# STRUCTURAL RELIABILITY ANALYSIS AND PREDICTION

THIRD EDITION

ROBERT E. MELCHERS
ANDRÉ T. BECK

WILEY

**Structural Reliability Analysis and Prediction**

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Third Edition

WILEY

This edition first published 2018


*Registered Office(s)*
John Wiley & Sons, Inc., 111 River Street, Hoboken, NJ 07030, USA
John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, UK

*Editorial Office*
The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, UK

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*Library of Congress Cataloging-in-Publication Data*

Names: Melchers, R. E. (Robert E.), 1945- author. | Beck, André T., author.
Title: Structural reliability analysis and prediction / by Robert E.
Melchers, University of Newcastle, AS, André T. Beck, University of São
Paulo, BR.
Description: Third edition. | Hoboken, NJ : Wiley, 2018. | Includes
bibliographical references and index. |
Identifiers: LCCN 2017031782 (print) | LCCN 2017032813 (ebook) | ISBN
9781119266075 (pdf) | ISBN 9781119266068 (epub) | ISBN 9781119265993 (pbk.)
Subjects: LCSH: Structural stability. | Reliability (Engineering) |
Probabilities.
Classification: LCC TA656.5 (ebook) | LCC TA656.5 .M45 2018 (print) | DDC
624.1/71–dc23
LC record available at https://lccn.loc.gov/2017031782

Cover Design: Wiley
Cover Image: © maxsattana/Gettyimages

Set in 10/12pt WarnockPro by SPi Global, Chennai, India

10 9 8 7 6 5 4 3 2 1

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# Preface

This third edition marks some 18 years since the second edition of this book appeared and what seems like half a lifetime ago—some 31 years—since the first edition was written. It has been extremely gratifying that the book has lasted this long, that it continues to be used by many and that a new edition was welcomed by Wiley.

Since the second edition the subject has consolidated and largely turned to more and more areas of application, including a renewed interest from the geotechnical engineering research community. But also in practice structural reliability increasingly is being applied, particularly for situations where quantitative, data-based risk assessment of non-elementary structural or other systems is required. Overviews of the papers contributed to conferences such as ICASP, ICOSSAR, IFIP, IALCCE and CSM shows much attention paid to applications and relatively little to sorting out some of the remaining really challenging theoretical problems such as how to deal with complex systems with a multitude of random variables or processes, and for which many potential failure modes and combination of such modes may exist. Fortunately, the availability of ever greater computational power has meant that enumeration methods, once thought to be the way forward for dealing with really complex problems, can be cast aside in favour of sheer brute force number crunching. In this sense Chapter 3 and the parts of Chapter 5 dealing with Monte Carlo methods are now more important, for practical problems, than the elegant but simpler FOSM/FOR/SOR methods that allow easier insight into 'what was driving what'.

The present edition follows much of the second edition but updates areas such as Monte Carlo methods, systems reliability, some aspects of load and resistance modelling, code calibration, analysis of existing structures and adds, for the first time, a chapter on optimization in the context of structural reliability. The co-author for this edition, André T. Beck, has contributed much to these changes, as well as to the worked examples provided where relevant for each chapter and collected together in Appendix F. We have had the good fortune to have at hand the many comments and corrections, principally supplied by Dr. Bill Gray during his post-doctoral days at The University of Newcastle. As before, we have had to be selective in our coverage and have had to make difficult decisions about what to include and what leave out.

Now, as 18 years ago, a surf or a beach run at Newcastle's wonderful Pacific Ocean beaches, a surf or a bike ride along the south-eastern Brazilian coast, seem better ways to spend one's time than revising a book. Our spouses tell us so, our colleagues tell us so, our minds tell us so, but what do we do?

10 February 2017

*Robert E. Melchers*
Bar Beach, Newcastle

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Florianópolis, SC

# Preface to the Second Edition

It is over ten years since the first edition of this book appeared and more than 12 years since the text was written. At the time structural reliability as a discipline was evolving rapidly but was also approaching a degree of maturity. Perhaps it is not surprising, then, that rather little of the first edition now seems out-dated.

This edition differs from the first mainly in matters of detail. The overall layout has been retained but all of the original text has been reviewed. Many sections have been partly rewritten to make them clearer and more complete and many, often small but annoying, errors and mistakes have been corrected. Hopefully not too many new ones have crept in. Many new references have been added and older, now less relevant, ones deleted. This is particularly the case in referring to applications, in which area there has been much progress.

The most significant changes in this edition include the up-dating of the sections dealing with Monte Carlo simulation, the addition of the Nataf transformation in the discussion of FOSM/FORM methods, some comments about asymptotic methods, additional discussion of structural systems subject to multiple loads and a new chapter devoted to the safety checking of existing structures, an area of increasing importance.

Other areas in which there have been rapid developments, such as simulation of random processes and random fields, applications in structural dynamics and fatigue and specialist refinements of theory are all of interest but beyond the scope of an introductory book. Readers might care to refer to the specialist literature, proceedings of conferences such as the ICASP, ICOSSAR and IFIP series and to journals such as Structural Safety, Probabilistic Engineering Mechanics and the Journals of Engineering Mechanics and Structural Engineering of ASCE. Overviews of various aspects of applications of structural reliability are given also in Progress in Structural Engineering and Mechanics. There are, of course, other places to look, but these should form a good starting point for keeping in touch with theoretical developments and applications.

In preparing this edition I had the good furtune to have at hand a range of comments, notes and advice. I am particularly indebted to my immediate colleagues Mark Stewart and Dimitry Val for their critical comments and their assistance with some of the new sections. Former research students have also contributed and I mention in this regard particularly H.Y. Chan, M. Moarefzadeh and X.L. Guan. Naturally, I owe a very significant debt to the international structural reliability community in general and to some key people in particular, including Ove Ditlevsen, Rudiger Rackwitz, Armen Der Kiureghian and Bruce Ellingwood—they, and many others, will know that I appreciate their forebearance and friendship.

The encouragement and generous comments from many sources is deeply appreciated. It has contributed to making the hard slog of revision a little less painful. Sometimes a beach run or a surf seemed a better alternative to spending an hour or so making more corrections to the text… As before, the forebearance of my family is deeply appreciated. Like many academic households they have learnt that academics are their own worst enemies and need occasionally to be dragged away from their Macintoshes to more socially acceptable activities.

August, 1998

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Bar Beach, Newcastle

## Preface to the First Edition

The aim of this book is to present a unified view of the techniques and theory for the analysis and prediction of the reliability of structures using probability theory. By reliability, in this context, will be understood not just reliability against extreme events such as structural collapse or facture, but against the violation of any structural engineering requirements which the structure is expected to satisfy.

In practice, two classes of problems may arise. In the first, the reliability of an existing structure at the 'present time' is required to be assessed. In the second, and much more difficult class, the likely reliability of some future, or as yet uncompleted, structure must be predicted. One common example of such a requirement is in structural design codes, which are essentially instruments for the prediction of structural safety and serviceability supported by previous experience and expert opinion. Another example is the reliability assessment of major structures such as large towers, offshore platforms and industrial or nuclear plants for which structural design codes are either not available or not wholly acceptable. In this situation, the prediction of safety both in absolute terms and in terms of its interrelation to project economics is becoming increasingly important. This class of assessment relies on the (usually reasonable but potentially dangerous) assumption that past experience can be extrapolated into the future.

It might be evident from these remarks that the analysis (and prediction) of structural reliability is rather different from the types of analysis normally performed in structural engineering. Concern is less with details of stress calculations, or member behaviour, but rather with the uncertainties in such behaviour and how this interacts with uncertainties in loading and in material strength. Because such uncertainties cannot be directly observed for any one particular structure, there is a much greater level of abstraction and conceptualization in reliability analysis than is conventionally the case for structural analysis or design. Modelling is not only concerned with the proper and appropriate representation of the physics of any structural engineering problem, but also with the need to obtain realistic, sufficiently simple and workable models or representations of both the loads and the material strengths, and also their respective uncertainties. How such modelling might be done and how such models can be used to analyse or predict structural reliability is the central theme of this book.

In one important sense, however, the subject matter has a distinct parallel with conventional structural engineering analysis and its continual refinement; that is, that ultimately concern is with costs. Such costs include not only those of design, construction, supervision and maintenance but also the possible cost of failure (or loss of serviceability). This theme, although not explicitly pursued throughout the book, is nevertheless a central one, as will become clear in Chapter 2. The assessment or predictions obtained using the methods outlined in this book have direct application in decision-making techniques such as cost-benefit analysis or, more precisely when probability is included, risk-benefit analysis. As will be seen in Chapter 9, one important area of application for the methods presented here is in structural design codes, which, it will be recognized, are essentially particular (if perhaps rather crude and intuitive) forms of risk-benefit methodology.

A number of other recent books have been devoted to the structural reliability theme. This book is distinct from the others in that it has evolved from a short course of lectures for undergraduate students as well as a 30-h graduate course of lectures which the author has given periodically to (mainly) practising structural engineers during the last 8 years. It is also different in that it does not attempt to deal with related topics such as spectral analysis for which excellent introductory texts are already available.

Other features of the present book are its treatment of structural system reliability (Chapter 5) and the discussion of both simulation methods (Chapter 3) and modern second-moment and transformation methods (Chapter 4). Also considered is the important topic of human error and human intervention in the relationship between calculated (or 'nominal') failure probabilities and those observed in populations of real structures (Chapter 2).

The book commences (Chapter 1) by reviewing traditional methods of defining structural safety such as the 'factor of safety', the 'load factor', 'partial factor' formats (i.e. 'limit state design' formats) and the 'return period'. Some consistency aspects of these methods are then presented and their limited use of available data noted, before a simple probabilistic safety measure, the 'safety margin' and the associated failure probability are introduced. This simple one-load one-resistance model is sufficient to introduce the fundamental ideas of structural reliability assessment. Apart from Chapter 2, the rest of the book is concerned with elaborating and illustrating the reliability analysis and prediction theme.

While Chapters 3, 4 and 5 deal with particular calculation techniques for time-independent situations, Chapter 6 is concerned with extending the 'return period' concept introduced in Chapter 1 to more general formulations for time-dependent problems. The three principal methods for introducing time, the time-integrated approach, the discrete time approach and the fully time-dependent approach, are each outlined and examples given. The last approach is considerably more demanding than the other two (classical) methods since it is necessary to introduce elements of stochastic process theory. First-time readers may well decide to skip rather quickly through much of this chapter. Applications to fatigue problems and structural vibrations are briefly discussed from the point of view of probability theory, but again the physics of these problems is outside the scope of the present book.

Modelling of wind and floor loadings is described in Chapter 7 whilst Chapter 8 reviews probability models generally accepted for steel properties. Both load and strength models are then used in Chapter 9. This deals with the theory of structural design codes and code calibration, an important area of application for probabilistic reliability prediction methodology.

It will be assumed throughout that the reader is familiar with modern methods of structural analysis and that he (or she) has a basic background in statistics and probability. Statistical data analysis is well described in existing texts; a summary of probability theory used is given in Appendix A for convenience.

Further, reasonable competence in applied mathematics is assumed since no meaningful discussion of structural reliability theory can be had without it. The level of presentation, however, should not be beyond the grasp of final-year undergraduate students in engineering. Nevertheless, particularly difficult theoretical sections which might be skipped on a first reading are marked with an asterisk (*).

For teaching purposes, Chapters 1 and 2 could form the basis for a short undergraduate course in structural safety. A graduate course could take up the topics covered in all chapters, with instructors having a bias for second-moment methods skipping over some of the sections in Chapter 3 while those who might wish to concentrate on simulation could spend less time on Chapter 4. For an emphasis on code writing, Chapters 3 and 5 could be deleted and Chapters 4 and 6 cut short.

In all cases it is essential, in the author's view, that the theoretical material be supplemented by examples from experience. One way of achieving this is to discuss particular cases of structural failure in quite some detail, so that students realize that the theory is only one (and perhaps the least important) aspect of structural reliability. Structural reliability assessment is not a substitute for other methods of thinking about safety, nor is it necessarily any better; properly used, however, it has the potential to clarify and expose the issues of importance.

