LEARNING MADE EASY

2nd Edition

Calculus

for dummies®

A Wiley Brand

An easy-to-follow introduction to calculus

Detailed explanations of differentiation and integration

Real-life examples of when calculus comes in handy

Mark Ryan

Founder and owner of The Math Center

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Calculus For Dummies®, 2nd Edition

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Contents at a Glance

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Introduction

The mere thought of having to take a required calculus course is enough to make legions of students break out in a cold sweat. Others who have no intention of ever studying the subject have this notion that calculus is impossibly difficult unless you happen to be a direct descendant of Einstein.

Well, I'm here to tell you that you *can* master calculus. It's not nearly as tough as its mystique would lead you to think. Much of calculus is really just very advanced algebra, geometry, and trig. It builds upon and is a logical extension of those subjects. If you can do algebra, geometry, and trig, you can do calculus.

But why should you bother — apart from being required to take a course? Why climb Mt. Everest? Why listen to Beethoven's Ninth Symphony? Why visit the Louvre to see the Mona Lisa? Why watch *South Park?* Like these endeavors, doing calculus can be its own reward. There are many who say that calculus is one of the crowning achievements in all of intellectual history. As such, it's worth the effort. Read this jargon-free book, get a handle on calculus, and join the happy few who can proudly say, "Calculus? Oh, sure, I know calculus. It's no big deal."

About This Book

Calculus For Dummies, 2nd Edition, is intended for three groups of readers: students taking their first calculus course, students who need to brush up on their calculus to prepare for other studies, and adults of all ages who'd like a good introduction to the subject.

If you're enrolled in a calculus course and you find your textbook less than crystal clear, this is the book for you. It covers the most important topics in the first year of calculus: differentiation, integration, and infinite series.

If you've had elementary calculus, but it's been a couple of years and you want to review the concepts to prepare for, say, some graduate program, *Calculus For Dummies,* 2nd Edition will give you a thorough, no-nonsense refresher course.

Non-student readers will find the book's exposition clear and accessible. *Calculus For Dummies,* 2nd Edition, takes calculus out of the ivory tower and brings it down to earth.

This is a user-friendly math book. Whenever possible, I explain the calculus concepts by showing you connections between the calculus ideas and easier ideas from algebra and geometry. I then show you how the calculus concepts work in concrete examples. Only later do I give you the fancy calculus formulas. All explanations are in plain English, not math-speak.

The following conventions keep the text consistent and oh-so-easy to follow:

- Variables are in *italics.*
- Calculus terms are italicized and defined when they first appear in the text.
- In the step-by-step problem-solving methods, the general action you need to take is in bold, followed by the specifics of the particular problem.

It can be a great aid to true understanding of calculus — or any math topic for that matter — to focus on the *why* in addition to the *how-to.* With this in mind, I've put a lot of effort into explaining the underlying logic of many of the ideas in this book. If you want to give your study of calculus a solid foundation, you should read these explanations. But if you're really in a hurry, you can cut to the chase and read only the important introductory stuff, the example problems, the step-by-step solutions, and all the rules and definitions next to the icons. You can then read the remaining exposition only if you feel the need.

I find the sidebars interesting and entertaining. (What do you expect? I wrote them!) But you can skip them without missing any essential calculus. No, you won't be tested on this stuff.

Minor note: Within this book, you may note that some web addresses break across two lines of text. If you're reading this book in print and want to visit one of these web pages, simply key in the web address exactly as it's noted in the text, as though the line break doesn't exist. If you're reading this as an e-book, you've got it easy — just click the web address to be taken directly to the web page.

Foolish Assumptions

Call me crazy, but I assume . . .

- You know at least the basics of algebra, geometry, and trig.

  If you're rusty, Part 2 (and the online Cheat Sheet) contains a good review of these pre-calculus topics. Actually, if you're not currently taking a calculus course, and you're reading this book just to satisfy a general curiosity about calculus, you can get a good conceptual picture of the subject without the nitty-gritty details of algebra, geometry, and trig. But you won't, in that case, be able to follow all the problem solutions. In short, without the pre-calculus stuff, you can see the calculus *forest*, but not the *trees*. If you're enrolled in a calculus course, you've got no choice — you've got to know the trees as well as the forest.

- You're willing to do some w_ _ _ .

  No, not the dreaded *w*-word! Yes, that's w-o-r-k, *work*. I've tried to make this material as accessible as possible, but it is calculus after all. You can't learn calculus by just listening to a tape in your car or taking a pill — not yet anyway.

Is that too much to ask?

Icons Used in This Book

Keep your eyes on the icons:

MATH RULES

Next to this icon are calculus rules, definitions, and formulas.

REMEMBER

These are things you need to know from algebra, geometry, or trig, or things you should recall from earlier in the book.

TIP

The bull's-eye icon appears next to things that will make your life easier. Take note.

WARNING

This icon highlights common calculus mistakes. Take heed.

Beyond the Book

There's some great supplementary calculus material online that you might want to check out:

- To view this book's Cheat Sheet, simply go to `www.dummies.com` and search for "Calculus For Dummies Cheat Sheet" in the Search box; you'll find a nice list of important formulas, theorems, definitions, and so on from algebra, geometry, trigonometry, and calculus. This is a great place to go if you forget a formula.
- At `www.dummies.com`, there are articles on some calculus topics that many calculus courses skip. For example, the online article "Finding Volume with the Matryoshka Doll Method (a.k.a. the Cylindrical Shell Method)" covers one of the methods for computing volume that used to be part of the standard calculus curriculum, but which is now often omitted. You'll also find other interesting, off-the-beaten-path calculus articles. Check them out if you just can't get enough calculus.

Where to Go from Here

Why, Chapter 1, of course, if you want to start at the beginning. If you already have some background in calculus or just need a refresher course in one area or another, then feel free to skip around. Use the table of contents and index to find what you're looking for. If all goes well, in a half a year or so, you'll be able to check calculus off your list:

___ Run a marathon

___ Go skydiving

___ Write a book

✔ Learn calculus

___ Swim the English Channel

___ Cure cancer

___ Write a symphony

___Pull an inverted 720° at the X-Games

For the rest of your list, you're on your own.

