fourth edition

Fuzzy Logic

With Engineering Applications

Timothy J. Ross

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FUZZY LOGIC WITH ENGINEERING APPLICATIONS

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Fourth Edition

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This book is dedicated to the memory of my loving mother, Phyllis; to my dear aunts, Glee, Ruth, and Virginia; and to my beloved cousin, Carol, all of whom have departed in the last five years; and to my partner, Laura, who is giving me much optimism for the future. And, to my dear US/Brazilian friends, Cassiano and Kenya, for their unconditional friendship since 2010. Finally, to my new grandson, Jackson, I give you my best wishes in life.

Contents

About the Author

Timothy J. Ross is Professor and Regents' Lecturer of Civil Engineering at the University of New Mexico. He received his PhD in Civil Engineering from Stanford University, his MS from Rice University, and his BS from Washington State University. Professor Ross has held previous positions as Senior Research Structural Engineer, Air Force Weapons Laboratory, from 1978 to 1986; and Vulnerability Engineer, Defense Intelligence Agency, from 1973 to 1978. Professor Ross has authored more than 150 publications and has been active in the research and teaching of fuzzy logic since 1983. He is the founding Co-Editor-in-Chief of the *International Journal of Intelligent and Fuzzy Systems*, the co-editor of *Fuzzy Logic and Control: Software and Hardware Applications*, and the co-editor of *Fuzzy Logic and Probability Applications: Bridging the Gap*. His sabbatical leaves in 1994–1995 at the US Army Environmental Policy Institute, Atlanta, Georgia; in 2001–2002 at the University of Calgary, Alberta, Canada; in 2008–2009 at Gonzaga University in Spokane, Washington; and most recently in 2015 at the Air Force Research Laboratory at Kirtland AFB, New Mexico, have resulted in the education of numerous additional students, faculty, and research professionals in the subject of fuzzy logic because he transferred this technology to all of these institutions. Ross continues to be active in applying fuzzy logic in his areas of research: decision support systems, reliability theory, data fusion, and structural engineering.

Preface to the Fourth Edition

My primary motivations for writing the fourth edition of this text have been to (1) reduce the length of the text, (2) correct the errata discovered since the publication of the third edition, and (3) introduce limited new material for the readers. The first motivation has been accomplished by eliminating some sections that are rarely taught in the classroom by various faculty using this text and by eliminating some sections that do not add to the utility of the text as a tool to learn basic fundamentals of the subject.

Since the first edition was published, in 1995, the technology of fuzzy set theory and its application to systems, using fuzzy logic, has moved rapidly. Developments in other theories such as possibility theory and evidence theory (both being elements of a larger collection of methods under the rubric “generalized information theories”) have shed more light on the real virtues of fuzzy logic applications, and some developments in machine computation have made certain features of fuzzy logic much more useful than in the past. In fact, it would be fair to state that some developments in fuzzy systems are quite competitive with other, linear algebra-based methods in terms of computational speed and associated accuracy.

There is some new material which is included in the fourth edition to try to capture some of the newer developments; the keyword here is *some* because it would be impossible to summarize or illustrate even a small fraction of the new developments of the last six years since the third edition was published. As with any book containing technical material, the third edition contained errata that have been corrected in this fourth edition. As with the first three editions, a solutions manual for all problems in the fourth edition and software can be obtained by qualified instructors who visit www.wiley.com/go/ross/fuzzy4e and provide official documentation of their teaching status. In addition to the solutions manual, a directory of software will be made available to all student users and faculty of the text on this same website. Most of the software routines make use of the MATLAB platform, and most of the routines have been written by my students, except for the standard routines that exist as MATLAB functions.

As I discussed in the preface of the third edition, the axioms of a probability theory referred to as the *excluded middle* are again referred to in this edition as axioms—never as laws.

The operations due to De Morgan are also not referred to as laws, but as *principles* because these principles do apply to some (*not all*) uncertainty theories (e.g., probability and fuzzy). The *excluded middle axiom* (and its dual, the *axiom of contradiction*) are not *laws*; Newton produced *laws*, Kepler produced *laws*, Darcy, Boyle, Ohm, Kirchhoff, Bernoulli, and many others too numerous to list here all developed *laws*. *Laws* are mathematical expressions describing the immutable realizations of nature. Definitions, theorems, and axioms collectively can describe a certain axiomatic foundation describing a particular kind of theory, and nothing more; in this case, the *excluded middle* and other axioms can be used to describe a probability theory. Hence, if a fuzzy set theory does not happen to be *constrained* by an *excluded middle axiom*, it is not a *violation* of some immutable law of nature like Newton's laws; fuzzy set theory simply does not happen to have an axiom of the excluded middle; it does not need, nor is *constrained by*, such an axiom. In fact, as early as 1905 the famous mathematician L. E. J. Brouwer defined this excluded middle axiom as a *principle* in his writings; he showed that the *principle of the excluded middle* was inappropriate in some logics, including his own, which he termed *intuitionism*. Brouwer observed that Aristotelian logic is only a part of mathematics, the special kind of mathematical thought obtained if one restricts oneself to relations of the whole and part. Brouwer had to specify in which sense the principles of logic could be considered "laws" because within his intuitionistic framework thought did not follow any rules, and, hence, "law" could no longer mean "rule" (see the detailed discussions on this in Chapters 5 and 13). In this regard, I continue to take on the cause advocated by Brouwer more than a century ago.

Also in this fourth edition, as in previous editions, I do not refer to "fuzzy measure theory" but instead describe it as "monotone measure theory"; the reader will see this in the title of Chapter 13. The former phrase still causes confusion when referring to fuzzy set theory; we hope to help in ending this confusion. In Chapter 13, in describing the monotone measure, m, I use the phrase describing this measure as a "basic evidence assignment (bea)," as opposed to the early use of the phrase "basic probability assignment (bpa)." Again, we attempt to avoid confusion with any of the terms typically used in probability theory.

As with the first three editions, this fourth edition is designed for the professional and academic audience interested primarily in applications of fuzzy logic in engineering and technology. I have found that the majority of students and practicing professionals are interested in the applications of fuzzy logic to their particular fields. Hence, the text is written for an audience primarily at the senior undergraduate and first-year graduate levels. With numerous examples throughout, this text is written to assist the learning process of a broad cross section of technical disciplines. It is primarily focused on applications, but each of the chapters begin with the rudimentary structure of the underlying mathematics required for a fundamental understanding of the methods illustrated.

Chapter 1 introduces the basic concept of fuzziness and distinguishes fuzzy uncertainty from other forms of uncertainty. It also introduces the fundamental idea of set membership, thereby laying the foundation for all material that follows, and presents membership functions as the format used for expressing set membership. The chapter summarizes a historical review of uncertainty theories and reviews the idea of "sets as points" in an n-dimensional Euclidean space as a graphical analog in understanding the relationship between classical (crisp) and fuzzy sets. A new section in the chapter addresses the intuition of propagating uncertainty by showing an example that compares the results of propagating probabilities on the one hand, or membership values on the other, through a simple nonlinear function. In this example there are some counterintuitive findings that readers will find both interesting and instructive.

Chapter 2 reviews classical set theory and develops the basic ideas of fuzzy sets. Operations, axioms, and properties of fuzzy sets are introduced by way of comparisons with the same entities for classical sets. Various normative measures to model fuzzy intersections (t-norms) and fuzzy unions (t-conorms) are summarized.

Chapter 3 develops the ideas of fuzzy relations as a means of mapping fuzziness from one universe to another. Various forms of the composition operation for relations are presented. Again, the epistemological approach in this chapter uses comparisons with classical relations in developing and illustrating fuzzy relations. Chapter 3 also illustrates methods to determine the numerical values contained within a specific class of fuzzy relations, called *similarity relations*. The section on a three-dimensional physical analogy of equivalence relations has been deleted.

Chapter 4 discusses the fuzzification of scalar variables and the defuzzification of membership functions. It introduces the basic features of a membership function and it discusses, briefly, the notion of interval-valued fuzzy sets. Defuzzification is necessary in dealing with the ubiquitous crisp (binary) world around us. The chapter details defuzzification of fuzzy sets and fuzzy relations into crisp sets and crisp relations, respectively, using lambda-cuts, and it describes a variety of methods to defuzzify membership functions into scalar values. Some of the defuzzification methods in the third edition have been deleted because they are seldom used in practice and because they are covered elsewhere in the literature. Examples of all methods are given in the chapter.

Chapter 5 introduces the precepts of fuzzy logic, again through a review of the relevant features of classical, or a propositional, logic. Various logical connectives and operations are illustrated. There is a thorough discussion of the various forms of the implication operation and the composition operation provided in this chapter. Three different inference methods, popular in the literature, are illustrated. Approximate reasoning, or reasoning under imprecise (fuzzy) information, is also introduced in this chapter. Basic IF–THEN rule structures are introduced and three graphical methods of inference are presented. The section on Natural Language has been shortened. A few more examples of the difficulties of using the axiom of the excluded middle are given in the summary of the chapter.

Chapter 6 provides several classical methods of developing membership functions, including methods that make use of the technologies of neural networks, genetic algorithms, and inductive reasoning.

Chapter 7 presents six automated methods that can be used to generate rules and membership functions from observed or measured input–output data. The procedures are essentially computational methods of learning. Examples are provided to illustrate each method. Many of the problems at the end of the chapter will require software; this software can be downloaded from www.wiley.com/go/ross/fuzzy4e.

Beginning the second category of chapters in the book highlighting applications, Chapter 8 continues with the rule-based format to introduce fuzzy nonlinear simulation and complex system modeling. In this context, nonlinear functions are seen as mappings of information "patches" from the input space to information "patches" of the output space, instead of the "point-to-point" idea taught in classical engineering courses. Fidelity of the simulation is illustrated with standard functions, but the power of the idea can be seen in systems too complex for an algorithmic description. This chapter formalizes fuzzy associative memories (FAMs) as generalized mappings.

Chapter 9 develops fuzzy decision making by introducing some simple concepts in ordering, preference and consensus, and multiobjective decisions. It introduces the powerful concept of Bayesian decision methods by fuzzifying this classic probabilistic approach. This chapter illustrates the power of combining fuzzy set theory with probability to handle random and nonrandom uncertainty in the decision-making process.

Chapter 10 discusses a few fuzzy classification methods by contrasting them with classical methods of classification and develops a simple metric to assess the goodness of the classification, or misclassification. This chapter also summarizes classification using equivalence relations. It now has a section on pattern recognition, gleaned from the third edition. This section introduces a useful metric in pattern recognition using the algebra of fuzzy vectors. A single-feature and a multiple-feature procedure are summarized in the chapter. The section on image processing has been deleted because other books have extensive coverage of this area.

The chapter in the third edition on fuzzy arithmetic and fuzzy numbers has been deleted. A summary of Zadeh's extension principle and a few simple examples of fuzzy arithmetic are included in Chapter 12.

Chapter 11 introduces the field of fuzzy control systems. A brief review of control system design and control surfaces is provided. Simple example problems in control are provided. Two sections in this chapter are worth noting: fuzzy engineering process control and fuzzy statistical process control, with examples on both provided. A discussion of the comparison of fuzzy and classical control is contained in the chapter summary, and a few more examples of fuzzy control in industrial systems and applications are also included.

Chapter 12 has been extensively changed by including more information on genetically evolved fuzzy cognitive maps, and new sections on the extension principle and fuzzy arithmetic and on fuzzy data fusion are also detailed. Previous sections on fuzzy optimization and fuzzy agent-based methods are still contained in this chapter.

Finally, Chapter 13 enlarges the reader's understanding of the relationship between fuzzy uncertainty and random uncertainty (and other general forms of uncertainty, for that matter) by illustrating the foundations of monotone measures. The chapter discusses monotone measures in the context of evidence theory, possibility theory, and probability theory. The chapter has a section on methods to develop approximate possibility distribution functions derived from both data intervals and scalar point data.

Most of the text can be covered in a one-semester course at the senior undergraduate level. In fact, most science disciplines and virtually all math and engineering disciplines contain the basic ideas of set theory, mathematics, and deductive logic, which form the only knowledge necessary for a complete understanding of the text. For an introductory class, instructors may want to exclude some or all of the material covered in the last section of Chapter 6 (neural networks, genetic algorithms, and inductive reasoning), Chapter 7 (automated methods of generation), and any of the final three chapters: Chapter 11 (fuzzy control), Chapter 12 (miscellaneous fuzzy applications), and Chapter 13 on alternative measures of uncertainty. I consider the applications in Chapter 8 on simulations, Chapter 9 on decision making, and Chapter 10 on classification to be important in the first course on this subject. The other topics could be used either as introductory material for a graduate-level course or for additional coverage for graduate students taking the undergraduate course for graduate credit.

In terms of organization, the first eight chapters of the text develop the foundational material necessary to get students in a position where they can generate their own fuzzy systems. The last five chapters use the foundation material from the first eight chapters to present specific applications.

Most of the problems at the end of each chapter have been revised with different numbers, and there are many new problems that have been added to the text and some problems from the third edition deleted. In reducing the length of the book, some old problems have been deleted from many chapters. The problems in this text are typically based on current and potential applications, case studies, and education in intelligent and fuzzy systems in engineering and related technical fields. The problems address the disciplines of computer science, electrical engineering, manufacturing engineering, industrial engineering, chemical engineering, petroleum engineering, mechanical engineering, civil engineering, environmental engineering, and engineering management, and a few related fields such as mathematics, medicine, operations research, technology management, the hard and soft sciences, and some technical business issues. The references cited in the chapters are listed toward the end of each chapter. These references provide sufficient detail for those readers interested in learning more about particular applications using fuzzy sets or fuzzy logic. The large number of problems provided in the text at the end of each chapter allows instructors a sizable problem base to afford instruction using this text on a multisemester or multiyear basis, without having to assign the same problems term after term.

