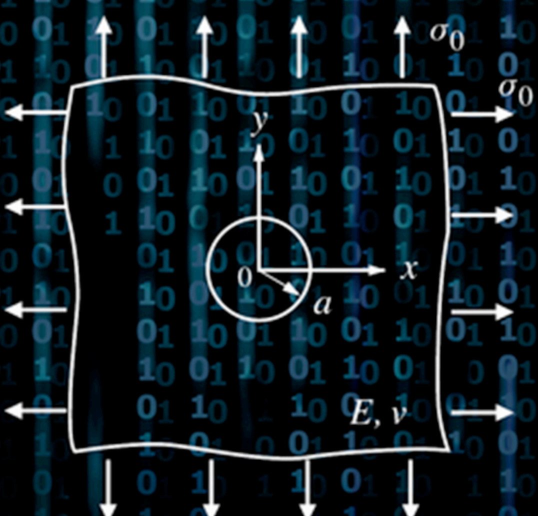
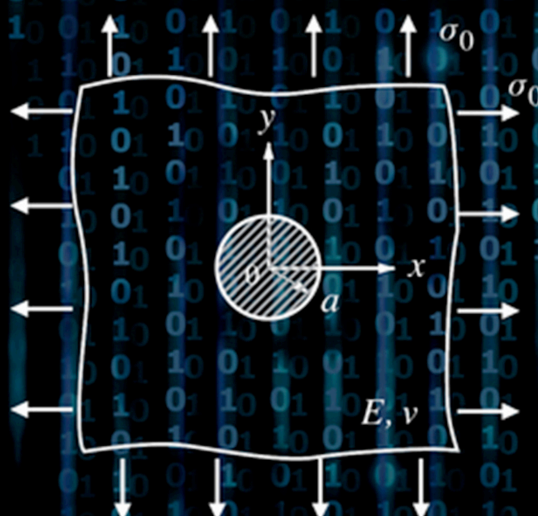
Theory of Elasticity and Stress Concentration





YUKITAKA MURAKAMI

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THEORY OF ELASTICITY AND STRESS CONCENTRATION

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Yukitaka Murakami
Kyushu University, Fukuoka, Japan

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Contents

Preface

The theory of elasticity is not applied mathematics. Solving differential equations and integral equations is not the objective of the theory of elasticity. Students and young researchers, who can use the modern commercial finite element method (FEM) software, are not attracted by the classical approach of applied mathematics. This situation is not good. Students, young researchers and young engineers skip directly from the elementary theory of the strength of materials to FEM without understanding the basic principles of the theory of elasticity. The author has seen many mistakes and judgement errors made by students, young researchers and young engineers in their applications of FEM to practical problems. These mistakes and judgement errors mostly come from a lack of basic knowledge of the theory of elasticity. Firstly, this book provides the basic but very important essence of the theory of elasticity. Second, many useful and interesting applications of the basic *way of thinking* are presented and explained. Readers do not need special mathematical knowledge to study this book. They will be able to understand the new approach of the theory of elasticity which is different from the classical mathematical theory of elasticity and will enjoy solving many interesting problems without using FEM.

The basic knowledge and engineering judgement acquired in Part I will encourage the readers to enter smoothly into Part II in which various important new *ways of thinking* and simple solution methods for stress concentration problems are presented. Approximate estimation methods for stress concentration will be very useful from the viewpoint of correct boundary conditions as well as the magnitude and relative importance of numerical variables. Thus, readers will be able to quickly find approximate solutions with practically sufficient accuracy and to avoid fatal mistakes produced by FEM calculations, performed without basic knowledge of the theory of elasticity and stress concentration.

The author believes with confidence that readers of this book will be able to develop themselves to a higher level of research and structural design.

Preface for Part I: Theory of Elasticity

Part I of this book presents a new *way of thinking* for the theory of elasticity. Several good quality textbooks on this topic have already been published, but they tend to be too mathematically based. Students can become confused by the very different approaches taken towards the elementary theory of strength of materials (ETSM) and the theory of elasticity and, therefore, believe that these two cannot be easily used cooperatively.

To study this book, readers do not need special mathematical knowledge such as differential equations, integral equations and tensor analysis. The concepts of stress field and strain are the most important themes in the study of the theory of elasticity. However, these concepts are not explored in sufficient depth within ETSM in order to teach engineers how to apply simple solutions using the theory of elasticity to solve practical problems. As various examples included in this book demonstrate, this book will help readers to understand not only the difference between ETSM and the theory of elasticity but also the essential relationship between them.

In addition to the concepts of field, the concepts of infinity and infinitesimal are also important. It is natural that everyone experiences difficulties in imagining infinity or infinitesimal. As a result, we must use caution when using unbounded or very small values, as the results are sometimes unexpected. We should be aware that infinity and infinitesimal are *relative* quantities.

Once the concepts of field and those of infinity and infinitesimal are mastered, the reader will become a true engineer having true engineering judgement, even if they cannot solve the problems using lengthy and troublesome differential or integral equations. However, the existing solutions must be used fully and care must be taken at times, very large values being treated as infinitesimal and very small values as infinite values depending on the specific problem. It will be seen in many cases treated in this book that small and large are only our impressions and that approximation is not only reasonable but very important.

Part I Nomenclature

Stresses and strains in an orthogonal coordinate system (x, y, z)	Normal stress (σ_x, σ_y, σ_z) Normal strain (ε_x, ε_y, ε_z) Shear stress (τ_{xy}, τ_{yz}, τ_{zx}) Shear strain (γ_{xy}, γ_{yz}, γ_{zx})
Stresses and strains in a cylindrical coordinate system (r, θ, z)	Normal stress (σ_r, σ_θ, σ_z) Normal strain (ε_r, ε_θ, ε_z) Shear stress ($\tau_{r\theta}$, $\tau_{\theta z}$, τ_{zr}) Shear strain ($\gamma_{r\theta}$, $\gamma_{\theta z}$, γ_{zr})
Rotation	ω
Normal stress and shear stress in a ξ-η-ζ coordinate system	Normal stress (σ_ξ, σ_η, σ_ζ) Shear stress ($\tau_{\xi\eta}$, $\tau_{\eta\zeta}$, $\tau_{\zeta\xi}$)
Remote stress	σ_0, τ_0 or $\sigma_{x\infty}$, $\sigma_{y\infty}$, $\tau_{xy\infty}$
Principal stresses	σ_1, σ_2, σ_3
Principal strains	ε_1, ε_2, ε_3
Direction cosines	l_i, m_i, n_i ($i = 1, 2, 3$)
Pressure	p or q
Concentrated force	P, Q
Body force	X, Y, Z or F_r, F_θ
Bending moment per unit length	M_x, M_y
Twisting moment per unit length	M_{xy} or M_{yx}
Twisting moment (torsional moment) or temperature	T
Torsional angle per unit length or crack propagation angle	θ_0
Surface tension	S
Airy's stress function or stress function in torsion	ϕ
Stress concentration factor	K_t
Stress intensity factor of Mode I	K_{I}
Stress intensity factor of Mode II	K_{II}
Stress intensity factor of Mode III	K_{III}

Radius of circle or major radius of ellipse or crack length	a
Minor radius of ellipse	b
Notch root radius or radius of curvature in membrane	ρ
Notch depth	t
Young's modulus	E
Poisson's ratio	ν
Shear modulus	G
Displacement in x, y, z coordinate system	u, v, w (Note: v looks the same as Poisson's ratio but is different.)
Displacement of membrane	z
Width of plate	W

Preface for Part II: Stress Concentration

Part II of this book is a compilation of the ideas on stress concentration which the author has developed over many years of teaching and research. This is not a handbook of stress concentration factors. This book guides a fundamental *way of thinking* for stress concentration. Fundamentals, typical misconceptions and new *ways of thinking* about stress concentration are presented. One of the motivations for writing this book is the concern about a decreasing basic knowledge of recent engineers about the nature of stress concentration.