1

An Overview of Calculus

IN THIS PART . . .

A brief and straightforward explanation of just what calculus is. ***Hint:*** It's got a lot to do with curves and with things that are constantly changing.

Examples of where you might see calculus at work in the real world: curving cables, curving domes, and the curving path of a spacecraft.

The first of the two big ideas in calculus: *differentiation,* which means finding a derivative. A derivative is basically just the fancy calculus version of a slope; and it's a simple rate — a this per that.

The second big calculus idea: *integration.* It's the fancy calculus version of adding up small parts of something to get the total.

An honest-to-goodness explanation of why calculus works: In short, it's because when you zoom in on curves (infinitely far), they become straight.

IN THIS CHAPTER

- **You're only in Chapter 1 and you're already going to get your first calc test**
- **Calculus — it's just souped-up regular math**
- **Zooming in is the key**
- **The world before and after calculus**

Chapter 1
What Is Calculus?

"My best day in Calc 101 at Southern Cal was the day I had to cut class to get a root canal."

— MARY JOHNSON

"I keep having this recurring dream where my calculus professor is coming after me with an axe."

— TOM FRANKLIN, COLORADO COLLEGE SOPHOMORE

"Calculus is fun, and it's so easy. I don't get what all the fuss is about."

— SAM EINSTEIN, ALBERT'S GREAT-GRANDSON

In this chapter, I answer the question "What is calculus?" in plain English, and I give you real-world examples of how calculus is used. After reading this and the following two short chapters, you *will* understand what calculus is all about. But here's a twist: Why don't you start out on the *wrong* foot by briefly checking out what calculus is *not*?

What Calculus Is Not

No sense delaying the inevitable. Ready for your first calculus test? Circle True or False.

> True or False: Unless you actually enjoy wearing a pocket protector, you've got no business taking calculus.
>
> True or False: Studying calculus is hazardous to your health.
>
> True or False: Calculus is totally irrelevant.

False, false, false! There's this mystique about calculus that it's this ridiculously difficult, incredibly arcane subject that no one in their right mind would sign up for unless it was a required course.

Don't buy into this misconception. Sure, calculus is difficult — I'm not going to lie to you — but it's manageable, doable. You made it through algebra, geometry, and trigonometry. Well, calculus just picks up where they leave off — it's simply the next step in a logical progression.

And calculus is not a dead language like Latin, spoken only by academics. It's the language of engineers, scientists, and economists. Okay, so it's a couple steps removed from your everyday life and unlikely to come up at a cocktail party. But the work of those engineers, scientists, and economists has a huge impact on your day-to-day life — from your microwave oven, cell phone, TV, and car to the medicines you take, the workings of the economy, and our national defense. At this very moment, something within your reach or within your view has been impacted by calculus.

So What Is Calculus, Already?

Calculus is basically just very advanced algebra and geometry. In one sense, it's not even a new subject — it takes the ordinary rules of algebra and geometry and tweaks them so that they can be used on more complicated problems. (The rub, of course, is that darn *other* sense in which it *is* a new and more difficult subject.)

Look at Figure 1-1. On the left is a man pushing a crate up a straight incline. On the right, the man is pushing the same crate up a curving incline. The problem, in both cases, is to determine the amount of energy required to push the crate to the top. You can do the problem on the left with regular math. For the one on the right, you need calculus (assuming you don't know the physics shortcuts).

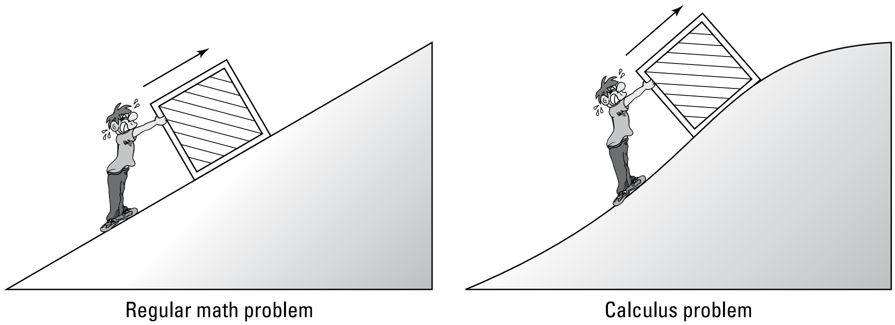


FIGURE 1-1: The difference between regular math and calculus: In a word, it's the *curve*.

For the straight incline, the man pushes with an *unchanging* force, and the crate goes up the incline at an *unchanging* speed. With some simple physics formulas and regular math (including algebra and trig), you can compute how many calories of energy are required to push the crate up the incline. Note that the amount of energy expended each second remains the same.

For the curving incline, on the other hand, things are constantly changing. The steepness of the incline is *changing* — and not just in increments like it's one steepness for the first 3 feet then a different steepness for the next 3 feet. It's *constantly changing.* And the man pushes with a *constantly changing* force — the steeper the incline, the harder the push. As a result, the amount of energy expended is also changing, not every second or every thousandth of a second, but *constantly changing* from one moment to the next. That's what makes it a calculus problem. By this time, it should come as no surprise to you that calculus is described as "the mathematics of change." Calculus takes the regular rules of math and applies them to fluid, evolving problems.

For the curving incline problem, the physics formulas remain the same, and the algebra and trig you use stay the same. The difference is that — in contrast to the straight incline problem, which you can sort of do in a single shot — you've got to break up the curving incline problem into small chunks and do each chunk separately. Figure 1-2 shows a small portion of the curving incline blown up to several times its size.