Again I wish to give credit to some of the individuals who have shaped my thinking about this subject since the first edition of 1995, and to others who by their simple association with me have caused me to be more circumspect about the use of the material contained in the text. Three colleagues at Los Alamos National Laboratory have continued to work with me on applications of fuzzy set theory, fuzzy logic, and generalized uncertainty theory: Dr. Greg Chavez (who wrote much of Chapter 7) and Drs. Jamie Langenbrunner and Jane Booker (retired), who both have worked extensively in an area known as *quantification of margins and uncertainty (QMU)* in assessing reliability of man-made systems; in this regard these three individuals have all explored the use of fuzzy logic and possibility theory in their work. I wish to acknowledge the organizational support of two individuals in the Brazilian Institute, Instituto de Pesquisas Engergéticas e Nucleares (IPEN), in São Paulo, Brazil. These two researchers, Drs. Francisco Lemos and Antônio Barroso, through their invitations and travel support, have enabled me to train numerous Brazilian scientists and engineers in fuzzy logic applications in their own fields of work, most notably nuclear waste management, knowledge management, and risk assessment. My discussions with them have given me ideas about where fuzzy logic can impact new fields of inquiry.

I wish to thank two of my recent graduate students who have undertaken MS theses or PhD dissertations related to fuzzy logic and whose diligent work has assisted me in writing this new edition: Clay Phillips, Sandia National Laboratory, and Donald Lincoln, NStone Corporation. These former students have helped me with additional material that I have added in Chapter 12 and have helped discover some errata in this text. There have been numerous students over the past five years who have found much of the errata I have corrected; unfortunately, too numerous to mention in this brief preface. I want to thank them all for their contributions.

Four individuals need specific mention because they have contributed some sections to this text. I would like to thank specifically Dr. Jerry Parkinson for his contributions to Chapter 11, in the areas of chemical process control and fuzzy statistical process control; Dr. Greg Chavez for his contributions in automated methods; Dr. Sunil Donald for his early work in possibility distributions in Chapter 13; and Dr. Jung Kim for his contribution in Chapter 13 of a new procedure to combine disparate interval data. I would like to acknowledge the work of two graduate students at the University of New Mexico: Rashid Ahmad who has helped in developing some equations and figures for one chapter in the text, and to Pradeep Paudel for the development of the solutions manual for the text.

One individual deserves my special thanks and praise, and that is Professor Mahmoud Taha, my colleague in Civil Engineering at the University of New Mexico. In the last five years he has continued with his work in fuzzy logic applications, pattern recognition, and applications using possibility theory; I am proud and grateful to have been his mentor. I am indebted to his collaborations, his quick adaptation in the application of these tools, and in being a proficient research colleague of mine.

I am grateful for support in the past from IPEN in Brazil to teach this subject in their facility in late 2012 and to work in the area of fuzzy cognitive maps and to the Fulbright Foundation of Brazil to support me for two summers in 2013–2014 to continue my work at the Pontifícia Universidade Católica (PUC) do Rio de Janeiro, Brazil. Although most of my research at PUC was in bamboo engineering, I taught a graduate course there in fuzzy logic. Three individuals at PUC deserve special note: Professors Khosrow Ghavami and Raul Rosas da Silva, who both made my visit to Rio de Janeiro a most personally rewarding visit, and graduate student, Marco Antônio da Cunha, whose energy allowed me to advance my understanding of the use of cognitive maps in modeling the creation of a new field in bamboo engineering and in the use of agent-based models in a new field of enzymatic catalysis. Most recently (2015) on my sabbatical to the Air Force Research Laboratory, Space Vehicles Directorate, Kirtland AFB, New Mexico, I want to thank Messrs. Paul Zetocha and Apoorva Bhopale for their interest, and collaboration in, a new thrust area of research in fuzzy data fusion in the area of satellite orbital control.

With so many texts covering specific niches of fuzzy logic it is not possible to summarize all the important facets of fuzzy set theory and fuzzy logic in a single text. The hundreds of edited works and tens of thousands of archival papers show clearly that this is a rapidly growing technology, where new discoveries are being published every month. It remains my fervent hope that this introductory text will assist students and practicing professionals to learn, to apply, and to be comfortable with fuzzy set theory and fuzzy logic. I welcome comments from all readers to improve this text as a useful guide for the community of engineers and technologists who will become knowledgeable about the potential of fuzzy system tools for their use in solving the complex problems that challenge us each day.

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Introduction

It is the mark of an instructed mind to rest satisfied with that degree of precision which the nature of the subject admits, and not to seek exactness where only an approximation of the truth is possible.

Aristotle, 384–322 B.C.
Ancient Greek philosopher

Precision is not truth.

Henri E. B. Matisse, 1869–1954
Impressionist painter

All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life but only to an imagined celestial existence.

Bertrand Russell, 1923
British philosopher and Nobel Laureate

We must exploit our tolerance for imprecision.

Lotfi Zadeh, 1973
Professor, Systems Engineering, UC–Berkeley

The preceding quotes, all of them legendary, have a common thread. That thread represents the relationship between precision and uncertainty. The more uncertainty in a problem, the less precise we can be in our understanding of that problem. It is ironic that the oldest quote is attributed to the philosopher who is credited with the establishment of Western logic—a binary logic that admits only the opposites of true and false, a logic that does not admit degrees of truth in between these two extremes. In other words, Aristotelian logic does not admit imprecision in truth. However, Aristotle's quote is so appropriate today; it is a quote that admits uncertainty.

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It is an admonishment that we should heed; we should balance the precision we seek with the uncertainty that exists. Most engineering texts do not address the uncertainty in the information, models, and solutions that are conveyed within the problems addressed therein. This text is dedicated to the characterization and quantification of uncertainty within engineering problems such that an appropriate level of precision can be expressed. When we ask ourselves why we should engage in this pursuit, one reason should be obvious: achieving high levels of precision costs time or money or both. Are we solving problems that require precision? The more complex a system is, the more imprecise or inexact is the information that we have to characterize that system. It seems, then, that precision and information and complexity are inextricably related in the problems we pose for eventual solution. However, for most of the problems that we face, the quote credited to Professor Zadeh suggests that we can do a better job in accepting some level of imprecision.

It seems intuitive that we should balance the degree of precision in a problem with the associated uncertainty in that problem. Hence, this text recognizes that uncertainty of various forms permeates all scientific endeavors, and it exists as an integral feature of all abstractions, models, and solutions. Hence, the intent of this book is to introduce methods to handle one of these forms of uncertainty in our technical problems, the form we have come to call *fuzziness*.

The Case for Imprecision

Our understanding of most physical processes is based largely on imprecise human reasoning. This imprecision (when compared to the precise quantities required by computers) is nonetheless a form of information that can be quite useful to humans. The ability to embed such reasoning in hitherto intractable and complex problems is the criterion by which the efficacy of fuzzy logic is judged. Undoubtedly, this ability cannot solve problems that require precision, problems such as shooting precision laser beams more than tens of kilometers in space; milling machine components to accuracies of parts per billion; or focusing a microscopic electron beam on a specimen the size of a nanometer. The impact of fuzzy logic in these areas might be years away, if ever. But not many human problems require such precision, problems such as parking a car, backing up a trailer, navigating a car among others on a freeway, washing clothes, controlling traffic at intersections, judging beauty contestants, and a preliminary understanding of a complex system.

There are many simple examples in our culture that illustrate the lack of necessity for precision in much of what we do. There is a joke that is a good illustration about the lack of information contained in a precise number (Paulos, 1995). "A natural history museum guard told a visitor that the dinosaur on exhibit was 90,000,006 years old. Upon questioning about the specific number he used, the guard explained that he was told the dinosaur was 90,000,000 years old when he was hired, six years before! One can easily see the folly in adding a precise number to an imprecise number.

Another example follows us on a daily basis (Rocha, Massad, and Pereira, 2005). In food preparation the older manuals provide recipes that are appropriate enough for cooking delectable foods. A typical recipe calls for "*about* a cup" of this, a "*few* tablespoons" of that, a "*smidgen*" of something, "four *or* five" slices of something else, a "*couple* of medium-sized" other things and "seasoning *to taste*." The recipe goes on to state that this will produce "*about* four servings." This vagueness and ambiguity is not objectionable, but the arithmetic that comes from it is. In italicized print at the end of this older recipe, it's affirmed that the content of the ingredients contains 761 calories, 428 milligrams of sodium, and 22.6 grams of fat per serving. It is inconceivable that these numbers are so precise, and that they could be consistent, in any way, with the recipe the way it is written.

As another example in literature, Marcel Proust was a famous French neuroscientist who wrote about our memories. The style of his sentences was often verbose. In his book about Proust, Jonah Lehrer (2007), states, "Proust covers vast distances within the space of periods (one sentence is 356 words long), and often begins with the obscure detail (the texture of a napkin or the noise of water in the pipes) and ends with an inductive meditation on all things." The reader would be reasonable to inquire, of what value in this statement is the precision of 356 words?

Requiring precision in engineering models and products translates to requiring high cost and long lead times in production and development. For other than simple systems, expense is proportional to precision: more precision entails higher cost. When considering the use of fuzzy logic for a given problem, an engineer or scientist should ponder the need for *exploiting the tolerance for imprecision*. Not only does high precision dictate high costs, but it also entails low tractability in a problem. Articles in the popular media illustrate the need to exploit imprecision. Take the "traveling sales rep" problem, for example. In this classic optimization problem, a sales representative wants to minimize total distance traveled by considering various itineraries and schedules between a series of cities on a particular trip. For a small number of cities, the problem is a trivial exercise in enumerating all the possibilities and choosing the shortest route. As the number of cities continues to grow, the problem quickly approaches a combinatorial explosion impossible to solve through an exhaustive search, even with a computer. For example, for 100 cities there are $100 \times 99 \times 98 \times 97 \times \ldots \times 2 \times 1$, or about 10^{200}, possible routes to consider! No computers exist today that can solve this problem through a brute-force enumeration of all the possible routes. There are real, practical problems analogous to the traveling sales rep problem. For example, such problems arise in the fabrication of circuit boards, in which precise lasers drill hundreds of thousands of holes in the board. Deciding in which order to drill the holes (where the board moves under a stationary laser) so as to minimize drilling time is a traveling sales rep problem (Kolata, 1991).

Thus, algorithms have been developed to solve the traveling sales rep problem in an optimal sense; that is, the exact answer is not guaranteed but an optimum answer is achievable; the optimality is measured as a percent accuracy, with 0% representing the exact answer and accuracies larger than zero representing answers of lesser accuracy. Suppose we consider a signal routing problem analogous to the traveling sales rep problem in which we want to find the optimum path (i.e., minimum travel time) between 100,000 nodes in a network to an accuracy within 1% of the exact solution; this requires significant CPU time on a supercomputer. If we take the same problem and increase the precision requirement a modest amount to an accuracy of 0.75%, the computing time approaches a few months! Now suppose we can live with an accuracy of 3.5% (quite a bit more accurate than most problems we deal with), and we want to consider an order-of-magnitude more nodes in the network, say 1,000,000; the computing time for this problem is on the order of several minutes (Kolata, 1991). This remarkable reduction in cost (translating time to dollars) is solely the result of the acceptance of a lesser degree of precision in the optimum solution. Can humans live with a little less precision? The answer to this question depends on the situation, but for the vast majority of problems we deal with every day the answer is a resounding yes.

A Historical Perspective

From a historical point of view, the issue of uncertainty has not always been embraced within the scientific community (Klir and Yuan, 1995). In the traditional view of science, uncertainty represents an undesirable state—a state that must be avoided at all costs. This was the state of science until the late nineteenth century when physicists realized that Newtonian mechanics did not address problems at the molecular level. Newer methods, associated with statistical mechanics, were developed, which recognized that statistical averages could replace the specific manifestations of microscopic entities. These statistical quantities, which summarized the activity of large numbers of microscopic entities, could then be connected in a model with appropriate macroscopic variables (Klir and Yuan, 1995). Now, the role of Newtonian mechanics and its underlying calculus, which considered no uncertainty, was replaced with statistical mechanics, which could be described by a probability theory—a theory that could capture a form of uncertainty, the type generally referred to as *random uncertainty*. After the development of statistical mechanics there has been a gradual trend in science during the past century to consider the influence of uncertainty on problems and to do so in an attempt to make our models more robust in the sense that we achieve credible solutions and at the same time quantify the amount of uncertainty.

Of course, the leading theory in quantifying uncertainty in scientific models from the late nineteenth century until the late twentieth century had been the probability theory. However, the gradual evolution of the expression of uncertainty using probability theory was challenged, first in 1937 by Max Black with his studies in vagueness and then with the introduction of fuzzy sets by Zadeh (1965). Zadeh's paper had a profound influence on the thinking about uncertainty because it challenged not only probability theory as the sole representation for uncertainty but also the foundations on which probability theory was based: classical binary (two-valued) logic (Klir and Yuan, 1995).

Probability theory dominated mathematics of uncertainty for more than five centuries. Probability concepts date back to the 1500s, to the time of Cardano when gamblers recognized the rules of probability in games of chance. The concepts were still much in the limelight in 1685, when the Bishop of Wells wrote a paper that discussed a problem in determining the truth of statements made by two witnesses who were both known to be unreliable to the extent that they tell the truth only with probabilities, p_1 and p_2, respectively. The Bishop's answer to this was based on his assumption that the two witnesses were independent sources of information (Lindley, 1987).

Probability theory was initially developed in the eighteenth century in such landmark treatises as Jacob Bernoulli's *Ars Conjectandi* (1713) and Abraham de Moivre's *Doctrine of Chances* (1738). Later in that century, a small number of articles appeared in the periodical literature that would have a profound effect on the field. Most notable of these were Thomas Bayes's "An essay towards solving a problem in the doctrine of chances" (1763) and Pierre Simon Laplace's formulation of the axioms relating to games of chance, "Memoire sur la probabilite des causes par les evenemens" (1774/1986). Laplace, only 25 years old at the time he began his work in 1772, wrote the first substantial article in mathematical statistics before the nineteenth century. Despite the fact that Laplace, at the same time, was heavily engaged in mathematical astronomy, his memoir was an explosion of ideas that provided the roots for modern decision theory, Bayesian inference with nuisance parameters (historians claim that Laplace did not know of Bayes's earlier work), and the asymptotic approximations of posterior distributions (Stigler, 1986).

By the time of Newton, physicists and mathematicians were formulating different theories of probability. The most popular ones remaining today are the relative frequency theory and the subjectivist or personalistic theory. The latter development was initiated by Thomas Bayes (1763), who articulated his powerful theorem for the assessment of subjective probabilities. The theorem specified that a human's degree of belief could be subjected to an objective, coherent, and measurable mathematical framework within subjective probability theory. In the early days of the twentieth century a formal framework for a conditional probability theory was developed.