It was reported in the United States and Europe [1–3] that the economic loss of fracture accidents reaches about 4% of GDP. Fracture accidents occur repeatedly regardless of the progress of science and technology. It seems that the number and severity of serious accidents is increasing. The author was involved in teaching strength of materials and theory of elasticity for many years in universities and industry and a recent impression based on the author's experience is that many engineers do not understand the fundamentals of the theory of elasticity.

How many engineers can give the correct answers to basic problems such as those in Figures 1 and 2?

The theory of elasticity lectures are likely to be abstract and mathematical. This trend is evident in the topics and emphasis of many text books. Such textbooks may be useful for some researchers but are almost useless for most practicing engineers. The author has been aware of this problem for many years and has changed the pedagogy of teaching the theory of elasticity by introducing various useful *ways of thinking* (see Part I). Engineers specializing in strength design and quality control are especially requested to acquire the fundamentals of theory of elasticity and afterwards to develop a sense about *stress concentration*. The subject is not difficult. Rather, as readers become familiar with the problems contained in this book, they will understand that the problems of stress concentration are full of interesting paradoxes.

Few accidents occur because of a numerical mistake or lack of precision in a stress analysis. A common attitude that analysis by FEM software will guarantee the correct answer and safety is the root cause of many failures. Most mistakes in the process of FEM analysis are made at the

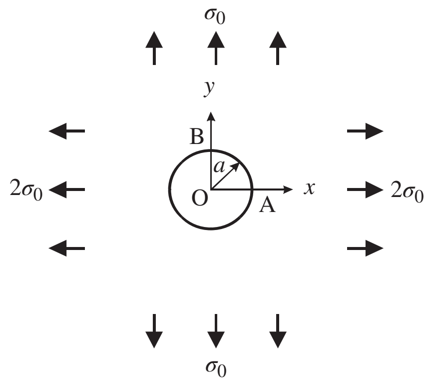

Figure 1 Stress concentration at a circular hole in a wide plate. How large is the maximum stress? (See Figure 1.2 in Example problem 1.1 in Part II, Chapter 1.)

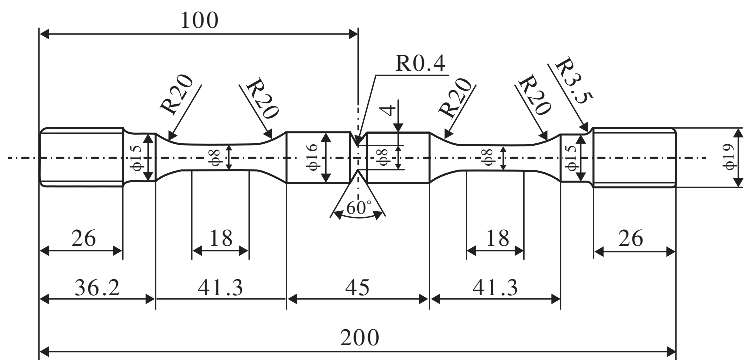

Figure 2 A cylindrical specimen for comparison of the fracture strengths at a smooth part and a notched part under tension (material is 0.13% annealed carbon steel, dimension unit is mm). Where does this specimen fracture from by tensile test? (See Figure 14.7 in the Example problem 14.1 in Part II, Chapter 14.)

beginning stage of determining boundary conditions regarding forces and displacements. Even worse, many users of FEM software are often not aware of such mistakes even after looking at strange results because they do not have a fundamental understanding of theory of elasticity and stress concentration.

The origin of fracture related accidents are mostly at the stress concentrations in a structure. As machine components and structures have various shapes for functional reasons, stress concentration cannot be avoided. Therefore, strength designers are required to evaluate stress concentration correctly and to design the shape of structures so that the stress concentration does not exceed the safety limits.

In this book, various elastic stress concentration problems are the main topic. The strains in an elastic state can be determined by Hooke's law in terms of stresses. In elastic–plastic conditions, the relationship between stresses and strains deviates from Hooke's law. Once plastic yielding occurs at a notch root, the stress concentration factor decreases compared to the elastic value and approaches one. However, the strain concentration factor increases and approaches the elastic value squared. Therefore, in elastic–plastic conditions, fatigue behavior is described in terms of strain concentration. However, if the stress and strain relationship at the notch root does not deviate much from Hooke's law or work hardening of material occurs after yielding, the description based on elastic stress concentration is valid. In general, in the case of high cycle fatigue, it is reasonable and effective for the solution of practical problems to consider only the elastic stress concentration. Thus, it is crucially important for strength design engineers to understand the nature of elastic stress concentration.

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[2] Battelle Columbus Laboratories (1983) *Economic Effects of Fracture in the United States. Part 2: A Report to NBS*. National Bureau of Standards and National Information Service, Washington, DC.
[3] Commission of the European Communities (1991) Economic Effects of Fracture in Europe, Final Report, Study Contract No. 320105. Commission of the European Communities, Brussels.

Part II Nomenclature

Stresses and strains in orthogonal coordinate system (x, y, z)	Normal stress (σ_x, σ_y, σ_z)
	Normal strain (ε_x, ε_y, ε_z)
	Shear stress (τ_{xy}, τ_{yz}, τ_{zx})
	Shear strain (γ_{xy}, γ_{yz}, γ_{zx})
Stresses and strains in cylindrical coordinate system (r, θ, z)	Normal stress (σ_r, σ_θ, σ_z)
	Normal strain (ε_r, ε_θ, ε_z)
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	Shear strain ($\gamma_{r\theta}$, $\gamma_{\theta z}$, γ_{zr})
Normal stress and shear stress in ξ–η coordinate system	Normal stress (σ_ξ, σ_η), shear stress $\tau_{\xi\eta}$
Remote stress	σ_0, τ_0 or $\sigma_{x\infty}$, $\sigma_{y\infty}$, $\tau_{xy\infty}$
Principal stresses	σ_1, σ_2
Pressure	p or q
Concentrated force	P, Q
Stress concentration factor	K_t
Stress concentration factor in elastic plastic state	K_σ
Strain concentration factor in elastic plastic state	K_ε
Stress intensity factor of Mode I	K_{I}
Stress intensity factor of Mode II	K_{II}
Stress intensity factor of Mode III	K_{III}
Radius of circle or major radius of ellipse	a
Minor radius of ellipse (or Burger's vector of dislocation)	b
Notch root radius	ρ
Notch depth	t
Young's modulus	E
Poisson's ratio	ν
Shear modulus	G
Displacement in x, y, z coordinate system	u, v, w
Shape parameter of ellipse	$R = \sqrt{(a+b)/(a-b)}$
Plastic zone size	R

Acknowledgments

This book is a joint version of two books originally published separately in Japanese by Yokendo, Tokyo: one is *Theory of Elasticity* and the other is *New Way of Thinking for Stress Concentration*. The Japanese version of *Theory of Elasticity* has been used in many universities since its publication in 1984 and it is regarded as one of the best elasticity textbooks in Japan. For the publication of the Japanese versions, the author was indebted especially to the late Prof. Tatsuo Endo and the late Prof. Makoto Isida for their invaluable comments and moreover to Mr. Kiyoshi Oikawa, the president of Yokendo Publishing Co. Ltd. for publishing the original Japanese versions of these two books and kindly approving the publication of this English version.

