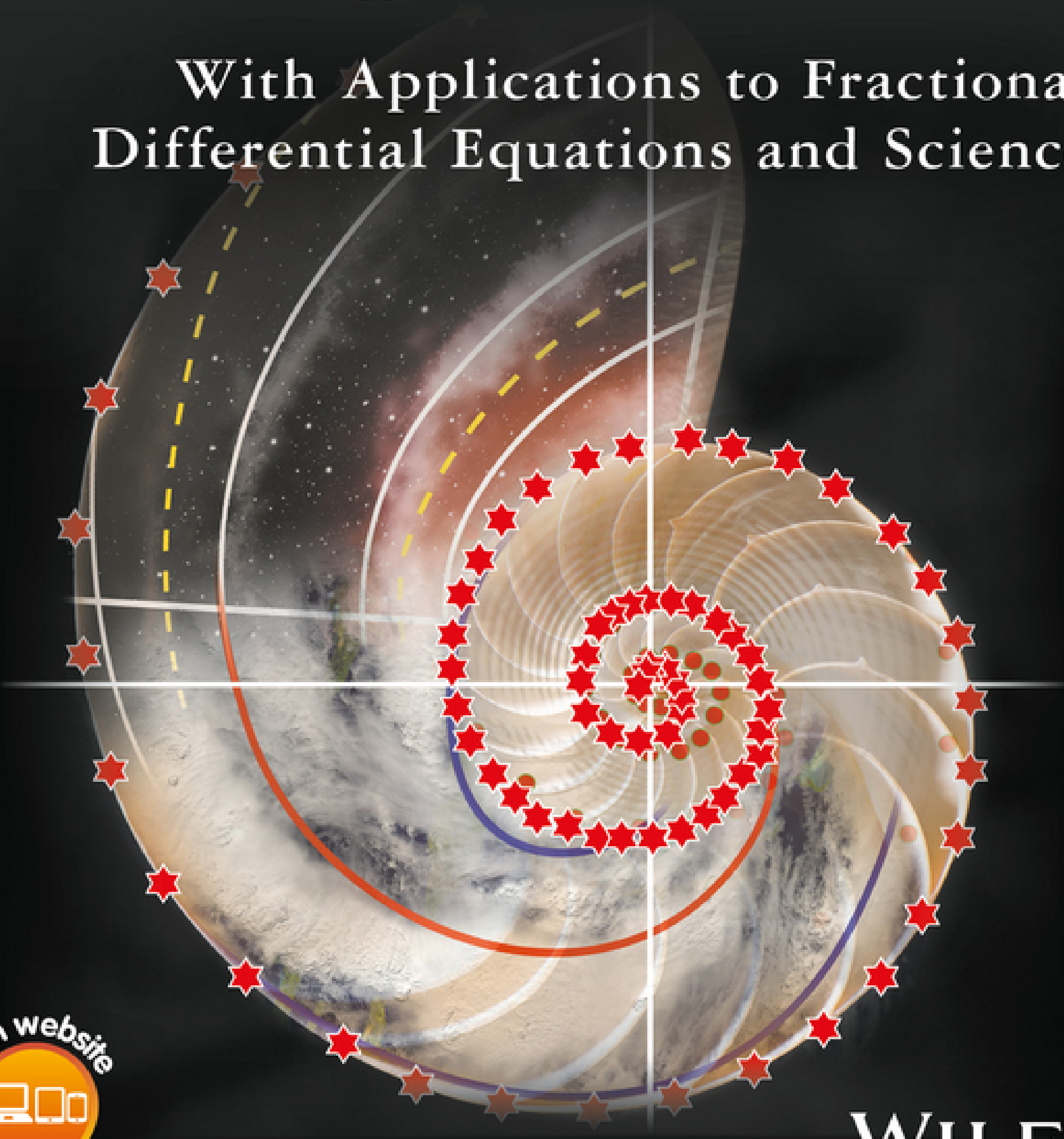
CARL F. LORENZO AND TOM T. HARTLEY

The Fractional Trigonometry

With Applications to Fractional Differential Equations and Science



with website

WILEY

The Fractional Trigonometry

The Fractional Trigonometry

With Applications to Fractional Differential Equations and Science

Carl F. Lorenzo
National Aeronautics and Space Administration
Glenn Research Center
Cleveland, Ohio

Tom T. Hartley
The University of Akron
Akron, Ohio

WILEY

Published simultaneously in Canada

Published by John Wiley & Sons, Inc., Hoboken, New Jersey

Library of Congress Cataloging-in-Publication Data:

Names: Lorenzo, Carl F. | Hartley, T. T. (Tom T.), 1964-
Title: The fractional trigonometry : with applications to fractional differential equations and science / Carl F. Lorenzo, National Aeronautics and Space Administration, Glenn Research Center, Cleveland, Ohio, Tom T. Hartley, The University of Akron, Akron, Ohio.
Description: Hoboken, New Jersey : John Wiley & Sons, Inc., [2017] | Includes bibliographical references and index.
Identifiers: LCCN 2016027837 (print) | LCCN 2016028337 (ebook) | ISBN 9781119139409 (cloth) | ISBN 9781119139423 (pdf) | ISBN 9781119139430 (epub)
Subjects: LCSH: Fractional calculus. | Trigonometry.
Classification: LCC QA314 .L67 2017 (print) | LCC QA314 (ebook) | DDC 516.24–dc23
LC record available at https://lccn.loc.gov/2016027837