The twentieth century saw the first developments of alternatives to probability theory and to classical Aristotelian logic as paradigms to address more kinds of uncertainty than just the random kind. Jan Lukasiewicz developed a multivalued, discrete logic (*circa* 1930). In the 1960s, Arthur Dempster (1967) developed a theory of evidence, which, for the first time, included an assessment of ignorance, or the absence of information. In 1965, Lotfi Zadeh introduced his seminal idea in a continuous-valued logic that he called *fuzzy set theory*. Glenn Shafer (1976) extended Dempster's work to produce a complete theory of evidence dealing with information from more than one source, and Lotfi Zadeh illustrated a possibility theory resulting from special cases of fuzzy sets. Later, in the 1980s, other investigators showed a strong relationship between evidence theory, probability theory, and possibility theory with the use of what was called *fuzzy measures* (Klir and Wierman, 1996), and what is now being termed *monotone measures*.

Uncertainty can be thought of in an epistemological sense as being the inverse of information. Information about a particular engineering or scientific problem may be incomplete, imprecise, fragmentary, unreliable, vague, contradictory, or deficient in some other way (Klir and Yuan, 1995). When we acquire more and more information about a problem, we become less and less uncertain about its formulation and solution. Problems that are characterized by little information are said to be ill-posed, complex, or not sufficiently known. These problems are imbued with a high degree of uncertainty. Uncertainty can be manifested in many forms; it can be fuzzy (not sharp, unclear, imprecise, approximate), it can be vague (not specific, amorphous), it can be ambiguous (too many choices, contradictory), it can be of the form of ignorance (dissonant, not knowing something), or it can be a form resulting from natural variability (conflicting, random, chaotic, unpredictable). Many other linguistic labels have been applied to these various forms, but for now these shall suffice. Zadeh (in Ross, Booker, and Parkinson, 2002) posed some simple examples of these forms in terms of a person's statements about when they shall return to a current place in time. The statement "I shall return soon" is vague, whereas the statement "I shall return in a few minutes" is fuzzy; the former is not known to be associated with any unit of time (seconds, hours, days), and the latter is associated with an uncertainty that is at least known to be on the order of minutes. The phrase, "I shall return within 2 minutes of 6 PM" involves an uncertainty that has a quantifiable imprecision; probability theory could address this form.

Vagueness can be used to describe certain kinds of uncertainty associated with linguistic information or intuitive information. Examples of vague information are that the data quality is "good" or that the transparency of an optical element is "acceptable." Moreover, in terms of semantics, even the terms *vague* and *fuzzy* cannot be generally considered synonyms, as explained by Zadeh (1995, p. 275): "usually a vague proposition is fuzzy, but the converse is not generally true."

Discussions about vagueness started with a famous work by the philosopher Max Black. Black (1937) defined a vague proposition as a proposition where the possible states (of the proposition) are not clearly defined with regard to inclusion. For example, consider the proposition that a person is young. Because the term *young* has different interpretations to different individuals, we cannot decisively determine the age(s) at which an individual is young compared with the age(s) at which an individual is not considered to be young. Thus, the proposition is vaguely defined. Classical (binary) logic does not hold under these circumstances, therefore we must establish a different method of interpretation.

Max Black, in writing his 1937 essay "Vagueness: An exercise in logical analysis," first cites remarks made by the ancient philosopher Plato about uncertainty in geometry and then embellishes on the writings of Bertrand Russell (1923) who emphasized that "all traditional logic habitually assumes that precise symbols are being employed." With these great thoughts as a prelude to his own arguments, he proceeded to produce his own, now-famous quote:

> It is a paradox, whose importance familiarity fails to diminish, that the most highly developed and useful scientific theories are ostensibly expressed in terms of objects never encountered in experience. The line traced by a draftsman, no matter how accurate, is seen beneath the microscope as a kind of corrugated trench, far removed from the ideal line of pure geometry. And the "point-planet" of astronomy, the "perfect gas" of thermodynamics, or the "pure-species" of genetics are equally remote from exact realization. Indeed the unintelligibility at the atomic or subatomic level of the notion of a rigidly demarcated boundary shows that such objects not merely are not but could not be encountered. While the mathematician constructs a theory in terms of "perfect" objects, the experimental scientist observes objects of which the properties demanded by theory are and can, in the very nature of measurement, be only approximately true.

Much later, in support of Black's work, Quine (1981) states:

> Diminish a table, conceptually, molecule by molecule: when is a table not a table? No stipulations will avail us here, however arbitrary. If the term 'table' is to be reconciled with bivalence, we must posit an exact demarcation, exact to the last molecule, even though we cannot specify it. We must hold that there are physical objects, coincident except for one molecule, such that one is a table and the other is not.

de Finetti (1974), publishing in his landmark book *Theory of Probability*, gets his readers' attention quickly by proclaiming, "Probability does not exist; it is a subjective description of a person's uncertainty. We should be normative about uncertainty and not descriptive" (p. x). He further emphasizes that the frequentist view of probability (objectivist view) "requires individual trials to be equally probable and stochastically independent" (p. x). In discussing the difference between possibility and probability, he states: "The logic of certainty furnishes us with the range of possibility (and the possible has no gradations); probability is an additional notion that one applies within the range of possibility, thus giving rise to graduations ('more or less' probable) that are meaningless in the logic of uncertainty" (p. 218). de Finetti gives us warnings: "The calculus of probability can say absolutely nothing about reality," and in referring to the dangers implicit in attempts to confuse certainty with high probability, he states:

> We have to stress this point because these attempts assume many forms and are always dangerous. In one sentence: to make a mistake of this kind leaves one inevitably faced with all sorts of

> fallacious arguments and contradictions whenever an attempt is made to state, on the basis of probabilistic considerations, that something must occur, or that its occurrence confirms or disproves some probabilistic assumptions (p. 215).

In a discussion about the use of such vague terms as *very probable* or *practically certain*, or *almost impossible*, de Finetti states:

> The field of probability and statistics is then transformed into a Tower of Babel, in which only the most naive amateur claims to understand what he says and hears, and this because, in a language devoid of convention, the fundamental distinctions between what is certain and what is not, and between what is impossible and what is not, are abolished. Certainty and impossibility then become confused with high or low degrees of a subjective probability, which is itself denied precisely by this falsification of the language. On the contrary, the preservation of a clear, terse distinction between certainty and uncertainty, impossibility and possibility, is the unique and essential precondition for making meaningful statements (which could be either right or wrong), whereas the alternative transforms every sentence into a nonsense (p. 213).

The Utility of Fuzzy Systems

Several sources have shown and proven that fuzzy systems are universal approximators (Kosko, 1994; Ying et al., 1999). These proofs stem from the isomorphism between two algebras—an abstract algebra (one dealing with groups, fields, and rings) and a linear algebra (one dealing with vector spaces, state vectors, and transition matrices)—and the structure of a fuzzy system, which comprises an implication between actions and conclusions (antecedents and consequents). The reason for this isomorphism is that both entities (algebra and fuzzy systems) involve a mapping between elements of two or more domains. Just as an algebraic function maps an input variable to an output variable, a fuzzy system maps an input group to an output group; in the latter these groups can be linguistic propositions or other forms of fuzzy information. The foundation on which fuzzy systems theory rests is a fundamental theorem from real analysis in algebra known as the *Stone–Weierstrass theorem*, which was first developed in the late nineteenth century by Weierstrass (1885), then simplified by Stone (1937).

In the coming years it will be the consequence of this isomorphism that will make fuzzy systems more and more popular as solution schemes, and it will make fuzzy systems theory a routine offering in the classroom as opposed to its previous status as a "new, but curious technology." Fuzzy systems, or whatever label scientists eventually come to call it in the future, will be a standard course in any science or engineering curriculum. It contains all of what algebra has to offer, plus more, because it can handle all kinds of information not just numerical quantities.

Although fuzzy systems are shown to be universal approximators to algebraic functions, it is not this attribute that actually makes them valuable to us in understanding new or evolving problems. Rather, the primary benefit of fuzzy systems theory is to approximate system behavior in which analytic functions or numerical relations do not exist. Hence, fuzzy systems have high potential to understand the systems that are devoid of analytic formulations: complex systems. Complex systems can be new systems that have not been tested; they can be systems involved with the human condition such as biological or medical systems; or they can be social, economic, or political systems, in which the vast arrays of inputs and outputs could not all possibly be captured analytically or controlled in any conventional sense. Moreover, the relationship between the causes and effects of these systems is generally not understood but often can be observed.

Alternatively, fuzzy systems theory can have utility in assessing some of our more conventional, less complex systems. For example, for some problems exact solutions are not always necessary. An approximate, but fast, solution can be useful in making preliminary design decisions; or as an initial estimate in a more accurate numerical technique to save computational costs; or in the myriad situations in which the inputs to a problem are vague, ambiguous, or not known at all. For example, suppose we need a controller to bring an aircraft out of a vertical dive. Conventional controllers cannot handle this scenario because they are restricted to linear ranges of variables; a dive situation is highly non-linear. In this case, we could use a fuzzy controller, which is adept at handling nonlinear situations albeit in an imprecise fashion, to bring the plane out of the dive into a more linear range, then hand off the control of the aircraft to a conventional, linear, highly accurate controller. Examples of other situations in which exact solutions are not warranted abound in our daily lives. For example, in the following quote from a popular science fiction movie,

C-3PO: Sir, the possibility of successfully navigating an asteroid field is approximately 3,720 to 1!
Han Solo: Never tell me the odds!
Characters in the movie *Star Wars: The Empire Strikes Back* (Episode V), 1980.

we have an illustration of where the input information (the odds of navigating through an asteroid field) is useless, so how does one make a decision in the presence of this information?

Hence, fuzzy systems are useful in two general contexts: (1) in situations involving highly complex systems whose behaviors are not well understood and (2) in situations where an approximate, but fast, solution is warranted.

As pointed out by Ben-Haim (2001), there is a distinction between models of systems and models of uncertainty. A fuzzy system can be thought of as an aggregation of both because it attempts to understand a system for which no model exists, and it does so with information that can be uncertain in a sense of being vague, or fuzzy, or imprecise, or altogether lacking. Systems whose behaviors are both understood and controllable are of the kind which exhibit a certain robustness to spurious changes. In this sense, robust systems are ones whose output (such as a decision system) does not change significantly under the influence of changes in the inputs because the system has been designed to operate within some window of uncertain conditions. It is maintained that fuzzy systems too are robust. They are robust because the uncertainties contained in both the inputs and outputs of the system are used in formulating the system structure itself, unlike conventional systems analysis that first poses a model, based on a collective set of assumptions needed to formulate a mathematical form, then uncertainties in each of the parameters of that mathematical abstraction are considered.

The positing of a mathematical form for our system can be our first mistake, and any subsequent uncertainty analysis of this mathematical abstraction could be misleading. We call this the *optimist's dilemma*: find out how a chicken clucks by first "assuming a spherical chicken." Once the sphericity of the chicken has been assumed, there are all kinds of elegant solutions that can be found; we can predict any number of sophisticated clucking sounds with our model. Unfortunately, when we monitor a real chicken it does not cluck the way we predict. The point being made here is that there are few physical and no mathematical abstractions that can be made to solve some of our complex problems, so we need new tools to deal with complexity; fuzzy systems and their associated developments can be one of these newer tools.

The power of fuzzy logic in terms of its impact on research and commercial markets is without debate. Thousands of researchers are working with fuzzy logic and producing patents and research papers. According to a report (Singh et al., 2013) on the impact of fuzzy logic as of March 4, 2013, there were 26 research journals on the theory or applications of fuzzy logic, there were 89,365 publications on theory or applications of fuzzy logic in the INSPEC database, there were 22,657 publications on theory or applications of fuzzy logic in the MathSciNet database, there were 16,898 patent applications and patents issued related to fuzzy logic in the United States, and there were 7,149 patent applications and patents issued related to fuzzy logic in Japan. The number of research contributions and commercial applications is growing daily and is growing at an increasing rate.

Limitations of Fuzzy Systems

However, this is not to suggest that we can now stop looking for additional tools to evaluate imprecision or to assess methods for achieving approximate but credible solutions to complex problems. Realistically, even fuzzy systems, as they are posed now, can be described as shallow models in the sense that they are primarily used in deductive reasoning. This is the kind of reasoning in which we infer the specific from the general. For example, in the game of tic-tac-toe, there are only a few moves for the entire game; we can deduce our next move from the previous move and our knowledge of the game. It is this kind of reasoning that we also called *shallow reasoning*, because our knowledge, as expressed linguistically, is of a shallow and meager kind. In contrast to this is the kind of reasoning that is inductive, where we infer the general from the particular; this method of inference is called *deep*, because our knowledge is of a deep and substantial kind—a game of chess would be closer to an inductive kind of model.

We should understand the distinction between using mathematical models to account for observed data and using mathematical models to describe the underlying process by which the observed data are generated or produced by nature (Arciszewski, Sauer, and Schum, 2003). Models of systems where the behavior can be observed, and whose predictions can only account for these observed data, are said to be shallow because they do not account for the underlying realities. Deep models, those of the inductive kind, are alleged to capture the physical process by which nature has produced the results we have observed. In his *Republic* (360 B.C./1991), Plato suggests the idea that things that are perceived are only imperfect copies of the true reality that can only be comprehended by pure thought. Plato was fond of mathematics, and he saw in its precise structure of logic idealized abstraction and separation from the material world. He thought of these things being so important that above the doorway to his Academy was placed the inscription "Let no one ignorant of mathematics enter here." In Plato's doctrine of forms, he argued that the phenomenal world was a mere shadowy image of the eternal, immutable real world, and that matter was docile and disorderly governed by a mind that was the source of coherence, harmony, and orderliness. He argued that if man was occupied with the things of the senses, then he could never gain true knowledge. In his work the *Phaedo*, he declares that as mere mortals we cannot expect to attain absolute truth about the universe, but instead must be content with developing a descriptive picture—a model (Barrow, 2000).