In advance of publication of the English version, the author gave lectures on the theory of elasticity and stress concentration at several universities in Europe and the United States. Especially at Aalto University in Finland, the author taught this subject for one complete semester and realized the importance of the new way of teaching the theory of elasticity. Throughout the author's long experience of teaching the theory of elasticity and its relationship with fatigue design, the author got useful comments and encouragements from Prof. Gary Marquis, Dean of the School of Engineering at Aalto University, Prof. Darrell Socie, University of Illinois, Prof. Stefano Beretta, Politecnico di Milano, Prof. Masahiro Endo, Fukuoka University and Prof. Hisao Matsunaga, Kyushu University. The author would like to express his sincere thanks to them.

Prof. Masahiro Endo's help is greatly appreciated for reading throughout the draft of the book and also for giving the author useful comments on the problems presented at the end of chapters.

Prof. Stefano Beretta organized the author's lecture on *Theory of Elasticity and Stress Concentration* for Italian PhD students in the summer school of the Italian Fracture Group in 2008. In the summer school, the author was deeply impressed with the students' attitudes and their strong curiosity toward the new *way of thinking* of *Theory of Elasticity*. The author would like to express his sincere thanks also to them.

The author would like to thank Ms. Hoshiko Utamaru of Kobe Materials Testing Laboratory, Co. Ltd. for her help with illustrating figures and typing equations. The author would like to thank Mr. Kazuo Takagi, the executive director of Kobe Materials Testing Laboratory, who kindly supported the author's writing work through the help of Ms. Hoshiko Utamaru's elaborative work. Without their help, the author could not have completed this book.

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Part I

Theory of Elasticity

1

Stress

1.1 Stress at the Surface of a Body

1.1.1 Normal Stress

When a body is in a liquid of pressure p, the surface of the body is subject to the same pressure p everywhere, irrespective of the material. Naturally the pressure acts perpendicular to the curved surface, unless the surface is subject to a frictional force. However, the action of frictional force is impossible because a liquid cannot sustain shear stress.

We describe this condition by saying that the *normal stress* σ_n at the surface of the body is $-p$, that is $\sigma_n = -p$. Thus, the normal stress is the force per unit area, when a force acts perpendicular to the surface (Figure 1.1).

1.1.2 Shear Stress

When a block, of weight W, is on a flat plate, there is a minimum force, F, which is necessary to move the block (Figure 1.2). This force is expressed by the equation

$$F = \mu W \tag{1.1}$$

where μ is the coefficient of static friction. Hence, both the bottom surface of the block and the top surface of the plate are subject to the same frictional force, F. In this situation both surfaces are subject to a shear stress τ. Denoting the average magnitude of the shear stress by τ_{ave}, we have

Theory of Elasticity and Stress Concentration, First Edition. Yukitaka Murakami.

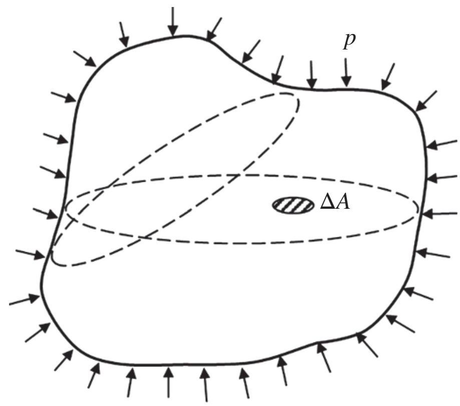

Figure 1.1

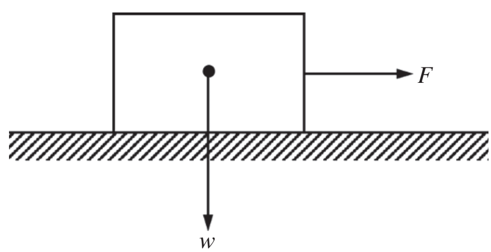

Figure 1.2

$$\tau_{\text{ave}} = \frac{F}{A} \tag{1.2}$$

where A is the area of the bottom surface of the block. In this way we can use the term *shear stress* to express the tangential force per unit area.

1.2 Stress in the Interior of a Body

If we consider the small area, ΔA, in the body shown in Figure 1.1, we can see that it is subjected to a force acting on the area ΔA. However, we cannot talk about the normal stress σ_n at that point yet, because we do not know either the magnitude or the direction of the internal force. However, supposing that there exists a normal component of the internal force, ΔF_n, we can use this to define the normal stress σ_n at that same point in the same way as we did at the surface by the limiting expression

$$\sigma_n = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A} \tag{1.3}$$

Similarly, supposing there exists a tangential component ΔF_t of the internal force, we can define the shear stress, τ, at ΔA by a similar equation,

$$\tau = \lim_{\Delta A \to 0} \frac{\Delta F_t}{\Delta A} \tag{1.4}$$

In the previous case shown in Figure 1.1 we see that, irrespective of the position and direction of the area, the only force acting is the normal one. That is to say, $\sigma_n = -p$ and $\tau = 0$ everywhere in the body. However, in general problems the conditions are different and the stress varies from point to point and the stress field is not uniform.

When we first meet a problem, we usually have no information about the stress state inside the body, as only the stresses at the *surface* are known. Hence, we can only use this information to solve the problem. *The stresses at the surface of the body are the keys to solving the problem.* These, already known stresses (or deformations), are called *boundary conditions*.

So now we can start to use the theory of elasticity using boundary conditions, but how can we use them to obtain the stresses inside the body?

1.3 Two Dimensional Stress, Three Dimensional Stress and Stress Transformation

1.3.1 Normal Stress

When a plate of uniform thickness having an arbitrary shape (Figure 1.3) is subjected to a constant pressure, p, along its periphery Γ, the normal stress σ_n and the shear stress τ at the periphery Γ are $\sigma_n = -p$ and $\tau = 0$, respectively. However, we still cannot determine the values of σ_n and τ at an arbitrary point A. How to find the stresses at point A will be explained later.

Now let us consider a rectangular plate of uniform thickness, as illustrated in Figure 1.4. Its boundary conditions are $\sigma_n = \sigma_{x0}$ along the side BC and AD, $\sigma_n = \sigma_{y0}$ along the side AB and CD

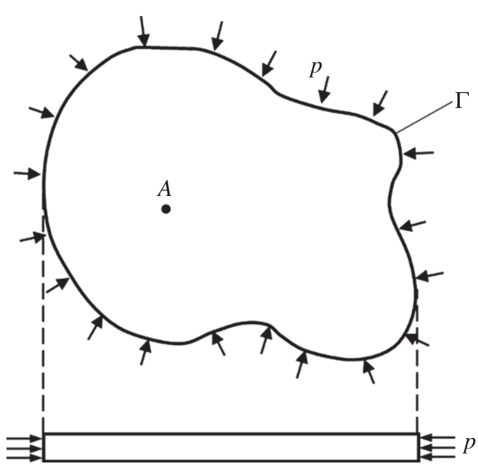

Figure 1.3

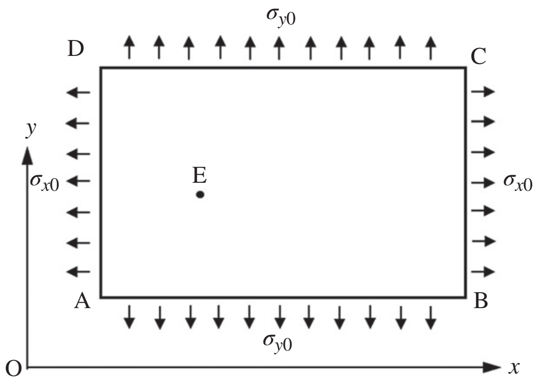

Figure 1.4

and $\tau = 0$ along all sides. The usual method of describing these conditions is to say, $\sigma_x = \sigma_{x0}$, $\tau_{xy} = 0$ along BC and AD, and $\sigma_y = \sigma_{y0}$, $\tau_{yx} = 0$ along AB and CD. The subscripts, x and y in σ_x and σ_y mean that σ_x and σ_y are normal stresses in the directions of the *x* and *y* axes respectively. The order of the double subscripts, like xy in τ_{xy}, have a universal meaning. The first indicates the face on which the stress acts and the second indicates the direction of the shear stress. Thus, τ_{xy} acts on a face normal to the *x* axis and acts in the *y* direction.