## Acknowledgements

This book has been a long time in the making. Throughout I have had the support and encouragement of Noel Murray, who first started me thinking seriously about structural safety, and also of Paul Grundy and Alan Holgate. In more recent times, research students Michael Harrington, Tang Liing Kiong, Mark Stewart and Chan Hon Ying have played an important part.

The first (and now unrecognizable) draft of part of the present book was commenced shortly after I visited the Technical University, Munich, during 1980 as a von Humboldt Fellow. I am deeply indebted to Gerhart Schueller, now of Universität Innsbruck, for arranging this visit, for his kind hospitality and his encouragement. During this time, and later, I was also able to have fruitful discussions with Rudiger Rackwitz.

Part of the last major revision of the book was written in the period November 1984-May 1985, when I visited the Imperial College of Science and Technology, London, with the support of the Science and Engineering Research Council. Working with Michael Baker was a most stimulating experience. His own book (with Thoft-Christensen) has been a valuable source of reference.

Throughout I have been extremely fortunate in having Mrs. Joy Helm and more recently Mrs. Anna Teneketzis turn my difficult manuscript into legible typescript. Their cheerful co-operation is very much appreciated, as is the efficient manner with which Rob Alexander produced the line drawings.

Finally the forbearance of my family was important, many a writing session being abruptly concluded with a cheerful 'How's Chapter 6 going, Dad?'

December 1985

*Robert E. Melchers*
Monash University

# 1

# Measures of Structural Reliability

## 1.1 Introduction

The manner in which an engineering structure will respond to loading depends on the type and magnitude of the applied load and the structural strength and stiffness. Whether the response is considered satisfactory depends on the requirements that must be satisfied. These include safety of the structure against collapse, limitations on damage, or on deflections or other criteria. Each such requirement may be termed a limit state. The 'violation' of a limit state can then be defined as the attainment of an undesirable condition for the structure. Some typical limit states are given in Table 1.1.

**Table 1.1** Typical limit states for structures.

| Limit state type | Description | Examples |
|---|---|---|
| Ultimate (safety) | Collapse of all or part of structure | Tipping or sliding, rupture, progressive collapse, plastic mechanism, instability, corrosion, fatigue, deterioration, fire. |
| Damage (often included in above) | | Excessive or premature cracking, deformation or permanent inelastic deformation. |
| Serviceability | Disruption of normal use | Excessive deflections, vibrations, local damage, etc. |

From observation it is known that very few structures collapse, or require major repairs, etc., so that the violation of the most serious limit states is a relatively rare occurrence. When violation of a limit state does occur, the consequences may be extreme, as exemplified by the spectacular collapses of structures such as the Tay Bridge (wind loading), Ronan Point Flats (gas explosion), Kielland Offshore Platform (local strength problems), Kobe earthquake (ductility), etc.

The study of structural reliability is concerned with the *calculation* and *prediction* of the probability of limit state violation for an engineered structural system at any stage during its life. In particular, the study of structural safety is concerned with the violation of the ultimate or safety limit states for the structure. More generally, the study of structural reliability is concerned with the violation of performance measures (of which ultimate or safety limit states are a subset). This broader definition allows the scope of application to move from structural criteria as specified in traditional design codes

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(Chapter 9) to broader-based performance requirements for structures, such as might be used in design optimization processes (Chapter 11).

In the simplest case, the probability of occurrence of an event such as limit state violation is a numerical measure of the chance of its occurrence. This measure either may be obtained from measurements of the long-term frequency of occurrence of the event for generally similar structures, or it may be simply a subjective estimate of the numerical value. In practice it is seldom possible to observe for a sufficiently long period of time, and a combination of subjective estimates and frequency observations for structural components and properties may be used to predict the probability of limit state violation for the structure.

In probabilistic assessments any uncertainty about a variable (expressed, as will be seen, in terms of its probability density function) is taken into account explicitly. This is not the case in traditional ways of measuring safety, such as the 'factor of safety' or 'load factor'. These are 'deterministic' measures, since the variables describing the structure, its strength and the applied loads are assumed to take on known (if conservative) values about which there is assumed to be no uncertainty. Precisely because of their traditional and really quite central position in structural engineering, it is appropriate to review the deterministic safety measures prior to developing probabilistic safety measures.

## 1.2 Deterministic Measures of Limit State Violation

### 1.2.1 Factor of Safety

The traditional method to define structural safety is through a 'factor of safety', usually associated with elastic stress analysis and which requires that:

$$\sigma_{\mathrm{i}}(\varepsilon) \leq \sigma_{p\,\mathrm{i}} \tag{1.1}$$

where $\sigma_i(\varepsilon)$ is the $i$ th applied stress component calculated to act at the generic point $\varepsilon$ in the structure, and $\sigma_{pi}$ is the permissible stress for the $i$ th stress component.

The permissible stresses $\sigma_{pi}$ are usually defined in structural design codes. They are derived from material strengths (ultimate moment, yield point moment, squash load, etc.), expressed in stress terms $\sigma_{ui}$ but reduced through a factor $F$:

$$\sigma_{pi} = \sigma_{ui}/F \tag{1.2}$$

where $F$ is the 'factor of safety'. The factor $F$ may be selected on the basis of experimental observations, previous practical experience, economic and, perhaps, political considerations. Usually, its selection is the responsibility of a code committee.

According to (1.1), failure of the structure should occur when any stressed part of it reaches the local permissible stress. Whether failure actually does occur depends entirely on how well $\sigma_i(\varepsilon)$ represents the actual stress in the real structure at $\varepsilon$ and how well $\sigma_{pi}$ represents actual material failure. It is well known that observed stresses do not always correspond well to the stresses calculated by linear elastic structural analysis (as commonly used in design). Stress redistribution, stress concentration and changes due to boundary effects and the physical size effect of members all contribute to the discrepancies.

Similarly, the permissible stresses that, commonly, are associated with linear elastic stress analysis are not infrequently obtained by linear scaling down, from well beyond the linear region, of the ultimate strengths obtained from tests. From the point of view of structural safety, this does not matter very much, provided that the designer recognizes that his calculations may well be quite fictitious and provided that (1.1) is a conservative safety measure.

By combining expressions (1.1) and (1.2) the condition of 'limit state violation' can be written as

$$\frac{\sigma_{ui}(\varepsilon)}{F} \leq \sigma_i(\varepsilon) \quad \text{or} \quad \frac{\sigma_{ui}(\varepsilon)}{F} \Big/ \sigma_i(\varepsilon) \leq 1 \tag{1.3}$$

Expressions (1.3) are 'limit state equations' when the inequality sign is replaced by an equality. These equations can be given also in terms of stress resultants, obtained by appropriate integration:

$$\frac{R_i(\varepsilon)}{F} \leq S_i(\varepsilon) \quad \text{or} \quad \frac{R_i(\varepsilon)}{F} \Big/ S_i(\varepsilon) \leq 1 \quad \text{for all } i \tag{1.4}$$

where $R_i$ is the $i$th resistance at location $\varepsilon$ and $S_i$ is the $i$th stress resultant (internal action). In general, the stress resultant $S_i$ are made up of the effects of one or more applied loads $Q_j$; typically

$$S_i = S_{iD} + S_{iL} + S_{iW}$$

where $D$ is the dead load, $L$ is the live load and $W$ is the wind load.

The term 'safety factor' also has been used in another sense, namely in relation to overturning, sliding, etc., of structures as a whole, or as in geomechanics (dam failure, embankment slip, etc.). In this application, expressions (1.3) are still valid provided that the stresses $\sigma_{ui}$ and $\sigma_i$ are interpreted appropriately.

### 1.2.2 Load Factor

The 'load factor' $\lambda$ is a special kind of safety factor developed for use originally in the plastic theory of structures. It is the theoretical factor by which a set of loads acting on the structure must be multiplied, just enough to cause the structure to collapse. Commonly, the loads are taken as those acting on the structure during service load conditions. The strength of the structure is determined from the idealized plastic material strength properties for structural members [Heyman, 1971].

For a given collapse mode (i.e. for a given ultimate 'limit state'), the structure is considered to have 'failed' or collapsed when the plastic resistances $R_{pi}$ are related to the factored loads $\lambda Q_j$ by

$$W_R(\mathbf{R}_R) \leq W_Q(\lambda \mathbf{Q}) \tag{1.5}$$

where $\mathbf{R}_p$ is the vector of all plastic resistances (e.g. plastic moments) and $\mathbf{Q}$ is the vector of all applied loads. Also, $W_R(\,)$ is the internal work function and $W_Q(\,)$ the external work function, both described by the plastic collapse mode being considered.

If proportional loading is assumed, as is usual, the load factor can be taken out of parentheses. Also the loads $Q_j$ usually consist of several components, such as dead, live, wind, etc. Thus (1.5) may be written in the form of a limit state equation:

$$\frac{W_R(R_{pi})}{\lambda W_Q(Q_D + Q_L + \ldots)} = 1$$

with 'failure' denoted by the left-hand side being less than unity.

Clearly there is much similarity in formulation between the factor of safety and the load factor as measures of structural safety. What *is* different is the reference level at which the two measures operate: the first at the level of working loads and at the 'member' level; the second at the level of collapse loads and at the 'structure level'.

### 1.2.3 Partial Factor ('Limit State Design')

A development of the above two measures of safety is the so-called 'partial factor' approach. For limit state $i$ it can be expressed at the level of stress resultants (i.e. member design level) as

$$\phi_i R_i \leq \gamma_{Di} S_{Di} + \gamma_{Li} S_{Li} + \ldots \tag{1.6}$$

where $R$ is the member resistance, $\phi$ is the partial factor on $R$ and $S_D, S_L$ are the dead and live load effects respectively with associated partial factors $\gamma_D, \gamma_L$. Expression (1.6) was originally developed during the 1960s for reinforced concrete codes. It enabled the live and wind loads to have greater 'partial' factors than the dead load, in view of the former's greater uncertainty, and it allowed a measure of workmanship variability and uncertainty about resistance modelling to be associated with the resistance $R$ [MacGregor, 1976]. This extension of earlier safety formats had considerable appeal since it allowed better representation of the factors and uncertainties associated with loadings and resistances.

For a plastic collapse analysis at the structure level, formulation (1.6) becomes

$$W_R(\phi \mathbf{R}) \leq W_Q(\gamma_D \mathbf{Q}_D + \gamma_L \mathbf{Q}_L + \ldots)$$

where $\mathbf{R}$ and $\mathbf{Q}$ are vectors of resistance and loads respectively. Clearly the partial factors $(\phi, \gamma)$ in this expression will be different from those of expression (1.6).

**Example 1.1** The simple portal frame of Figure 1.1(a) is subject to loads $Q_1$ and $Q_2$. If the relative moments of inertia of the members are known, the elastic bending moment diagram can be found as in Fig 1.1(b). The 'limit states' for bending capacity are then

section 2: $$\phi M_{C2} = \gamma_1 \frac{3l}{16} Q_1 + \gamma_2 \frac{3l}{16} Q_2$$

sections 1 and 3: $$\phi M_{C1,3} = \gamma_1 \frac{l}{16} Q_1 + \gamma_2 \frac{l}{16} Q_2$$

where $\phi, \gamma_1$ and $\gamma_2$ are partial factors described by a structural design code. The $M_{Ci}$ are the ultimate moment capacities required at sections $i (i = 1, 2, 3)$ for the structure to be considered 'just safe'.

If the frame is to be designed or analysed assuming rigid-plastic theory, the relative distribution of the plastic moments $M_{pi}$ around the frame must be known or assumed.

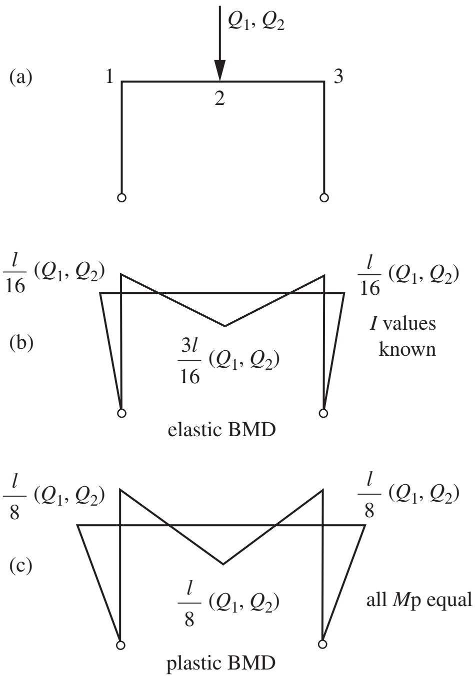



**Figure 1.1** Bending moment diagrams for Example 1.1.

If they all are equal, the plastic bending moment diagram of [Figure 1.1(c) is obtained and only one limit state equation is needed for sections 1–3:]

$$\phi_p M_{pi} = \gamma_{p1}\frac{l}{8}Q_1 + \gamma_{p2}\frac{l}{8}Q_2$$

where now $M_{pi}$ is the required plastic moment capacity at sections 1, 2 and 3 and where $\phi_p, \gamma_{p1}$ and $\gamma_{p2}$ are now code-prescribed partial factors for plastic structural systems.

