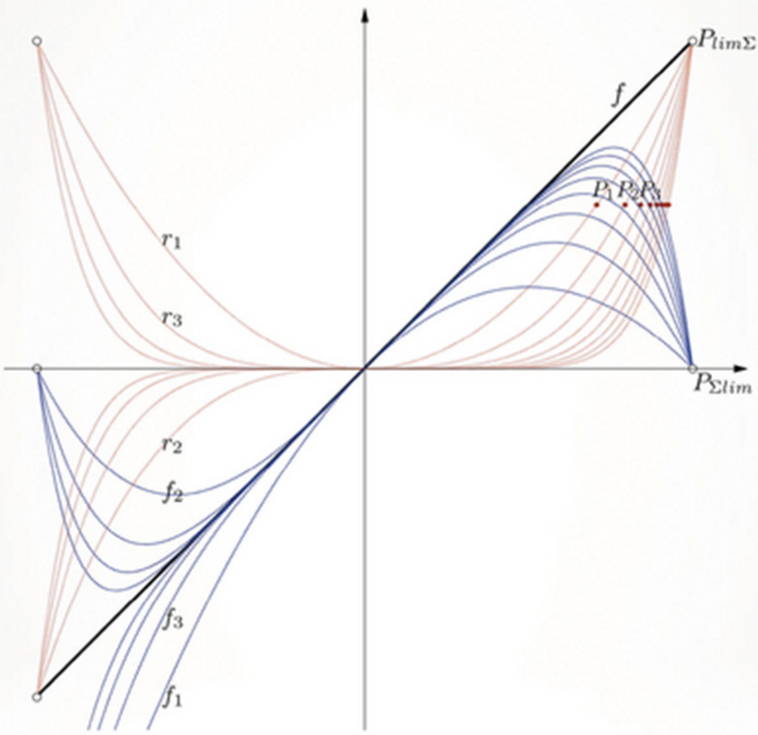
COUNTEREXAMPLES ON UNIFORM CONVERGENCE

SEQUENCES, SERIES, FUNCTIONS, AND INTEGRALS



ANDREI BOURCHTEIN AND LUDMILA BOURCHTEIN

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Counterexamples on Uniform Convergence

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Sequences, Series, Functions, and Integrals

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To Haim and Maria with the warmest memories;
To Maxim with love and inspiration;
To Valentina for always yummy breakfast;
To Victoria for amazing rainbow flower

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Preface

Looking for counterexamples is one of important things a mathematician does. Once a conjecture is proposed, if no proof can be found, the next step is to look for a counterexample. If no counterexample can be found, the next step is to try to find a proof again, and so on. On a more routine level, counterexamples play an important role for students as they learn new mathematical concepts. To best understand a theorem, it can be useful to see why each of the hypotheses of the theorem is necessary by finding counterexample when the hypotheses fails.

In this book, we present counterexamples related to different concepts and results on the uniform convergence usually studied in advanced calculus and real analysis courses. It includes the convergence of sequences, series and families of functions, and also proper and improper integrals depending on a parameter. The corresponding false statements are not formulated explicitly, but instead are invoked implicitly by the form of counterexamples.

The text is divided into six parts: the introductory chapter and five chapters of counterexamples. The first part contains some introductory material such as comments on notations, presentation form, and background theory. Chapter 1 considers conditions of uniform convergence. Chapter 2 deals with such properties of the limit functions as boundedness, existence of the limit and continuity. Chapter 3 analyzes the conditions of differentiability and integrability of the limit functions. Chapters 4 and 5 consider the properties of integrals (proper and improper) depending on a parameter.

The goal of the book is threefold. First, it provides a brief survey and discussion of principal results of the theory of uniform convergence in real analysis. Second, it supplies a material for a deeper study of the concepts and theorems on uniform convergence using counterexamples as a main technique. Finally, the text shows to the reader how such important mathematical tool as counterexamples can be used in different situations. We restricted our exposition to the main definitions and theorems in order to explore different versions (wrong and correct) of the fundamental concepts. Hence, many interesting (but more specific and applied) problems not related directly to the main notions and results are left out of the scope of this manuscript.

The selection and exposition of the material are directed, in the first place, to those advanced calculus and analysis students who are interested in a deeper understanding and broader knowledge of the topics of uniform convergence. We think the presented material may also be used by instructors that wish to go through the examples (or their variations) in class or assign them as homework or extracurricular projects. To this end, the main text is accompanied by the Instructor's Solutions Manual containing the detailed solutions to all the exercises proposed at the end of each chapter.