If $\sigma_{y0} = 0$, we can easily see that $\sigma_x = \sigma_{x0}$ at an arbitrary point, E, inside the plate. Likewise, if $\sigma_{x0} = 0$, then $\sigma_y = \sigma_{y0}$ everywhere in the plate. Therefore, when either σ_{x0} or σ_{y0} is not zero, we can easily see that $\sigma_x = \sigma_{x0}$ and $\sigma_y = \sigma_{y0}$ in the plate. This interpretation comes from considering the equilibrium of forces within the plate.

1.3.2 Shear Stress

Considering the case where shear stresses are the only boundary conditions, as shown in Figure 1.5, how do we find the stresses inside the plate? From the equilibrium of forces we can see that if τ_{yx0} acts along AB in the negative *x* direction, τ_{yx0} along CD must act in the positive *x* direction, otherwise the plate would not be in equilibrium. By the same reasoning we can see that the shear stresses along BC and AD will act in opposite directions.

In addition, the plate must be in equilibrium from the viewpoint of *rotation* as well. There must be no effective moments to cause rotation. This is known as the *condition of rotation.* Considering the condition of rotation we can use any *z* axis which pierces the plate. If we choose the *z* axis which pierces the plate at the point, A, then the equilibrium condition for rotation is written as follows.

$$\mathrm{BC} \cdot \tau_{xy0} \cdot \mathrm{AB} - \mathrm{CD} \cdot \tau_{yx0} \cdot \mathrm{AD} = 0 \tag{1.5}$$

and as CD = AB and AD = BC then this simplifies to

$$\tau_{xy0} = \tau_{yx0} \tag{1.6}$$

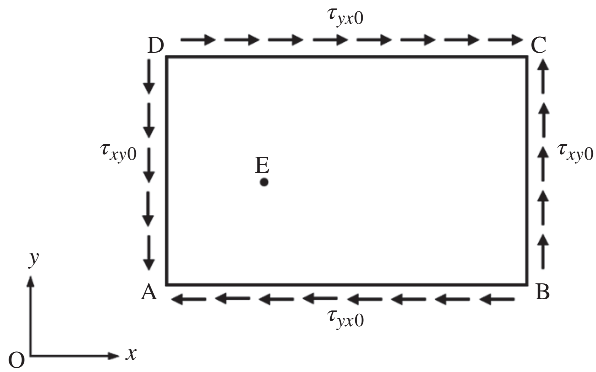

Figure 1.5

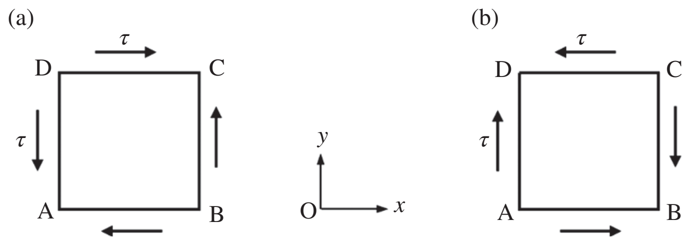

Figure 1.6

This is simple, but a very important relationship. Equation (1.6) means that *shear stresses exist only as shown in Figure 1.6a, b*. All the shear stresses are equal in magnitude and a shear stress *cannot* exist only on a single side (e.g. AB) or only on a couple of parallel sides (e.g. a couple of AB and CD). This rule also holds inside the plate. Although students and engineers sometimes underestimate this rule, an exact understanding of it will help to solve advanced problems later.

1.3.3 Stress in an Arbitrary Direction

1.3.3.1 Two Dimensional Stress Transformation

If the boundary conditions of a rectangular plate are like those shown in Figure 1.7, we can immediately see that the stresses at an arbitrary point inside the plate are $\sigma_x = \sigma_{x0}$, $\sigma_y = \sigma_{y0}$ and $\tau_{xy} = \tau_{xy0}$. However, these stresses are those defined in the *x*-*y* coordinate system. In many practical problems the stresses in different coordinate systems are needed.

Now let us determine the stresses σ_ξ, σ_η and $\tau_{\xi\eta}$ defined in the coordinate system (ξ, η) which is rotated by the angle θ from the x axis in the counterclockwise direction.

If we imagine a right angle triangle ABC as in Figure 1.8, inside the rectangular plate of Figure 1.7, the stresses σ_x, σ_y and τ_{xy} along the sides AB and AC are $\sigma_x = \sigma_{x0}$ (along AC) $\sigma_y = \sigma_{y0}$ (along AB) and $\tau_{xy} = \tau_{xy0}$ (along AB and AC) respectively. For the sake of simplicity we take the length of the side BC to be unity, that is $|BC| = 1$.

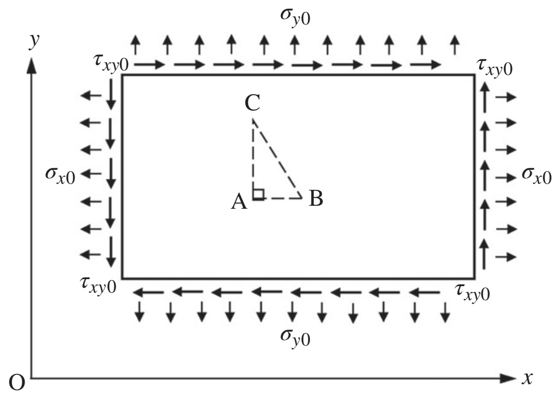

Figure 1.7

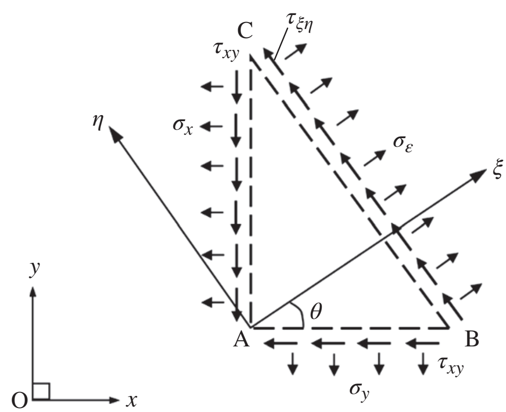

Figure 1.8

Using the *direction cosines* of the ξ and η axes with respect to the x and y axes, we can express the equilibrium conditions of the triangle ΔABC in the ξ and η directions.