### 1.2.4 A Deficiency in Some Safety Measures: Lack of Invariance

From Example 1.1, it will be evident that the partial factors $\phi$ and $\gamma_i (i = 1, \ldots)$ in (1.6) depend on the limit state being considered. Hence they depend on the definitions of $R, S_D$ and $S_L$. However, even for a given limit state, these definitions are not necessarily unique, and therefore the partial factors may not be unique either. This phenomenon is termed the 'lack of invariance' of the safety measure. It arises because there are different ways in which the relationships between resistances and loads may be defined. Some examples of this are given below. Ideally, the safety measure should not depend on the way in which the loads and resistances are defined.

**Example 1.2** The structure shown in Figure 1.2 is supported on two columns. The capacity of column B is $R = 24$ in compression. The safety of the structure can be measured in three different ways using the traditional 'factor of safety' $F$:

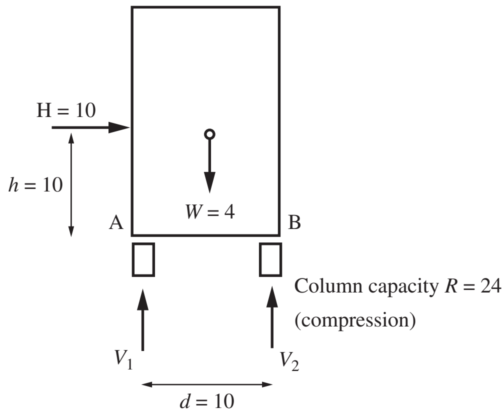



**Figure 1.2** Example 1.2: Structure subject to overturning under lateral load $H$ and with vertical load $W$ and supported by two columns applying vertical forces $V_1$ and $V_2$.

(a) *Overturning resistance about A*

$$F_1 = \frac{\textit{resisting moment about A}}{\textit{overturning moment about A}} = \frac{dR}{Hh + Wd/2} = \frac{10 \times 24}{10 \times 10 + 4 \times 5} = 2.0$$

(b) *Capacity of column B*

$$F_2 = \frac{\textit{compression resistance of column B}}{\textit{compressive load on column B}} = \frac{R}{Hh/d + W/2} = 2.0$$

(c) *Net capacity of column B* (resistance minus load effect of W)

$$F_3 = \frac{\textit{net compression resistance of column B}}{\textit{net compressive load on column B}} = \frac{R - W/2}{Hh/d} = 2.2$$

All three of these factors of safety $F_i (i = 1,\ 2,\ 3)$ for column B apply to the same structure and the same loading, so that the difference in the values of $F_i$ is due entirely what is considered to represent the resistance of the structure and what is considered to be the applied load. In general such a difference in outcomes is not helpful for the unique definition of a factor of safety. However, for some special cases of the partial factors the outcomes can be made the same. Thus it is easily verified that the calculations give the identical result $F_1 = F_2 = F_3 = 1.0$ if a partial safety factor $\phi = \frac{1}{2}$ is applied to the resistance $R$, thus:

$$F_1 = \frac{d\phi R}{Hh + Wd/2}, \quad F_2 = \frac{\phi R}{Hh/d + W/2} \quad F_3 = \frac{\phi R - W/2}{Hh/d}$$

Similarly, the result $F_1 = F_2 = F_3 = 1.0$ would be achieved if the loads $H$ and $W$ were factored by $\gamma = 2$. More generally, any choice of combination of $\phi$ and $\gamma$ could be appropriate, provided that $F_i = 1$. This can be expressed as:

$$F_1 = \frac{d\phi R}{\gamma(Hh + Wd/2)}, \quad F_2 = \frac{\phi R}{\gamma(Hh/d + W/2)}, \quad F_3 = \frac{\phi R - \gamma W/2}{\gamma Hh/d}$$

with $F_1 = F_2 = F_3 = 1$

A different way of defining a measure of safety is the 'safety margin'. It measures the excess of resistance compared with the stress resultant (or loading); thus:

$$Z = R - S \tag{1.7}$$

For the present example, the safety margins are

$$Z_1 = dR - (Hh + Wd/2) \tag{1.7a}$$

$$Z_2 = R - (Hh/d + W/2) \tag{1.7b}$$

and

$$Z_3 = R - W/2 - Hh/d \tag{1.7c}$$

It is readily verified that when $Z = 0$, i.e. at the point of failure, these three expressions are equivalent. This shows that the safety margin concept of safety is 'invariant' with respect to the limit state functions (1.7a–c).

**Example 1.3** [adapted from Ditlevsen, 1973] The reinforced concrete beam shown in Figure 1.3(a) has a moment capacity $R$ when it is subject to an axial force $N$ and a moment $M$ applied at the beam centroid $\xi = 0$. Both $N$ and $M$ are composed of the effects of a dead load and a live load: $N = N_D + N_L$ and $M = M_D + M_L$. The moment capacity calculated about $\xi = a$ is $R_1 = R + aN$, from simple statics. (Note that the actual moment capacity of the beam is not changed!) Also, at $\xi = a$, the applied moment is given by $M_1 = M + aN$. The state 'just safe' can now be defined for given moment capacity $R$, and given axial force $N$, by the factor of safety as:

$$F_0 = \frac{R}{M} \qquad \text{at } \xi = 0 \tag{1.8a}$$

$$F_1 = \frac{R_1}{M_1} = \frac{R + aN}{M + aN} \qquad \text{at } \xi = a \tag{1.8b}$$

In this format $F_1 = F_0$ is true only when $\xi = a = 0$. This means that the factor of safety depends on the convention chosen for the origin of the applied actions and of the resistance. If, as in Example 1.2, $R$ is replaced by the factored term $\phi R$, such that $F_0 = 1$, then it follows readily that $F_1$ is also unity. Hence, provided that 'partial factor' $\phi$ is chosen in such a way that the 'factor of safety' $F$ is unity, the origin chosen to define $R$, $N$ and $M$ is immaterial. A similar result holds if $N$ and $M$ are replaced by $\gamma N$ and $\gamma M$, where $\gamma$ is an appropriately chosen partial factor on the loading.

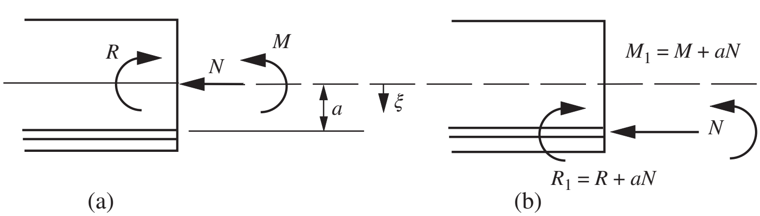



**Figure 1.3** Reinforced concrete beam: Example 1.3.

The state 'just safe' can be written also in the partial factor format of (1.6). Indeed, noting that $M = M_D + M_L$ and $N = N_D + N_L$, at $\xi = 0$ it follows that

$$\phi R = \gamma_D M_D + \gamma_L M_L \tag{1.9a}$$

and at $\xi = a$, treating, as before, $R_1 = R + aN$ as the resistance to bending,

$$\phi(R + aN_D + aN_L) = \gamma_D(M_D + aN_D) + \gamma_L(M_L + aN_L) \tag{1.9b}$$

Subtracting (1.9a) from (1.9b) and dividing out by $a$ leaves

$$(\phi - \gamma_D)N_D + (\phi - \gamma_L)N_L = 0 \tag{1.10}$$

Since in general $N_D$, $N_L > 0$, it follows that (1.10) will be satisfied only if either $\gamma_D \leq \phi \leq \gamma_L$ or $\gamma_L \leq \phi \leq \gamma_D$. Except for $\phi = \gamma_\mathrm{D} = \gamma_\mathrm{L} = 1$ these expressions are both inconsistent with the conventional interpretation that $\phi \leq 1$ (to reduce the calculated resistance) and $\gamma_\mathrm{D}$, $\gamma_\mathrm{L} \geq 1$ (to increase the loads or applied stresses).

The reason for this result should be clear. In (1.9b) the term $(aN_D + aN_L)$ on the left-hand side was treated as a resistance, per se, whereas it is strictly a resistance effect caused directly by the applied loading (note that it is not affected by workmanship, material strength, etc., as is $R$). The key to an invariant safety measure is thus at hand. Partial factors such as $\phi$ should be applied directly to resistances only, and partial factors such as $\gamma$ to loads only, and the direct application of (1.6) to a mixed variable $R_1 = R + aN$ is not correct.

It is important to note that the safety margin $Z$ (Equation 1.7) is invariant for both definitions of resistance in this example. In the first case $Z_0 = R - M$, while in the second case $Z_1 = (R + aN) - (M + aN) = R - M$.

### 1.2.5 Invariant Safety Measures

As can be seen from the above examples, one form of invariant safety measure is obtained if the resistances $R_i$ and the loads $Q_j$ acting on the structure are so factored that the ratio between any relevant pair $\phi_i R_i$ and $\gamma_j Q_j$ is unity at the point of limit state violation. In simple terms, this requires that all variables be reduced to a common base before being compared. This is the case for the permissible stress measure of structural safety expressed by equation (1.3). Another and important form of invariant safety measure is the safety margin $Z = R - S$ defined in equation (1.7). It will be used extensively in the sections to follow because of its invariant properties.

Some readers may recognize a parallel between the above discussion and the decision criteria in cost-benefit analysis. The safety margin corresponds to the 'net present value' criterion and the problem of safety factor invariance to the 'numerator-denominator' problem [e.g. Prest and Turvey, 1965].

## 1.3 A Partial Probabilistic Safety Measure of Limit State Violation—The Return Period

In the historical development of engineering design, loads due to natural phenomena such as winds, waves, storms, floods, earthquakes, etc. were recognized quite early as having randomness in time as well as in space. The randomness in time was considered in terms of the 'return period'. The return period is defined as the average (or expected) time between two successive statistically independent events. Of course, the actual time $T$ between events is a random variable.

In most practical applications an 'event' constitutes the exceedance of a certain threshold, for example as associated with loading (e.g. wind velocity > 100 m/s). Such an event may be used to define a 'design load' and the design of the structure itself is then usually considered deterministically, i.e. using conventional design procedures. Hence this approach is only a partially probabilistic method.

The return period may be defined as follows. For independent samples from a population (i.e. for a Bernoulli trial sequence), the trial $T$ on which the first occurrence of an event takes place is given by the geometric distribution (A.23), which states that the probability that the first occurrence occurs on the $t$th trial is:

$$P(T = t) = p\ (1 - p)^{t-1} \qquad t = 1,\ 2, \ldots \tag{1.11}$$

where $p$ is the probability of occurrence of the event (e.g. $X > x$) in any one trial and $1 - p$ is the probability that the event does not occur. If trials are now interpreted as time intervals, during each of which only the occurrence of events $X > x$ is recorded, the first occurrence of an event becomes the 'first occurrence time', given by expression (1.11). The 'mean recurrence time' or the 'return period' is then the expected value of $T$ (see A. 10):

$$\begin{aligned} E(T) = \overline{T} &= \sum_{t=1}^{\infty} tp(1-p)^{t-1} = p[1 + 2(1-p) + 3(1-p)^2 + \ldots] \\ &= \frac{p}{[1-(1-p)]^2} \qquad \text{for } (1-p) < 1.0 \\ &= \frac{1}{p} \qquad \text{or} \qquad = [1 - F_x(x)]^{-1} \end{aligned} \tag{1.12}$$

where $F_X(x) = P(X \leq x)$ is the cumulative distribution function of $X$.