Centuries later, Galileo was advised by his inquisitors that he must not say that his mathematical models were describing the realities of nature, but rather that they simply were adequate models of the observations he made with his telescope (Drake, 1957); hence, that they were solely deductive. In this regard, models that only attempt to replicate some phenomenological behavior are considered shallow models or models of the deductive kind, and they lack the knowledge needed for true understanding of a physical process. The system that emerges under inductive reasoning will have connections with both evolution and complexity. How do humans reason in situations that are complicated or ill-defined? Modern psychology tells us that as humans we are only moderately good at deductive logic, and we make only moderate use of it. Bu, we are superb at seeing or recognizing or matching patterns—behaviors that confer obvious evolutionary benefits. In problems of complication then, we look for patterns, and we simplify the problem by using these to construct temporary internal models or hypotheses or schemata to work with (Bower and Hilgard, 1981). We carry out localized deductions based on our current hypotheses and we act on these deductions. Then, as feedback from the environment comes in, we may strengthen or weaken our beliefs in our current hypotheses, discarding some when they cease to perform, and replacing them as needed with new ones. In other words, where we cannot fully reason or lack full definition of the problem, we use simple models to fill the gaps in our understanding; such behavior is inductive.

Some sophisticated models may, in fact, be a complex weave of deductive and inductive steps. But even our so-called "deep models" may not be deep enough. An illustration of this comes from a recent popular decision problem, articulated as the El Farol problem by Arthur (1994). This problem involves a decision-making scenario in which inductive reasoning is assumed and modeled, and its implications are examined. El Farol is a bar in Santa Fe, New Mexico, where on one night of the week in particular there is popular Irish music offered. Suppose N bar patrons decide independently each week whether to go to El Farol on this certain night. For simplicity, we set $N = 100$. Space in the bar is limited, and the evening is enjoyable if things are not too crowded, specifically, if fewer than 60% of the possible 100 are present. There is no way to tell the number coming for sure in advance, therefore a bar patron goes—deems it worth going—if he expects fewer than 60 to show up or stays home if he expects more than 60 to go; there is no need that utilities differ much above and below 60. Choices are unaffected by previous visits; there is no collusion or prior communication among the bar patrons and the only information available is the number who came in past weeks. Of interest is the dynamics of the number of bar patrons attending from week to week.

There are two interesting features of this problem. First, if there was an obvious model that all bar patrons could use to forecast attendance and on which to base their decisions, then a deductive solution would be possible. But no such model exists in this case. Given the numbers attending in the recent past, a large number of expectational models might be reasonable and defensible. Thus, not knowing which model other patrons might choose, a reference patron cannot choose his in a well-defined way. There is no deductively rational solution, that is, no "correct" expectational model. From the patrons' viewpoint, the problem is ill-defined and they are propelled into a realm of induction. Second, any commonality of expectations gets disintegrated: if everyone believes few will go, then all will go. But this would invalidate that belief. Similarly, if all believe most will go, nobody will go, invalidating that belief. Expectations will be forced to differ, but not in a methodical, predictive way.

Scientists have long been uneasy with the assumption of perfect, deductive rationality in decision contexts that are complicated and potentially ill-defined. The level at which humans can apply perfect rationality is surprisingly modest. Yet, it has not been clear how to deal with imperfect or bounded rationality. From the inductive example given in the El Farol problem, it would be easy to suggest that as humans in these contexts we use inductive reasoning: we induce a variety of working hypotheses, act on the most credible, and replace hypotheses with new ones if they cease to work. Such reasoning can be modeled in a variety of ways. Usually, this leads to a rich psychological world in which peoples' ideas or mental models compete for survival against other peoples' ideas or mental models, a world that is both evolutionary and complex. And, although this seems the best course of action for modeling complex questions and problems, this text reveals a few ideas about models which go beyond those of the rule-based kind. These are briefly introduced in Chapter 12 (genetically evolved fuzzy cognitive maps and fuzzy agent-based models).

The Illusion: Ignoring Uncertainty and Accuracy

A slight variation in the axioms at the foundation of a theory can result in huge changes at the frontier.

Stanley P. Gudder, 1988
Author, *Quantum Probability*

The uninitiated often claim that fuzzy set theory is just another form of probability theory in disguise. This statement, of course, is simply not true. Gaines (1978) does an eloquent job of addressing this issue. Historically, probability and fuzzy sets have been presented as distinct theoretical foundations for reasoning and decision making in situations involving uncertainty. Yet, when one examines the underlying axioms of both probability and fuzzy set theories, the two theories differ by only one axiom in a total of 16 axioms needed for a complete representation! Gaines established a common basis for both forms of logic of uncertainty in which a basic uncertainty logic is defined in terms of valuation on a lattice of propositions. Addition of the axiom of the excluded middle to the basic logic gives a standard probability logic. Alternatively, addition of a requirement for strong truth-functionality gives a fuzzy logic. The quote by Stanley Gudder is quite instructive in this case: probability theory and fuzzy set theory each satisfy a different set of axioms; hence, neither theory should be held to the standards of the others' axiomatic constraints.

Basic statistical analysis is founded on probability theory or stationary random processes, whereas most experimental results contain both random (typically noise) and nonrandom processes. One class of random processes—stationary random processes—exhibits the following three characteristics: (1) The sample space on which the processes are defined cannot change from one experiment to another; that is, the outcome space cannot change. (2) The frequency of occurrence, or probability, of an event within that sample space is constant and cannot change from trial to trial or experiment to experiment. (3) The outcomes must be repeatable from experiment to experiment. The outcome of one trial does not influence the outcome of a previous or future trial. There are more general classes of random processes than the class mentioned here. However, fuzzy sets are not governed by these characteristics.

Stationary random processes are those that arise out of chance, in which the chances represent frequencies of occurrence that can be measured. Problems like picking colored balls out of an urn, coin and dice tossing, and many card games are good examples of stationary random processes. How many of the decisions that humans must make every day could be categorized as random? How about the uncertainty in the weather, is this random? How about your uncertainty in choosing clothes for the next day, or which car to buy, or your preference in colors, are these random uncertainties? How about the risk in whether a substance consumed by an individual now will cause cancer in that individual 15 years from now, is this a form of random uncertainty? Although it is possible to model all of these forms of uncertainty with various classes of random processes, the solutions may not be reliable. Treatment of these forms of uncertainty using fuzzy set theory should also be done with caution. One needs to study the character of the uncertainty and then choose an appropriate approach to develop a model of the process. Features of a problem that vary in time and space should be considered. For example, when the weather report suggests that there is a 60% chance of rain tomorrow, does this mean that there has been rain on tomorrow's date for 60 of the last 100 years? Does it mean that somewhere in your community 60% of the land area will receive rain? Does it mean that 60% of the time it will be raining and 40% of the time it will not be raining? Humans often deal with these forms of uncertainty linguistically, such as, "It will likely rain tomorrow." And, with this crude assessment of the possibility of rain, humans can still make appropriately accurate decisions about the weather.

Random errors will generally average out over time or space. Nonrandom errors, such as some unknown form of bias (often called a *systematic error*) in an experiment, will not generally average out and will likely grow larger with time. The systematic errors generally arise from causes about which we are ignorant, for which we lack information, or that we cannot control. Distinguishing between random and nonrandom errors is a difficult problem in many situations, and to quantify this distinction often results in the illusion that the analyst knows the extent and character of each type of error. In all likelihood, nonrandom errors can increase without bounds. Moreover, variability of the random kind cannot be reduced with additional information, although it can be quantified. By contrast, nonrandom uncertainty, which too can be quantified with various theories, can be reduced with the acquisition of additional information.

It is historically interesting that the word *statistics* is derived from the now-obsolete term *statist*, which means *an expert in statesmanship*. Statistics were the numerical facts that statists used to describe the operations of states. To many people, statistics, and other recent methods to represent uncertainty such as evidence theory and fuzzy set theory, are still the facts by which politicians, newspapers, insurance sellers, and other broker occupations approach us as potential customers for their services or products! The air of sophistication that these methods provide to an issue should not be the basis for making a decision; it should be made only after a good balance has been achieved between the information content in a problem and the proper representation tool to assess it.

Popular lore suggests that the various uncertainty theories allow engineers to fool themselves in a highly sophisticated way when looking at relatively incoherent heaps of data (computational or experimental), as if this form of deception is any more palatable than just plain ignorance. All too often, scientists and engineers are led to use these theories as a crutch to explain vagaries in their models or in their data. For example, in probability applications the assumption of independent random variables is often assumed to provide a simpler method to prescribe joint probability distribution functions. An analogous assumption, called *noninteractive sets* (see Chapter 2 in Ross, 2004), is used in fuzzy applications to develop joint membership functions from individual membership functions for sets from different universes of discourse. Should one ignore apparently aberrant information or consider all information in the model whether or not it conforms to the engineers' preconceptions? Additional experiments to increase understanding cost money, and yet, they might increase the uncertainty by revealing conflicting information. It could best be said that statistics alone, or fuzzy sets alone, or evidence theory alone, are individually insufficient to explain many of the imponderables that people face every day. Collectively they could be powerful. A poem by Cunningham (1971) titled "Meditation on Statistical Method" provides a good lesson in caution for any technologist pondering the thought that ignoring uncertainty (again, using statistics because of the era of the poem) in a problem will somehow make its solution seem more accurate.

Plato despair!
We prove by norms
How numbers bear
Empiric forms,

How random wrongs
Will average right
If time be long
And error slight;

But in our hearts
Hyperbole
Curves and departs
To infinity.

Error is boundless.
Nor hope nor doubt,
Though both be groundless,
Will average out.

Uncertainty and Information

Information is the resolution of uncertainty.

Claude Shannon, twentieth century mathematician

Only a small portion of knowledge (information) for a typical problem might be regarded as certain or deterministic. Unfortunately, the vast majority of the material taught in engineering classes is based on the presumption that knowledge involved is deterministic. Most processes are neatly and surreptitiously reduced to closed-form algorithms: equations and formulas. When students graduate, it seems that their biggest fear upon entering the real world is "forgetting the correct formula." These formulas typically describe a deterministic process, one where there is no uncertainty in the physics of the process (i.e., the right formula) and there is no uncertainty in the parameters of the process (i.e., the coefficients are known with impunity). It is only after we leave the university, it seems, that we realize we were duped in academia and that the information we have for a particular problem virtually always contains uncertainty. For how many of our problems can we say that the information content is known absolutely, that is, with no ignorance, no vagueness, no imprecision, or no element of chance? Uncertain information can take on many different forms. There is uncertainty that arises because of complexity, for example, the complexity in the reliability network of a nuclear reactor. There is uncertainty that arises from ignorance, from various classes of randomness, from the inability to perform adequate measurements, from lack of knowledge, or from the fuzziness inherent in our natural language.

The nature of uncertainty in a problem is an important point that engineers should ponder before their selection of an appropriate method to express the uncertainty. Fuzzy sets provide a mathematical way to represent vagueness and fuzziness in humanistic systems. For example, suppose you are teaching your child to bake cookies and you want to give instructions about when to take the cookies out of the oven. You could say to take them out when the temperature inside the cookie dough reaches 375° F, or you could advise your child to take them out when the tops of the cookies turn *light brown*. Which instruction would you give? Most likely, you would use the second of the two instructions. The first instruction is too precise to implement practically; in this case precision is not useful. The vague term *light brown* is useful in this context and can be acted on even by a child. We all use vague terms, imprecise information, and other fuzzy data just as easily as we deal with situations governed by chance, where probability techniques are warranted and useful. Hence, our sophisticated computational methods should be able to represent and manipulate a variety of uncertainties. Other representations of uncertainties resulting from ambiguity, nonspecificity, beliefs, and ignorance are introduced in Chapter 13. The one uncertainty that is not addressed in this text is the one termed *unknown*. The statement by a recent U.S. politician, is an interesting diversion that suggests why a method to quantify *unknownness* is perhaps a bit premature.

> The Unknown
> As we know,
> There are known knowns.
> There are things we know we know.
> We also know
> There are known unknowns.
> That is to say
> We know there are some things
> We do not know.
> But there are also unknown unknowns,
> The ones we don't know
> We don't know.
> —Feb. 12, 2002, Donald Rumsfeld, U.S. Secretary of Defense

Fuzzy Sets and Membership

The foregoing sections discuss the various elements of uncertainty. Making decisions about processes that contain nonrandom uncertainty, such as the uncertainty in natural language, has been shown to be less than perfect. The idea proposed by Lotfi Zadeh suggested that *set membership* is the key to decision making when faced with uncertainty. In fact, Zadeh made the following statement in his seminal paper of 1965:

> The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and, potentially, may prove to have a much wider scope of applicability, particularly in the fields of pattern classification and information processing. Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables (p. 339).

As an example, we can easily assess whether someone is taller than 6 feet. In a binary sense, the person either is or is not, based on the accuracy, or imprecision, of our measuring device. For example, if "tall" is a set defined as heights equal to or greater than 6 feet, a computer would not recognize an individual of height $5'11.999''$ as being a member of the set "tall." But how do we assess the uncertainty in the following question: Is the person *nearly* 6 feet tall? The uncertainty in this case is the result of the vagueness or ambiguity of the adjective *nearly*. A $5'11''$ person could clearly be a member of the set of "nearly 6 feet tall" people. In the first situation, the uncertainty of whether a person, whose height is unknown, is 6 feet or not is binary; the person either is or is not, and we can produce a probability assessment of that prospect based on height data from many people. But the uncertainty of whether a person is nearly 6 feet is non-random. The degree to which the person approaches a height of 6 feet is fuzzy. In reality, "tallness" is a matter of degree and is relative. Among peoples of the Tutsi tribe in Rwanda and Burundi, a height for a male of 6 feet is considered short. So, 6 feet can be tall in one context and short in another. In the real (fuzzy) world, the set of tall people can overlap with the set of not-tall people, an impossibility when one follows the precepts of classical binary logic (this is discussed in Chapter 5).