Table 1.1 defines the direction cosines as follows:

$$l_1 = \cos\theta, \quad m_1 = \sin\theta$$

$$l_2 = -\sin\theta, \quad m_2 = \cos\theta$$

$$(\sigma_\xi \cdot 1) \cdot 1 = (\sigma_x \cdot l_1) \cdot l_1 + (\sigma_y \cdot m_1) \cdot m_1 + (\tau_{xy} \cdot l_1) \cdot m_1 + (\tau_{xy} \cdot m_1) \cdot l_1 \tag{1.7}$$

$$(\tau_{\xi\eta} \cdot 1) \cdot 1 = (\sigma_x \cdot l_1) \cdot l_2 + (\sigma_y \cdot m_1) \cdot m_2 + (\tau_{xy} \cdot l_1) \cdot m_2 + (\tau_{xy} \cdot m_1) \cdot l_2 \tag{1.8}$$

In the above equations, the quantities in the parentheses are forces acting on the side of the triangle, (stress · area), and the direction cosines that follow the parentheses are the operators for

Table 1.1 Direction cosine

	x	y
ξ	l_1	m_1
η	l_2	m_2

obtaining the components of the forces. A common mistake students make when using the *stress transformation* is that they multiply stresses and direction cosines *only once*. This is because they use the wrong condition of *stress* instead of *force*[1] when they are trying to solve for equilibrium conditions.

Arranging Equations 1.7 and 1.8 and adding that for σ_η gives:

$$\sigma_\xi = \sigma_x {l_1}^2 + \sigma_y {m_1}^2 + 2\tau_{xy} l_1 m_1 \tag{1.9}$$

$$\sigma_\eta = \sigma_x {l_2}^2 + \sigma_y {m_2}^2 + 2\tau_{xy} l_2 m_2 \tag{1.10}$$

$$\tau_{\xi\eta} = \sigma_x l_1 l_2 + \sigma_y m_1 m_2 + \tau_{xy}(l_1 m_2 + l_2 m_1) \tag{1.11}$$

Rewriting these equations using the angle θ in Figure 1.8 gives:

$$\left.\begin{aligned} \sigma_\xi &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\cos\theta\cdot\sin\theta \\ \sigma_\eta &= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\cos\theta\cdot\sin\theta \\ \tau_{\xi\eta} &= (\sigma_y - \sigma_x)\cos\theta\cdot\sin\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \end{aligned}\right\} \tag{1.12}$$

When students look at Equation 1.12, they often forget that it was derived using the equilibrium condition of *forces*. This is a very important equation and later we will use it often to solve various problems.

On deriving Equation 1.12, we usually draw a diagram similar to that in Figure 1.8. However, it should be noted that in Figure 1.8 we know the stresses on two sides (AB, AC) of the triangle and on only one side, (BC), is the stress unknown.

If we draw the figure like Figure 1.9 instead of Figure 1.8 then we cannot derive Equation 1.12 because we know only the stresses on the side (AB) and we do not know the stresses σ_ξ, σ_η and $\tau_{\xi\eta}$ acting on BC and AC. This is another mistake that confuses many students.

Equation 1.12 was derived assuming that the plate of Figure 1.7 was subjected to a uniform stress. However, as we did not talk about the size of the plate we can also use Equation 1.12 in the cases of non-uniform stress, simply by imagining a sufficiently small rectangle inside the arbitrarily shaped plate, over which the stresses do not vary. As we shall see, Equation 1.12 is used often in such cases.

[1] The author believes that the teaching of Mohr's circle is another major cause of misunderstanding. This is because the use of Mohr's circle is usually taught after the derivation of Equation 1.12 and cosine and sine appear only once in its analysis.

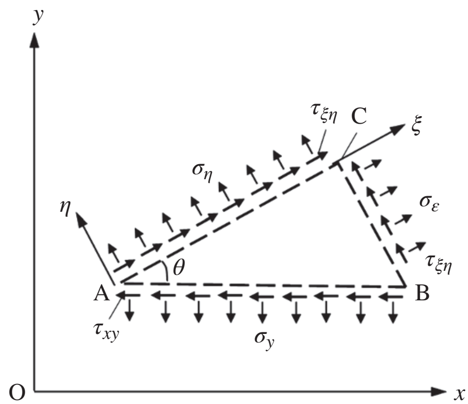

Figure 1.9

Example problem 1.1
When an arbitrarily shaped plate of uniform thickness is subjected to a constant pressure, p along its periphery, Γ, verify that the normal stress, σ, and the shear stress, τ, are $\sigma = -p$ and $\tau = 0$ throughout the plate (Figure 1.10).

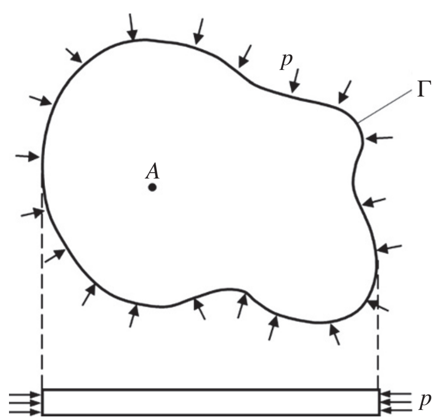

Figure 1.10

Solution
We cannot solve this problem by dividing the plane into small parts. We will solve the problem by visualizing the plate as an arbitrary shaped part within a larger, known plate instead.

Look at Figure 1.11. We know that the plate is subject to a constant pressure p along its periphery. We also know that the stresses inside the plate will be $\sigma_x = -p$, $\sigma_y = -p$ and $\tau_{xy} = 0$ when the x–y coordinate system is used. Now, drawing in the periphery Γ, as shown with the dotted line in Figure 1.11, we take the ξ axis to be normal to periphery at the arbitrary point B on the periphery Γ. Now we take an arbitrary point on the axis, the point O′, and define a ξ-η

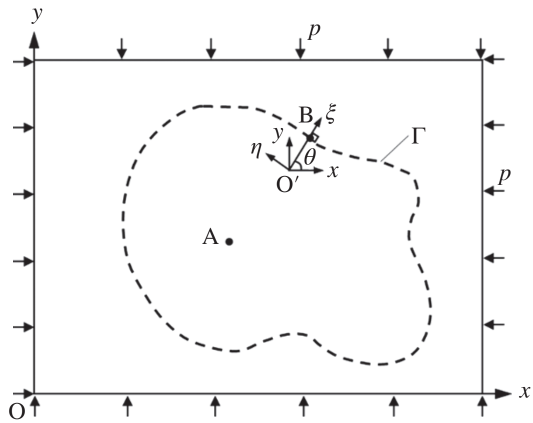

Figure 1.11

coordinate system with the origin at the point O′. When the angle between the x axis and the ξ axis is θ, the stresses at the point B can be obtained from Equation 1.12 and are as follows:

$$\sigma_\xi = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\cos\theta\cdot\sin\theta = -p\cos^2\theta - p\sin^2\theta + 0 = -p$$

$$\begin{aligned}\tau_{\xi\eta} &= \left(\sigma_y - \sigma_x\right)\cos\theta\cdot\sin\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)\\ &= (-p+p)\cos\theta\cdot\sin\theta + 0\left(\cos^2\theta - \sin^2\theta\right) = 0\end{aligned}$$

Point B is not a special point on the periphery of the plate and so we can see that the normal stress and shear stress will be the same everywhere along the periphery. This means that the conditions on the boundary of the plate in Figure 1.10, correspond to those of the periphery of the plate indicated by Γ, described within the rectangular plate in Figure 1.11. Consequently, to determine the stress state at point A in Figure 1.10, we have only to think of the stress state of the identical point A in Figure 1.11.

As was noted before, point B is not a special point in the rectangular plate and so the stresses at point A can be expressed in the same way as those at point B. Thus we can see that the stresses will be $\sigma = -p$ and $\tau = 0$ at any point and direction within Γ.