Thus the return period $\overline{T}$ is equal to the reciprocal of the probability of the occurrence of the event in any one (or a single) time interval. For most engineering problems, the chosen time interval is one year, so that $p$ is the probability of occurrence of the event $X > x$ in any one year (e.g. the probability that a load $> x$ will occur (at least once) during the year). Then $\overline{T}$ is the number of years, on average, between events.

Because the exceedance events that occur during a time period (e.g. during a year) are associated with the end of that period, $\overline{T}$ is dependent on the time period chosen [Borgman, 1963]. This is illustrated in Figure 1.4, where four exceedance events, A, B, C and D are shown occurring after an arbitrary initial event 0. The mean recurrence time $\overline{T}_1$ for the actual observations is shown in Figure 1.4(a) and is given by the average of the distance (i.e. time) between the events, i.e. by $\overline{T}_1 \approx 1.5$ years.

In Figure 1.4(b) with the time period taken as 1 year, and the events counted at the end of each time period, it follows easily that $\overline{T}_2 = 7/4 = 1.75$ years. Similarly, for $\overline{T}_3 = 2$ years. However, when a 4-year time period is used (Figure 1.4(d)) two of the

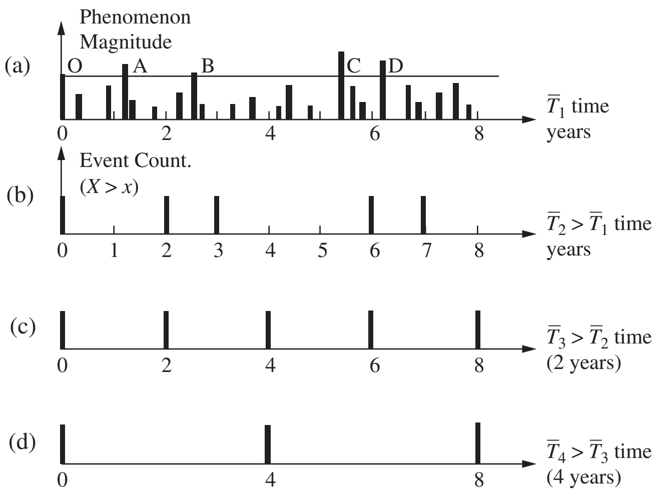



**Figure 1.4** Idealizations of actual load phenomenon for the 'return period' concept.

events in each period are counted as one at the end of the period, and $\overline{T}_4$ in this case becomes 4 years.

This somewhat artificial example shows three things. Firstly, that the return period depends, as noted, on the definition of the time scale, and secondly that the possible occurrence of more than one event within a time period is ignored. This means that, where event occurrence is relatively frequent compared with the time period employed, the return period measure is not accurate.

The third and a most important point is that the probability distribution of the magnitude of $X$ (i.e. the phenomenon being considered) is not considered. Only magnitudes $X > x$ are counted. This means that the return period is a probabilistic measure in terms of time only, but not in terms of the magnitude of the loading and its interaction with the resistance.

It should be clear that in practice the events may not be independent, as postulated, particularly if the events occur rather frequently. Fortunately, the return period concept is used mainly for rather rare events (i.e. the level $X$ is quite high), and it is then reasonable to assume event independence. Time scale dependence is then also not a significant issue. Chapter 6 gives a much more detailed discussion.

**Example 1.4** For a structure subject to a '50-year wind' of 60 km/h velocity:

(a) the return period for a 60-km/h wind = $\overline{T}$ = 50 years
(b) the probability of exceeding 60 km/h in any one year is

$$p = 1/\overline{T} = 1/50 = 2\%;$$

(c) the probability of exceeding the design wind velocity (i.e. $V > 60$) for the first time during the fourth year, is (geometric distribution A.23):

$$P_T(T = 4) = (0.02)(0.98)^3 = 0.01882$$

(d) the probability of exceeding the design wind velocity in only one of the years in a 4-year period is given by the binomial distribution (A.17):

$$P_X(x=1) = \binom{4}{1}(0.02)(0.98)^3 = \frac{4\times 3\times 2\times 1}{1(3\times 2\times 1)}(0.02)(0.98)^3 = 0.0753$$

(e) the probability of exceeding the design wind velocity (i.e. $V > 60$) during any of the years in a 4 year period is given by the geometric distribution (A.23):

$$\begin{aligned}P_T(T\le 4) &= \sum_{t=1}^{4} P_T(T=t) = \sum_{t=1}^{4}(0.02)(0.98)^{t-1}\\ &= 0.02 + 0.0196 + 0.01921 + 0.01883\\ &= 0.0776\end{aligned}$$

or alternatively,

$$P_T(T\le 4) = 1 - [P(V<60)]^4 = 1-(1-0.02)^4 = 0.0776$$

Note that the period 4 years can be generalized to 'design life' $t_L$ and the question rephrased to 'the probability of exceeding the design velocity within the design life':

$$P_T(T\le t_L) = \sum_{t=1}^{t_L} P_T(T=t) \qquad \text{or} \qquad = 1-(1-p)^{t_L} \tag{1.13}$$

Some typical values for the relationship between the exceedance probability $P_T(T\le t_L)$ the return period $\overline{T} = 1/p$ and the design life $t_L$ are given in Table 1.2 [Borgman, 1963].

(f) the probability of exceeding the design wind velocity within the return period is

$$P_T(T\le \overline{T}) = 1-[P(V<60)]^{\overline{T}}$$

but $P(V<60) = 1-P(V\ge 60) = 1-p$ where $p = 1/\overline{T}$. Hence

$$\begin{aligned}P_T(T\le \overline{T}) &= 1-(1-p)^{\overline{T}}\\ &= 1-\left(1-\overline{T}p+\frac{\overline{T}(\overline{T}-1)}{2!}\,p^2-\ldots\right)\\ &\approx 1-\exp(-\overline{T}p) \qquad \text{for large } \overline{T} \text{ (i.e. small } p)\\ &\approx 1-\exp(-1) = 1-0.3679 = 0.6321\end{aligned}$$

Note that even for smaller $\overline{T}$, this result is a good approximation; thus, for $\overline{T} = 5$,

$$P_T(T\le 5) = 1-\left(1-\frac{1}{5}\right)^5 = 0.6723$$

This shows that there is a chance of about 2 in 3 that the exceedance event will occur within a design life equal to the return period.

**Table 1.2** Return period $\overline{T}$ as function of design life $t_L$ and exceedance probability $P_T$ $(T \leq t_L)$.

| | Return period $\overline{T}$ for the following exceedance probabilities (see 1.13) | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Design life $t_L$** | **0.02** | **0.05** | **0.10** | **0.15** | **0.20** | **0.30** | **0.40** | **0.50** | **0.70** |
| 1 | 50 | 20 | 10 | 7 | 5 | 3 | 3 | 2 | 1 |
| 2 | 99 | 39 | 19 | 13 | 9 | 6 | 4 | 3 | 2 |
| 3 | 149 | 59 | 29 | 29 | 14 | 9 | 6 | 5 | 3 |
| 4 | 198 | 78 | 38 | 25 | 18 | 12 | 8 | 6 | 4 |
| 5 | 248 | 98 | 48 | 31 | 23 | 15 | 10 | 8 | 5 |
| 6 | 297 | 117 | 57 | 37 | 27 | 17 | 12 | 9 | 6 |
| 7 | 347 | 137 | 67 | 44 | 32 | 20 | 14 | 11 | 6 |
| 8 | 396 | 156 | 76 | 50 | 36 | 23 | 16 | 12 | 7 |
| 9 | 446 | 176 | 86 | 56 | 41 | 26 | 18 | 13 | 8 |
| 10 | 495 | 195 | 95 | 62 | 45 | 29 | 20 | 15 | 9 |
| 12 | 594 | 234 | 114 | 74 | 54 | 34 | 24 | 18 | 10 |
| 14 | 693 | 273 | 133 | 87 | 63 | 40 | 28 | 21 | 12 |
| 16 | 792 | 312 | 152 | 99 | 72 | 45 | 32 | 24 | 14 |
| 18 | 892 | 351 | 171 | 111 | 81 | 51 | 36 | 26 | 15 |
| 20 | 990 | 390 | 190 | 124 | 90 | 57 | 40 | 29 | 17 |
| 25 | 1238 | 488 | 238 | 154 | 113 | 71 | 49 | 37 | 21 |
| 30 | 1485 | 585 | 285 | 185 | 135 | 85 | 59 | 44 | 25 |
| 35 | 1733 | 683 | 333 | 216 | 157 | 99 | 69 | 51 | 30 |
| 40 | 1981 | 780 | 380 | 247 | 180 | 113 | 79 | 58 | 34 |
| 45 | 2228 | 878 | 428 | 277 | 202 | 127 | 89 | 65 | 38 |
| 50 | 2475 | 975 | 475 | 308 | 225 | 141 | 98 | 73 | 42 |

## 1.4 Probabilistic Measure of Limit State Violation

### 1.4.1 Introduction

The return period concept considers only the probability that a loading exceeds a set limit and assumes such exceedances (or 'level crossings' – see Chapter 6) to be randomly distributed in time. This is a useful improvement over deterministic descriptions of loading but ignores the fact that, even at a given point in time, the actual value of the loading is uncertain. This is illustrated in Figure 1.5 for floor loading.

The histogram of Figure 1.5 shows, for example, that the probability that the floor loading lies between 0.6 and 0.7 kPa is about 7%. Such information is obtained from actual surveys of floor loads (see Chapter 7), and can be represented by the probability density function $f_Q(q)$. (Recall that $f_Q(\,)$ denotes the probability that the load $Q$ will take on a value between $q$ and $q + \Delta q$ as $\Delta q \rightarrow 0$ - see also Section A.3.) The load $Q$ can be converted to a load effect $S$ by conventional structural analysis procedures. Using the same transformation(s), the probability density function $f_S(\,)$ can be obtained also,

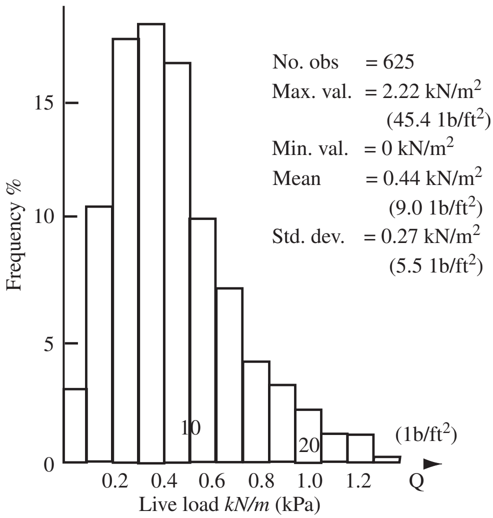



**Figure 1.5** Histogram of private office live loads [after Culver, 1976].

if necessary, using methods such as outlined in Section A.10. However, details of this need not be of concern for the present.

Resistance, geometric and workmanship variables and many others may be described similarly in probabilistic terms. For example, a typical resistance histogram and the inferred probability distribution for the yield strength of steel are shown in Figure 1.6. Naturally, material strengths such as steel yield strength can be converted to member resistance $R$ by multiplying by section properties (such as $A$, the cross-sectional area). Then it is possible to determine a probability density function $f_R(\,)$.

In general, the loads applied to a structure fluctuate with time and are of uncertain value at any one point in time. This is carried over directly to the load effects (or internal actions) $S$. Somewhat similarly the structural resistance $R$ will be a function of time

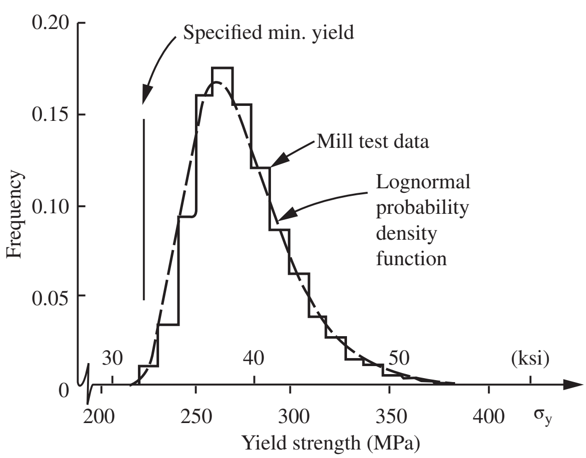



**Figure 1.6** Histogram and inferred distribution for structural steel yield strength [adapted from Alpsten, 1972 with permission of ASCE].