This notion of set membership, then, is central to the representation of objects within a universe by sets defined on the universe. Classical sets contain objects that satisfy precise properties of membership; fuzzy sets contain objects that satisfy imprecise properties of membership, that is, membership of an object in a fuzzy set can be approximate. For example, the set of heights *from 5 to 7 feet* is precise (crisp); the set of heights in the region *around 6 feet* is imprecise, or fuzzy. To elaborate, suppose we have an exhaustive collection of individual elements (singletons) x, which make up a universe of information (discourse), X. Further, various combinations of these individual elements make up sets, say A, on the universe. For crisp sets, an element x in the universe X is either a member of some crisp set A or not. This binary issue of membership can be represented mathematically with the indicator function,

$$\chi_A(x) = \begin{cases} 1, & x \in \mathrm{A} \\ 0, & x \notin \mathrm{A} \end{cases}, \tag{1.1}$$

where the symbol $\chi_\mathrm{A}(x)$ gives the indication of an unambiguous membership of element x in set A, and the symbols $\in$ and $\notin$ denote contained in and not contained in, respectively. For our example of the universe of heights of people, suppose set A is the crisp set of all people with $5.0 \le x \le 7.0$ feet, shown in Figure 1.1a. A particular individual, x_1, has a height of 6.0 feet. The membership of this individual in crisp set A is equal to 1, or full membership, given symbolically as $\chi_\mathrm{A}(x_1) = 1$. Another individual, say x_2, has a height of 4.99 feet. The membership of this individual in set A is equal to 0, or no membership, hence $\chi_\mathrm{A}(x_2) = 0$, also seen in Figure 1.1a. In these cases the membership in a set is binary, either an element is a member of a set or it is not.

Zadeh extended the notion of binary membership to accommodate various "degrees of membership" on the real continuous interval [0, 1], where the endpoints of 0 and 1 conform to no membership and full membership, respectively, just as the indicator function does for crisp sets, but where the infinite number of values in between the endpoints can represent various degrees of membership for an element x in some set on the universe. The sets on the universe X that can

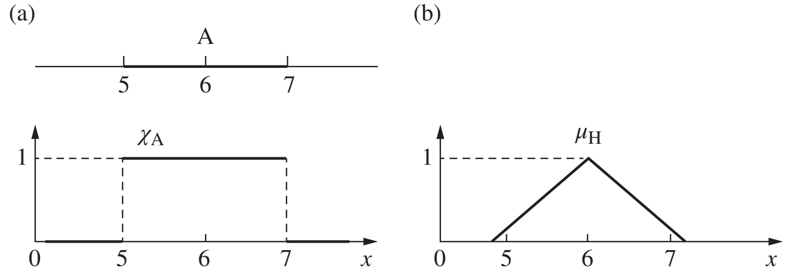

Figure 1.1 Height membership functions for (a) a crisp set A and (b) a fuzzy set H.

accommodate "degrees of membership" were termed by Zadeh as *fuzzy sets*. Continuing further on the example on heights, consider a set H consisting of heights *near 6 feet*. Because the property *near 6 feet* is fuzzy, there is no unique membership function for H. Rather, the analyst must decide what the membership function, denoted μ_H, should look like. Plausible properties of this function might be (1) normality ($\mu_H(6) = 1$), (2) monotonicity (the closer H is to 6, the closer μ_H is to 1), and (3) symmetry (numbers equidistant from 6 should have the same value of μ_H) (Bezdek, 1993). Such a membership function is illustrated in Figure 1.1b. A key difference between crisp and fuzzy sets is their membership function; a crisp set has a unique membership function, whereas a fuzzy set can have an infinite number of membership functions to represent it. For fuzzy sets, the uniqueness is sacrificed, but flexibility is gained because the membership function can be adjusted to maximize the utility for a particular application. It should be noted that a crisp set is a *special case* of a fuzzy set; it is a fuzzy set with no ambiguity on its boundaries.

James Bezdek (1993) provided one of the most lucid comparisons between crisp and fuzzy sets. It bears repeating here. Crisp sets of real objects are equivalent to, and isomorphically described by, a unique membership function, such as χ_A in Figure 1.1a. But there is no set-theoretic equivalent of "real objects" corresponding to χ_A. Fuzzy sets are always *functions*, which map a universe of objects, say X, onto the unit interval [0, 1]; that is, the fuzzy set H is the *function* μ_H that carries X into [0, 1]. Hence, *every* function that maps X onto [0, 1] is a fuzzy set. Although this statement is true in a formal mathematical sense, many functions that qualify on the basis of this definition cannot be suitable fuzzy sets. But, they *become* fuzzy sets when, and only when, they match some intuitively plausible semantic description of imprecise properties of the objects in X.

The membership function embodies the mathematical representation of membership in a set, and the notation used throughout this text for a fuzzy set is a set symbol with a tilde underscore, say $\underset{\sim}{A}$, where the functional mapping is given as

$$\mu_{\underset{\sim}{A}}(x) \in [0, 1], \tag{1.2}$$

and the symbol $\mu_{\underset{\sim}{A}}(x)$ is the degree of membership of element x in fuzzy set $\underset{\sim}{A}$. Therefore, $\mu_{\underset{\sim}{A}}(x)$ is a value on the unit interval that measures the degree to which element x belongs to fuzzy set $\underset{\sim}{A}$; equivalently, $\mu_{\underset{\sim}{A}}(x)$ = degree to which $x \in \underset{\sim}{A}$.

Chance versus Fuzziness

Suppose you are a basketball recruiter and are looking for a “very tall” player for the center position on a men’s team. One of your information sources tells you that a hot prospect in Oregon has a 95% chance of being taller than 7 feet. Another of your sources tells you that a good player in Louisiana has a high membership in the set of “very tall” people. The problem with the information from the first source is that it is a probabilistic quantity. There is a 5% chance that the Oregon player is not taller than 7 feet and could, conceivably, be someone of extremely short stature. The second source of information would, in this case, contain a different kind of uncertainty for the recruiter; it is a fuzziness resulting from the linguistic qualifier *very tall* because if the player turned out to be shorter than 7 feet tall there is still a high likelihood that he would be quite tall.

Another example involves a personal choice. Suppose you are seated at a table on which rest two glasses of liquid. The liquid in the first glass is described to you as having a 95% chance of being healthful and good. The liquid in the second glass is described as having a 0.95 membership in the class of “healthful and good” liquids. Which glass would you select, keeping in mind that the first glass has a 5% chance of being filled with nonhealthful liquids, including poisons (Bezdek, 1993)?

What philosophical distinction can be made regarding these two forms of information? Suppose we are allowed to measure the basketball players’ heights and test the liquids in the glasses. The prior probability of 0.95 in each case becomes a posterior probability of 1.0 or 0; that is, either the player is or is not taller than 7 feet and the liquid is either benign or not. However, the membership value of 0.95, which measures the extent to which the player’s height is taller than 7 feet or the drinkability of the liquid is “healthful and good,” remains 0.95 after measuring or testing. These two examples illustrate clearly the difference in the information content between chance and fuzziness.

This brings us to the clearest distinction between fuzziness and chance. Fuzziness describes the lack of distinction of an event, whereas chance describes the uncertainty in the occurrence of the event. The event will occur or not occur; but is the description of the event clear enough to measure its occurrence or nonoccurrence? Consider the following geometric questions, which serve to illustrate our ability to address fuzziness (lack of distinctiveness) with certain mathematical relations. The geometric shape in Figure 1.2a can resemble a disk, a cylinder, or a

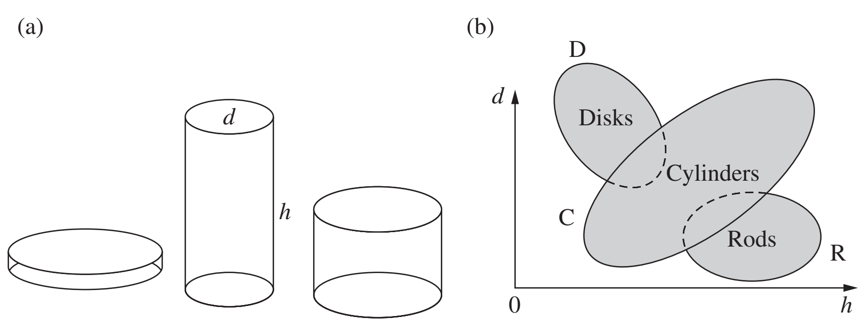

Figure 1.2 Relationship between (a) mathematical terms and (b) fuzzy linguistic terms.

rod depending on the aspect ratio of d/h. For $d/h \ll 1$, the shape of the object approaches a long rod; in fact, as $d/h \to 0$ the shape approaches a line. For $d/h \gg 1$, the object approaches the shape of a flat disk; as $d/h \to \infty$ the object approaches a circular area. For other values of this aspect ratio, for example, for $d/h \approx 1$, the shape is typical of what we would call a *right circular cylinder* (see Figure 1.2b).

The geometric shape in Figure 1.3a is an ellipse, with parameters a and b. Under what conditions of these two parameters will a general elliptic shape become a circle? Mathematically, we know that a circle results when $a/b = 1$, and hence this is a specific, crisp geometric shape. We know that when $a/b \ll 1$ or $a/b \gg 1$, we clearly have an elliptic shape, and as $a/b \to \infty$, a line segment results. Using this knowledge, we can develop a description of the membership function to describe the geometric set we call an approximate circle. Without a theoretical development, the following expression describing a Gaussian curve (for this membership function all points on the real line have nonzero membership; this can be an advantage or disadvantage depending on the nature of the problem) offers a good approximation for the membership function of the fuzzy set "approximate circle," denoted $\underset{\sim}{C}$:

$$\mu_{\underset{\sim}{C}}\left(\frac{a}{b}\right) = \exp\left[-3\left(\frac{a}{b}-1\right)^2\right] \tag{1.3}$$

Figure 1.3b is a plot of the membership function given in Equation (1.3). As the elliptic ratio a/b approaches a value of unity, the membership value approaches unity; for $a/b = 1$, we have an unambiguous circle. As $a/b \to \infty$ or $a/b \to 0$, we get a line segment; hence, the membership of the shape in the fuzzy set $\underset{\sim}{C}$ approaches zero because a line segment is not similar in shape to a circle. In Figure 1.3b, we see that as we get farther from $a/b = 1$ our membership in the set "approximate circle" gets smaller and smaller. All values of a/b, which have a membership value of unity, are called the *prototypes*; in this case $a/b = 1$ is the only prototype for the set "approximate circle," because at this value it is exactly a circle.

Suppose we were to place in a bag a large number of generally elliptical two-dimensional shapes and ask the question: What is the probability of randomly selecting an "approximate circle" from the bag? We would not be able to answer this question without first assessing the two different kinds of uncertainty. First, we would have to address the issue of fuzziness

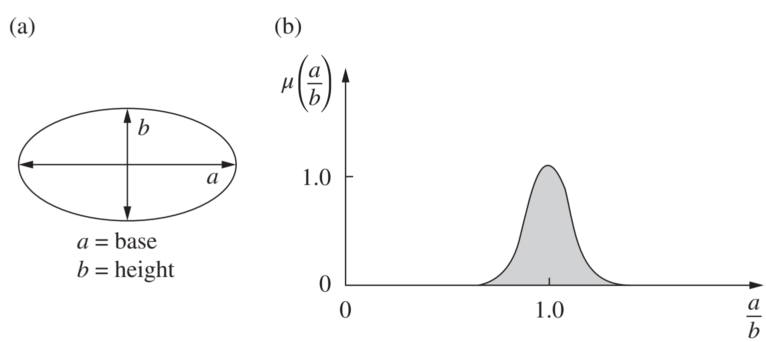

Figure 1.3 The (a) geometric shape and (b) membership function for an approximate circle.

in the meaning of the term *approximate circle* by selecting a value of membership, which we would be willing to call the shape an approximate circle; for example, any shape with a membership value above 0.9 in the fuzzy set "approximate circle" would be considered a circle. Second, we would have to know the proportion of the shapes in the bag that have membership values above 0.9. The first issue is one of assessing fuzziness and the second relates to the frequencies required to address questions of chance.

Intuition of Uncertainty: Fuzzy versus Probability

It is instructive to see how the propagation of uncertainty in a simple nonlinear model can reveal vast differences in the results between a probability model and a fuzzy model, and whether these would conform to our intuition. Suppose we have the simple model, $y = \sin(x)$, and we know that the input parameter, x, is uncertain. We want to model the uncertainty in the input x using a probability density function and also to model the uncertainty in x using a fuzzy membership function. It is important that these two functions look the same, geometrically. Figure 1.4 shows the modeling issues, and results. In Figure 1.4a, the uncertainty in the input is modeled as a uniform probability density function; each element in the universe of the input has equal frequency of occurrence. In Figure 1.4b, the uncertainty in the input is modeled as a fuzzy membership function; here each element in the universe of the input ($-\pi/2$ to $+\pi/2$) has an equal membership of unity.