Set in 10/12pt Warnock by SPi Global, Chennai, India

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

Contents

Preface

There has been a strong resurgence of interest in the fractional calculus over the last two or three decades. This expansion of the classical calculus to derivatives and integrals of fractional order has given rise to the hope of a new understanding of the behavior of the physical world. The hope is that problems that have resisted solution by the integer-order calculus will yield to this greatly expanded capability. As a result of our work in the fractional calculus, and more particularly, in functions for the solutions of fractional differential equations, an interest was fostered in the behavior of generalized exponential functions for this application. Our work with the fundamental fractional differential equation had developed a function we named the F-function. This function, which had previously been mentioned in a footnote by Oldham and Spanier, acts as the fractional exponential function. It was a natural step from there to an interest in a fractional trigonometry. At that time, only a few pages of work were available in the literature and were based on the Mittag-Leffler function. These are shown in Appendix A.

This book brings together our research in this area over the past 15 years and adds much new unpublished material.

The classical trigonometry plays a very important role relative to the integer-order calculus; that is, it, together with the common exponential function, provides solutions for linear differential equations. We will find that the fractional trigonometry plays an analogous role relative to the fractional calculus by providing solutions to linear fractional differential equations. The importance of the classical trigonometry goes far beyond the solutions of triangles. Its use in Fourier integrals, Fourier series, signal processing, harmonic analysis, and more provided great motivation for the development of a fractional trigonometry to expand application to the fractional calculus domain.

Because we are engineers, this book has been written in the style of the engineering mathematical books rather than the more rigorous and compact style of definition, theorem, and proof, found in most mathematical texts. We, of course, have made every effort to assure the derivations to be correct and are hopeful that the style has made the material accessible to a larger audience. We are also hopeful that this will not detract the interest of the mathematical community in the area since their skills will be needed to develop this important new area. Most of the materials of this book should be accessible to an undergraduate student with a background in Laplace transforms.

After an introductory chapter, which offers a brief insight into the fractional calculus, the book is organized in two major parts. In Chapters 2–11, the definitions and theory of the fractional exponential and the fractional trigonometry are developed. Chapters 12–19 provide insight into various areas of potential application.

Chapter 2 develops the F-function from consideration of the fundamental fractional differential equation. It generalizes the common exponential function for application in the fractional calculus. The F-function, the fractional eigenfunction, together with its generalization, the

R-function (Chapter 3), will later form the theoretical basis of the fractional trigonometry. Properties of these functions are developed in these two chapters. Their relationship to other functions for the fractional calculus is presented. An important characteristic of the R-function is that it contains the F-function as a special case and also contains its derivatives and integrals. In later chapters, it is shown that many of the newly defined fractional trigonometric functions inherit this important property. Chapter 4 further develops properties of the R-function that expose the character of this fractional exponential function.

In Chapter 5, the R-function, $R_{q,v}(a,t)$, with real arguments for a and t, is used to define the fractional hyperbolic functions. These functions generalize the classical hyperbolic functions. Fractional exponential forms of the hyperbolic functions are derived as well as their Laplace transforms. Furthermore, fractional differintegrals of the functions are determined. An example demonstrates the use of the Laplace transform in the solution of fractional hyperbolic differential equations. The fractional hyperbolic functions are found to be closely related to the R_1-trigonometric functions defined in Chapter 6.

Chapters 6–8 present three fractional trigonometries. We have tried to make each of these chapters as stand-alone developments, at the expense of minor repetition. Chapter 6 develops the R_1-trigonometry. It is based on the R-function with imaginary parameter a, namely $R_{q,v}(ia,t)$. Multiplication of the parameter by i toggles the R_1-hyperbolic functions to the R_1-trigonometric functions, and so on.

A fractional trigonometry, the R_2-trigonometry based on an imaginary time variable, $R_{q,v}(a,it)$, is developed in Chapter 7. It is found that these functions are characterized by their attraction to circles when plotted in phase plane format.

The obvious extension of these two trigonometries, the R_3-trigonometry of Chapter 8, sets both the a parameter and the t variable to be imaginary, $R_{q,v}(ia,it)$. It was thought at the time that this trigonometry would behave as an hyperbolic analog to the R_2-trigonometry. However, such simple relationships between the two were not found.

Chapter 9 presents the heart of the fractional trigonometry, namely the fractional meta-trigonometry. Here, both a and t are allowed to be fully complex, by choosing as the basis $R_{q,v}(i^\alpha a, i^\beta t)$. This chapter generalizes the results of the previous four chapters. Laplace transforms for the generalized functions are determined along with their fractional differintegrals. Fractional exponential forms for the functions are also determined.

In Chapter 10, the ratio and reciprocal functions associated with the generalized fractional sines and cosines of Chapter 9, that is, $Sin_{q,v}(a,\alpha,\beta,k,t)$ and $Cos_{q,v}(a,\alpha,\beta,k,t)$, as well as the generalized parity functions are considered. The parity functions represent a new set of fractional trigonometric functions with no counterpart in the classical trigonometry. Because of the large number of possible ratio and reciprocal functions, the treatment in this chapter is cursory.

Because of the newness of this material, we have tried to be generous with the graphic forms of the functions. In spite of this attitude, we have found that because of different behavior over various domains of the functions and the number of parameters in the functions that complete coverage in this regard to be impossible and the reader is encouraged to program some of the functions and to experiment for themselves.