It is assumed that a reader has knowledge of a traditional university course of calculus. In order to make the majority of the examples and solutions accessible to calculus and analysis students, we tried to keep the level of reasoning as simple as possible. As in the majority of the mathematics books, the logical sequence of the material just follows the chapter sequence, that is, the content of the next chapter may depend on the previous text, but not vice-versa.

The book is not appropriate as the main textbook for a course, but rather, it can be used as a supplement that can help students to master important concepts and theorems. So we think the best way to use the book is to read its parts while taking a respective calculus/analysis course. On the other hand, the students already familiarized with the subjects of university calculus can find here deeper interpretation of the results and finer relation between concepts than in standard presentations. Also, more experienced students will better understand provided examples and ideas behind their construction.

To facilitate the reading of the main text (containing counterexamples) and make the text self-contained, and also to fix terminology, notation, and concepts, we gather the relevant definitions and results in the introductory chapter. For many examples, we make explicit references to the concepts/theorems to which they are related.

A short (but representative) list of bibliography can be found at the end of the book, including both collections of problems and textbooks in calculus/analysis. On the one hand, these references are the sources of some examples collected here, although it was out of our scope to trace all the original sources. On the other hand, they may be used for finding further information (examples and theory) on various topics. Some of these references are classic collections of the problems, such as that by Demidovich [6] and by Gelbaum and Olmsted [8]. Our preparation of the text was inspired, in the first place, by the latter book. We tried to extend its approach to the specialized topics of the uniform convergence, which frequently are sources of misunderstanding and confusion for fresh mathematics students. We hope that both students and professionals will find our book useful and (at least partly) challenging.

List of Examples

Chapter 1. Conditions of Uniform Convergence

Example 1. A function $f(x, y)$ defined on $X \times Y$ converges pointwise on X as y approaches y_0, but this convergence is nonuniform on X. 1

A sequence of functions converges (pointwise) on a set, but this convergence is nonuniform. 1

A series of functions converges (pointwise) on a set, but this convergence is nonuniform. 2

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Example 3. A sequence of functions converges on X and there exists its subsequence that converges uniformly on X, but the original sequence does not converge uniformly on X. 5

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A series of functions converges uniformly on any interval $[c, d] \subset (a, b)$, but the convergence is nonuniform on (a, b). 6

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A series converges on X, but it does not converge uniformly on any subinterval of X. 9

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Suppose functions $f_n(x)$ map X on Y and function $g(y)$ is continuous on Y; the sequence $f_n(x)$ converges nonuniformly on X, but the sequence $g_n(x) = g(f_n(x))$ converges uniformly on X 23

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Both series $\sum u_n(x)$ and $\sum v_n(x)$ converge uniformly on X, but the series $\sum u_n(x)v_n(x)$ does not converge (even pointwise) on X. 30

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Chapter 2. Properties of the Limit Function: Boundedness, Limits, Continuity

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A sequence of unbounded and continuous on X functions converges on X to an unbounded and continuous function, but the convergence is nonuniform on X. 49

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Example 7. A series $\sum u_n(x)$ converges on X and $\lim\limits_{x\to x_0} u_n(x)$ exists for each n, but $\lim\limits_{x\to x_0} \sum u_n(x) \neq \sum \lim\limits_{x\to x_0} u_n(x)$. 51

A sequence $f_n(x)$ converges on X and $\lim\limits_{x\to x_0} f_n(x)$ exists for each n, but $\lim\limits_{x\to x_0} \lim\limits_{n\to\infty} f_n(x) \neq \lim\limits_{n\to\infty} \lim\limits_{x\to x_0} f_n(x)$. 52

A function $f(x,y)$ is defined on $X \times Y$, converges on X to a limit function $\varphi(x)$, as $y \to y_0$, and $\lim\limits_{x\to x_0} f(x,y)$ exists for each $y \in Y$, but $\lim\limits_{x\to x_0} \lim\limits_{y\to y_0} f(x,y) \neq \lim\limits_{y\to y_0} \lim\limits_{x\to x_0} f(x,y)$. 52