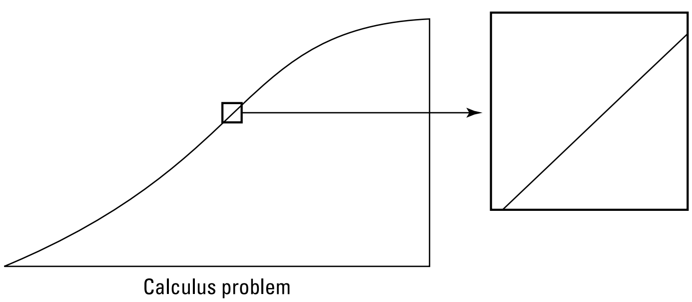


FIGURE 1-2: Zooming in on the curve — voilà, it's straight (almost).

When you zoom in far enough, the small length of the curving incline becomes practically straight. Then, because it's straight, you can solve that small chunk just like the straight incline problem. Each small chunk can be solved the same way, and then you just add up all the chunks.

That's calculus in a nutshell. It takes a problem that can't be done with regular math because things are constantly changing — the changing quantities show up on a graph as curves — it zooms in on the curve till it becomes straight, and then it finishes off the problem with regular math.

What makes the invention of calculus such a fantastic achievement is that it does what seems impossible: it zooms in *infinitely.* As a matter of fact, everything in calculus involves infinity in one way or another, because if something is constantly changing, it's changing infinitely often from each infinitesimal moment to the next.

Real-World Examples of Calculus

So, with regular math you can do the straight incline problem; with calculus you can do the curving incline problem. Here are some more examples.

With regular math you can determine the length of a buried cable that runs diagonally from one corner of a park to the other (remember the Pythagorean theorem?). With calculus you can determine the length of a cable hung between two towers that has the shape of a *catenary* (which is different, by the way, from a simple circular arc or a parabola). Knowing the exact length is of obvious importance to a power company planning hundreds of miles of new electric cable. See Figure 1-3.

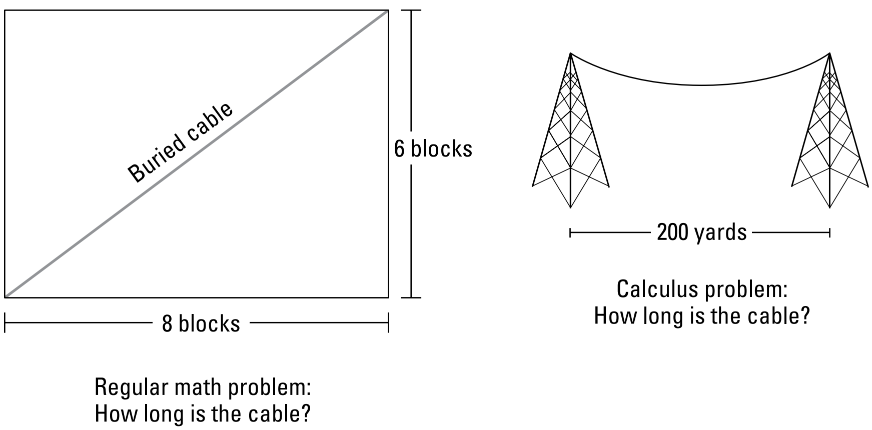


FIGURE 1-3: Without and with calculus.

You can calculate the area of the flat roof of a home with ordinary geometry. With calculus you can compute the area of a complicated, nonspherical shape like the dome of the Minneapolis Metrodome. Architects designing such a building need to know the dome's area to determine the cost of materials and to figure the weight of the dome (with and without snow on it). The weight, of course, is needed for planning the strength of the supporting structure. Check out Figure 1-4.

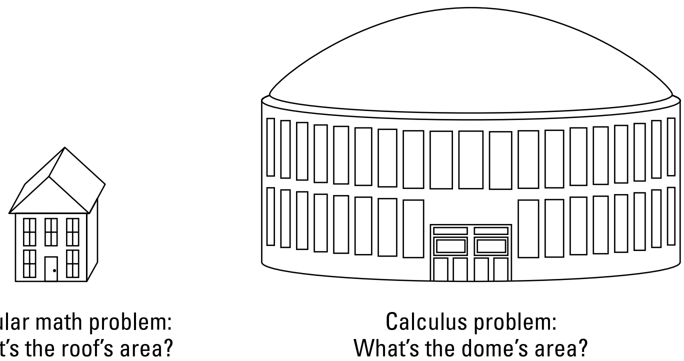


FIGURE 1-4: Sans and avec calculus.

With regular math and some simple physics, you can calculate how much a quarterback must lead his receiver to complete a pass. (I'm assuming here that the receiver runs in a *straight* line and at a *constant* speed.) But when NASA, in 1975, calculated the necessary "lead" for aiming the Viking I at Mars, it needed calculus because both the Earth and Mars travel on *elliptical* orbits (of different shapes) and the speeds of both are *constantly changing* — not to mention the fact that on its way to Mars, the spacecraft is affected by the different and *constantly changing* gravitational pulls of the Earth, the moon, Mars, and the sun. See Figure 1-5.

You see many real-world applications of calculus throughout this book. The differentiation problems in Part 4 all involve the steepness of a curve — like the steepness of the curving incline in Figure 1-1. In Part 5, you do integration problems like the cable-length problem shown back in Figure 1-3. These problems involve breaking up something into little sections, calculating each section, and then adding up the sections to get the total. More about that in Chapter 2.