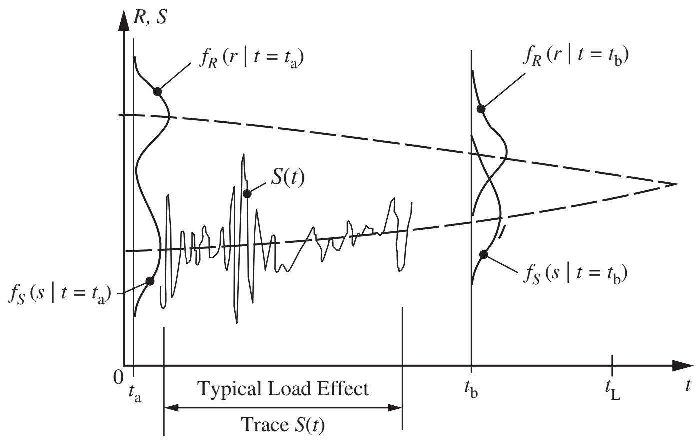



**Figure 1.7** Schematic time-dependent reliability problem.

(but not usually a fluctuating one) owing to fatigue, deterioration and similar actions. Loads have a tendency to increase, and resistances to decrease, with time. It is likely also that the uncertainty in both quantities increases with time, particularly if they have to be predicted. This means that the probability density functions $f_S(\ )$ and $f_R(\ )$ become wider and flatter with time and that the mean values of $S$ and $R$ also change with time. As a result, the general reliability problem can be represented as in Figure 1.7.

The safety limit state will be violated whenever, at any time $t$,

$$R(t) - S(t) < 0 \qquad \text{or} \qquad \frac{R(t)}{S(t)} < 1 \tag{1.14}$$

The probability that this occurs for any one (single) load application (or load cycle) is the probability of limit state violation, or simply the probability of failure $p_f$. Roughly, it may be represented by, but is not actually equal to, the amount of overlap of the probability density functions $f_R$ and $f_S$ in Figure 1.7. Since this overlap may vary with time, $p_f$ also may be a function of time.

To make the problem more tractable, it is convenient for many situations to assume that $Q$ and $R$ are 'time-invariant', that is they are not functions of time. An example of this is the case when the load $Q$ is applied to the structure only once and the probability of limit state violation is sought for that particular load application only.

However, if the load is applied many times (e.g. a single time-varying load might be considered this way) and $R$ is taken as constant, then the maximum value of that load (within a given time interval $[0, T]$) is of interest if it is assumed that the structure will fail under the (once-only) application of this maximum load. One way to properly represent this maximum load is through the use of an extreme value distribution, such as the Gumbel (EV-I) or Frechet (EV-II) distributions (see Appendix A). If this is done, the effect of time may be ignored in the reliability calculations. This approach is not satisfactory when more than one load is involved or when the resistance changes with time. Discussion of these matters and the more general reliability problem is deferred to Chapter 6.

### 1.4.2 The Basic Reliability Problem

The basic structural reliability problem considers only one load effect $S$ resisted by one resistance $R$. Each is described by a known probability density function, $f_S(\ )$ and

$f_R(\ )$ respectively. As noted, $S$ may be obtained from the applied loading $Q$ through a structural analysis (either deterministic or with random components). It is important that $R$ and $S$ be expressed in the same units.

For convenience, but without loss of generality, only the safety of a structural element will be considered here and, as usual, that structural element will be considered to have failed if its resistance $R$ is less than the stress resultant $S$ acting on it. The probability $p_f$ of failure of the structural element can then be stated in any of the following ways:

$$p_f = P(R \leq S) \tag{1.15a}$$

$$= P(R - S \leq 0) \tag{1.15b}$$

$$= P\left(\frac{R}{S} \leq 1\right) \tag{1.15c}$$

$$= P(\ln R - \ln S \leq 1) \tag{1.15d}$$

or in general

$$= P[G(R,\ S) \leq 0] \tag{1.15e}$$

where $G(\ )$ is termed the 'limit state function' and the probability of failure is identical with the probability of limit state violation. Equations (1.15) could, of course, also have been written in terms of $R$ and $Q$ for the structure as a whole.

Quite general (marginal) density functions $f_R$ and $f_S$ for $R$ and $S$ respectively are shown in Figure 1.8 together with the joint (bivariate) density function $f_{RS}(r, s)$ (see also Section A.6). For any infinitesimal element $(\Delta r\ \Delta s)$, the latter represents the probability that $R$ takes on a value between $r$ and $r + \Delta r$ and $S$ a value between $s$ and $s + \Delta s$ as $\Delta r$ and $\Delta s$ each approach zero. In Figure 1.8, Equations (1.15) are represented by the hatched failure domain $D$, so that the failure probability may be written as:

$$p_f = P(R - S \leq 0) = \int_D \int f_{RS}(r, s)\ dr\, ds \tag{1.16}$$

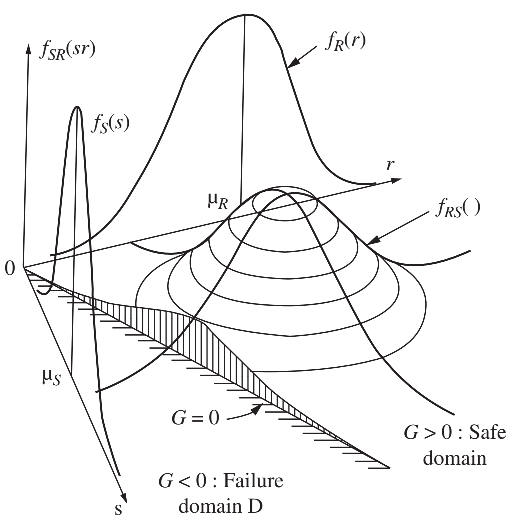



**Figure 1.8** Space of the two random variable $(r,\ s)$ and the joint density function $f_{RS}(r,\ s)$, the marginal density functions $f_R$ and $f_S$ and the failure domain D.

When $R$ and $S$ are independent, $f_{RS}(r, s) = f_R(r)f_S(s)$ (see A.6.3), and (1.16) becomes:

$$p_f = P(R - S \leq 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{S \geq r} f_R(r) f_S(s)\, dr\, ds \tag{1.17}$$

Noting that for any random variable $X$, the cumulative distribution function is given by (A.8):

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(y)\, dy$$

provided $x \geq y$, it follows that for the common, but special, case when $R$ and $S$ are independent, (1.17) can be written in the single integral form:

$$p_f = P(R - S \leq 0) = \int_{-\infty}^{\infty} F_R(x) f_S(x)\, dx \tag{1.18}$$

This is also known as a 'convolution integral' with meaning easily explained by reference to Figure 1.9. $F_R(x)$ is the probability that $R \leq x$ or the probability that the actual resistance $R$ of the member is less than some value $x$. This represents failure if the loading is $\geq x$. The probability that this is the case is given by the term $f_S(x)$ that represents the probability that the load effect $S$ acting in the member has a value between $x$ and $x + \Delta x$ in the limit as $\Delta x \to 0$. By considering all possible values of $x$, i.e. by taking the integral over all $x$, the total failure probability is obtained. This is also seen in Figure 1.10 where the (marginal) density functions $f_R$ and $f_S$ have been drawn along the same axis.

Through integration of $f_R(\,)$ in (1.17), the order of integration was reduced by one. This is convenient and useful, but not general. It was only possible because $R$ was assumed independent of $S$. In general, dependence between variables should be considered. This more complex situation is discussed further is Section 1.5 and Chapters 3 and 4.

For the present, restricting attention to simpler formulations, an alternative to expression (1.18) is:

$$p_f = \int_{-\infty}^{\infty} [1 - F_S(x)] f_R(x) dx \tag{1.19}$$

This can be seen to be simply the 'sum' of the failure probabilities over all the cases of resistance for which the load exceeds the resistance.

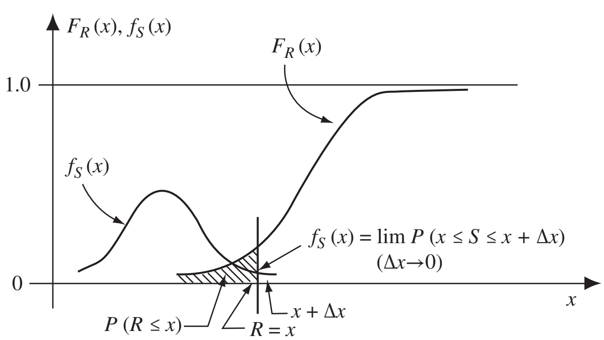



**Figure 1.9** Basic $R - S$ problem: $F_R(\,)f_S(\,)$ representation.

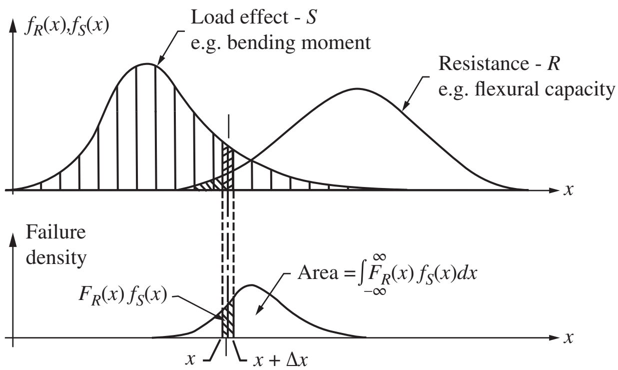



**Figure 1.10** Basic $R - S$ problem: $f_R(\,)\,f_S(\,)$ representation.

The lower limit of integration shown in Expressions (1.17) to (1.19) may not be totally satisfactory, since a 'negative' resistance usually is physically not possible. The lower integration limit therefore strictly should be zero, although this may be inconvenient and slightly inaccurate if $R$ or $S$ or both are modelled by distributions unlimited in the lower tail (such as the Normal or Gaussian distribution). The inaccuracy arises strictly from the modelling of $R$ and/or $S$, and not from the theory involved with (1.17) to (1.19). This important point is sometimes overlooked in discussions about appropriate distributions to represent random variables.

### 1.4.3 Special Case: Normal Random Variables

For a few distributions of $R$ and $S$ it is possible to integrate the convolution integral (1.18) analytically. The most notable example is when both $R$ and $S$ are normal random variables with means $\mu_R$ and $\mu_S$ and variances $\sigma_R^2$ and $\sigma_S^2$ respectively. The safety margin $Z = R - S$ then has a mean and variance given by well-known rules for addition (subtraction) of normal random variables:

$$\mu_Z = \mu_R - \mu_S \tag{1.20a}$$

$$\sigma_Z^2 = \sigma_R^2 + \sigma_S^2 \tag{1.20b}$$

Equation (1.15b) then becomes

$$p_f = P(R - S \leq 0) = P(Z \leq 0) = \Phi\left(\frac{0 - \mu_Z}{\sigma_Z}\right) \tag{1.21}$$

where $\Phi(\,)$ is the standard normal distribution function (zero mean and unit variance) extensively tabulated in statistics texts (see also Appendix D). The random variable $Z = R - S$ is shown in Figure 1.11, in which the failure region $Z \leq 0$ is shown shaded. Using (1.20) and (1.21) it follows that [Cornell, 1969a]

$$p_f = \Phi\left[\frac{-(\mu_R - \mu_S)}{(\sigma_S^2 + \sigma_R^2)^{1/2}}\right] = \Phi(-\beta) \tag{1.22}$$

where $\beta = \mu_Z/\sigma_Z$ is defined as the 'safety index' (1.21).