Example 8. A series $\sum u_n(x)$ converges on X and $\lim\limits_{x\to x_0} u_n(x)$ exists for each n, but $\lim\limits_{x\to x_0} \sum u_n(x) \neq \sum \lim\limits_{x\to x_0} u_n(x)$ since the left-hand side limit is infinite. 53

A sequence $f_n(x)$ converges on X and $\lim\limits_{x\to x_0} f_n(x)$ exists for each n, but $\lim\limits_{x\to x_0} \lim\limits_{n\to\infty} f_n(x) \neq \lim\limits_{n\to\infty} \lim\limits_{x\to x_0} f_n(x)$ since the left-hand side limit is infinite. 54

A function $f(x,y)$ is defined on $X \times Y$, converges on X to a limit function $\varphi(x)$, as $y \to y_0$, and $\lim\limits_{x\to x_0} f(x,y)$ exists for each $y \in Y$, but $\lim\limits_{x\to x_0} \lim\limits_{y\to y_0} f(x,y) \neq \lim\limits_{y\to y_0} \lim\limits_{x\to x_0} f(x,y)$ since the left-hand side limit is infinite. 55

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converges or $\lim\limits_{x\to x_0} \sum u_n(x)$ exists, but nevertheless the remaining condition does not hold. 55

Suppose $f_n(x)$ converges on X and $\lim\limits_{x\to x_0} f_n(x)$ exists for each n. Even if one of the limits—$\lim\limits_{n\to\infty}\lim\limits_{x\to x_0} f_n(x)$ or $\lim\limits_{x\to x_0}\lim\limits_{n\to\infty} f_n(x)$—exists, another one may not exist. 56

Assume a function $f(x,y)$ is defined on $X \times Y$, converges on X to a limit function $\varphi(x)$, as $y \to y_0$, and $\lim\limits_{x\to x_0} f(x,y)$ exists for each $y \in Y$. Even though one of the two iterated limits—$\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y)$ or $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(x,y)$—exists, another one may not exist. 57

Example 10. A series converges nonuniformly on X, but still a limit of this series can be calculated term by term. 58

A sequence $f_n(x)$ converges nonuniformly on X, but still $\lim\limits_{x\to x_0}\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty}\lim\limits_{x\to x_0} f_n(x)$. 59

A function $f(x,y)$ defined on $X \times Y$ converges nonuniformly on X to a limit function $\varphi(x)$, as $y \to y_0$, but $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(x,y) = \lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y)$. 59

Example 11. A sequence of discontinuous functions converges uniformly on X, but the limit function is continuous on X. 60

All the terms of a uniformly convergent on X series are discontinuous at some point $x_0 \in X$, but the series sum is continuous at x_0. 60

A function $f(x,y)$ defined on $X \times Y$ has a discontinuity at $x_0 \in X$ for any $y \in Y$ and converges uniformly on X to a limit function $\varphi(x)$, as $y \to y_0$, but $\varphi(x)$ is continuous at x_0. 61

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A series of continuous functions converges on X, but the sum of this series is a discontinuous function. 65

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A function $f(x,y)$ defined on $X \times Y$ is differentiable in $x \in X$ at any fixed $y \in Y$, and $f(x,y)$ converges uniformly on X to a function $\varphi(x)$ as y approaches y_0, but $\varphi'(x) \neq \lim\limits_{y\to y_0} f_x(x,y)$, $\forall x \in X$. ... 100

A series of differentiable functions converges uniformly on X, but this series cannot be differentiated term by term on X. ... 100

Example 3. A sequence $f_n(x)$ of differentiable on X functions converges on X to a function $f(x)$ and $f'(x) = \lim\limits_{n\to\infty} f_n'(x)$, $\forall x \in X$, however, the convergence of $f_n'(x)$ to $f'(x)$ is nonuniform on X. ... 101

A function $f(x,y)$ defined on $X \times Y$ is differentiable in $x \in X$ at any fixed $y \in Y$, $f(x,y)$ converges on X to a function $\varphi(x)$, as y approaches y_0, and $\varphi'(x) = \lim\limits_{y\to y_0} f_x(x,y)$, $\forall x \in X$, however, the convergence of $f_x(x,y)$ is not uniform. ... 103

A series $\sum u_n(x)$ of differentiable functions converges on X and can be differentiated term by term on X, however, the series $\sum u_n'(x)$ converges nonuniformly on X. 104

Example 4. A sequence $f_n(x)$ of differentiable functions converges nonuniformly on X to a function $f(x)$, but nevertheless $f'(x) = \lim_{n\to\infty} f_n'(x)$ on X. 106