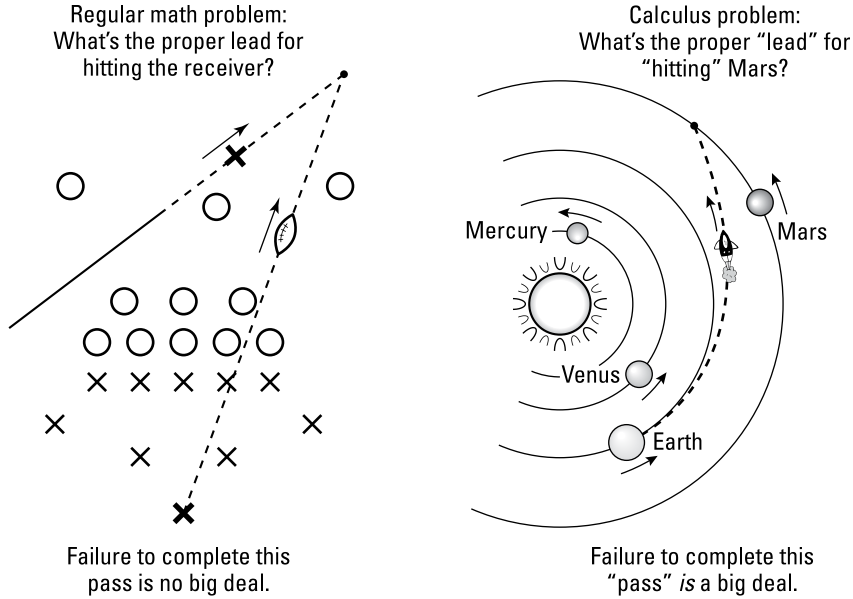


FIGURE 1-5: B.C.E. (Before the Calculus Era) and C.E. (the Calculus Era).

IN THIS CHAPTER

Delving into the derivative: It's a rate and a slope

Investigating the integral — addition for experts

Infinite series: Achilles versus the tortoise — place your bets

Chapter 2

The Two Big Ideas of Calculus: Differentiation and Integration — plus Infinite Series

This book covers the two main topics in calculus — differentiation and integration — as well as a third topic, infinite series. All three topics touch the earth and the heavens because all are built upon the rules of ordinary algebra and geometry and all involve the idea of infinity.

Defining Differentiation

Differentiation is the process of finding the *derivative* of a curve. And the word "derivative" is just the fancy calculus term for the curve's slope or steepness. And because the slope of a curve is equivalent to a simple rate (like *miles per hour* or *profit per item*), the derivative is a rate as well as a slope.

The derivative is a slope

In algebra, you learned about the slope of a line — it's equal to the ratio of the *rise* to the *run*. In other words, $Slope = \frac{rise}{run}$. See Figure 2-1. Let me guess: A sudden rush of algebra nostalgia is flooding over you.

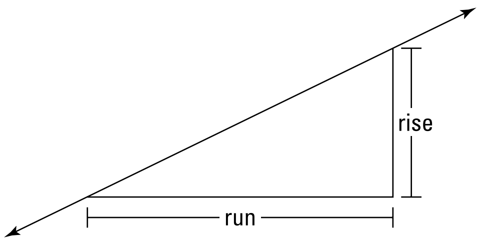


FIGURE 2-1: The *slope* of a line equals the *rise* over the *run*.

In Figure 2-1, the *rise* is half as long as the *run*, so the line has a slope of 1/2.

On a curve, the slope is constantly *changing*, so you need calculus to determine its slope. See Figure 2-2.

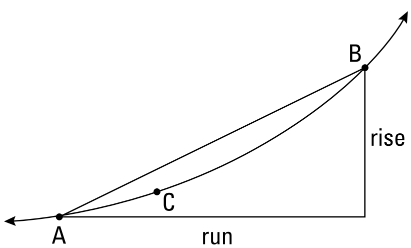


FIGURE 2-2: The slope of a curve isn't so simple.

Just like the line in Figure 2-1, the straight line between A and B in Figure 2-2 has a slope of 1/2. And the slope of this line is the same at every point between A and B. But you can see that, unlike the line, the steepness of the curve is changing between A and B. At A, the curve is less steep than the line, and at B, the curve is steeper than the line. What do you do if you want the exact slope at, say, point C? Can you guess? Time's up. Answer: You zoom in. See Figure 2-3.

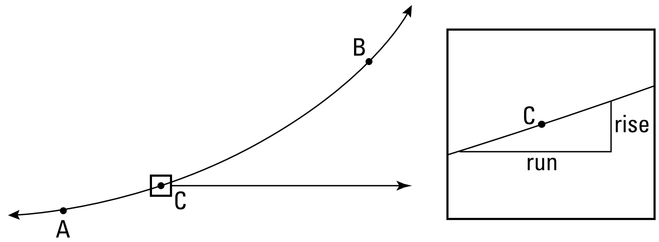


FIGURE 2-3: Zooming in on the curve.

When you zoom in far enough — really far, actually *infinitely* far — the little piece of the curve becomes straight, and you can figure the slope the old-fashioned way. That's how differentiation works.

The derivative is a rate

Because the derivative of a curve is the slope — which equals $\frac{rise}{run}$ or *rise per run* — the derivative is also a rate, a *this per that* like *miles per hour* or *gallons per minute* (the name of the particular rate simply depends on the units used on the *x*- and *y*-axes). The two graphs in Figure 2-4 show a relationship between distance and time — they could represent a trip in your car.

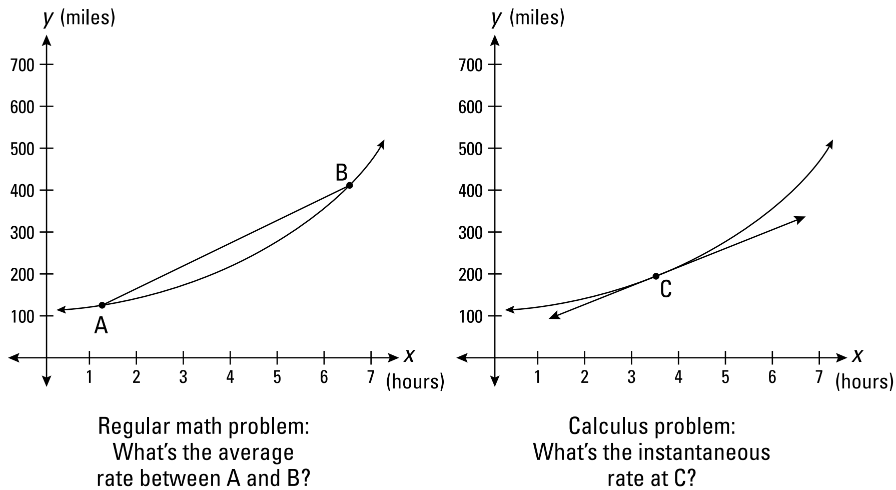


FIGURE 2-4: Average rate and instantaneous rate.