This example demonstrates how we can determine the stress state inside a plate, knowing only the boundary conditions and the stress transformation equation. We should notice the fact that we did not mention the material of the plate besides its uniform thickness. Never forget the conclusion gained from this problem. It is very important not only in elasticity but also in various problems of plasticity.

1.3.3.2 Three Dimensional Stress Transformation

This section describes the method to find the stresses in an arbitrary direction, when a solid, brick-like body is subjected to a uniform stress (Figure 1.12).

Suppose that the stresses σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx} (note: $\tau_{yx} = \tau_{xy}$, $\tau_{yz} = \tau_{zy}$, $\tau_{xz} = \tau_{zx}$) are known and the stresses in the $\xi - \eta - \zeta$ coordinate system are required. We denote the direction cosines between these two coordinate systems as shown in Table 1.2.

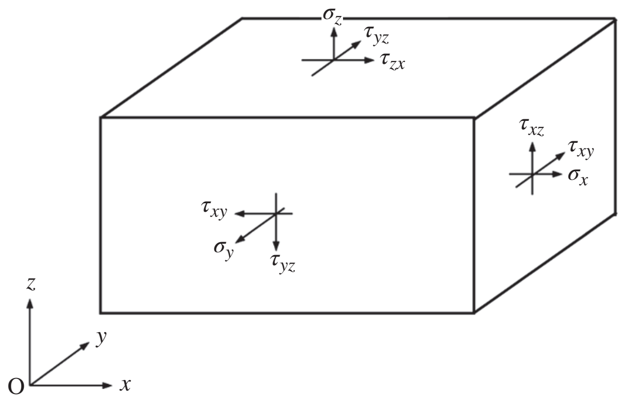

Figure 1.12

Table 1.2 Direction cosines

	x	y	z
ξ	l_1	m_1	n_1
η	l_2	m_2	n_2
ζ	l_3	m_3	n_3

Even before we find the solution it is possible to guess the form of the final solution. It is similar to the method adopted for the two dimensional (2D) case (Section 1.3.3.1) but is lengthier. So, before taking the lengthy approach, let us look at Equations 1.9 to 1.11 carefully, specifically by paying attention to the regularity of the direction cosines. In addition, we should be aware of the fact that *the 2D case is just a special case of a 3D case*, that is in 2D cases the z and ζ axes are perpendicular to the 2D plane (x–y and ξ–η planes). Thus, from these considerations and Equations 1.9 to 1.11, we can predict the following equations for the 3D case:

$$\left.\begin{aligned}
\sigma_\xi &= \sigma_x l_1{}^2 + \sigma_y m_1{}^2 + \sigma_z n_1{}^2 + 2\left(\tau_{xy} l_1 m_1 + \tau_{yz} m_1 n_1 + \tau_{zx} n_1 l_1\right)\\
\tau_{\xi\eta} &= \sigma_x l_1 l_2 + \sigma_y m_1 m_2 + \sigma_z n_1 n_2 + \tau_{xy}(l_1 m_2 + l_2 m_1)\\
&\quad + \tau_{yz}(m_1 n_2 + m_2 n_1) + \tau_{zx}(n_1 l_2 + n_2 l_1)\\
\tau_{\xi\zeta} &= \sigma_x l_1 l_3 + \sigma_y m_1 m_3 + \sigma_z n_1 n_3 + \tau_{xy}(l_1 m_3 + l_3 m_1)\\
&\quad + \tau_{yz}(m_1 n_3 + m_3 n_1) + \tau_{zx}(n_1 l_3 + n_3 l_1)\\
&\ldots
\end{aligned}\right\} \tag{1.13}$$

Actually, Equation 1.13 is an exact expression. The orthodox, and lengthy, derivation of Equation 1.13 was obtained by considering a triangular pyramid (tetrahedron) instead of the triangle that was used for the 2D case (Figure 1.8). We take the ξ axis to be perpendicular to the plane ABC and the area of ΔABC to be 1. Denoting the areas of ΔOBC, ΔOAC and

ΔOAB by A_x, A_y and A_z, respectively, we can see that $A_x = l_1$, $A_y = m_1$ and $A_z = n_1$. Using these relationships and by following a similar method based on the equilibrium condition of force, we can obtain Equation 1.13.

How the stresses of Equation 1.13 act on the plane ABC is illustrated in Figure 1.13. In Figure 1.14, we denote σ_ξ by σ and express the *resultant shear stress*, τ, on the plane ABC (ξ–η plane), as follows:

$$\tau^2 = \tau_{\xi\eta}{}^2 + \tau_{\xi\zeta}{}^2 \tag{1.14}$$

We also express the *resultant stress*, p, on the ξ–η plane by the following equation:

$$p^2 = \sigma^2 + \tau^2 \tag{1.15}$$

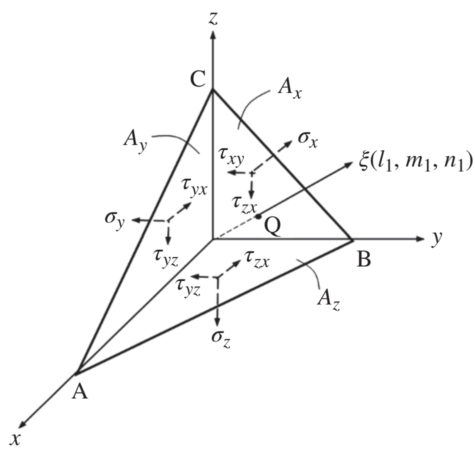

Figure 1.13

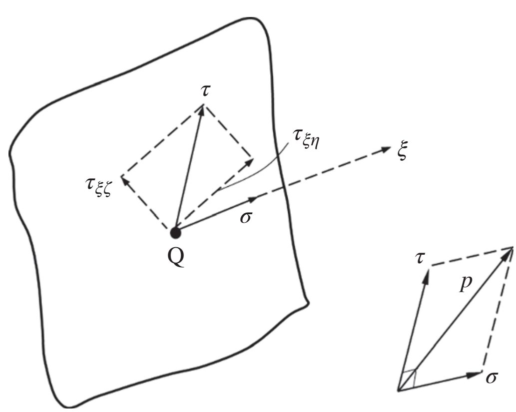

Figure 1.14

When $\tau = 0$, we have $p = \sigma$ and the resultant force acts perpendicular to the ξ–η plane. In this case, we call ξ a principal axis. As will be shown later, there are three principal axes.

1.3.4 Principal Stresses

1.3.4.1 Principal Stresses in 2D Stress State

By looking at Figures 1.7 and 1.8, we can see that in general there will be two kinds of stresses acting on the plane BC: normal stress σ_ξ and shear stress $\tau_{\xi\eta}$. However, if we vary the angle θ continuously from 0 to 2π, we will find that $\tau_{\xi\eta}$ vanishes at certain values of θ.

In Figure 1.15, p is the resultant stress, where $p^2 = {\sigma_\xi}^2 + {\tau_{\xi\eta}}^2$. If we take the length $|BC| = 1$ then we have $|AB| = m_1$ and $|AC| = l_1$. Now, by considering the equilibrium conditions of the element ABC and denoting the x and y components of the force acting on the side BC by p_x and p_y, respectively, we have:

$$\left.\begin{aligned} p_x &= \sigma_x l_1 + \tau_{xy} m_1 \\ p_y &= \tau_{xy} l_1 + \sigma_y m_1 \end{aligned}\right\} \tag{1.16}$$

As was mentioned before, at certain values of θ, $\tau_{\xi\eta}$ vanishes, so we can write

$$\left.\begin{aligned} p_x &= \sigma_\xi l_1 \\ p_y &= \sigma_\xi m_1 \end{aligned}\right\} \tag{1.17}$$

because $p = \sigma_\xi$ at those special angles.