A function $f(x, y)$ defined on $X \times Y$ is differentiable in $x \in X$ at any fixed parameter $y \in Y$, $f(x, y)$ converges nonuniformly on X to a function $\varphi(x)$ as y approaches y_0, but nevertheless $\varphi'(x) = \lim_{y\to y_0} f_x(x, y)$ on X. 107

A series of differentiable functions converges nonuniformly on X, but nevertheless this series can be differentiated term by term on X. 107

A series of differentiable functions converges nonuniformly on X and both the partial sums and the sum of the series cannot be expressed through elementary functions, but nevertheless this series can be differentiated term by term on X. 109

Example 5. A sequence $f_n(x)$ of infinitely differentiable functions converges uniformly on X, but the sequence $f_n'(x)$ diverges at each point of X. 111

Example 6. A sequence $f_n(x)$ of infinitely differentiable functions converges uniformly on X, but the sequence of the derivatives $f_n'(x)$ is convergent and divergent at infinitely many points of X. 111

Example 7. A series of continuous functions converges uniformly on X, but the sum of this series is not differentiable at an infinite number of points in X. (A function is continuous on an interval, but it can be nondifferentiable at infinitely many points of this interval.) 113

Example 8. A series $\sum u_n(x)$ of continuous functions converges uniformly on X, but nevertheless the sum of this series is nondifferentiable at any point of X. (There exist continuous on an interval functions $f(x)$ that are nondifferentiable at any point of this interval.) 114

Example 9. A sequence of Riemann integrable on $[a, b]$ functions converges on $[a, b]$, but the limit function is not Riemann integrable on $[a, b]$. 117

Example 10. A sequence $f_n(x)$ converges on $[a, b]$ to $f(x)$, and each of the functions $f_n(x)$ is not Riemann integrable on $[a, b]$, but $f(x)$ is Riemann integrable on $[a, b]$. 118

Example 11. A sequence of continuous functions $f_n(x)$ converges on $[a, b]$ to $f(x)$, but $\lim_{n\to\infty} \int_a^b f_n(x)dx \neq \int_a^b f(x)dx$. 118

A function $f(x,y)$ defined on $X \times Y = [a,b] \times Y$ is continuous in $x \in X$ at any fixed $y \in Y$, and there exists $\lim\limits_{y \to y_0} f(x,y) = \varphi(x)$, but $\int_a^b \varphi(x)dx \neq \lim\limits_{y \to y_0} \int_a^b f(x,y)dx$. 119

A series of continuous functions converges on $[a,b]$, but it cannot be integrated term by term on $[a,b]$. 120

Example 12. A sequence of continuous functions $f_n(x)$ converges on $[a,b]$ to $f(x)$ and $\lim\limits_{n \to \infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$, but the convergence is nonuniform on $[a,b]$. 123

A function $f(x,y)$ defined on $X \times Y = [a,b] \times Y$ is continuous in $x \in X$ at any fixed $y \in Y$, there exists $\lim\limits_{y \to y_0} f(x,y) = \varphi(x)$, and $\int_a^b \varphi(x)dx = \lim\limits_{y \to y_0} \int_a^b f(x,y)dx$, but $f(x,y)$ converges nonuniformly on X to $\varphi(x)$. 123

A series of continuous functions converges on $[a,b]$ and it can be integrated term by term, but this convergence is nonuniform on $[a,b]$. 124

Example 13. The elements of a convergent on X sequence and its limit function have infinitely many discontinuity points on X, but the formula of term-by-term integration holds. 127

The terms of a convergent on X series and its sum have infinitely many discontinuity points on X, but the formula of term-by-term integration holds. 127

Chapter 4. Integrals Depending on a Parameter

Example 1. A function $f(x,y)$ defined on $[a,b] \times Y$ is continuous in $x \in [a,b]$ at any fixed $y \in Y$, and there exists $\lim\limits_{y \to y_0} f(x,y) = \varphi(x)$, but $\varphi(x)$ is not integrable on $[a,b]$ or $\int_a^b \varphi(x)dx \neq \lim\limits_{y \to y_0} \int_a^b f(x,y)dx$. 133