A regular algebra problem is shown on the left in Figure 2-4. If you know the *x*- and *y*-coordinates of points A and B, you can use the slope formula $\left(Slope = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} \right)$ to calculate the slope between A and B, and, in this problem, that slope gives you the *average* rate in *miles per hour* for the interval from A to B.

For the problem on the right, on the other hand, you need calculus. (You can't use the slope formula because you've only got one point.) Using the derivative of the curve, you can determine the *exact* slope or steepness at point C. Just to the left of C on the curve, the slope is slightly lower, and just to the right of C on the curve, the slope is slightly higher. But precisely at C, for a single infinitesimal moment, you get a slope that's different from the neighboring slopes. The slope for this single infinitesimal point on the curve gives you the *instantaneous* rate in *miles per hour* at point C.

Investigating Integration

Integration is the second big idea in calculus, and it's basically just fancy addition. Integration is the process of cutting up an area into tiny sections, figuring the areas of the small sections, and then adding up the little bits of area to get the whole area. Figure 2-5 shows two area problems — one that you can do with geometry and one where you need calculus.

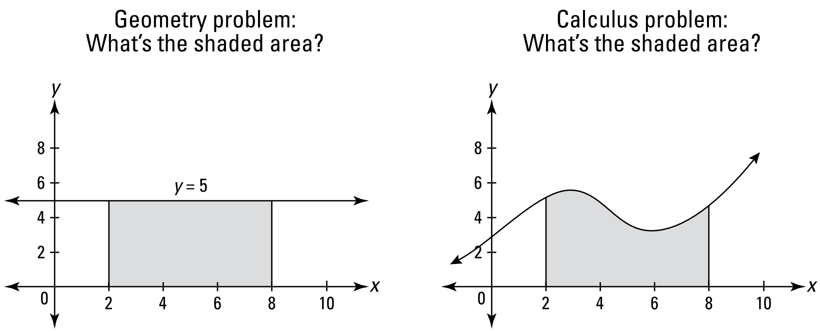


FIGURE 2-5: If you can't determine the area on the left, hang up your calculator.

The shaded area on the left is a simple rectangle, so its area, of course, equals length times width. But you can't figure the area on the right with regular geometry because there's no area formula for this funny shape. So what do you do? Why, zoom in, of course. Figure 2-6 shows the top portion of a narrow strip of the weird shape blown up to several times its size.

When you zoom in as shown in Figure 2-6, the curve becomes practically straight, and the further you zoom in, the straighter it gets. After zooming in, you get the shape on the right in Figure 2-6, which is practically an ordinary trapezoid (its top is still slightly curved). Well, with the magic of integration, you zoom in *infinitely* close (sort of — you can't really get infinitely close, right?). At that point, the shape is exactly an ordinary trapezoid — or, if you want to get really basic, it's a triangle sitting on top of a rectangle. Because you can compute the areas of rectangles, triangles, and trapezoids with ordinary geometry, you can get the area of this and all the other thin strips and then add up all these areas to get the total area. That's integration.

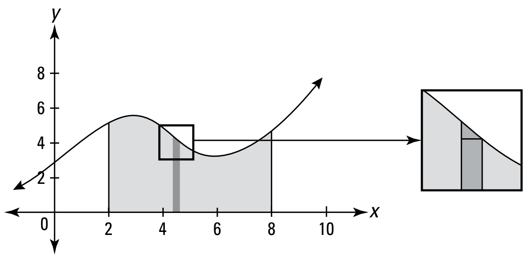


FIGURE 2-6: For the umpteenth time, when you zoom in, the curve becomes straight.

Figure 2-7 has two graphs of a city's electrical energy consumption on a typical summer day. The horizontal axes show the number of hours after midnight, and the vertical axes show the amount of power (in kilowatts) used by the city at different times during the day.

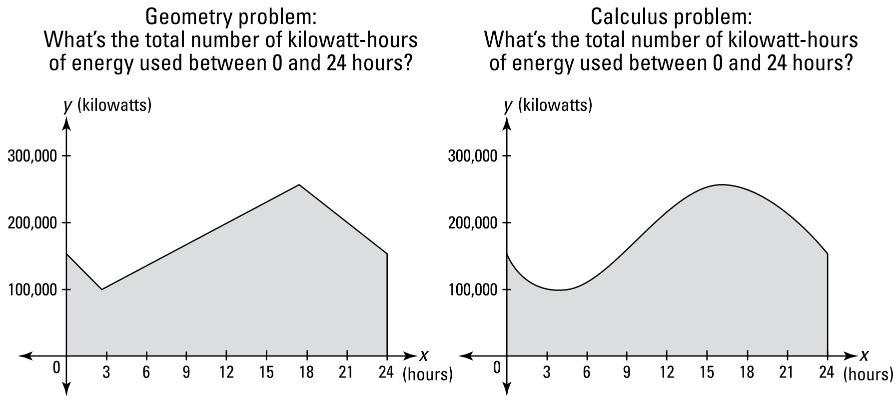


FIGURE 2-7: Total kilowatt-hours of energy used by a city during a single day.

The crooked line on the left and the curve on the right show how the number of kilowatts of power depends on the time of day. In both cases, the shaded area gives the number of kilowatt-hours of energy consumed during a typical 24-hour period. The shaded area in the oversimplified and unrealistic problem on the left can be calculated with regular geometry. But the true relationship between the amount of power used and the time of day is more complicated than a crooked straight line. In a realistic energy-consumption problem, you'd get something like the graph on the right. Because of its weird curve, you need calculus to determine the shaded area. In the real world, the relationship between different variables is rarely as simple as a straight-line graph. That's what makes calculus so useful.