To show how the uncertainty in the output, y, is determined we make use of two standard propagation approaches. In probability theory this propagation from uncertainty in the input to uncertainty in the output is made by using what is called *derived distributions* (Benjamin and Cornell, 1970). In fuzzy set theory, the propagation of uncertainty in the input to uncertainty in the output is developed using the *extension principle* (Zadeh, 1975). For our model, we have the propagation model, $y = \sin(x)$; we define the uncertainty in the input by a function $f(x)$, and we define the uncertainty in the output by a function $f(y)$. In probability theory we will use the uniform density function, $f(x) = 1/\pi$, to model the input uncertainty. Using the *derived distribution* method, we get the calculus relation,

$$f(x)dx = f(y)dy = \frac{1}{\pi}dx \tag{1.4}$$

Equation (1.4) then becomes, when considering both monotonically increasing and decreasing functions,

$$f(y) = f(x)\left|\frac{1}{dy/dx}\right| \tag{1.5}$$

Now, we define the inverse function of the output, y, as

$$y = \sin x;\ \text{hence, } x = \sin^{-1} y \tag{1.6}$$

Combining Equation (1.6) with Equation (1.5) yields,

$$f(y) = f\left(\sin^{-1}y\right)\left|\frac{d\left[\sin^{-1}y\right]}{dy}\right| = f\left(\sin^{-1}y\right)\left|\frac{d[1/\sin y]}{dy}\right| = f(x)\left|\frac{1}{\sqrt{1-y^2}}\right| = \frac{1}{\pi}\left|\frac{1}{\sqrt{1-y^2}}\right| \tag{1.7}$$

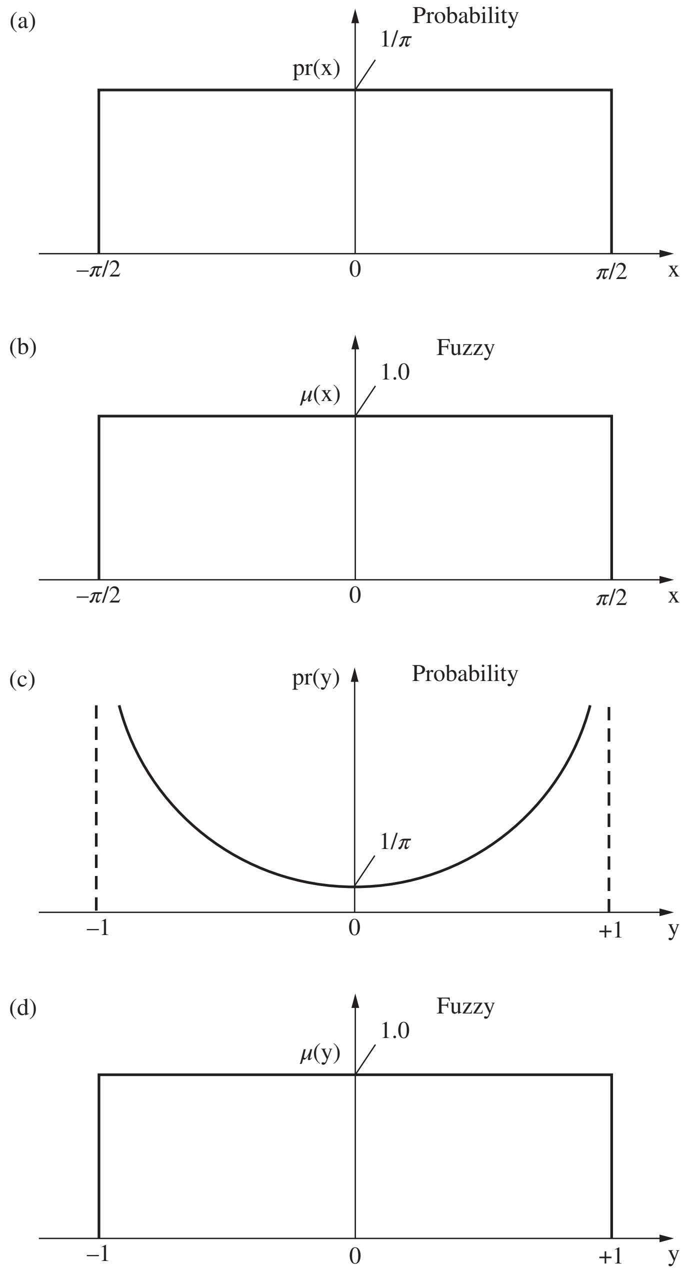

Figure 1.4 Comparison of probability and fuzzy approaches to uncertainty propagation and intuitive understanding of results: (a) input x is uniformly distributed, (b) input x is a fuzzy set with no ambiguity, (c) output y is distributed as a saddle function, and (d) output y is a fuzzy set with no ambiguity.

Now, we want to evaluate Equation (1.7) for the output at the points $y=0$ and $y=1$,

$$y(0) = f(0)\left|\frac{1}{\sqrt{1-0}}\right| = \frac{1}{\pi}\cdot 1 = \frac{1}{\pi} \tag{1.8a}$$

$$y(1) = y(-1) = \sin^{-1}(1)\left|\frac{1}{\sqrt{1-1}}\right| = \infty \tag{1.8b}$$

As can be seen in Figure 1.4c the derived output probability density function for y is a saddle function (see Equation 1.7). As seen in Figure 1.4d the output for the fuzzy case is a uniform membership function, which looks just like the input membership function; this fuzzy result comes from Zadeh's *extension principle* which is illustrated in detail in Chapter 12 for a harmonic function. The extension principle is used extensively in the area of fuzzy arithmetic, which is briefly summarized in Chapter 12 of this text.

With regard to the problem just addressed in Figure 1.4 and Equations (1.4 to 1.8) we now encounter the following question. Based on knowing that the input is an uncertain function that is equally frequent at any value in the universe of the input, which of the approaches seems more intuitive? In the case of the probabilistic model, the output with the lowest density value ($f(y) = 1/\pi$) occurs at the mean value ($x=0$) of the input, and the highest density of the output (∞) occurs at the extremes ($y = \pm 1$) of the output (see Figure 1.4c). In the fuzzy model, the same uncertainty in the input results in the same uncertainty in the output (see Figure 1.4d). It might be clear to the reader which of these models is more intuitive, but at the very least it shows how some models can produce counterintuitive results!

Sets as Points in Hypercubes

There is an interesting geometric analog for illustrating the idea of set membership (Kosko, 1992). Heretofore, we have described a fuzzy set $\underset{\sim}{A}$ defined on a universe X. For a universe with only one element, the membership function is defined on the unit interval [0, 1]; for a two-element universe, the membership function is defined on the unit square; and for a three-element universe, the membership function is defined on the unit cube. All of these situations are shown in Figure 1.5. For a universe of n elements, we define the membership on the unit hypercube, $I^n = [0, 1]^n$.

The endpoints on the unit interval in Figure 1.5a, and the vertices of the unit square and the unit cube in Figure. 1.5b and c, respectively, represent the possible crisp subsets, or collections, of the elements of the universe in each figure. This collection of possible crisp (nonfuzzy) subsets of elements in a universe constitutes the power set of the universe. For example, in Figure 1.5c the universe comprises three elements, $X = \{x_1, x_2, x_3\}$. The point (0, 0, 1) represents the crisp subset in three-space, where x_1 and x_2 have no membership and element x_3 has full membership, that is, the subset $\{x_3\}$; the point (1, 1, 0) is the crisp subset where x_1 and x_2 have full membership and element x_3 has no membership, that is, the subset $\{x_1, x_2\}$; and so on for the other six vertices in Figure 1.5c. In general, there are 2^n subsets in the power set of a universe with n elements; geometrically, this universe is represented by a hypercube in n-space, where the 2^n vertices represent the collection of sets constituting the power set. Two points in the diagrams bear special note, as illustrated in Figure 1.5c. In this figure the point (1, 1, 1),

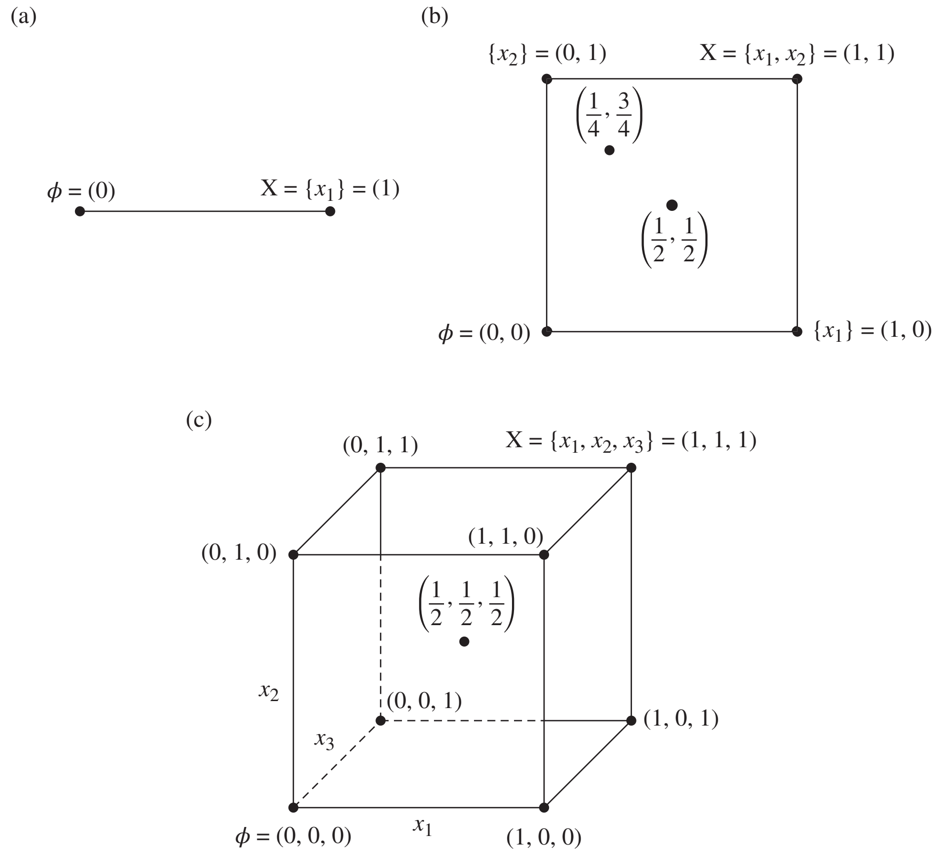

Figure 1.5 "Sets as points" (Kosko, 1992): (a) one-element universe, (b) two-element universe, (c) three-element universe.

where all elements in the universe have full membership, is called the *whole set*, X, and the point (0, 0, 0), where all elements in the universe have no membership, is called the *null set*, ∅.

The centroids of each of the diagrams in Figure 1.5 represent single points where the membership value for each element in the universe equals $\frac{1}{2}$. For example, the point $(\frac{1}{2}, \frac{1}{2})$ in Figure 1.5b is in the midpoint of the square. This midpoint in each of the three figures is a special point; it is the set of maximum "fuzziness." A membership value of $\frac{1}{2}$ indicates that the element belongs to the fuzzy set as much as it does not, that is, it holds equal membership in both the fuzzy set and its complement. In a geometric sense, this point is the location in the space that is farthest from any of the vertices and yet equidistant from all of them. In fact, all points interior to the vertices of the spaces represented in Figure 1.5 represent fuzzy sets, where the membership value of each variable is a number between 0 and 1. For example, in Figure 1.5b, the point $(\frac{1}{4}, \frac{3}{4})$ represents a fuzzy set where variable x_1 has a 0.25 degree of membership in the set and variable x_2 has a 0.75 degree of membership in the set. It is obvious by inspection of the diagrams in Figure 1.5 that, although the number of subsets in the power set is enumerated by the 2^n vertices, the number of fuzzy sets on the universe is infinite, as represented by the infinite number of points on the interior of each space.

Finally, the vertices of the cube in Figure 1.5c are the identical coordinates found in the value set, $V\{P(X)\}$, developed in Example 2.4 of the next chapter.

Summary

This chapter has discussed models with essentially two different kinds of information: fuzzy membership functions, which represent similarities of objects to nondistinct properties, and probabilities, which provide knowledge about relative frequencies. The value of either of these kinds of information in making decisions is a matter of preference; popular, but controversial, contrary views have been offered (Ross et al., 2002). Fuzzy models are *not* replacements for probability models. As seen in Figure 1.1, every crisp set is fuzzy, but the converse does not hold. An example (Fig. 1.4) was given that illustrates that the choice of an uncertainty model can lead to some strange counterintuitive results, and the reader is cautioned to exercise judgment in the selection of the most appropriate model that conforms to the actual uncertainty present in the problem. The idea that crisp sets are special forms of fuzzy sets was illustrated graphically in the section on sets as points, in which crisp sets are represented by the vertices of a unit hypercube. All other points within the unit hypercube, or along its edges, are graphically analogous to a fuzzy set. Fuzzy models are not that different from more familiar models. Sometimes they work better, and sometimes they do not. After all, the efficacy of a model in solving a problem should be the only criterion used to judge that model. Lately, a growing body of evidence suggests that fuzzy approaches to real problems are an effective alternative to previous, traditional methods.

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Problems

1.1 Develop a reasonable membership function for the following fuzzy sets based on moving vehicles on a freeway for the speed range of 0–75 mph.
   a. Fast
   b. Moderate
   c. Slow

1.2 A region experiences three seasons during a year. Each of those seasons last for four months starting with winter, then spring, and summer. Develop a membership function for the winter and the summer season on a scale of calendar months.

1.3 For the cylindrical shapes shown in Figure 1.2, develop a membership function for each of the following shapes using the ratio *d/h*, and discuss the reason for any overlapping among the three membership functions:

a. Rod
b. Cylinder
c. Disk

1.4 The question of whether a glass of water is half-full or half-empty is an age-old philosophical issue. Such descriptions of the volume of liquid in a glass depend on the state of mind of the person asked the question. Develop membership functions for the fuzzy sets "half-full," "full." and "half-empty" using a scale of percent of total volume. Assume the maximum volume of water in the glass is V_0. Discuss whether the terms *half-full and half-empty* should have identical membership functions. Does your answer solve this ageless riddle?

1.5 The pH level for a drinking water standard should be between 6 and 8.5. Draw a reasonable membership function for drinking water with an optimum pH standard value on a scale of 0–14.

1.6 To generate electricity, a turbine should rotate "at least at a speed of 40 rpm." Draw a membership function to show the effect of speed on generating electricity using
a. Crisp membership function
b. Fuzzy membership function

1.7 According to Hooke's law: within the elastic limit, stress is directly proportional to strain. A mild steel shows this behavior for a stress up to 335 MPa. Draw both crisp and fuzzy membership functions showing that "mild steel" is within the elastic limit.

1.8 Develop algorithms for the following membership function shapes:
a. Triangular
b. Trapezoid
c. Gaussian

1.9 Water can be classified into three states: solid (ice), liquid (water), and gas (water vapor). These three states are functions of temperature. Draw membership functions for these three states of water in terms of their temperature; use either Celsius or Fahrenheit for your temperature scale.

1.10 A circular column loaded axially is assumed to be eccentric when the load is acting 5% off the axis, depending on the diameter of the column, d. As shown in Figure P1.10 we

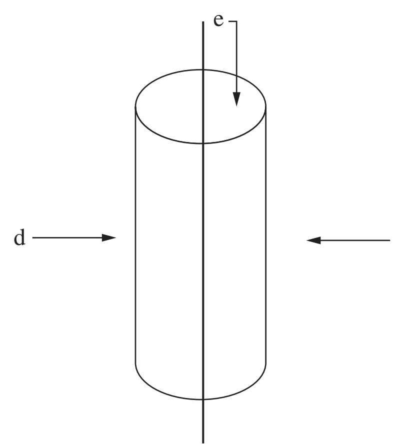

Figure P1.10

have the following conditions: *e/d* = 0.05 eccentric; *e/d* <0.05 not very eccentric; *e/d* >0.05 very eccentric. Develop a membership function for “eccentricity” on the scale of *e/d* ratios.