Example 2. A function $f(x,y)$ defined on $[a,b] \times Y$ is continuous in x for any fixed y, and the function $\varphi(x) = \lim\limits_{y \to y_0} f(x,y)$ is continuous on $[a,b]$, but $\lim\limits_{y \to y_0} \int_a^b f(x,y)dx \neq \int_a^b \lim\limits_{y \to y_0} f(x,y)dx$. 134

Example 3. A function $f(x,y)$ defined on $[a,b] \times Y$ is continuous in $x \in [a,b]$ at any fixed $y \in Y$, there exists $\lim\limits_{y \to y_0} f(x,y) = \varphi(x)$ and $\int_a^b \varphi(x)dx = \lim\limits_{y \to y_0} \int_a^b f(x,y)dx$, but $f(x,y)$ converges nonuniformly on $[a,b]$. 135

Example 4. A function $f(x, y)$ is continuous on $\mathbb{R}^2$ except at only one point, but $F(y) = \int_a^b f(x, y)dx$ is not continuous. 137

Example 5. A function $f(x, y)$ is continuous in y for any fixed x and also in x for any fixed y, but $F(y) = \int_a^b f(x, y)dx$ is not continuous. 137

Example 6. A function $f(x, y)$ is discontinuous at infinitely many points of $[a, b] \times Y$, but $F(y) = \int_a^b f(x, y)dx$ is continuous on Y. 139

A function $f(x, y)$ is discontinuous and unbounded at infinitely many points of $[a, b] \times Y$, but still $F(y) = \int_a^b f(x, y)dx$ is continuous on Y. 140

A function $f(x, y)$ is discontinuous at each point of $[a, b] \times Y$, but nevertheless $F(y) = \int_a^b f(x, y)dx$ is continuous on Y. 143

Example 7. Both $f(x, y)$ and $f_y(x, y)$ are continuous in y for any fixed x and also in x for any fixed y, but $\frac{d}{dy} \int_a^b f(x, y)dx \neq \int_a^b f_y(x, y)dx$. 144

Example 8. Both $f(x, y)$ and $f_y(x, y)$ are continuous in $\mathbb{R}^2$ except at one point, but $\frac{d}{dy} \int_a^b f(x, y)dx \neq \int_a^b f_y(x, y)dx$. 146

Example 9. A function $f(x, y)$ is continuous on $\mathbb{R}^2$ and $f_y(x, 0)$ exists for $\forall x \in \mathbb{R}$, but $\frac{d}{dy} \int_a^b f(x, y)dx|_{y=0} \neq \int_a^b f_y(x, 0)dx$. 147

Example 10. A function $f(x, y)$ has infinitely many points of discontinuity on $[a, b] \times Y$, but $\frac{d}{dy} \int_a^b f(x, y)dx = \int_a^b f_y(x, y)dx, \forall y \in Y$. 151

Example 11. Although $f_y(x, y)$ is discontinuous at (x_0, y_0), but $\frac{d}{dy} \int_a^b f(x, y)dx|_{y=y_0} = \int_a^b f_y(x, y_0)dx$. 151

Both the function and its partial derivative are discontinuous at some points, but still the integration and partial differentiation can be interchanged. .. 153

Example 12. A function $f(x, y)$, defined on $[a, b] \times Y$, converges on $[a, b]$ to a function $\varphi(x)$ as y approaches y_0, and $f(x, y)$ is not Riemann integrable on $[a, b]$ for each fixed $y \in Y$, but $\varphi(x)$ is Riemann integrable on $[a, b]$. 154

Example 13. A function $f(x, y)$ is defined on $[a, b] \times [c, d]$ and one of the iterated integrals—$\int_c^d dy \int_a^b f(x, y)dx$ or $\int_a^b dx \int_c^d f(x, y)dy$—exists, but another does not exist. 154

A function $f(x, y)$ is discontinuous at only one point of $[a, b] \times [c, d]$, but both iterated integrals—$\int_c^d dy \int_a^b f(x, y)dx$ and $\int_a^b dx \int_c^d f(x, y)dy$—do not exist. 156

Example 14. A function $f(x, y)$ is defined on $[a, b] \times [c, d]$ and both iterated integrals $\int_c^d dy \int_a^b f(x, y)dx$ and $\int_a^b dx \int_c^d f(x, y)dy$ exist, but they assume different values. 156

Example 15. A function $f(x, y)$ is defined on $[a, b] \times [c, d]$ and $\int_c^d dy \int_a^b f(x, y)dx = \int_a^b dx \int_c^d f(x, y)dy$, but $f(x, y)$ is discontinuous on $[a, b] \times [c, d]$. 158