Sorting Out Infinite Series

Infinite series deal with the adding up of an infinite number of numbers. Don't try this on your calculator unless you've got a lot of extra time on your hands. Here's a simple example. The following sequence of numbers is generated by a simple doubling process — each term is twice the one before it:

$$1, 2, 4, 8, 16, 32, 64, 128, \ldots$$

The infinite *series* associated with this *sequence* of numbers is just the sum of the numbers:

$$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \ldots$$

Divergent series

The preceding series of doubling numbers is *divergent* because if you continue the addition indefinitely, the sum will grow bigger and bigger without limit. And if you could add up "all" the numbers in this series — that's all *infinitely many* of them — the sum would be infinity. *Divergent* usually means — there are exceptions — that the series adds up to infinity.

Divergent series are rather uninteresting because they do what you expect. You keep adding more numbers, so the sum keeps growing, and if you continue this forever, the sum grows to infinity. Big surprise.

Convergent series

Convergent series are much more interesting. With a convergent series, you also keep adding more numbers, the sum keeps growing, but even though you add numbers forever and the sum grows forever, the sum of all the infinitely many terms is a *finite* number. This surprising result brings me to Zeno's famous paradox of Achilles and the tortoise. (That's Zeno of Elea, of course, from the 5th century B.C.)

Achilles is racing a tortoise — some gutsy warrior, eh? Our generous hero gives the tortoise a 100-yard head start. Achilles runs at 20 mph; the tortoise "runs" at 2 mph. Zeno used the following argument to "prove" that Achilles will never catch or pass the tortoise. If you're persuaded by this "proof," by the way, you've really got to get out more.

Imagine that you're a journalist covering the race for *Spartan Sports Weekly*, and you're taking a series of photos for your article. Figure 2-8 shows the situation at the start of the race and your first two photos.

You take your first photo the instant Achilles reaches the point where the tortoise started. By the time Achilles gets there, the tortoise has "raced" forward and is now 10 yards ahead of Achilles. (The tortoise moves a tenth as fast as Achilles, so in the time it takes Achilles to travel 100 yards, the tortoise covers a tenth as much ground, or 10 yards.) If you do the math, you find that it took Achilles about 10 seconds to run the 100 yards. (For the sake of argument, let's call it exactly 10 seconds.)

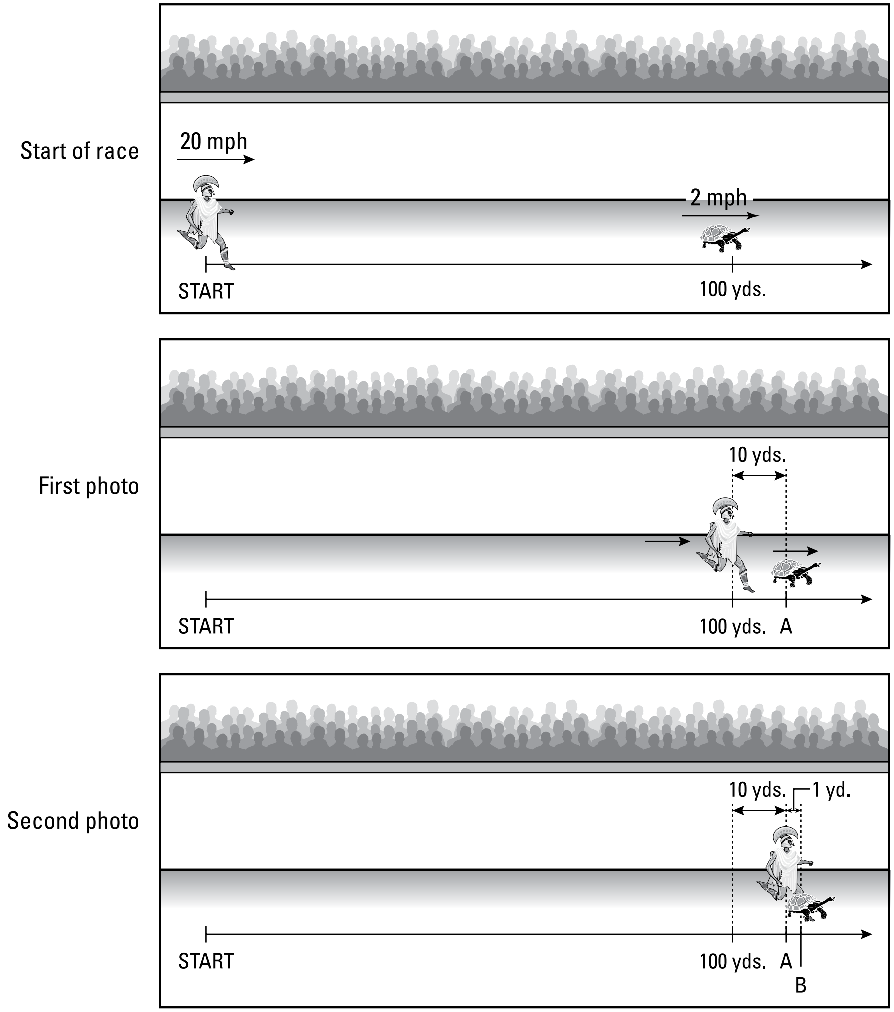


FIGURE 2-8: Achilles versus the tortoise — it's a photo finish.

You have a cool app that allows you to look at your first photo and note precisely where the tortoise is as Achilles crosses the tortoise's starting point. The tortoise's position is shown as point A in the middle image in Figure 2-8. Then you take your second photo when Achilles reaches point A, which takes him about one more second. In that second, the tortoise has moved ahead 1 yard to point B. You take your third photo (not shown) when Achilles reaches point B and the tortoise has moved ahead to point C.