In Chapter 3, two new functions, the G- and H-functions, are introduced. These functions are generalizations of the R-function with multiple real and complex roots in the denominators of their Laplace transforms. Because of the great generality of these functions, consideration of these functions as the basis for further generalization of the fractional trigonometry is discussed in Chapter 11. In Chapter 12, these functions are needed for the solution of linear fractional differential equations with repeated roots.

Part II of the book is largely dedicated to applications and potential application of the fractional trigonometry.

The most important application is the use of the fractional trigonometry for the solution of linear constant-coefficient commensurate-order fractional differential equations. In Chapter 12, specialized Laplace transforms for the meta-trigonometric functions are developed and applied to the solution of these linear fractional differential equations. Examples showing the solution of fractional differential equations with unrepeated roots and with repeated real and complex roots are given.

Chapter 13 studies the time- and frequency-domain responses for linear fractional systems based on the *R*-function and the meta-trigonometric functions. The stability of the basic fractional elements is also considered.

Unlike the classical trigonometric functions, the fractional counterparts do not generally share the periodicity property. As a practical result, we are limited to evaluation of the defining infinite series for function evaluation. This presents numerical difficulties as the time and/or order variables increase. Chapter 14 discusses this problem and establishes series convergence.

Phase plane plots of pairs of the fractional trigonometric functions define a new and unique family of spirals, the *fractional* spirals. Chapter 15 studies these spirals and their relationship to some of the classical spirals.

Linear oscillators are often used in the study of ordinary differential equations and in the modeling of physical systems. Chapter 16 identifies those linear fractional trigonometric oscillators that are neutrally stable. This chapter also explores possible application of coupled fractional trigonometric oscillators.

Chapters 17 – 19 study the possible application of the fractional spirals and thus the fractional trigonometry and fractional differential equations. The potential applications include sea shell growth and morphology, mathematical classification of spiral galaxy morphology, and various weather phenomena such as hurricanes and tornados.

Finally, Chapter 20 looks at some of the many remaining challenges and opportunities relative to the fractional exponential function and the fractional trigonometry, in particular, the need for a fractional field equation as it relates to spatial fractional spiral morphology.

For the professional with a background in the fractional calculus, a quick coverage of the essence of the book may be had from Chapters 2, 3, 9, and 12, with selected attention to the applications of interest contained in Chapters 15 – 19.

Carl F. Lorenzo
Tom T. Hartley
Cleveland OH,
July 2016

Acknowledgments

It is with pleasure that the authors extend their gratitude to the National Aeronautics and Space Administration, Glenn Research Center for their long-term and continued support of our work in both fractional calculus and fractional trigonometry. While many people have helped us in many ways, we are deeply indebted to the managers that have supported our work, including Dr. Norman Wenger, Dr. Walter C. Merrill, Dr. Jih-Fen Lei, Dr. Gary T. Seng of NASA Glenn Research Center, and Dr. Alex DeAbreu of the University of Akron. Also from NASA Glenn Research Center, we are grateful to Dr. Ten-Huei Guo for his review of many of our papers, Mr. Robert C. Anderson for his help when the computers become balky, and Dr. John Lekki for his enlightening and helpful discussions on photon entanglement.

We are grateful to our editors at John Wiley & Sons, Susanne Steitz-Filler and Allison McGinniss, for their support and guidance through the publication process.

We are most appreciative of Dr. Mary V. Zeller from NASA Glenn Research Center, whose support and encouragement during the writing of this book were most helpful.

We would like to express our gratitude to Professor Rachid Malti of Bordeaux University whose work on stability of elementary transfer functions of the second kind provides an important contribution to Chapter 13. We are grateful to Professor Diego Dominici of New York State University at New Paltz for his helpful discussions on inverse functions.

We would like to thank in advance any kind readers who forward to us any suggestions for improvement or corrections in the book.

Finally, we must recognize the support and love of our wives Margaret Lorenzo and Karen Hartley and our children Mary (Lorenzo) Stafford, Ann (Lorenzo) Wahlay, James Lorenzo, Carol (Lorenzo) Palko, Margaret (Lorenzo) Strekal, and John Hartley who give meaning to our work.

About the Companion Website

This book is accompanied by a companion website:

www.wiley.com/go/Lorenzo/Fractional_Trigonometry

The website includes:

- Figures from the book appearing in color.

1

Introduction

The ongoing development of the fractional calculus and the related development of fractional differential equations have created new opportunities and new challenges. The need for a generalized exponential function applicable to fractional-order differential equations has given rise to new functions. In the traditional integer-order calculus, the role of the exponential function and the trigonometric functions is central to the solution of linear ordinary differential equations. Such a supporting structure is also needed for the fractional calculus and fractional differential equations.

The purpose of this book is the development of the fractional trigonometries and hyperboletries that generalize the traditional trigonometric and hyperbolic functions based on generalizations of the common exponential function. The fundamental idea is that through the development of the fractional calculus, which generalizes the integer-order calculus, generalizations of the exponential function have been developed. The exponential function in the integer-order calculus provides the basis for the solution of linear fractional differential equations. Also, it may be thought of as the basis of the trigonometry.