1.11 If the level of water in a dam is below 110 m height it is said to be a “safe” height. But if the level rises to more than 120 m, which is considered as “dangerous” height, then immediate opening of a gate is required. Draw a membership function for a “safe” water level.

1.12 Probability distributions can be shown to exist on certain planes that intersect the regions shown in Figure 1.5. Draw the points, lines, and planes on which probability distributions exist for the one-, two-, and three-element cases shown in Figure 1.5.

2

Classical Sets and Fuzzy Sets

Philosophical objections may be raised by the logical implications of building a mathematical structure on the premise of fuzziness, since it seems (at least superficially) necessary to require that an object be or not be an element of a given set. From an aesthetic viewpoint, this may be the most satisfactory state of affairs, but to the extent that mathematical structures are used to model physical actualities, it is often an unrealistic requirement.… Fuzzy sets have an intuitively plausible philosophical basis. Once this is accepted, analytical and practical considerations concerning fuzzy sets are in most respects quite orthodox.

James Bezdek, 1981
Professor, Computer Science

Quantum mechanics brought an unexpected fuzziness into physics because of quantum uncertainty, the Heisenberg uncertainty principle.

Edward Witten, twentieth-century American Mathematician

As alluded to in Chapter 1, the *universe of discourse* is the universe of all available information on a given problem. Once this universe is defined we are able to define certain events on this information space. I will describe sets as mathematical abstractions of these events and of the universe itself. Figure 2.1a shows an abstraction of a universe of discourse, say X, and a crisp (classical) set A somewhere in this universe. A classical set is defined by *crisp* boundaries, that is, there is no uncertainty in the prescription or location of the boundaries of the set, as shown in Figure 2.1a, where the boundary of crisp set A is an unambiguous line. A fuzzy set, on the other hand, is prescribed by vague or ambiguous properties; hence, its boundaries are ambiguously specified, as shown by the fuzzy boundary for set $\underset{\sim}{A}$ in Figure 2.1b.

The notion of set membership was introduced, from a one-dimensional viewpoint in Chapter 1. Figure 2.1 again helps to explain this idea, but from a two-dimensional perspective.

Fuzzy Logic with Engineering Applications, Fourth Edition. Timothy J. Ross.

Companion website: www.wiley.com/go/ross/fuzzy4e

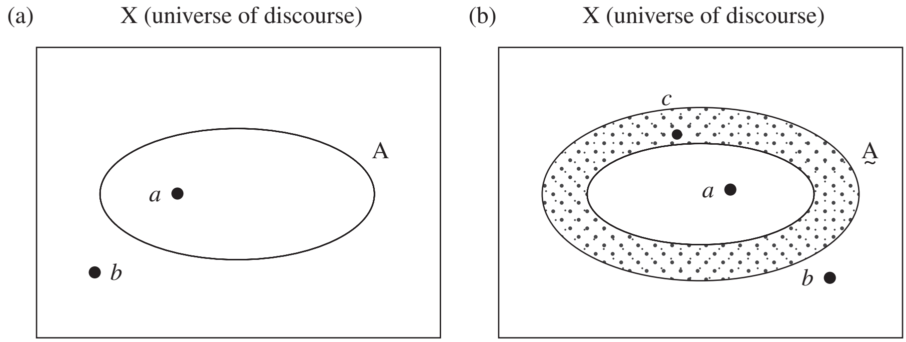

Figure 2.1 Diagrams for (a) crisp set boundary and (b) fuzzy set boundary.

Point a in Figure 2.1a is clearly a member of crisp set A; point b is unambiguously *not* a member of set A. Figure 2.1b shows the vague, ambiguous boundary of a fuzzy set $\underset{\sim}{A}$ on the same universe X: the shaded boundary represents the boundary region of $\underset{\sim}{A}$. In the central (unshaded) region of the fuzzy set, point a is clearly a full member of the set. Outside the boundary region of the fuzzy set, point b is clearly not a member of the fuzzy set. However, the membership of point c, which is on the boundary region, is ambiguous. If complete membership in a set (such as point a in Figure 2.1b) is represented by the number 1, and no membership in a set (such as point b in Figure 2.1b) is represented by 0, so then point c in Figure 2.1b must have some intermediate value of membership (partial membership in fuzzy set $\underset{\sim}{A}$) on the interval [0, 1]. Presumably, the membership of point c in $\underset{\sim}{A}$ approaches a value of 1 as it moves closer to the central (unshaded) region in Figure 2.1b of $\underset{\sim}{A}$ and the membership of point c in $\underset{\sim}{A}$ approaches a value of 0 as it moves closer to leaving the boundary region of $\underset{\sim}{A}$.

In this chapter, the precepts and operations of fuzzy sets are compared with those of classical sets. Several good books are available for reviewing this basic material (see for example, Dubois and Prade, 1980; Klir and Folger, 1988; Zimmermann, 1991; Klir and Yuan, 1995). Fuzzy sets embrace virtually all (with one exception, as will be seen) of the definitions, precepts, and axioms that define classical sets. As indicated in Chapter 1, crisp sets are a special case of fuzzy sets; they are sets without ambiguity in their membership (i.e., they are sets with unambiguous boundaries). It will be shown that fuzzy set theory is a mathematically rigorous and comprehensive set theory useful in characterizing concepts (sets) with natural ambiguity. It is instructive to introduce fuzzy sets by first reviewing the elements of classical (crisp) set theory.

Classical Sets

Define a universe of discourse, X, as a collection of objects all having the same characteristics. The individual elements in the universe X will be denoted as x. The features of the elements in X can be discrete, countable integers, or continuous valued quantities on the real line. Examples of elements of various universes might be as follows:

the clock speeds of computer CPUs;
the operating currents of an electronic motor;
the operating temperature of a heat pump (in degrees Celsius);

the Richter magnitudes of an earthquake;
the integers 1 to 10.

Most real-world engineering processes contain elements that are real and nonnegative. The first four items just named are examples of such elements. However, for purposes of modeling, most engineering problems are simplified to consider only integer values of the elements in a universe of discourse. So, for example, computer clock speeds might be measured in integer values of megahertz and heat pump temperatures might be measured in integer values of degree Celsius. Further, most engineering processes are simplified to consider only finite-sized universes. Although Richter magnitudes may not have a theoretical limit, we have not historically measured earthquake magnitudes much above 9; this value might be the upper bound in a structural engineering design problem. As another example, suppose you are interested in the stress under one leg of the chair in which you are sitting. You might argue that it is possible to get an infinite stress on one leg of the chair by sitting in the chair in such a manner that only one leg is supporting you and by letting the area of the tip of that leg approach zero. Although this is theoretically possible, in reality the chair leg will either buckle elastically as the tip area becomes small or yield plastically and fail because materials that have infinite strength have not yet been developed. Hence, choosing a universe that is discrete and finite or one that is continuous and infinite is a modeling choice; the choice does not alter the characterization of sets defined on the universe. If elements of a universe are continuous, then sets defined on the universe will be composed of continuous elements. For example, if the universe of discourse is defined as all Richter magnitudes up to a value of 9, then we can define a set of "destructive magnitudes," which might be composed of (1) all magnitudes greater than or equal to a value of 6 in the crisp case or (2) all magnitudes "approximately 6 and higher" in the fuzzy case.

A useful attribute of sets and the universes on which they are defined is a metric known as the *cardinality*, or the *cardinal number*. The total number of elements in a universe X is called its *cardinal number*, denoted n_x, where x again is a label for individual elements in the universe. Discrete universes that are composed of a countably finite collection of elements will have a finite cardinal number; continuous universes comprises an infinite collection of elements will have an infinite cardinality. Collections of elements within a universe are called *sets*, and collections of elements within sets are called *subsets*. Sets and subsets are terms that are often used synonymously because any set is also a subset of the universal set X. The collection of all possible sets in the universe is called the *whole set*.

For crisp sets A and B consisting of collections of some elements in X, the following notation is defined:

$x \in \mathrm{X}$	$\rightarrow$	x belongs to X
$x \in \mathrm{A}$	$\rightarrow$	x belongs to A
$x \notin \mathrm{A}$	$\rightarrow$	x does not belong to A

For sets A and B on X, we also have

$\mathrm{A} \subset \mathrm{B}$	$\rightarrow$	A is fully contained in B (if $x \in \mathrm{A}$, then $x \in \mathrm{B}$)
$\mathrm{A} \subseteq \mathrm{B}$	$\rightarrow$	A is contained in or is equivalent to B
$(\mathrm{A} \leftrightarrow \mathrm{B})$	$\rightarrow$	$\mathrm{A} \subseteq$ Band $\mathrm{B} \subseteq \mathrm{A}$ (A is equivalent to B)

We define the *null set*, ∅, as the set containing no elements, and the whole set, X, as the set of all elements in the universe. The null set is analogous to an impossible event, and the whole set is analogous to a certain event. All possible sets of X constitute a special set called the *power set*, P(X). For a specific universe X, the power set P(X) is enumerated in the following example.

Example 2.1
We have a universe composed of three elements, $X = \{a, b, c\}$, so the cardinal number is $n_x = 3$. The power set is

$$\mathrm{P(X)} = \{\, \emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{a,b\},\ \{a,c\},\ \{b,c\},\ \{a,b,c\}\}.$$

The cardinality of the power set, denoted $n_{\mathrm{P(X)}}$, is found as

$$n_{\mathrm{P(X)}} = 2^{n_{\mathrm{X}}} = 2^3 = 8.$$

Note that if the cardinality of the universe is infinite, then the cardinality of the power set is also infinity, that is, $n_{\mathrm{X}} = \infty \Rightarrow n_{\mathrm{P(X)}} = \infty$.

Operations on Classical Sets

Let A and B be two sets on the universe X. The union between the two sets, denoted A ∪ B, represents all those elements in the universe that reside in (or belong to) the set A, the set B, or both sets A and B. (This operation is also called the *logical or*; another form of the union is the *exclusive or* operation. The *exclusive or* is described in Chapter 5.) The intersection of the two sets, denoted A ∩ B, represents all those elements in the universe X that simultaneously reside in (or belong to) both sets A and B. The complement of a set A, denoted $\bar{\mathrm{A}}$, is defined as the collection of all elements in the universe that do not reside in the set A. The difference of a set A with respect to B, denoted A | B, is defined as the collection of all elements in the universe that reside in A and that do not reside in B simultaneously. These operations are shown below in set-theoretic terms.

Union $$\mathrm{A} \cup \mathrm{B} = \{x \mid x \in \mathrm{A} \text{ or } x \in \mathrm{B}\}. \tag{2.1}$$

Intersection $$\mathrm{A} \cap \mathrm{B} = \{x \mid x \in \mathrm{A} \text{ and } x \in \mathrm{B}\}. \tag{2.2}$$

Complement $$\overline{\mathrm{A}} = \{x \mid x \notin \mathrm{A},\ x \in \mathrm{X}\}. \tag{2.3}$$

Difference $$\mathrm{A} \mid \mathrm{B} = \{x \mid x \in \mathrm{A} \text{ and } x \notin \mathrm{B}\}. \tag{2.4}$$

These four operations are shown in terms of Venn diagrams in Figures 2.2–2.5.

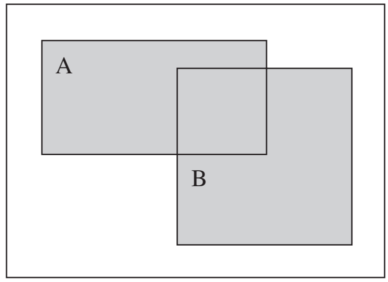

Figure 2.2 Union of sets A and B (logical or).

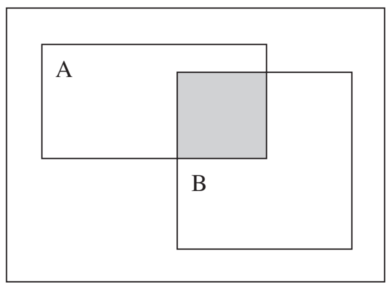

Figure 2.3 Intersection of sets A and B.

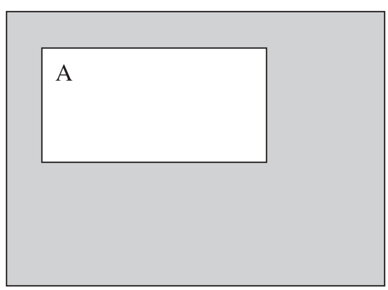

Figure 2.4 Complement of set A.

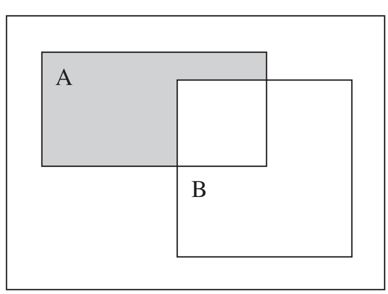

Figure 2.5 Difference operation A|B.

Properties of Classical (Crisp) Sets

Certain properties of sets are important because of their influence on the mathematical manipulation of sets. The most appropriate properties for defining classical sets and showing their similarity to fuzzy sets are as follows:

Commutativity

$$\begin{aligned} &A \cup B = B \cup A \\ &A \cap B = B \cap A. \end{aligned} \tag{2.5}$$

Associativity

$$\begin{aligned} &A \cup (B \cup C) = (A \cup B) \cup C \\ &A \cap (B \cap C) = (A \cap B) \cap C. \end{aligned} \tag{2.6}$$

Distributivity

$$\begin{aligned} &A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ &A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \end{aligned} \tag{2.7}$$

Idempotency

$$\begin{aligned} &A \cup A = A \\ &A \cap A = A. \end{aligned} \tag{2.8}$$

Identity

$$\begin{aligned} &A \cup \emptyset = A \\ &A \cap X = A \\ &A \cap \emptyset = \emptyset. \\ &A \cup X = X. \end{aligned} \tag{2.9}$$

Transitivity

$$\text{If } A \subseteq B \text{ and } B \subseteq C, \quad \text{then } A \subseteq C. \tag{2.10}$$

Involution

$$\overline{\overline{A}} = A. \tag{2.11}$$

Two special properties of set operations are known as the *excluded middle axioms* and *De Morgan's principles*. These properties are en umerated here for two sets A and B. The *excluded middle axioms* are important because these are the only set operations described here that are *not* valid for both classical sets and fuzzy sets. There are two excluded middle axioms, which are given in Equation (2.12). The first, called the *axiom of the excluded middle*, deals with the union of a set A and its complement; the second, called the *axiom of contradiction*, represents the intersection of a set A and its complement.