Every time Achilles reaches the point where the tortoise was, you take another photo. There is no end to this series of photographs. Assuming you and your camera can work infinitely fast, you will take an infinite number of photos. And *every single time* Achilles reaches the point where the tortoise was, the tortoise has covered more ground — even if only a millimeter or a millionth of a millimeter. This process never ends, right? Thus, the argument goes, because you can never get to the end of your infinite series of photos, Achilles can never catch or pass the tortoise.

Well, as everyone knows, Achilles does in fact reach and pass the tortoise —thus the paradox. The mathematics of infinite series explains how this infinite series of time intervals sums to a *finite* amount of time — the precise time when Achilles passes the tortoise. Here's the sum for those who are curious:

$$\begin{aligned}&10 \text{ sec.} + 1 \text{ sec.} + 0.1 \text{ sec.} + 0.01 \text{ sec.} + 0.001 \text{ sec.} + \ldots \\ &= 11.111\ldots \text{ sec., or } 11\frac{1}{9} \text{ seconds.}\end{aligned}$$

Achilles passes the tortoise after $11\frac{1}{9}$ seconds at the $111\frac{1}{9}$-yard mark.

Infinite series problems are rich with bizarre, counterintuitive paradoxes. You see more of them in Part 5.

IN THIS CHAPTER

Using limits to zoom in on curves

Slope equals rise over run

Area of a triangle equals one-half base times height

The Pythagorean theorem: $a^2 + b^2 = c^2$

Chapter 3
Why Calculus Works

In Chapters 1 and 2, I talk a lot about the process of zooming in on a curve till it looks straight. The mathematics of calculus works because of this basic nature of curves — that they're *locally straight* — in other words, curves are straight at the microscopic level. The earth is round, but to us it looks flat because we're sort of at the microscopic level when compared to the size of the earth. Calculus works because after you zoom in and curves look straight, you can use regular algebra and geometry with them. The zooming-in process is achieved through the mathematics of limits.

The Limit Concept: A Mathematical Microscope

The mathematics of *limits* is the microscope that zooms in on a curve. Here's how a limit works. Say you want the exact slope or steepness of the parabola $y = x^2$ at the point (1, 1). See Figure 3-1.

With the slope formula from algebra, you can figure the slope of the line between (1, 1) and (2, 4). From (1, 1) to (2, 4), you go over 1 and up 3, so the slope is $\frac{3}{1}$, or just 3. But you can see in Figure 3-1 that this line is steeper than the tangent line at (1, 1) that shows the parabola's steepness at that specific point. The limit process sort of lets you slide the point that starts at (2, 4) down toward (1, 1) till it's a thousandth of an inch away, then a millionth, then a billionth, and so on down to the microscopic level. If you do the math, the slopes between (1, 1) and your moving point would look something like 2.8, then 2.6, then 2.4, and so on, and then, once you get to a thousandth of an inch away, 2.001, 2.000001, 2.000000001, and so on. And with the almost magical mathematics of limits, you can conclude that the slope at (1, 1) is precisely 2, even though the sliding point never reaches (1, 1). (If it did, you'd only have one point left and you need two separate points to use the slope formula.) The mathematics of limits is all based on this zooming-in process, and it works, again, because the further you zoom in, the straighter the curve gets.

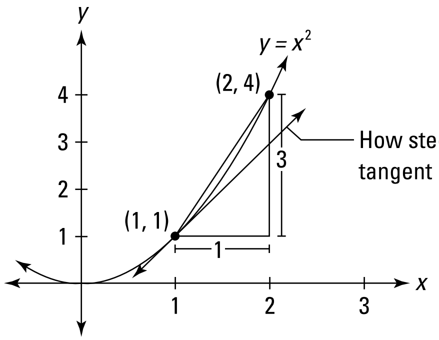


FIGURE 3-1: The parabola $y = x^2$ with a tangent line at (1, 1).

What Happens When You Zoom In

Figure 3-2 shows three diagrams of one curve and three things you might like to know about the curve: 1) the exact slope or steepness at point C, 2) the area under the curve between A and B, and 3) the exact length of the curve from A to B. You can't answer these questions with regular algebra or geometry formulas because the regular formulas for *slope*, *area*, and *length* work for straight lines (and simple curves like circles), but not for weird curves like this one.

The first row of Figure 3-3 shows a magnified detail from the three diagrams of the curve in Figure 3-2. The second row shows further magnification, and the third row yet another magnification. For each little window that gets blown up (like from the first to the second row of Figure 3-3), I've drawn in a new dotted diagonal line to help you see how with each magnification, the blown up pieces of the curves get straighter and straighter. This process is continued indefinitely.

Finally, Figure 3-4 shows the result after an "infinite" number of magnifications — sort of. After zooming in forever, an infinitely small piece of the original curve and the straight diagonal line are now one and the same. You can think of the lengths 3 and 4 in Figure 3-4 (no pun intended) as 3 and 4 millionths of an inch, no, make that 3 and 4 billionths of an inch, no, trillionths, no, gazillionths, . . .

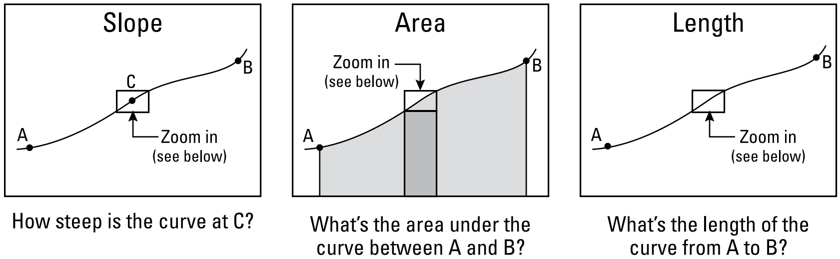


FIGURE 3-2: One curve — three questions.