A high-level summary of the flow of the development of the book is given in Figure 1.1. The generalized exponential functions that we use, the F-function and the R-function, are fractional eigenfunctions; that is, they return themselves on fractional differintegration. The F-function is the solution to the fundamental fractional differential equation

$$_0d_t^q x(t) + ax(t) = \delta(t)$$

when driven by a unit impulse. The R-function, $R_{q,v}(a,t)$, generalizes the F-function by including its integrals and derivatives as well. First, we show that these functions provide solutions to the fundamental fractional-order differential equation. Then, we explore the properties of the generalized exponential functions and develop some properties of the functions that will aid in the development and understanding of the fractional trigonometries and hyperboletries. This development follows a few mathematical preliminaries.

The R_1, R_2, and R_3 trigonometries along with the R_1 hyperboletry are developed by replacing a and t in the R-function with various combinations of real and purely imaginary variables. Based on the newly defined functions, a variety of basic properties and identities are determined. Furthermore, the Laplace transforms of the functions are determined and the fractional derivatives and fractional integrals of the functions elucidated.

The following chapters develop an overarching fractional trigonometry called the fractional meta-trigonometry that contains all of the fractional trigonometries shown in Figure 1.1 and infinitely many more. This is accomplished by replacing a and t in the R-function with general complex variables. We find that the fractional trigonometric functions lead to a generalization

The Fractional Trigonometry: With Applications to Fractional Differential Equations and Science, First Edition.
Carl F. Lorenzo and Tom T. Hartley.

Companion Website: www.wiley.com/go/Lorenzo/Fractional_Trigonometry

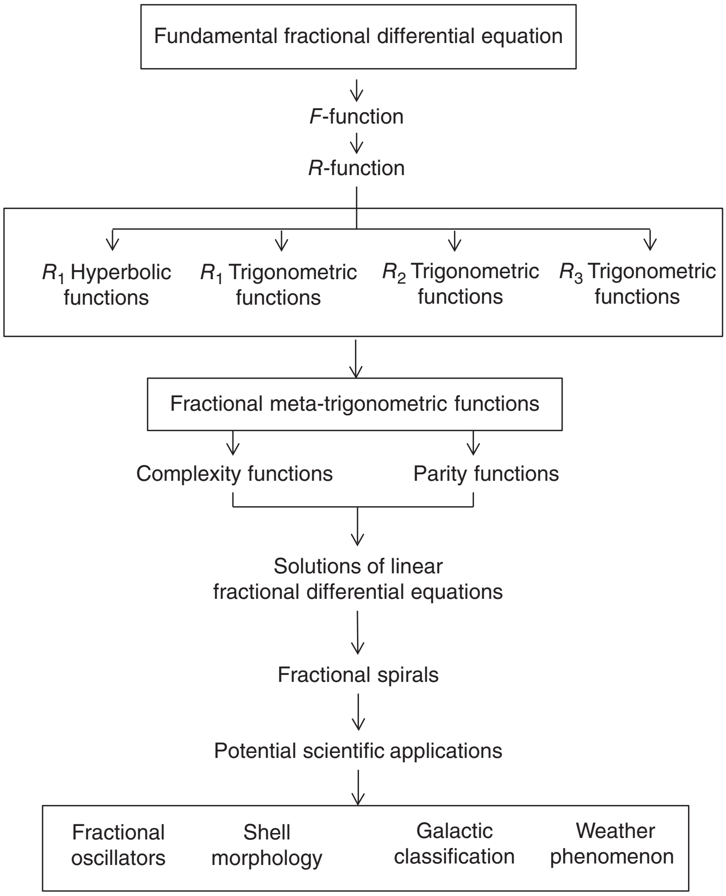


Figure 1.1 Overview of the development of the fractional trigonometry and its applications.

of the circular functions, which we have called the *fractional spiral functions*. These functions appear to model various natural phenomena, and preliminary applications of these functions to the properties of fractional oscillators, sea shells, galaxies, and more are explored. An important aspect of this modeling is that we can infer from the spiral functions the underlying fractional differential equations describing the phenomena, which is demonstrated. More importantly, the new fractional functions provide the solutions to classes of linear fractional differential equations.

1.1 Background

Because of the close association of the fractional calculus and the fractional trigonometry to be developed, we present here a brief introduction to the concepts of the fractional calculus for the reader who is unfamiliar with the area.

Several important textbooks have been written that are extremely helpful to someone entering the field. Perhaps the most useful from the engineering and scientific viewpoint, are the textbooks by Oldham and Spanier, "*The Fractional Calculus*" [104], and by Igor Podluby entitled "*Fractional Differential Equations*" [109]. A more mathematically oriented treatment is given in the book by Miller and Ross [95]. An encyclopedic reference volume written by Samko et al. [114] has also been published. Furthermore, a very good engineering book has been written by Oustaloup [105] and is available in French and Bush [19].

There are a growing number of physical systems whose behavior can be compactly described using fractional differential equations theory. Areas include long lines, electrochemical processes, diffusion, dielectric polarization, noise, viscoelasticity, chaos, creep, rheology, capacitors, batteries, heat conduction, percolation, cylindrical waves, cylindrical diffusion, water through a weir notch, Boussinesq shallow water waves, financial systems, biological systems, semiconductors, control systems, electrical machinery, and more.

1.2 The Fractional Integral and Derivative

The first question we need to address is "just what is a fractional derivative?" There are two separate but equivalent definitions of the fractional differintegral (Oldham and Spanier [104]), known as the Grünwald definition and the Riemann–Liouville definition. We present the Grünwald definition first, as it most recognizably generalizes the standard calculus. We then follow with the Riemann–Liouville definition as it is most easily used in practice.