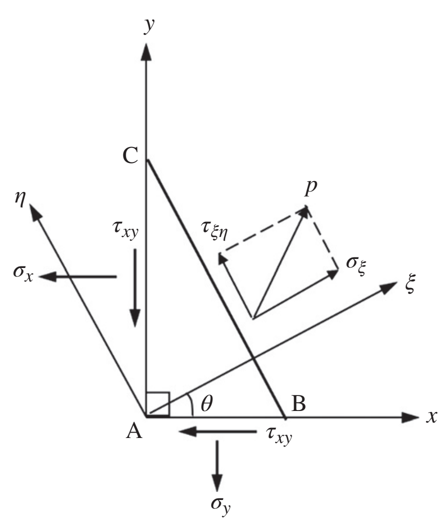

Figure 1.15

From Equations 1.16 and 1.17 and denoting σ_ξ by σ, we have

$$\left.\begin{aligned} \sigma_x l_1 + \tau_{xy} m_1 = \sigma l_1 \\ \tau_{xy} l_1 + \sigma_y m_1 = \sigma m_1 \end{aligned}\right\} \tag{1.18}$$

Rewriting Equation 1.18 gives

$$\left.\begin{aligned} (\sigma_x - \sigma) l_1 + \tau_{xy} m_1 = 0 \\ \tau_{xy} l_1 + (\sigma_y - \sigma) m_1 = 0 \end{aligned}\right\} \tag{1.19}$$

And by considering the properties of the direction cosines so that we know l_1 and m_1 cannot both be zero at the same angle, (i.e. $l_1{}^2 + m_1{}^2 \neq 0$) then the following equation must hold.

$$\begin{vmatrix} (\sigma_x - \sigma) & \tau_{xy} \\ \tau_{xy} & (\sigma_y - \sigma) \end{vmatrix} = 0 \tag{1.20}$$

Solving Equation 1.20 gives us two roots of σ. Denoting these roots by σ_1 and σ_2, we obtain,

$$\sigma_1, \sigma_2 = \frac{(\sigma_x + \sigma_y) \pm \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}{}^2}}{2} \tag{1.21}$$

σ_1 and σ_2 are the *principal stresses* (we define $\sigma_1 > \sigma_2$). In a 2D stress state, such as this, there are usually two principal stresses.

From the above discussion, *the principal stresses are the normal stresses acting on the planes with no shear stress.* We can see that the process to obtain Equation 1.21 from Equation 1.16 is identical with that to obtain the eigenvalues in linear algebra. This means that σ_1 and σ_2 correspond to the eigenvalues and, therefore, are the maximum and minimum normal stresses, respectively. The directions they act in are called the *principal axes*. The direction cosines for the principal axes can be obtained from Equation 1.19, that is

$$\frac{m_1}{l_1} = \tan\theta = \frac{\sigma - \sigma_x}{\tau_{xy}} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{\sigma - \sigma_x}{\tau_{xy}}\right) \tag{1.22}$$

In linear algebra, two eigenvectors, (i.e. the two principal axes), orthogonally intersect one another. This means $(l_1, m_1) \cdot (l_1', m_1') = 0$. The same result can be derived by differentiating σ_ξ with respect to θ in Equation 1.12 and using the condition $\mathrm{d}\sigma_\xi/\mathrm{d}\theta = 0$. However, the author recommends readers to memorize Equation 1.20 rather than Equation 1.21 or the conventional derivation by $\mathrm{d}\sigma_\xi/\mathrm{d}\theta = 0$, because Equation 1.20 is not only easy to memorize but it also helps us to see the physical meaning in a compact way. This method for the interpretation of the physical meaning of the principal stresses helps us to move easily into 3D problems, in which the conventional method of reduction becomes very lengthy and boring.

Example problem 1.2
Verify that the principal stresses σ_1 and σ_2 are the maximum and minimum values which normal stress can take.

Solution
Denoting the direction cosines between the principal axis 1 and the axes (ξ, η) by (l_1, l_2) and the direction cosines between the principal axis 2 and (ξ, η) by (m_1, m_2), from Equation 1.9:

$$\sigma_\xi = \sigma_1 {l_1}^2 + \sigma_2 {m_1}^2 \le \sigma_1 {l_1}^2 + \sigma_1 {m_1}^2 = \sigma_1$$
$$\sigma_\xi = \sigma_1 {l_1}^2 + \sigma_2 {m_1}^2 \ge \sigma_2 {l_1}^2 + \sigma_2 {m_1}^2 = \sigma_2$$

Hence, $\sigma_2 \le \sigma_\xi \le \sigma_1$.

Example problem 1.3
Find the principal stresses on the surface of a uniform round bar which is subjected to torsion, by determining a shear stress τ on the surface.

Solution

$$\begin{vmatrix} (\sigma_x - \sigma) & \tau_{xy} \\ \tau_{xy} & (\sigma_y - \sigma) \end{vmatrix} = \begin{vmatrix} -\sigma & \tau \\ \tau & -\sigma \end{vmatrix} = \sigma^2 - \tau^2 = 0$$
$$\sigma_1 = \tau, \quad \sigma_2 = -\tau$$
$$\theta_1 = \tan^{-1}\frac{\sigma_1 - \sigma_x}{\tau_{xy}} = \tan^{-1}\frac{\sigma_1}{\tau} = \tan^{-1}1 = \frac{\pi}{4}$$
$$\theta_2 = \tan^{-1}\frac{\sigma_2 - \sigma_x}{\tau_{xy}} = \tan^{-1}\frac{\sigma_2}{\tau} = \tan^{-1}(-1) = -\frac{\pi}{4}$$

This result means that, as shown in Figure 1.16, the stress state of $\sigma_x = \sigma_y = 0$, $\tau_{xy} = \tau$ is identical to the stress state of $\sigma_1 = \tau$ (tension) and $\sigma_2 = -\tau$ (compression) if looked from $\pm 45°$. The phenomena of shear fracture of ductile materials along the plane perpendicular to the axis of

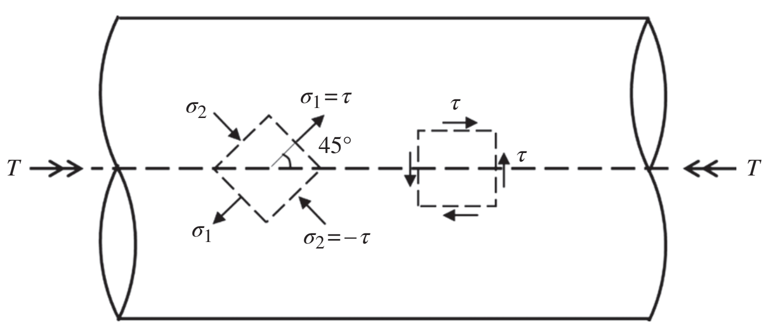

Figure 1.16

twisting moment and spiral shape fracture of brittle materials along the plane of $\sim 45^\circ$ are related to this stress state.