A function $f(x, y)$ is defined on $[a, b] \times [c, d]$ and both iterated integrals exist and are equal, but nevertheless the double integral does not exist. 159

Example 16. A function $f(x, y)$ has infinitely many discontinuity points in $[a, b] \times [c, d]$, but $\int_c^d dy \int_a^b f(x, y)dx = \int_a^b dx \int_c^d f(x, y)dy$. 160

A function $f(x, y)$ is discontinuous and unbounded at infinitely many points in $[a, b] \times [c, d]$, but nevertheless $\int_c^d dy \int_a^b f(x, y)dx = \int_a^b dx \int_c^d f(x, y)dy$. 161

Chapter 5. Improper Integrals Depending on a Parameter

Example 1. An integral $\int_a^{+\infty} f(x, y)dx$ converges on a set Y, but it does not converge uniformly on this set. 167

Example 2. An integral $\int_a^{+\infty} f(x, y)dx$ converges on a (finite or infinite) interval (c, d) and converges uniformly on any interval $[c_1, d_1] \subset (c, d)$, but it does not converge uniformly on (c, d). 167

Example 3. An integral $\int_a^{+\infty} f(x, y)dx$ converges on a (finite or infinite) interval (c, d), but it does not converge uniformly on an interval $[c_1, d_1] \subset (c, d)$. 167

Example 4. An integral $\int_a^{+\infty} f(x, y)dx$ converges absolutely on an interval (c, d), but it does not converge uniformly on this interval. 170

Example 5. An integral $\int_a^{+\infty} f(x, y)dx$ converges uniformly on an interval Y, but it does not converge absolutely on this interval. 171

Example 6. An integral $\int_a^{+\infty} f(x, y)dx$ converges uniformly and absolutely on an interval Y, but $\int_a^{+\infty} |f(x, y)|dx$ does not converge uniformly on this interval. 172

Example 7. An integral $\int_a^{+\infty} f(x, y)dx$ converges uniformly and absolutely on Y, but there is no bound of $f(x, y)$ such that $|f(x, y)| \le \varphi(x)$, $\forall y \in Y$, where the improper integral $\int_a^{+\infty} \varphi(x)dx$ converges. 173

Example 8. Suppose functions $f(x,y)$ and $g(x,y)$ are positive and continuous on $[a,+\infty)\times Y$, and $\lim\limits_{x\to+\infty}\frac{f(x,y)}{g(x,y)}=1$, $\forall y\in Y$. One of the integrals $\int_a^{+\infty}f(x,y)dx$ or $\int_a^{+\infty}g(x,y)dx$ converges uniformly on Y, but another converges nonuniformly. 174

Example 9. Improper integrals $\int_a^{+\infty}f(x,y)dx$ and $\int_a^{+\infty}g(x,y)dx$ converge nonuniformly on Y, but the integral $\int_a^{+\infty}f(x,y)+g(x,y)dx$ converges uniformly on Y. 176

Improper integrals $\int_a^{+\infty}f(x,y)dx$ and $\int_a^{+\infty}g(x,y)dx$ converge nonuniformly on Y, and the integral $\int_a^{+\infty}f(x,y)+g(x,y)dx$ also converges nonuniformly on Y. 177

Example 10. Improper integrals $\int_a^{+\infty}f(x,y)dx$ and $\int_a^{+\infty}g(x,y)dx$ converge nonuniformly on Y, but the integral $\int_a^{+\infty}f(x,y)g(x,y)dx$ converges uniformly on Y. 178

An integral $\int_a^{+\infty}f(x,y)dx$ converges nonuniformly on Y, but the integral $\int_a^{+\infty}f^2(x,y)dx$ converges uniformly on Y. 179

Improper integrals $\int_a^{+\infty}f(x,y)dx$ and $\int_a^{+\infty}g(x,y)dx$ converge nonuniformly on Y, and the integral $\int_a^{+\infty}f(x,y)g(x,y)dx$ also converges nonuniformly on Y. 179

Example 11. Improper integrals $\int_a^{+\infty}f(x,y)dx$ and $\int_a^{+\infty}g(x,y)dx$ converge uniformly on Y, but the integral $\int_a^{+\infty}f(x,y)g(x,y)dx$ converges nonuniformly on Y. 179