1.2.1 Grünwald Definition

The Grünwald definition of the fractional-order differintegral is essentially a generalization of the derivative definition that most of us learned in introductory calculus, namely

$$\left.\frac{d^q f(t)}{[d(t-a)]^q}\right|_{GRUN} \equiv \lim_{N\to\infty} \frac{\left(\frac{t-a}{N}\right)^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(t - j\left(\frac{t-a}{N}\right)\right), \tag{1.1}$$

or in a slightly more familiar form

$$\left.\frac{d^q f(t)}{[d(t-a)]^q}\right|_{GRUN} \equiv \lim_{N\to\infty} \sum_{j=0}^{N-1} b_j(q) \frac{f(t - j\Delta t_N)}{(\Delta t_N)^q}, \tag{1.2}$$

where

$$\Delta t_N = \frac{t-a}{N}, \quad b_j(q) = \frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}.$$

In this definition, q is not limited to the integers and may be any real or complex number, and a is the starting time of the fractional differintegration, not to be confused with a in the differential equation in the introduction. Also, $q > 0$ defines differentiation, and $q < 0$ integration. Furthermore, $\Gamma(\circ)$ is the gamma function, or the generalized factorial function. It basically acts as a calibration constant here to properly interpolate the operators for values of q between the integers. In terms of notation, Oldham and Spanier [104] provide a development of equation (1.2) and generalize the fractional differintegral as

$$\frac{d^q x(t)}{[d(t-a)]^q}, \tag{1.3}$$

where it should be noticed that the differential in the denominator starts at some time a, and ends at a final time t. Thus, we see that the fractional derivative is defined on an interval and is no longer a local operator except for integer orders. Interestingly, the gamma functions force the series to terminate with a finite number of terms whenever q is any integer greater than or equal to zero, which represent the usual integer-order derivatives. When q is a negative integer, this series also contains the single and multiple integrals as well (which have always had infinite memory). The important aspect to be recognized is that there exists an uncountable infinity of fractional derivatives and integrals between the integers. The Grünwald definition is also equivalent to the more often used Riemann–Liouville definition, which is discussed in the following section.

1.2.2 Riemann–Liouville Definition

The Riemann–Liouville definition is easiest to present for fractional integrals first, and then generalize that to the fractional derivatives. The qth-order integral is defined as (see, e.g., Oldham and Spanier [104], Podlubny [109])

$$ {}_a d_t^{-q} x(t) \equiv \frac{d^{-q} x(t)}{[d(t-a)]^{-q}} \equiv \int_a^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau, \quad t \ge a, \tag{1.4} $$

It is important to note that this is the *key definition* of the fractional integral and is used by most investigators. Miller and Ross [95] provide four separate developments of this important equation. It can be shown that whenever q is a positive integer, this equation becomes a standard integer-order multiple integral. The Riemann–Liouville fractional derivative is defined as the integer-order derivative of a fractional integral

$$ {}_a d_t^{q} x(t) \equiv \frac{d^m}{dt^m} ({}_a d_t^{q-m} x(t)), \quad t \ge a, \tag{1.5} $$

where m is typically chosen as the smallest integer such that $q-m$ is negative, and the integer-order derivatives are those as defined in the traditional calculus. These equations define the *uninitialized* fractional integral and derivative.

For most engineering problems, system components, by virtue of their histories, are placed into some initial configuration or are initialized. Using mechanical systems as an example, the initial conditions are often mass positions and velocities at time zero. Fractional-order components, however, require a time-varying initialization Lorenzo [77] and Hartley [85], as they inherently have a fading infinite memory. Considering the aforementioned fractional-order integral, we assume that the fractional-order integration was started in the past, beginning at some time a, while the given problem begins at time $c > a$, where c is usually taken to be zero. Consider two fractional integrals of the same order acting on $x(t)$, where $x(t)$ and all of its derivatives are zero for all $t < a$. If the integral starting at c is to continue the integral starting at a, we must add an initialization ψ, thus

$$ \begin{aligned} {}_a d_t^{-q} x(t) = {}_c d_t^{-q} x(t) + \psi \Rightarrow & \int_a^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau \\ &= \int_c^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau + \psi, \quad t \ge c, \quad q > 0, \end{aligned} \tag{1.6} $$

therefore

$$ \psi = \int_a^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau - \int_c^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau = \int_a^c \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau, \quad t \ge c,\ q > 0, \tag{1.7} $$

clearly a time-varying function. This term represents the historical effect (Lorenzo and Hartley [68, 71]) or the initialization required for the fractional integral. The *initialized* fractional-order integration operator then is defined as

$$ {}_cD_t^{-q}x(t) \equiv {}_cd_t^{-q}x(t) + \psi(x_i, -q, a, c, t), \quad t \geq c, \tag{1.8}$$

where

$$\psi(x_i, -q, a, c, t) \equiv \int_a^c \frac{(t-\tau)^{q-1}}{\Gamma(q)} x_i(\tau)\, d\tau, \quad t \geq c. \tag{1.9}$$

$\psi(x_i, -q, a, c, t)$ is called the initialization function and is generally a time-varying function that must be added to the fractional-order operator to account for the effects of the past. This is a generalization of the constant of integration that is usually added to the normal order-one integral. The subscript i is appended to x to indicate that x_i is not necessarily the same as x. Clearly then, ${}_cD_t^{-q}x(t) = {}_ad_t^{-q}x(t), \quad t \geq c$. The initialization function is a time-varying function and is required to properly bring the historical effects of the fractional-order integral into the future. Similar considerations also apply for fractional-order derivatives [68, 71], that is, for any real value of q. Again, for convenience, $c = 0$ is typically chosen.