1.3.4.2 Principal Stresses in 3D Stress State

In a manner similar to that used with 2D problems, we can obtain an equation which determines the three principal stresses σ_1, σ_2 and σ_3, ($\sigma_1 > \sigma_2 > \sigma_3$) in 3D problems:

$$\begin{vmatrix} (\sigma_x - \sigma) & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & (\sigma_y - \sigma) & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & (\sigma_z - \sigma) \end{vmatrix} = 0 \tag{1.23}$$

Resolving Equation 1.23:

$$\begin{aligned} &\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma \\ &\quad - (\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2) = 0 \end{aligned} \tag{1.24}$$

If we denote the roots by σ_1, σ_2 and σ_3, Equation 1.24 can be rewritten in the following form:

$$(\sigma - \sigma_1) \cdot (\sigma - \sigma_2) \cdot (\sigma - \sigma_3) = 0 \tag{1.25}$$

And resolving again we have

$$\sigma^3 - J_1\sigma^2 - J_2\sigma - J_3 = 0 \tag{1.26}$$

where

$$\begin{aligned} J_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ J_2 &= -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \\ &= \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 - 2(\sigma_1 + \sigma_2 + \sigma_3)^2\right] \\ J_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned}$$

Since the principal stresses σ_1, σ_2 and σ_3 are independent of the coordinates chosen, J_1, J_2 and J_3 are constant in a certain stress state and are called the first, second and third *stress invariants*. If we note the equivalence of Equations 1.24 and 1.26, we will understand that

$$J_1 = \sigma_1 + \sigma_2 + \sigma_3 = \sigma_x + \sigma_y + \sigma_z = \sigma_\xi + \sigma_\eta + \sigma_\zeta \tag{1.27}$$

$$\begin{aligned} J_2 &= -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = -(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2) \\ &= -(\sigma_\xi\sigma_\eta + \sigma_\eta\sigma_\zeta + \sigma_\zeta\sigma_\xi - \tau_{\xi\eta}^2 - \tau_{\eta\zeta}^2 - \tau_{\zeta\xi}^2) \end{aligned} \tag{1.28}$$

$$\begin{aligned} J_3 = \sigma_1\sigma_2\sigma_3 &= \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}{}^2 - \sigma_y\tau_{zx}{}^2 - \sigma_z\tau_{xy}{}^2 \\ &= \sigma_\xi\sigma_\eta\sigma_\zeta + 2\tau_{\xi\eta}\tau_{\eta\zeta}\tau_{\zeta\xi} - \sigma_\xi\tau_{\eta\zeta}{}^2 - \sigma_\eta\tau_{\zeta\xi}{}^2 - \sigma_\zeta\tau_{\xi\eta}{}^2 \end{aligned} \tag{1.29}$$

We have obtained these important equations (Equations 1.27 to 1.29) without solving Equations 1.23 and 1.24. In general we cannot solve Equation 1.24 in the traditional manner, using a compass and triangles (which are used to find the principal stresses in 2D problems). This means that we cannot draw a so-called Mohr's circle from the stresses (σ_x, σ_y, σ_z, τ_{xy} τ_{yz}, τ_{zx}). However, we should not be disappointed, because Mohr's circles are not as important as they are widely assumed to be. J_1 is the quantity related to the component of hydrostatic compression of a stress state and, as explained later, it is the quantity related to the volume change in terms of strains through Hooke's law. J_2 is the quantity related to the cause of plastic deformation. The meaning of J_3 is not clear as J_1 and J_2. Equation 1.23 is simple and impressive, and as engineers know there is beauty in simplicity. This equation will also be very useful in later problems.

Example problem 1.4
Verify that the three principal axes intersect each other orthogonally in 3D problems.

Hint
Pay attention to the definition of the principal stress and the nature of shear stress.

1.3.5 Principal Shear Stresses

In the case of 2D stress states, the principal shear stress is called the maximum shear stress. Materials can fail by tensile normal stress or by shear stress depending on their ductility or brittleness. Failure of materials by shear stress occurs mostly in a plane with the maximum shear stress.

First we shall consider the maximum shear stress in a 2D stress state. From Equation 1.11, when σ_x, σ_y and τ_{xy} are known, the shear stress on the plane ξ = constant, or η = constant, can be expressed as

$$\tau_{\xi\eta} = \sigma_x l_1 l_2 + \sigma_y m_1 m_2 + \tau_{xy}(l_1 m_2 + l_2 m_1)$$

Since σ_x, σ_y and τ_{xy} are known, we can determine the principal stresses σ_1 and σ_2 and the direction cosines. Now, supposing that l_1', l_2' and so on are direction cosines defined between the ξ and η axes and the principal axes (direction of the principal stresses), we have

$$\tau_{\xi\eta} = \sigma_1 l_1' l_2' + \sigma_2 m_1' m_2', \quad \sigma_1 > \sigma_2 \tag{1.30}$$

and by considering the nature of the direction cosines (see Appendix A.1 of Part I),

$$l_1' l_2' + m_1' m_2' = 0,$$

we obtain,

$$\tau_{\xi\eta} = (\sigma_1 - \sigma_2) l_1' l_2' \tag{1.31}$$

Now we need to know the direction in which $l_1'l_2'$ becomes maximum. Since $(\sigma_1 - \sigma_2)$ is a quantity that is constant irrespective of the direction, we must make $|l_1'l_2'|$ the maximum.

Since $l'^2_1 + l'^2_2 = 1$, we have the absolute maximum value of $\tau_{\xi\eta}$ when $|l_1'| = |l_2'|$ and so we have the angle at which the maximum shear stress acts:

$$\theta = \pi/4 \quad \text{or} \quad (\pi/2 + \pi/4) \tag{1.32}$$

$$\tau_{\max} = \mp \frac{1}{2}(\sigma_1 - \sigma_2) \tag{1.33}$$

In Equation 1.33, the plus and minus signs in front of the parentheses are not important, they only indicate the direction of the shear stress. This result means that the maximum shear stress is in the plane at an angle of $\pi/4$ from both the principal axes. Looking at the stress state from the direction of one of the principal axes and using a similar method, we can derive the *principal shear stresses* in three dimensions that correspond to $\tau_{\max}$ in 2D problems as follows:

$$\tau_1 = \frac{1}{2}|\sigma_2 - \sigma_3|, \quad \tau_2 = \frac{1}{2}|\sigma_3 - \sigma_1|, \quad \tau_3 = \frac{1}{2}|\sigma_1 - \sigma_2| \tag{1.34}$$

One of the shear stresses is algebraically maximum, another minimum and the last one is between these. τ_1 acts on the plane which divides equally the angle made by two planes on which σ_2 and σ_3 act. The same rule applies to τ_2 and τ_3 respectively.

Problems of Chapter 1

1. A tensile test was carried out on a belt which was made by bonding the straight ends, as shown in Figure 1.17a. A tensile fracture occurred from the bonded end at 1/400 of the original strength of the belt. In order to increase the strength of the bonded end, the belt was sliced in an inclined direction and was bonded again with the same bond. Determine the approximate angle of slicing in Figure 1.17b to prevent fracture at the bonded interface. Assume that the bonded interface fractures only by tensile normal stress and the fracture is not influenced by shear stress.

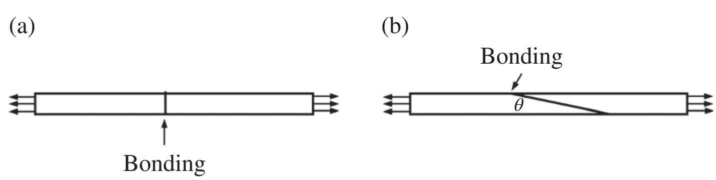

Figure 1.17

2. Determine the stress inside a cylinder which is subjected to the same internal pressure p_i and external pressure p_o, that is $p_i = p_o = -p_0$.
3. When the normal stress σ_ξ and shear stress $\tau_{\xi\eta}$ in the direction of $\theta = \theta_0$ against the axis of the bar at the outer radius of a cylindrical bar under a twisting moment are $\sigma_\xi = \sigma_0$ and