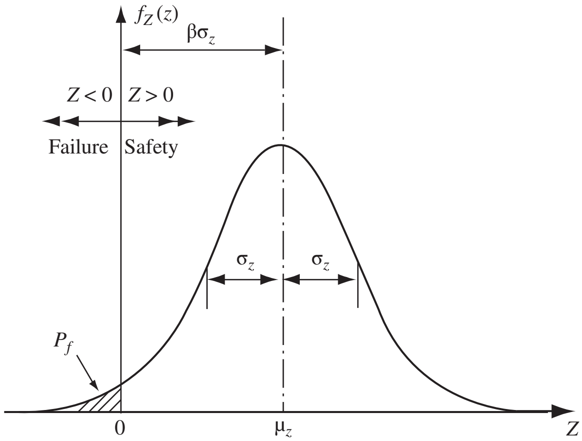



**Figure 1.11** Distribution of safety margin $Z = R - S$.

If either of the standard deviations $\sigma_S$ or $\sigma_R$ or both is increased, the term in square brackets in (1.22), will become smaller and hence $p_f$ will increase, as might be expected. Similarly if the difference between the mean of the load effect and the mean of the resistance is reduced, $p_f$ increases. These observations may be deduced also from Figure 1.7, taking the amount of overlap of $f_R(\ )$ and $f_S(\ )$ as a rough indicator of $p_f$ at any point in time.

**Example 1.5** A simply supported timber beam of length 5 m is loaded with a central load $Q$ having mean $\mu_Q = 3$ kN and variance $\sigma_Q^2 = 1$ (kN)$^2$. The bending strength of similar beams has been found to have a mean strength $\mu_R = 10$ kNm with a coefficient of variation (COV) of 0.15. It is desired to evaluate the probability of failure.

Assume that the beam self-weight and any variation in the length of the beam can be ignored. From basic structural theory, the applied moment (the load effect $S$) at the centre of the beam (due to the load $Q$) is given by $S = (QL)/4$. Since $L = 5$ it follows that the mean load effect and the variance of $S$ are:

$$\mu_S = \frac{5}{4}\mu_Q = \frac{5}{4} \times 3 = 3.75 \text{ kNm} \qquad \text{(see A.160)}$$

$$\sigma_S^2 = \left(\frac{5}{4}\right)^2 \sigma_Q^2 = \frac{25}{16} \times 1 = 1.56 \text{ (kNm)}^2 \qquad \text{(see A.162)}$$

Also, the mean resistance and its variance are:

$$\mu_R = 10 \text{ kNm}$$
$$\sigma_R^2 = [(COV)\mu_R]^2 = (0.15 \times 10)^2 = 2.25 \text{ (kNm)}^2$$

Hence

$$\mu_Z = \mu_R - \mu_S = 10 - 3.75 = 6.25$$
$$\sigma_Z^2 = \sigma_R^2 + \sigma_S^2 = 2.25 + 1.56 = 3.81$$

Therefore $\beta = \frac{\mu_Z}{\sigma_Z} = \frac{6.25}{1.95} = 3.20$ and from (1.21) and Appendix D

$$p_f = \Phi(-3.20) = 7 \times 10^{-4}.$$

### 1.4.4 Safety Factors and Characteristic Values

The traditional deterministic measures of limit state violation, namely the factor of safety and the load factor, can be related directly to the probability $p_f$ of limit state violation. Analytically this is demonstrated most easily for the basic 'one-resistance one-load-effect' case, when $R$ and $S$ (or $Q$) are each normally distributed.

Consider a convenient simple safety measure sometimes referred to as the 'central' safety factor $\lambda_0$ and defined as

$$\lambda_0 = \frac{\mu_R}{\mu_S} \quad \text{or} \quad = \frac{\mu_R}{\mu_Q} \tag{1.23}$$

This definition does not accord with conventional usage, since generally some upper range value of applied load or stress is compared with some lower range value of strength of material. Such values might be termed 'characteristic' values, reflecting that in conventional usage (e.g. in design) the load or strength is described only by this value. Thus the characteristic yield strength of steel bars is the strength that most (say 95%) bars will exceed. There is a finite (but small) probability that some bars will have a lower strength.

For resistances, the design or 'characteristic' values are defined on the low side of the mean resistance (see Figure 1.12):

$$R_k = \mu_R(1 - k_R V_R) \tag{1.24}$$

where $R_k$ is the characteristic resistance, $\mu_R$ the mean resistance, $V_R$ the coefficient of variation for $R$ and $k_R$ a constant. This description is based on the Normal distribution. $R_k$ is the value of resistance below which only, say 5% of samples will fail. Also, for the standardized Normal distribution function (see Section A.5.7), it follows that

$$0.05 = \Phi\left(-\frac{R_k - \mu_R}{\sigma_R}\right)$$

and for a 5% 'one-sided tail', $k_{0.05} = 1.645 = (\mu_R - R_k)/\sigma_R$ (e.g. see Appendix D). Expression (1.24) now follows directly, noting that the standard deviation can be expressed as $\sigma_R = \mu_R V_R$.

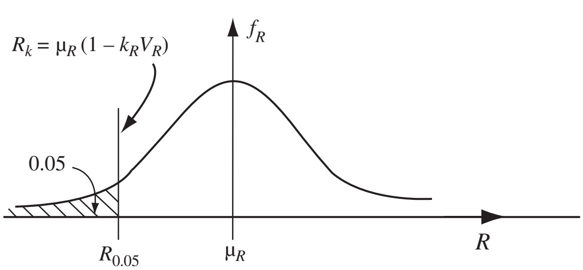



**Figure 1.12** Definition of characteristic resistance.

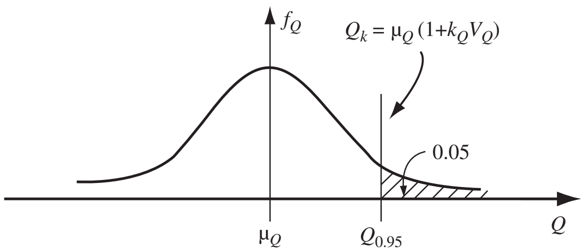



**Figure 1.13** Definition of characteristic load.

Similarly, for the load effect the characteristic value is estimated on the high side of the mean:

$$S_k = \mu_S(1 + k_S V_S) \tag{1.25}$$

where $S_k$ is the characteristic load effect (a design value), $\mu_S$ the mean load effect, $V_S$ the coefficient of variation for $S$ and $k_S$ is a constant. If design values are defined, for example, as not being exceeded 95% of the time a load effect is applied, then $k_S = 1.645$ if $S$ is Normally distributed (see Figure 1.13). Where loads (actions) are used, $Q$ replaces $S$ in (1.25).

In codified design, the percentiles used (such as 5% and 95% above) either are explicitly specified or may be deduced from the characteristic value specified in existing codes or documents. Other percentile characteristic values can be obtained in the manner indicated above for Normal distributions, and also for non-Normal distributions. Example 1.6 below shows a typical calculation, while Table 1.3 summarizes 5 and 95

**Table 1.3** 5% and 95% values for $X_k / \mu_X$.

| | | $X_k / \mu_X$ for the following coefficients of variation | | | | |
|---|---|---|---|---|---|---|
| **Distribution type** | $q$ **%** | **0.1** | **0.2** | **0.3** | **0.4** | **0.5** |
| Normal | 5 | 0.8355 | 0.6710 | 0.5065 | 0.3421 | 0.1176 |
| | 95 | 1.164 | 1.329 | 1.493 | 1.658 | 1.822 |
| Lognormal | 5 | 0.8445 | 0.7080 | 0.5910 | 0.4927 | 0.4112 |
| | 95 | 1.172 | 1.358 | 1.552 | 1.750 | 1.945 |
| Gumbel | 5 | 0.8694 | 0.7389 | 0.6083 | 0.4778 | 0.3472 |
| | 95 | 1.187 | 1.373 | 1.560 | 1.746 | 1.933 |
| Frechet | 5 | 0.8802 | 0.7809 | 0.6999 | 0.6344 | 0.5818 |
| | 95 | 1.187 | 1.367 | 1.534 | 1.681 | 1.809 |
| Weibull | 5 | 0.8169 | 0.6470 | 0.4979 | 0.3736 | 0.2747 |
| | 95 | 1.142 | 1.305 | 1.489 | 1.689 | 1.903 |
| Gamma | 5 | 0.8414 | 0.6953 | 0.5608[a)] | 0 4355[a)] | 0.3416 |
| | 95 | 1.170 | 1.350 | 1.541[a)] | 1.752[a)] | 1.938 |

a) Note that values are for $V_X = 0.302$ and 0.408 respectively, since the Gamma distribution usually only allows discrete values of $V_X^2 (= k)$ (see Section A.5.6).

percentile values for some common distributions. Similar results may be derived for other percentile values.

In some situations it may be appropriate to use the mean of the extreme value distribution to define a design load value. The precise choice is rather arbitrary and need not be of specific concern provided it is consistent. The important point is that the characteristic values are derived values and are convenient and useful for practical design rules but they have no fundamental meaning.

Using the characteristic values for the basic variables, it is now possible to define the so-called 'characteristic safety factor' $\lambda_k$:

$$\lambda_k = \frac{R_k}{S_k} \quad \text{or} \quad = \frac{R_k}{Q_k} \tag{1.26}$$

which corresponds closely to the conventional understanding of the factor of safety if the characteristic values are taken to correspond to the usual design values.

A relationship can be established between the characteristic safety factor $\lambda_k$ (and the central $\lambda_0$) and the probability $p_f$ of limit state violation. Obviously this relationship will depend on the probability distributions for $R$ and $S$, so that no general result can be given. Again, a particularly simple but quite useful case is when both $R$ and $S$ are described by Normal distributions. From (1.22), the probability of failure is

$$p_f = \Phi\left[\frac{-(\mu_R - \mu_S)}{(V_R^2\mu_R^2 + V_S^2\mu_S^2)^{1/2}}\right] \tag{1.27}$$

and dividing through by $\mu_S$

$$p_f = \Phi\left[\frac{-(\lambda_0 - 1)}{(V_R^2\lambda_0^2 + V_S^2)^{1/2}}\right] = \Phi(-\beta)\text{ say} \tag{1.28}$$

where $\lambda_0$ is given by (1.23) and $\beta$ is the 'safety index' as before. It follows that

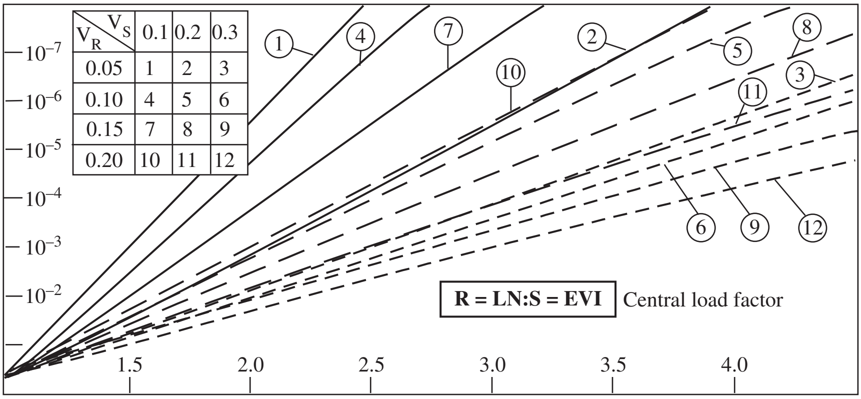
$$\lambda_0 = \frac{1 + \beta(V_R^2 + V_S^2 - \beta^2 V_R^2 V_S^2)^{1/2}}{1 - \beta^2 V_R^2} \tag{1.29}$$

Also (1.24)–(1.26) give

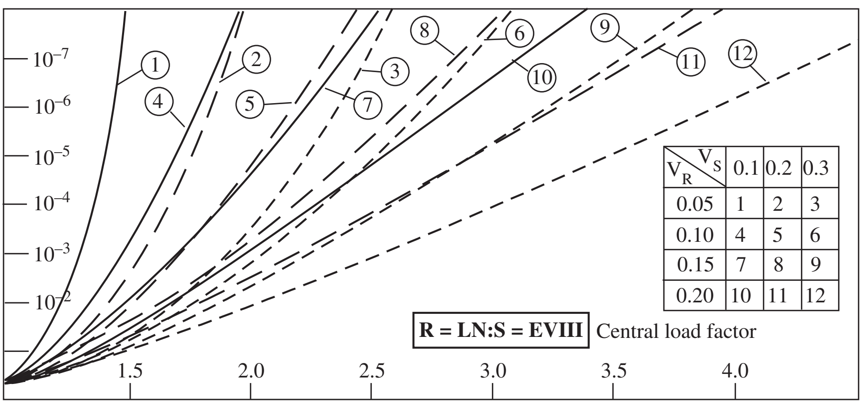
$$\lambda_k = \frac{1 - k_R V_R}{1 + k_S V_S}\ \lambda_0 \tag{1.30}$$

so that a relationship between $p_f$, $\lambda_0$ and $\lambda_k$ for given $V_R$, $V_S$, $k_R$ and $k_S$ follows immediately. Some typical relationships obtained by numerical integration are given in Figure 1.14.