1.2.3 The Nature of the Fractional-Order Operator

The important properties of integer-order integration and differentiation have been shown to hold for initialized fractional-order operators (Lorenzo and Hartley [68] and [71]), including linearity and the index law. Physical insight into the nature of the fractional operators may be found in Hartley and Lorenzo [44, 47]. The fractional differintegral operator is a linear operator, and all the properties associated with linear operators hold for them. Also of considerable importance is the index law; that is, ${}_ad_t^{u+v}x(t) = {}_ad_t^u\,{}_ad_t^v x(t)$. The index law essentially allows us to state, for example, that the half-derivative of the half-derivative of a function is the same as the first-derivative of that function.

Laplace transforms are standard tools for integer-order operators and can still be readily used for fractional-order operators. In this regard, the Laplace transform of the initialized fractional-order differintegral is shown in Lorenzo and Hartley [68, 71] to be

$$L\left\{{}_0D_t^q x(t)\right\} = L\left\{{}_0d_t^q x(t) + \psi(x, q, a, 0, t)\right\} = s^q X(s) + L\left\{\psi(x, q, a, 0, t)\right\} \quad \text{for all real } q. \tag{1.10}$$

It is important to note that $L\{{}_0d_t^q x(t)\} = s^q X(s)$, for all q, as this is the generalization of the derivative rule for the integer-order situation. Also, note that $L^{-1}\{s^{-q}\} = t^{q-1}/\Gamma(q), q > 0$, which leads directly to the Riemann–Liouville definition via convolution

$${}_0d_t^{-q}x(t) \Leftrightarrow s^{-q}X(s) \;\Leftrightarrow\; \int_0^t \frac{(t)^{q-1}}{\Gamma(q)} x(t-\tau)\, d\tau = \int_0^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau. \tag{1.11}$$

The Laplace transform for the fractional integral is given [78] as

$$\begin{aligned} L\left\{{}_0D_t^{-q}h(t)\right\} &= L\left\{{}_0d_t^{-q}f(t)\right\} + L\left\{\psi\left(f_i, -q, -a, 0, t\right)\right\} \\ &= \frac{1}{s^q} L\{f(t)\} + \frac{1}{s^q\Gamma(q)} \int_{-a}^{0} e^{-\tau s}\Gamma(q+1, -\tau s) f_i(\tau) d\tau. \quad q \geq 0, \end{aligned} \tag{1.12}$$

where $q \geq 0$ and

$$h(t) = \begin{cases} f_i(t) & -a < t \leq 0, \\ f(t) & 0 < t, \end{cases}$$

and where f_i may differ from f during the initialization period. More detailed forms are presented in Ref. [78].

The transform for the fractional derivative of order u, where $u = n - q$, is given by

$$L\left\{{}_0D_t^u f(t)\right\} = s^{n-q} L\left\{f(t)\right\} - \sum_{j=0}^{n-1} s^{n-1-j} \psi^{(j)}(f_i, -q, -a, 0, t)|_{t=0+} + s^n L\left\{\psi(f_i, -q, -a, 0, t)\right\}, \tag{1.13}$$

where $u = n - q \geq 0$, $n = 1, 2, 3, \ldots$, $q \geq 0$, $f_i(t) = 0, \forall t < -a$, and

$$s^n L\{\psi(f_i, -q, -a, 0, t)\} = \frac{s^{n-q-1}}{\Gamma(q+1)} \left[e^{as}\Gamma(q+1, as) f_i(-a) - \Gamma(q+1) f_i(0) + \int_{-a}^{0} e^{-\tau s}\, \Gamma(q+1, -\tau s) f_i'(\tau) d\tau \right]. \tag{1.14}$$

In this relationship, $\psi(f_i, -q, -a, 0, t)$ is the initialization function for the fractional integral part of the operator. An alternative form of equation (1.14) where the integration is based on $f_i(t)$ rather than $f_i'(t)$ is given by

$$L\left\{{}_0D_t^u f(t)\right\} = s^{n-q} L\{f(t)\} - \sum_{j=0}^{n-1} s^{n-1-j} \psi^{(j)}(f_i, -q, -a, 0, t)|_{t=0+} + \frac{s^{n-q}}{\Gamma(q)} \int_{-a}^{0} e^{-\tau s} \Gamma(q, -\tau s) f_i(\tau) d\tau, \tag{1.15}$$

where $u = n - q \geq 0, \quad n = 1, 2, 3, \ldots, \quad q \geq 0, f_i(t) = 0, \quad \forall t < -a.$

These forms simplify for constant initialization [78], that is, when f_i = constant = b,

$$L\{{}_0D_t^u f(t)\} = s^{n-q} L\{f(t)\} + b s^{n-q-1} \left[\frac{e^{as}\Gamma(q-n+1, as)}{\Gamma(q-n+1)} - 1 \right],$$
$$q \text{ not integer}, \quad 0 \leq u = (n-q) \leq n, \quad n = 1, 2, 3, \ldots. \tag{1.16}$$

1.3 The Traditional Trigonometry

The application of the traditional integer-order trigonometry to analysis as well as engineering and science goes well beyond the calculation of triangles and triangulation. The applications include Fourier analysis, spectral analysis, solutions to ordinary and partial differential equations, and more. The trigonometric functions are found in nearly every branch of mathematics. The traditional trigonometry was originated for the solution of plane triangles. However, an additional way of interpreting the integer-order trigonometry is based on its relationship to the exponential function. The connections between the trigonometric functions and the exponential functions are very close. These relationships center on the Euler equation; that is, for $z = x + iy$

$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y), \tag{1.17}$$

as well as the definitions

$$\cos(t) \equiv \frac{e^{it} + e^{-it}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \tag{1.18}$$

and

$$\sin(t) \equiv \frac{e^{it} - e^{-it}}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \tag{1.19}$$

for the sine and cosine functions. In fact, the exponential and trigonometric functions are fundamental to complex numbers and complex computation.