An integral $\int_a^{+\infty}f(x,y)dx$ converges uniformly on Y, but the integral $\int_a^{+\infty}f^2(x,y)dx$ does not converge uniformly on Y. 180

Improper integrals $\int_a^{+\infty}f(x,y)dx$ and $\int_a^{+\infty}g(x,y)dx$ converge uniformly on Y, and $\int_a^{+\infty}f(x,y)g(x,y)dx$ also converges uniformly on Y. 180

Example 12. An improper integral $\int_a^{+\infty}f(x,y)dx$ converges uniformly and $\int_a^{+\infty}g(x,y)dx$ converges nonuniformly on Y, but the improper integral $\int_a^{+\infty}f(x,y)g(x,y)dx$ converges uniformly on Y. 181

An improper integral $\int_a^{+\infty}f(x,y)dx$ converges uniformly and $\int_a^{+\infty}g(x,y)dx$ converges nonuniformly on Y, but the improper integral $\int_a^{+\infty}f(x,y)g(x,y)dx$ converges nonuniformly on Y. 182

Example 13. Improper integrals $\int_a^{+\infty}f(x,y)dx$ and $\int_a^{+\infty}g(x,y)dx$ diverge on Y, but nevertheless the integral $\int_a^{+\infty}f(x,y)g(x,y)dx$ converges uniformly on Y. 182

Example 14. Improper integrals $\int_a^{+\infty} f(x,y)dx$ and $\int_a^{+\infty} g(x,y)dx$ converge uniformly on Y, but the improper integral $\int_a^{+\infty} f(x,y)g(x,y)dx$ diverges on Y. 183

Example 15. For each fixed $y \in Y$, integral $\int_a^A f(x,y)dx$ is bounded and function $g(x,y)$ is monotone in $x \in [a,+\infty)$; additionally, $g(x,y)$ converges uniformly to 0 on Y as $x \to +\infty$, but the improper integral $\int_a^{+\infty} f(x,y)g(x,y)dx$ does not converge uniformly on Y. 185

Example 16. The integral $\int_a^A f(x,y)dx$ is uniformly bounded on Y and $g(x,y)$ converges uniformly to 0 on Y as $x \to +\infty$, but the improper integral $\int_a^{+\infty} f(x,y)g(x,y)dx$ does not converge uniformly on Y. 186

Example 17. The integral $\int_a^A f(x,y)dx$ is uniformly bounded on Y, and the function $g(x,y)$ is monotone in x and converges to 0 for $\forall y \in Y$ as $x \to +\infty$, but the improper integral $\int_a^{+\infty} f(x,y)g(x,y)dx$ does not converge uniformly on Y. 187

The violation of one of the conditions of Dirichlet's theorem can lead not only to nonuniform convergence of $\int_a^{+\infty} f(x,y)g(x,y)dx$ but even to divergence. 187

Example 18. All the conditions of Dirichlet's theorem are violated, but nevertheless the improper integral $\int_a^{+\infty} f(x,y)g(x,y)dx$ converges uniformly on Y. 188

Example 19. For each fixed $y \in Y$, integral $\int_a^{+\infty} f(x,y)dx$ is convergent and function $g(x,y)$ is monotone in $x \in [a,+\infty)$; additionally, $g(x,y)$ is uniformly bounded on $[a,+\infty) \times Y$, but the improper integral $\int_a^{+\infty} f(x,y)g(x,y)dx$ does not converge uniformly on Y. 189

Example 20. The integral $\int_a^{+\infty} f(x,y)dx$ is uniformly convergent and $g(x,y)$ is uniformly bounded on Y, but the improper integral $\int_a^{+\infty} f(x,y)g(x,y)dx$ does not converge uniformly on Y. 190

Example 21. The integral $\int_a^{+\infty} f(x,y)dx$ converges uniformly on Y, and the function $g(x,y)$ is monotone in x and bounded for each fixed $y \in Y$, but the improper integral $\int_a^{+\infty} f(x,y)g(x,y)dx$ does not converge uniformly on Y. 190

The violation of only one condition in Abel's theorem results in the divergence of $\int_a^{+\infty} f(x,y)g(x,y)dx$. 191

Example 22. All the conditions of Abel's theorem are violated, but nevertheless the improper integral $\int_a^{+\infty} f(x,y)g(x,y)dx$ converges uniformly on Y. 192