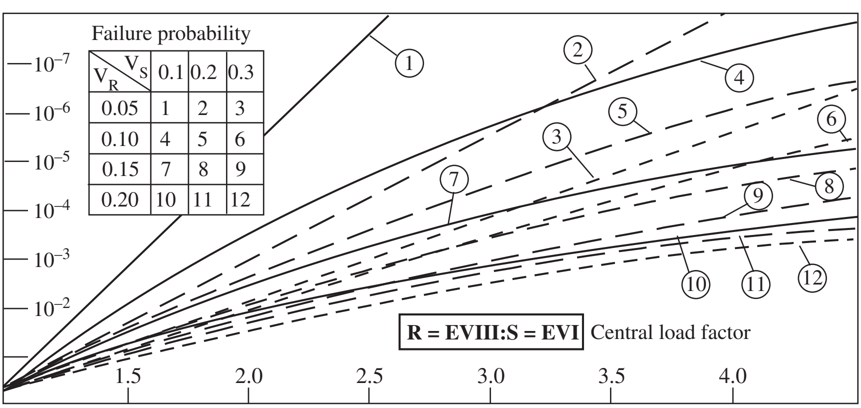
Expressions (1.29) and (1.30) indicate that the factors $\lambda_0$ and $\lambda_k$ depend on the variability or uncertainty associated with $R$ and $S$; with greater $V_R$ and $V_S$ requiring greater factors if the failure probability $p_f$ is to be kept constant (Figure 1.14). This demonstrates again the deficiencies of the deterministic measures of limit state violation. They ignore much information that may be available about uncertainties in structural strengths or applied loads.




(a)




(b)




(c)

**Figure 1.14** Failure probability $p_f$ versus central safety factor $\lambda_0$ for lognormal (LN) and extreme value (EV-) distributions and different coefficients of variation.

**Example 1.6** For a random variable $S$ with $\mu_S = 60$, $V_S = 0.2$, the 95 percentile for the Gumbel (EV-I) distribution, for example, may be determined as follows (see A.77)

$$0.95 = F_Y(y) = \exp[-e^{-\alpha(y-u)}]$$

where, from (A.79), $\alpha^2 = \pi^2/6\,\sigma_Y^2$ and, from (A.78), $u = \mu_Y - \gamma/\alpha$ with $\gamma = 0.57722$.

Now $\sigma_Y = \sigma_S = 0.2 \times 60 = 12$, $\mu_Y = \mu_S = 60$, so that $\alpha = (\pi/\sqrt{6})/12 = 0.1069$ and $u = 60 - 0.57722/0.1069 = 54.60$. Hence

$$0.95 = \exp[-e^{-0.1069(S-54.60)}]$$

or

$$S_{0.95} = 82.38$$

Alternatively, Table 1.3 shows that, for the Gumbel distribution, $S_{0.95}/\mu_S = 1.373$. Thus the 95 percentile value of $S$ is $S_{0.95} = 1.373\mu_S = 82.38$.

### 1.4.5 Numerical Integration of the Convolution Integral

As noted above, closed-form integration of Expressions (1.16) or (1.18) is only possible for some special cases. One of these cases, when both $R$ and $S$ are normally distributed, has already been considered (see Section 1.4.3). When both $R$ and $S$ are lognormal, and failure is defined as $Z = R/S < 1$, an exactly parallel result is obtained (see Example 1.7 below).

In general, however, to evaluate (1.16) or (1.18) for non-normal distributions, recourse must be made to numerical integration. The simplest approach, using the trapezoidal rule, is often quite effective [e.g. Dahlquist and Bjorck, 1974; Davis and Rabinowitz, 1975]. Step sizes around $x = 0.2\sigma_R$ have proved sufficiently accurate together with an integration range of about $\pm\ 5\ \sigma_Z$ instead of $\pm\infty$ [Ferry-Borges and Castenheta, 1971].

Some typical results obtained by numerical integration are given in Figure 1.14. Other, and similar, results have been given by Freudenthal (1964) and Ferry-Borges and Castenheta (1971).

**Example 1.7** As an exercise for readers, use the probability density function (A.61) for the lognormal variable $Z = R/S$, where $R$ and $S$ are each lognormal, to show that

$$p_f = \Phi(-\beta_1) = \Phi\left\{-\frac{\ln\left\{\dfrac{\mu_R}{\mu_S}[(1+V_S^2)/(1+V_R^2)]^{1|2}\right\}}{\{\ln[(1+V_R^2)(1+V_S^2)]\}^{1/2}}\right\}$$

Also show that this simplifies to

$$p_f = \Phi(-\beta_1) \approx \Phi\left\{-\frac{\ln(\mu_R/\mu_S)}{(V_R^2+V_S^2)^{1|2}}\right\} \qquad \text{for } V_R < 0.3,\ V_S < 0.3$$

Finally, show that the expression for the central safety factor $\lambda_0 = \mu_R/\mu_S$ simplifies to

$$\lambda_0 \approx \exp[\beta_1(V_R^2 + V_S^2)^{1/2}]$$

## 1.5 Generalized Reliability Problem

For many problems the simple formulations (1.15a)–(1.15e) are not entirely adequate, since it may not be possible to reduce the structural reliability problem to a simple $R$ versus $S$ formulation with $R$ and $S$ independent random variables.

In general, $R$ is a function of material properties and element or structure dimensions, while $S$ is a function of applied loads $Q$, material densities and perhaps dimensions of the structure, each of which may be a random variable. Also, $R$ and $S$ may not be independent, such as when some loads act to oppose failure (e.g. overturning) or when the same dimensions affect both $R$ and $S$. In this case it is not valid to use the convolution integral (1.18). It is also not valid when there is more than one applied stress resultant acting at a section, or more than one factor contributing to the resistance of the structure. A more general formulation is required. The first step is to define the variables involved in the generalized reliability problem.

### 1.5.1 Basic Variables

The fundamental variables that define and characterize the behaviour and safety of a structure may be termed the 'basic' variables. Usually they are the variables employed in conventional structural analysis and design. Typical examples are dimensions, densities or unit weights, materials, loads, material strengths. The compressive strength of concrete would be considered a basic variable even though it can be related to more fundamental variables such as cement content, water-to-cement ratio, aggregate size, grading and strength, etc. However, structural engineers do not normally use these latter variables in strength or safety calculations.

It is very convenient to choose the basic variables such that they are independent. However, this may not always be possible. Thus the compressive and tensile strengths and the elastic modulus of concrete are related; yet in a particular analysis they might each be treated as a basic variable. Dependence between basic variables usually adds some complexity to a reliability analysis. It is important that the dependence structure between dependent variables be known and expressible in some form. Usually this will be through a correlation matrix; however, as noted in Appendix A, this can at best provide only limited information.

The probability distributions to be assigned to the basic variables depend on the knowledge that is available. If it can be assumed that past observations and experience for similar structures can be used, validly, for the structure under consideration the probability distributions might be inferred directly from such observed data. More generally, subjective information may be employed or some combination of techniques may be required. Thus, in practice some subjective influence is nearly always present, since only seldom are sufficient data available to identify unambiguously only one distribution as the most appropriate.

Sometimes physical reasoning may be used to suggest an appropriate probability distribution. Thus, where a basic variable consists of the sum of many other variables (which are not explicitly considered), the central limit theorem (see Section A.5.8) can be invoked to suppose that a normal distribution (see Section A.5.7) is appropriate. This reasoning would be appropriate for the compressive strength of concrete (many component strengths) and for the dead load of a beam or slab (again many components of weight and several dimensions). In another example, the maximum wind velocity per year might be represented by the Gumbel (EV-I) distribution (see Section A.5.11), as this is based on an underlying wind phenomenon that, at any instantaneous point in time can be considered described as essentially Normal in probability distribution (see Chapter 7).

The parameters of the distribution may be estimated from the data using one of the usual methods, e.g. methods of moments, maximum likelihood, or order statistics. These are well described in standard statistics texts and will not be considered here [e.g. Ang and Tang, 1975]. However, it must be emphasized that such techniques should not be used blindly. Critical examination of the data for trends and outliers is always necessary, and the reasons for these phenomena should be established. It is quite possible for such behaviour to be the result of data recording and storage procedures rather than the behaviour of the variable itself.

Finally, when model parameters have been selected, the model should be compared with the data if at all possible. A graphical plot on appropriate probability paper is often very revealing, but analytical 'goodness of fit' tests (e.g. Kolmogorov-Smirnov test) can be used also.

It may not be possible, always, to describe each basic variable by an appropriate probability distribution. The required information may not be available. In such circumstances a 'point estimate' of the value of the basic variable might be used, i.e. the best estimate, given the known information. If some uncertainty information about the variable is also available, it might be appropriate to represent it by an estimate of its mean and its variance only. This is then known as a 'second moment' representation. One way in which such a representation might be interpreted is that in the absence of more precise data, the variable might be assumed to have a normal distribution (as this is completely described by the mean and variance, i.e. the first two moments (see Section A.5.7)). However, other probability distributions might be more appropriate, even if only the first two moments are known or can be determined.

### 1.5.2 Generalized Limit State Equations

With the basic variables and their probability distributions established, the next step is to replace the simple $R - S$ form of limit state function with a generalized version, expressed directly in terms of basic variables.

Let the vector $\mathbf{X}$ represent all the basic variables involved in the problem. Then the resistance $R$ can be expressed as $R = G_R(\mathbf{X})$ and the loading or load effect as $S = G_S(\mathbf{X})$. Since the functions $G_R$ and $G_S$ may be non-linear, the cumulative distribution function $F_R(\ )$, for example, must be obtained by multiple integration over the relevant basic variables (see A.155):

$$F_R(r) = \int_r \cdots \int f_{\mathbf{X}}(x)dx$$

A similar expression would apply for $S$ and $F_S(\ )$. These could then be used in (1.18) or (1.19). Fortunately, it is seldom necessary to follow this somewhat complex and piecemeal approach. Instead it is noted that in (1.15e) the limit state function $G(R, S)$ itself can be generalized. When the functions $G_R(\mathbf{X})$ and $G_S(\mathbf{X})$ are used in $G(R, S)$, the resulting limit state function can be written simply as $G(\mathbf{X})$, where $\mathbf{X}$ is the vector of all relevant basic variables and $G(\ )$ is some function expressing the relationship between the limit state and the basic variables. The limit state equation $G(\mathbf{x}) = 0$ now defines the boundary between the satisfactory or 'safe' domain $G > 0$ and the unsatisfactory or 'unsafe' domain $G \leq 0$ in $n$-dimensional basic variable space. Usually the limit state equation(s) is derived from the physics of the problem. (Note that $\mathbf{X}$ is the vector of random variables and that $\mathbf{X} = \mathbf{x}$ defines a particular 'point' $\mathbf{x}$ in the basic variable space.)

Where some loads may influence resistance (e.g. in overturning situations (see Figure 1.2)) care should be taken that $G(\mathbf{X})$ is defined properly. Again by analogy with the simple case of Figure 1.2 a useful rule is that any basic variable adding resistance to the limit state should have a positive gradient, that is: $\partial G/\partial X_i > 0$.

**Example 1.8** Consider a simple pin-ended strut supporting one end of a simply supported beam of length $L_1$, loaded at midpoint by a load $Q$. The actual load on the strut is thus $QL_1/2$. The strength of the strut is governed by its length $L_2$, its radius $r$ of gyration, its cross-sectional area $A$ and either the yield strength $\sigma_Y$ of the steel or some combination of axial load capacity and bending capacity, usually expressed by an interaction rule in structural design codes. Such rules are based, usually, on experimental observations and are then modified for code users by adding conservative assumptions and factors of safety. It is apparent, therefore, that code rules must be used with great caution in reliability analyses. A better approach is to use the original data and/or original relationships for ultimate strength.