Axiom of the excluded middle

$$A \cup \overline{A} = X. \tag{2.12a}$$

Axiom of the contradiction

$$A \cap \overline{A} = \emptyset. \tag{2.12b}$$

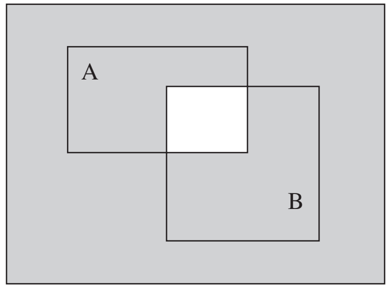

Figure 2.6 De Morgan's principle $\left(\overline{\mathrm{A} \cap \mathrm{B}}\right)$.

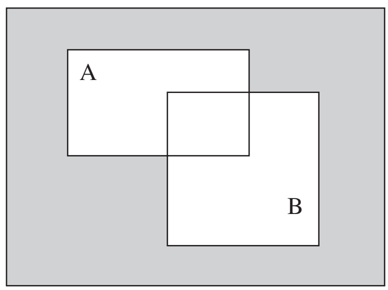

Figure 2.7 De Morgan's principle $\left(\overline{\mathrm{A} \cup \mathrm{B}}\right)$.

De Morgan's principles are important because of their usefulness in proving tautologies and contradictions in logic, as well as in a host of other set operations and proofs. De Morgan's principles are displayed in the shaded areas of the Venn diagrams in Figures 2.6 and 2.7 and described mathematically in Equation (2.13).

$$\overline{\mathrm{A} \cap \mathrm{B}} = \overline{\mathrm{A}} \cup \overline{\mathrm{B}}. \tag{2.13a}$$

$$\overline{\mathrm{A} \cup \mathrm{B}} = \overline{\mathrm{A}} \cap \overline{\mathrm{B}}. \tag{2.13b}$$

In general, De Morgan's principles can be stated for n sets, as provided here for events, E_i:

$$\overline{\mathrm{E}_1 \cup \mathrm{E}_2 \cup \cdots \cup \mathrm{E}_n} = \overline{\mathrm{E}_1} \cap \overline{\mathrm{E}_2} \cap \cdots \cap \overline{\mathrm{E}_n}. \tag{2.14a}$$

$$\overline{\mathrm{E}_1 \cap \mathrm{E}_2 \cap \cdots \cap \mathrm{E}_n} = \overline{\mathrm{E}_1} \cup \overline{\mathrm{E}_2} \cup \cdots \cup \overline{\mathrm{E}_n}. \tag{2.14b}$$

From the general equations, which are given in Equation (2.14), for De Morgan's principles, there is a duality relation: the complement of a union or an intersection is equal to the intersection or union, respectively, of the respective complements. This result is powerful in dealing with set structures because we often have information about the complement of a set (or event) or the complement of combinations of sets (or events), rather than information about the sets themselves.

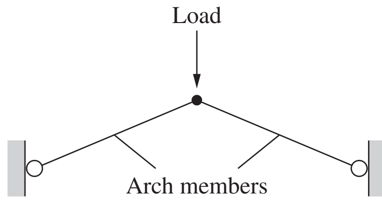

Figure 2.8 A two-member arch.

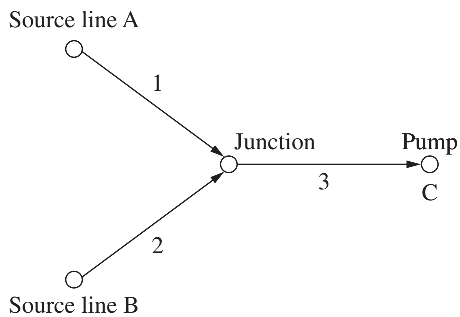

Figure 2.9 Hydraulic hose system.

Example 2.2
A shallow arch consists of two slender members as shown in Figure 2.8. If either of the members fails, then the arch will collapse. If E_1 = survival of member 1 and E_2 = survival of member 2, then survival of the arch = $E_1 \cap E_2$, and, conversely, collapse of the arch = $\overline{E_1 \cap E_2}$. Logically, collapse of the arch will occur if either of the members fails, that is, when $\overline{E_1} \cup \overline{E_2}$. Therefore,

$$\overline{E_1 \cap E_2} = \overline{E_1} \cup \overline{E_2},$$

which is an illustration of De Morgan's principle.

As Equation (2.14) suggests, De Morgan's principles are useful for compound events, as illustrated in the following example.

Example 2.3
For purposes of safety, the fluid supply for a hydraulic pump C in an airplane comes from two redundant source lines, A and B. The fluid is transported by high-pressure hoses consisting of branches 1, 2, and 3, as shown in Figure 2.9. Operating specifications for the pump indicate that either source line alone is capable of supplying the necessary fluid pressure to the pump. Denote E_1 = failure of branch 1, E_2 = failure of branch 2, and E_3 = failure of branch 3. Then insufficient pressure to operate the pump would be caused by $(E_1 \cap E_2) \cup E_3$, and sufficient pressure would

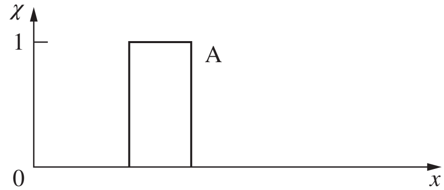

Figure 2.10 Membership function is a mapping for crisp set A.

be the complement of this event. Using De Morgan's principles, we can calculate the condition of sufficient pressure to be

$$\overline{(E_1 \cap E_2) \cup E_3} = \left(\overline{E_1} \cup \overline{E_2}\right) \cap \overline{E_3},$$

in which $\left(\overline{E_1} \cup \overline{E_2}\right)$ means the availability of pressure at the junction, and $\overline{E_3}$ means the absence of failure in branch 3.

Mapping of Classical Sets to Functions

Mapping is an important concept in relating set-theoretic forms to function-theoretic representations of information. In its most general form it can be used to map elements or subsets in one universe of discourse to elements or sets in another universe. Suppose X and Y are two different universes of discourse (information). If an element x is contained in X and corresponds to an element y contained in Y, it is generally termed a mapping from X to Y, or f: X → Y. As a mapping, the characteristic (indicator) function χ_{A} is defined as

$$\chi_{\mathrm{A}}(x) = \begin{cases} 1, & x \in \mathrm{A} \\ 0, & x \notin \mathrm{A} \end{cases}, \tag{2.15}$$

where χ_{A} expresses "membership" in set A for the element x in the universe. This membership idea is a mapping from an element x in universe X to one of the two elements in universe Y, that is, to the elements 0 or 1, as shown in Figure 2.10.

For any set A defined on the universe X, there exists a function-theoretic set, called a *value set*, denoted V(A), under the mapping of the characteristic function, χ. By convention, the null set ∅ is assigned the membership value 0 and the whole set X is assigned the membership value 1.

Example 2.4
Continuing with the example (Example 2.1) of a universe with three elements, $X = \{a, b, c\}$, we desire to map the elements of the power set of X, that is, P(X), to a universe, Y, consisting of only two elements (the characteristic function),

$$\mathrm{Y} = \{0, 1\}.$$

As before, the elements of the power set are enumerated.

$$P(X) = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}.$$

Thus, the elements in the value set V(A) as determined from the mapping are

$$V\{P(X)\} = \{\{0, 0, 0\}, \{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 1\}, \{1, 1, 0\}, \{0, 1, 1\}, \{1, 0, 1\}, \{1, 1, 1\}\}.$$

For example, the third subset in the power set P(X) is the element b. For this subset there is no a, so a value of 0 goes in the first position of the data triplet; there is a b, so a value of 1 goes in the second position of the data triplet; and there is no c, so a value of 0 goes in the third position of the data triplet. Hence, the third subset of the value set is the data triplet, $\{0, 1, 0\}$, as already seen. The value set has a graphical analog that is described in Chapter 1 in the section Sets as Points in Hypercubes.

Now, define two sets, A and B, on the universe X. The union of these two sets in terms of function-theoretic terms is given as follows (the symbol $\vee$ is the maximum operator and $\wedge$ is the minimum operator):

Union $$A \cup B \rightarrow \chi_{A\cup B}(x) = \chi_A(x) \vee \chi_B(x) = \max(\chi_A(x), \chi_B(x)). \tag{2.16}$$

The intersection of these two sets in function-theoretic terms is given as follows:

Intersection $$A \cap B \rightarrow \chi_{A\cap B}(x) = \chi_A(x) \wedge \chi_B(x) = \min(\chi_A(x), \chi_B(x)). \tag{2.17}$$

The complement of a single set on universe X, say A, is given as follows:

Complement $$\overline{A} \rightarrow \chi_{\overline{A}}(x) = 1 - \chi_A(x). \tag{2.18}$$

For two sets on the same universe, say A and B, if one set (A) is contained in another set (B), then

Containment $$A \subseteq B \rightarrow \chi_A(x) \leq \chi_B(x). \tag{2.19}$$

Function-theoretic operators for union and intersection (other than maximum and minimum, respectively) are discussed in the literature (Gupta and Qi, 1991).

Fuzzy Sets

In classical, or crisp, sets the transition for an element in the universe between membership and non-membership in a given set is abrupt and well defined (said to be *crisp*). For an element in a universe that contains fuzzy sets, this transition can be gradual. This transition among various degrees of membership can be thought of as conforming to the fact that the boundaries of the fuzzy sets are vague and ambiguous. Hence, membership of an element from the universe in this set is measured by a function that attempts to describe vagueness and ambiguity.

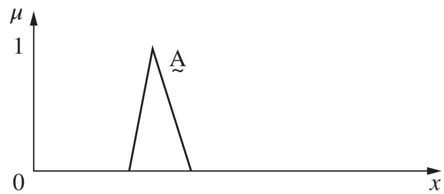

Figure 2.11 Membership function for fuzzy set $\underset{\sim}{A}$.

A fuzzy set, then, is a set containing elements that have varying degrees of membership in the set. This idea is in contrast with classical, or crisp, sets because members of a crisp set would not be members unless their membership is full, or complete, in that set (i.e., their membership is assigned a value of 1). Elements in a fuzzy set, because their membership need not be complete, can also be members of other fuzzy sets on the same universe.

Elements of a fuzzy set are mapped to a universe of *membership values* using a function-theoretic form. As mentioned in Chapter 1, Equation (1.2), fuzzy sets are denoted in this text by a set symbol with a tilde understrike. So, for example, $\underset{\sim}{A}$ would be the *fuzzy set A*. This function maps elements of a fuzzy set $\underset{\sim}{A}$ to a real numbered value on the interval 0–1. If an element in the universe, say x, is a member of fuzzy set $\underset{\sim}{A}$, then this mapping is given by Equation (1.2), or $\mu_{\underset{\sim}{A}}(x) \in [0, 1]$. This mapping is shown in Figure 2.11 for a typical fuzzy set.

A notation convention for fuzzy sets when the universe of discourse, X, is discrete and finite, is as follows for a fuzzy set $\underset{\sim}{A}$:

$$\underset{\sim}{A} = \left\{ \frac{\mu_{\underset{\sim}{A}}(x_1)}{x_1} + \frac{\mu_{\underset{\sim}{A}}(x_2)}{x_2} + \cdots \right\} = \left\{ \sum_i \frac{\mu_{\underset{\sim}{A}}(x_i)}{x_i} \right\}. \tag{2.20}$$

When the universe, X, is continuous and infinite, the fuzzy set $\underset{\sim}{A}$ is denoted as

$$\underset{\sim}{A} = \left\{ \int \frac{\mu_{\underset{\sim}{A}}(x)}{x} \right\}. \tag{2.21}$$

In both notations, the horizontal bar is not a quotient but rather a delimiter. The numerator in each term is the membership value in set $\underset{\sim}{A}$ associated with the element of the universe indicated in the denominator. In the first notation, the summation symbol is not for algebraic summation, but rather denotes the collection or aggregation of each element; hence, the "+" signs in the first notation are not the algebraic "add" but are an aggregation or collection operator. In the second notation, the integral sign is not an algebraic integral but a continuous function-theoretic aggregation operator for continuous variables. Both notations are due to Zadeh (1965).

Fuzzy Set Operations

Define three fuzzy sets $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$ on the universe X. For a given element x of the universe, the following function-theoretic operations for the set-theoretic operations of union, intersection, and complement are defined for $\underset{\sim}{A}$, $\underset{\sim}{B}$,and $\underset{\sim}{C}$ on X:

Union $$\mu_{\underset{\sim}{A}\cup\underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \vee \mu_{\underset{\sim}{B}}(x). \tag{2.22}$$

Intersection $$\mu_{\underset{\sim}{A}\cap\underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(x). \tag{2.23}$$

Complement $$\mu_{\overline{\underset{\sim}{A}}}(x) = 1 - \mu_{\underset{\sim}{A}}(x). \tag{2.24}$$

Venn diagrams for these operations, extended to consider fuzzy sets, are shown in Figures 2.12–2.14. The operations given in Equations (2.22) to (2.24) are known as the

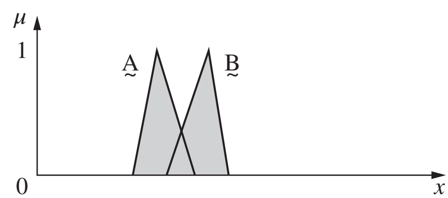

Figure 2.12 Union of fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$.

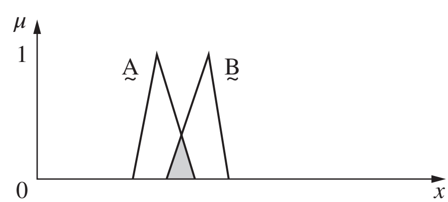

Figure 2.13 Intersection of fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$.

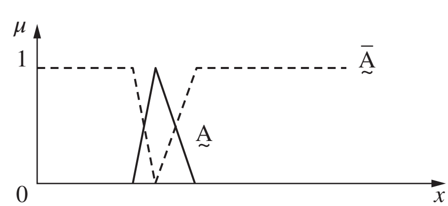

Figure 2.14 Complement of fuzzy set $\underset{\sim}{A}$.