The hyperbolic functions are also based on the exponential function; these are given in the following relationships:

$$\cosh(t) \equiv \frac{e^t + e^{-t}}{2} = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}, \tag{1.20}$$

and

$$\sinh(t) \equiv \frac{e^t - e^{-t}}{2} = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}. \tag{1.21}$$

The development of the fractional calculus has involved new functions that generalized the common exponential function. These functions allow the opportunity to generalize both the hyperbolic functions and the trigonometric functions to "fractional" or "generalized" versions. Two of these functions, to be detailed later in the book, are the F-function (Hartley and Lorenzo [45]), which is the solution of the fundamental fractional differential equation

$${}_cD_t^q x(t) = -ax(t) + bu(t) \tag{1.22}$$

and its generalization, the R-function (Lorenzo and Hartley [69, 70]). They are defined as

$$F_q(a, t) \equiv \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)q-1}}{\Gamma((n+1)q)}, \quad t > 0 \tag{1.23}$$

and its νth differintegral

$$R_{q,\nu}(a, t) \equiv \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)q-1-\nu}}{\Gamma((n+1)q-\nu)}, \quad t > 0. \tag{1.24}$$

The Laplace transforms of these functions are determined in Ref. [69] as

$$L\left\{F_q(a, t)\right\} = \frac{1}{s^q - a} \quad \text{and} \quad L\left\{R_{q,\nu}(a, t)\right\} = \frac{s^\nu}{s^q - a}, \quad \mathrm{Re}(q - \nu) \ge 0. \tag{1.25}$$

It can be seen from the series definitions of these functions that they contain the exponential function

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots = \sum_{0}^{\infty} \frac{(at)^n}{\Gamma(n+1)} \tag{1.26}$$

as the $q = 1,\ \nu = 0$ special case. This result and the fact that the F- and R-functions are eigenfunctions for the qth-order derivative are powerful drivers toward a new generalized trigonometry based on the fractional (or generalized) exponential function, that is, the F- or the R-function. The expectation and hope is that such a trigonometry will lead also to new generalizations of all the products of the integer-order trigonometry, a situation that will be broadly useful. These expectations and more derive from the usefulness of the ordinary trigonometry.

It is well known, and follows from equation (1.26), that

$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots \tag{1.27}$$

$$e^{it} = \left\{1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right\} + i\left\{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right\}. \tag{1.28}$$

These series are, of course, recognized as representing the circular functions giving the well-known Euler equation

$$e^{it} = \cos(t) + i\sin(t). \tag{1.29}$$

It is important to note that $\cos(t)$ is a summation of terms that are simultaneously both the real part of e^{it} and are even powers of t. Also, that $\sin(t)$ is a summation of terms that are simultaneously both the imaginary part of e^{it} and are odd powers of t. This observation will prove important in the development to follow. Not all of the new fractional trigonometric functions will share this property.

The R-function, since it includes within it the fractional differintegrals of the F-function, and is a representation of the fractional eigenfunction, is used as the generalizing basis of the exponential function. Based on the R-function, parallels with the integer-order trigonometry are used to generate related fractional trigonometries. The properties of these new trigonometries and identities flowing from the definitions are then developed.

The trigonometries derived from these generalizations will be jointly termed "The Fractional Trigonometry." The definitions for the fractional trigonometries can be based on several different parallels between various properties of the integer-order trigonometry and the proposed fractional-order trigonometries. For example, parallels based on equations (1.17)–(1.19) each provide a basis for definitions. Laplace transforms of the new functions are determined. Fractional differential equations for which the functions are solutions and various intra- and interrelationships of the new trigonometric functions are studied.

1.4 Previous Efforts

There have been previous definitions offered for fractional trigonometric functions. These efforts, each amounting to a page or so of definitions, have been based on the Mittag-Leffler function and are discussed in Appendix A. In all cases, the definitions are subsets of those to be presented here.

1.5 Expectations of a Generalized Trigonometry and Hyperboletry

There are some characteristics that a generalized trigonometry should have and additional characteristics that may be desirable. We require that any fractional trigonometry should

contain the traditional, integer-order, plane trigonometry as a special case,
have an eigenfunction basis,
exhibit series compatibility between defined functions and generalized exponentials, and
form a basis for the solution of fractional-order linear differential equations.

These requirements are essentially self-explanatory. The first requires backward compatibility to the ordinary trigonometry. The second and fourth requirements are a way of saying that the new generalized trigonometry should be closely coupled to the solution of fractional differential equations and that the solutions should be expressible as linear combinations of the functions. The expectation flowing from this is that we expect insight into the solutions of fractional differential equations from the fractional trigonometry to be similar to that obtained from the trigonometric solutions associated with the solutions of ordinary differential equations.