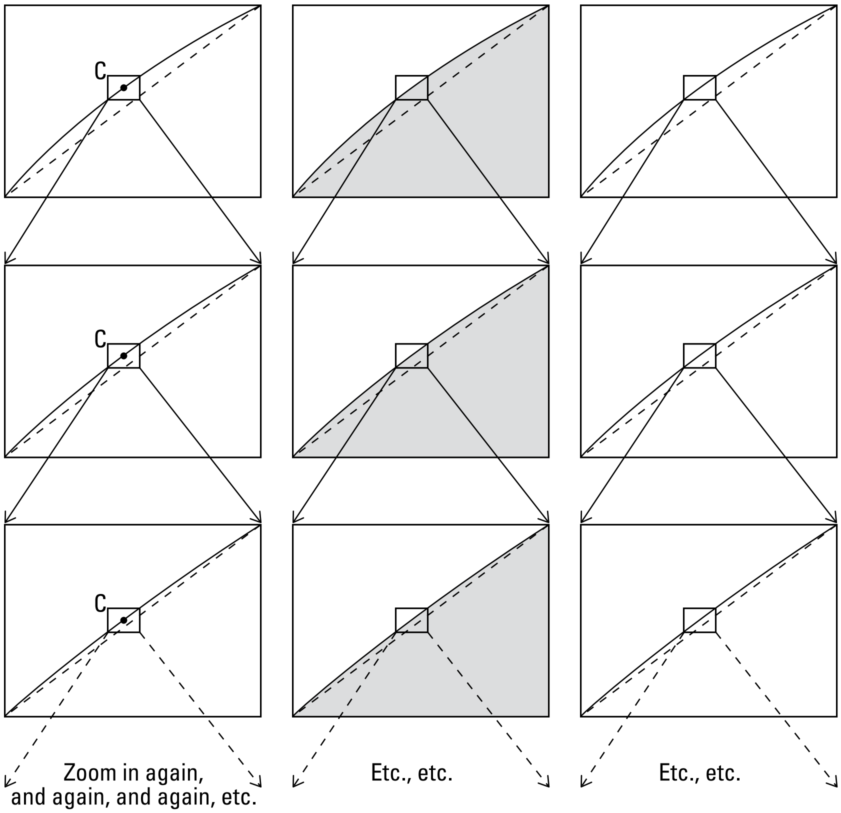
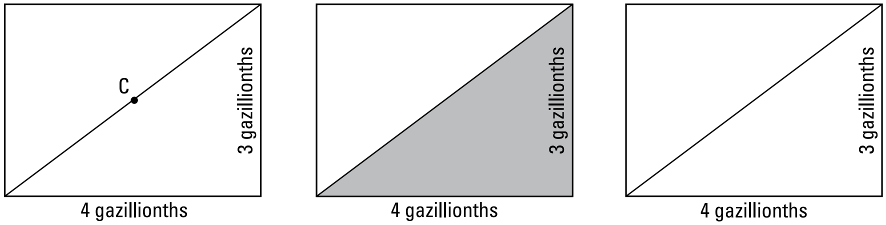


FIGURE 3-3: Zooming in to the microscopic level.

Now that you've zoomed in "forever," the curve is perfectly straight and you can use regular algebra and geometry formulas to answer the three questions about the curve in Figure 3-2.

For the diagram on the left in Figure 3-4, you can now use the regular *slope* formula from algebra to find the slope at point C. It's exactly $\frac{3}{4}$ — that's the answer to the first question in Figure 3-2. This is how differentiation works.



Slope equals $\frac{rise}{run}$ so the slope at point C is $\frac{3}{4}$.

The area of a triangle equals $\frac{1}{2}$ *base* × *height,* so this area is 6 square gazillionths.

The Pythagorean theorem ($a^2 + b^2 = c^2$) gives you the length of the hypotenuse – it's 5 gazillionths.

FIGURE 3-4: Your final destination — the sub, sub, sub . . . subatomic level.

For the diagram in the middle of Figure 3-4, the regular triangle formula from geometry gives you an area of 6. Then you can get the shaded area inside the strip shown in Figure 3-2 by adding this 6 to the area of the thin rectangle under the triangle (the dark-shaded rectangle in Figure 3-2). Then you repeat this process for all the other narrow strips (not shown), and finally just add up all the little areas. This is how integration works.

And for the diagram on the right of Figure 3-4, the Pythagorean theorem from geometry gives you a length of 5. Then to find the total length of the curve from A to B in Figure 3-2, you do the same thing for the other minute sections of the curve and then add up all the little lengths. This is how you calculate arc length (another integration problem).

Well, there you have it. Calculus uses the limit process to zoom in on a curve till it's straight. After it's straight, the rules of regular-old algebra and geometry apply. Calculus thus gives ordinary algebra and geometry the power to handle complicated problems involving *changing* quantities (which on a graph show up as *curves*). This explains why calculus has so many practical uses, because if there's something you can count on — in addition to death and taxes — it's that things are always changing.

Two Caveats; or, Precision, Preschmidgen

Not everything in this chapter (or this book for that matter) will satisfy the high standards of the Grand Poobah of Precision in Mathematical Writing.

I may lose my license to practice mathematics

With regard to the middle diagrams in Figures 3-2 through 3-4, I'm playing a bit fast and loose with the mathematics. The process of integration — finding the area under a curve — doesn't exactly work the way I explained. My explanation isn't really wrong, it's just a bit sideways. But — I don't care what anybody says — that's my story and I'm stickin' to it. Actually, it's not a bad way to think about how integration works, and, anyhow, this is only an introductory chapter.

What the heck does "infinity" really mean?

The second caveat is that whenever I talk about infinity — like in the last section where I discussed zooming in an infinite number of times — I do something like put the word "infinity" in quotes or say something like "you *sort of* zoom in forever." I do this to cover my butt. Whenever you talk about infinity, you're always on shaky ground. What would it mean to zoom in forever or an infinite number of times? You can't do it; you'd never get there. We can imagine — sort of — what it's like to zoom in forever, but there's something a bit fishy about the idea — and thus the qualifications.