For the squash load limit state, it follows easily that the relevant limit state equation is:

$$G_1(\mathbf{X}) = \sigma_Y A - \frac{QL_1}{2}$$

Here usually all the variables may be considered to be random variables, although some might be considered closely deterministic, for example the variable $A$ since usually there is little uncertainty about its value.

For the interaction case, the limit state equation is:

$$G_2(\mathbf{X}) = FN\left(\sigma_Y A, \frac{L_2}{r}\right) - \frac{QL_1}{2}$$

where $FN(\ )$ is an appropriate interaction equation for ultimate strength of pin-ended struts.

### 1.5.3 Generalized Reliability Problem Formulation

With the limit state function expressed as $G(\mathbf{X})$, the generalization of (1.16) becomes:

$$p_f = P[G(\mathbf{X}) \leq 0] = \int \dots \int_{G(\mathbf{X}) \leq 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{1.31}$$

Here $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function for the $n$-dimensional vector $\mathbf{X}$ of basic variables. Note that the resistance $R$ and load effect $S$ are not shown in the formulation — they are implicit in $\mathbf{X}$. Moreover, even if $\mathbf{X}$ were dissected, $R$ and $S$

may not show up explicitly and may be represented by the variables of which they are composed. If the basic variables are all independent, formulation (1.31) is simplified, with (see A.117):

$$f_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n} f_{X_i}(x_i) = f_{X_1}(x_1).f_{X_2}(x_2).f_{X_3}(x_3)\dots \tag{1.32}$$

Here $f_{X_i}(x_i)$ is the 'marginal' probability density function for the basic variable $X_i$.

The region of integration $G(\mathbf{X}) \le 0$ in (1.31) denotes the (hyper-)space in which limit state violation occurs. It is directly analogous to the failure domain $D$ shown in Figure 1.8. Except for some special cases, the integration of (1.31) over the failure domain $G(\mathbf{X}) \le 0$ cannot be performed analytically. However, the solution of (1.31) can be made more tractable by simplification or by numerical treatment (or both) of (i) the integration process, (ii) the integrand $f_{\mathbf{X}}(\,)$ and (iii) the definition of the failure domain. Each approach has been explored in the literature. Two dominant approaches have emerged:

(a) using numerical approximations such as simulation to perform the multidimensional integration required in (1.31)—the so-called 'Monte Carlo' methods;
(b) sidestepping the integration process completely by transforming $f_{\mathbf{X}}(\mathbf{x})$ in (1.31) to a multi-Normal probability density function and using some remarkable properties which may then be used to determine, approximately, the probability of failure—the so-called 'First Order Second Moment' methods and developments thereof.

These methods are described in more detail in Chapters 3 and 4 respectively. Some special results are given also in Appendix C.

### 1.5.4 Conditional Reliability Problems*

The probability estimate given by (1.31) becomes conditional when complete statistical information about the random variables $\mathbf{X}$ is not available. For example, the means or the variances might be estimated or not known with precision. In this case the probability expressed by (1.31) is a 'point estimate', given a particular set of assumptions about the probability distributions for $\mathbf{X}$. If the relevant statistical parameters are denoted $\boldsymbol{\theta}$ and are considered as random variables, the probability estimate becomes a conditional estimate and is a function of $\boldsymbol{\theta}$. Further, the limit state function now will be a function of $\boldsymbol{\theta}$ as well, i.e. $G(\mathbf{x}, \boldsymbol{\theta}) = 0$ and the joint probability function in $\mathbf{X}$ will be a function of $\boldsymbol{\theta}$ also, thus $f_{\mathbf{X}|\boldsymbol{\theta}}(\,)$. It should be noted that the nature of the uncertainties for the basic random variables $\mathbf{X}$ are different from the uncertainties in $\boldsymbol{\theta}$, the first expressing inherent variability (see Chapter 2) and the latter expressing uncertainty, which can be influenced by the collection of additional data (and perhaps by the use of alternative probability models). The net result is that the probability can now be expressed as a conditional probability estimate:

$$p_f(\boldsymbol{\theta}) = \int_{G(\mathbf{x},\boldsymbol{\theta})\le 0} f_{\mathbf{x}|\boldsymbol{\theta}}(\mathbf{x}\,|\,\boldsymbol{\theta})d\mathbf{x} \tag{1.33}$$

Of course, for decision-making an unconditional probability estimate is required. This can be done by invoking the total probability theorem (see A.6). For the present this can be done by taking the expected value of the conditional probability estimate [Der Kiureghian, 1990]:

$$p_f = E[p_f(\boldsymbol{\theta})] = \int_{\theta} p_f(\boldsymbol{\theta}) f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta} \tag{1.34}$$

where $E[\ ]$ is the expectation operator and $f_{\Theta}(\boldsymbol{\theta})$ is the joint probability density function of the parameters $\boldsymbol{\theta}$. Substitution of (1.33) into (1.34) then yields the unconditional probability estimate.

In passing it is noted that the integral of $f_{\mathbf{X}|\boldsymbol{\Theta}}(\ )$ over $\boldsymbol{\theta}$ sometimes is referred to, in the present context, as a 'predictive' distribution (since it takes into account uncertainties in $\boldsymbol{\theta}$), defined as:

$$f_{\mathbf{X}}(\mathbf{x}) = \int_{\theta} f_{\mathbf{X}|\boldsymbol{\Theta}}(\mathbf{x} \mid \boldsymbol{\theta}) f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta} \tag{1.35}$$

Methods to solve for these integrals are the subject of discussion in Chapters 3 and 4 and are relevant also to Bayesian updating in Chapter 10.

Another way in which the probability estimate (1.31) can be conditional is if the limit state function is given a more general interpretation. Consider, for convenience, the indicator function $I(\ )$ defined such that (Figure 1.15(a)):

$$\begin{aligned} I(x) &= 0 \qquad \text{if } x \le 0 \\ &= 1 \qquad \text{if } x > 0 \end{aligned} \tag{1.36}$$

It follows that $I[G(\mathbf{X})]$ may then be interpreted as a 'utility function' with the failure state $G(\mathbf{X}) \le 0$ having a 'utility' of zero, and the safe state $G(\mathbf{X}) > 0$ having unit 'utility'.

In practice, such as in problems involving serviceability considerations, the distinction between full utility and zero utility may not always be clear cut, and values between zero and unity may be appropriate (see Figure 1.15). Thus it may be that utility depends inversely on concrete crack size, with no cracks having a utility of 1, cracks < 0.1 mm a utility of 0.5 and greater cracks zero utility. Clearly many other possibilities and applications exist [e.g. Reid and Turkstra, 1980; Stewart, 1996b; Augusti and Ciampoli, 2008; Barbato *et al.*, 2013].

If now $J(x)$ denotes the above more general interpretation of the indicator function (see Figure 1.15(b)) and $J^c(\ ) = 1 - J(\ )$ defines the complement of $J$, the generalization of (1.31) becomes:

$$p_f = P\{J^c[G(\mathbf{X})]\} = \int\int_{\mathbf{X}} \{J^c[G(\mathbf{x})] f_{\mathbf{x}}(\mathbf{x})\}\, d\mathbf{x} \tag{1.37}$$

As might be imagined, evaluation of (1.34) is not necessarily a simple matter.

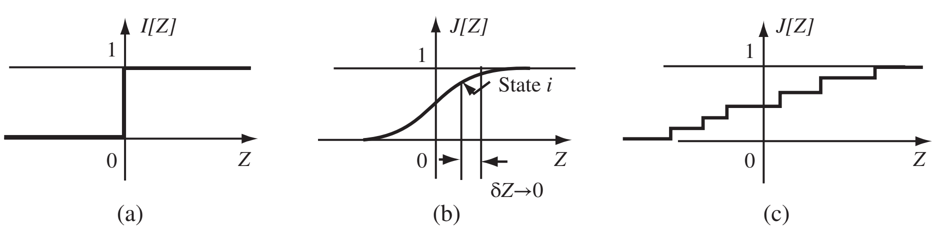



**Figure 1.15** Limit state violation indicators.

If $J$ (or $J^c$) is defined as a probability distribution function (see A.8), then (1.37) may be interpreted also as a total probability (see A.6) with (1.31) providing only the conditional probability of failure for a given realization of the limit state function. This more general interpretation is useful in structural reliability problems that are part of more general risk and reliability studies. For these, a range of possible limit state functions, not all structural, might arise. A good example is that of Probabilistic Safety Analysis (PSA) for nuclear facilities such as power stations or Probabilistic Risk Analysis (PRA) for other potentially hazardous facilities. In the case of nuclear power stations suffering a 'loss of coolant accident' (LOCA), the ability of the reactor block and its building to contain the resulting run-away reaction and its products is critical. The conditional probability of failure under a LOCA can be estimated for a defined external event such as for a level of (earthquake-induced) ground shaking. Repeating this for other levels of ground shaking and using the probability density for these levels together with the conditional probabilities of plant failure in the theorem of total probability allows the total probability of failure to be estimated. Similarly the occurrence of a LOCA may by the result of failure of critical pipework, and this may depend on the relative dynamic response of the reactor block and the reactor building under earthquake conditions (and others). More details of integrating structural reliability estimates as conditional events in larger risk assessments are discussed elsewhere [e.g. Stewart and Melchers, 1997].

## 1.6 Conclusion

Various ways in which structural reliability may be defined have been reviewed in this chapter. To do so it was necessary to introduce the concept of 'limit states'. This was seen to be a formalization of the possibly multiple criteria under which the structure can be considered to have 'failed' or have reached an unsatisfactory state.

Traditional measures of limit state violation were reviewed, including the factor of safety, the load factor and possible 'limit state design' concepts. It was shown that care is required in their definition; otherwise the safety measure might depend on how safety is defined, i.e. the formulation might not be 'invariant'.

Another common measure, the return period, was reviewed prior to the introduction of a fully probabilistic measure of limit state violation. Several aspects of this were then outlined and generalizations given.

# 2

# Structural Reliability Assessment

## 2.1 Introduction

Before proceeding to elaborate the concepts introduced in Chapter 1, it is necessary to address fundamental questions about the meaning of the calculated probability of limit state violation (whether for ultimate limit states or otherwise). Specifically, what does the calculated probability $p_f$ mean? Can it be related to observed rates of failure for real structures? How can knowledge of $p_f$ help in achieving better (safer?) or more economical structures? And how does it relate to failure probabilities for other constructed or existing facilities? Surprising as it may seem, a degree of controversy and disagreement still remains about these important questions.

It will be helpful to examine the meaning of some terms. 'Probability' has already been used in Chapter 1. It denotes the chance that a particular, predefined event occurs. Classically, the probability of event occurrence was considered to be obtainable only from many repeated observations of the process that led to the event, the so-called 'frequentist' (or objective) definition. Obviously, the events must be observed. The observation process itself immediately adds an element of subjectivity, even to an otherwise frequentist meaning, in much the same way that observations in, say, physics are always partly subjective [de Finetti, 1974; Popper, 1959; Blockley, 1980; Jeffrey, 2004]. This aspect is sometimes (erroneously) ignored, and relative frequency data assumed to be purely 'objective' information.

An alternative interpretation is that probability expresses a 'degree of belief' about the occurrence of an event, rather than the actual (but unknown) frequency. It is therefore a 'subjective' or 'personal' probability. This interpretation is much wider than the relative frequency definition, and in its extreme form could be based on no previous data or experience of any sort to express degree of belief. Subjective probabilities are often referred to as 'Bayesian' probabilities, reflecting the tract on probability theory expressing these ideas, written by the Rev. Thomas Bayes in the early 1700s and since developed by many others. It is sometimes noted that a subjective probability estimate reflects the degree of ignorance about the phenomenon under consideration. There is a large literature on subjective probabilities [e.g. as summarized in Lindley, 1972; Jeffrey, 2004]. Reconciliation of the various interpretations of the meaning of probability still has interesting practical and sometimes quite controversial issues, although the latter are more philosophical than practical in nature [e.g. Fishburn, 1964; Kyburg, 1978; Hasofer, 1984; Lind, 1996; CIRIA 2014].

*Structural Reliability Analysis and Prediction,* Third Edition. Robert E. Melchers and André T. Beck.