In general, our requirements and expectations for the generalized hyperbolic functions parallel those listed for the fractional trigonometry. For example, we require backward compatibility with the traditional hyperbolic functions, and so on. In addition, we expect to maintain or generalize the relationships between the traditional integer-order trigonometric functions and the traditional integer-order hyperbolic functions when the fractional versions are defined.

2

The Fractional Exponential Function via the Fundamental Fractional Differential Equation

2.1 The Fundamental Fractional Differential Equation

This chapter develops the F-function as the solution of the fundamental fractional differential equation equation (2.1). A similar function was first used by Robotnov [112, 113]. The F-function was given in a footnote by Oldham and Spanier [104], p. 122, and later independently found by Hartley and Lorenzo [45, 48] as the solution to equation (2.1). This function is the foundation to the development of the fractional trigonometries. Following our study of the F-function, we introduce the R-function, which generalizes the F-function by including its fractional derivatives and integrals. The derivations of this chapter are abstracted from Hartley and Lorenzo, NASA [45]:

> The problem to be addressed is the solution of the uninitialized fractional-order differential equation
>
> $$_cD_t^q x(t) = {}_cd_t^q x(t) = -ax(t) + bu(t), \quad q > 0 \tag{2.1}$$
>
> where $_cD_t^q x(t)$ is the initialized qth-order derivative and $_cd_t^q x(t)$ represents the uninitialized qth-order derivative. Here, it is assumed for simplicity that the problem starts at $t = 0$, which sets $c = 0$. It is also assumed that all initial conditions, or initialization functions, are zero; thus, $_cD_t^q x(t) = {}_cd_t^q x(t)$. We are primarily concerned with the forced response. Rewriting equation (2.1) with these assumptions gives
>
> $$_0d_t^q x(t) = -ax(t) + bu(t). \tag{2.2}$$
>
> We use Laplace transform techniques to obtain the solution of this differential equation. In order to do so for this problem, the Laplace transform of the fractional differential is required. Using equation (1.11) and ignoring initialization terms, equation (2.2) can be Laplace transformed as
>
> $$s^q = X(s) = -aX(s) + bU(s). \tag{2.3}$$
>
> This equation is rearranged to obtain the transfer function
>
> $$\frac{X(s)}{U(s)} = G(s) = \frac{b}{s^q + a}. \tag{2.4}$$

The Fractional Trigonometry: With Applications to Fractional Differential Equations and Science, First Edition.
Carl F. Lorenzo and Tom T. Hartley.

Companion Website: www.wiley.com/go/Lorenzo/Fractional_Trigonometry

This transfer function of the fundamental linear fractional-order differential equation contains the fundamental "fractional" pole (discussed later) and is a basis element for fractional differential equations of higher order. Specifically, transfer functions can be inverse transformed to obtain the impulse response of a differential equation. The impulse response can then be used with the convolution approach to obtain the solution of fractional differential equations with arbitrary forcing functions. In general, if $U(s)$ is given, the product $G(s)U(s)$ can be expanded using partial fractions, and the forced response obtained by inverse transforming each term separately. To accomplish these tasks, it is necessary to obtain the inverse transform of equation (2.4), which is the impulse response of the fundamental fractional-order differential equation.

To obtain the solution for arbitrary q, it is necessary to derive the generalized fundamental impulse response for the fractional-order differential equation equation (2.4), as this is not available in the standard tables of transforms, such as Oberhettinger and Badii [103] or Erdelyi et al. [34]. This is derived in the following section.

2.2 The Generalized Impulse Response Function

Continuing from Ref. [45]:

To obtain the generalized impulse response, we expand the right-hand side of equation (2.4) in descending powers of s, and then inverse transform the series term-by-term. It is assumed that $q > 0$. As the constant b in equation (2.4) is a constant multiplier, it can be assumed, with no loss of generality, to be unity. Then, expanding the right-hand side of equation (2.4) about $s = \infty$ using long division gives

$$F(s) = \frac{1}{s^q + a} = \frac{1}{s^q} - \frac{a}{s^{2q}} + \frac{a^2}{s^{3q}} - \cdots = \frac{1}{s^q}\sum_{n=0}^{\infty} \frac{(-a)^n}{s^{nq}}. \tag{2.5}$$

This power series, in s^{-q}, can now be inverse transformed term-by-term using the transform pair $L\{t^{k-1}\} = \Gamma(k)/s^k$ for $k > 0$. The result is

$$\begin{aligned} L^{-1}\{F(s)\} &= L^{-1}\left\{\frac{1}{s^q} - \frac{a}{s^{2q}} + \frac{a^2}{s^{3q}} - \cdots\right\} \\ &= \frac{t^{q-1}}{\Gamma(q)} - \frac{at^{2q-1}}{\Gamma(2q)} + \frac{a^2t^{3q-1}}{\Gamma(3q)} - \cdots, \quad q > 0. \end{aligned} \tag{2.6}$$

The right-hand side can now be collected into a summation and used as the definition of the generalized impulse response function

$$F_q[a,t] \equiv t^{q-1}\sum_{n=0}^{\infty} \frac{(a)^n t^{nq}}{\Gamma((n+1)q)}, \quad q > 0. \tag{2.7}$$

This function is the generalization of the common exponential function that is needed for the fractional calculus. Furthermore, the important Laplace transform identity

$$L\{F_q[a,t]\} = \frac{1}{s^q - a}, \quad q > 0 \tag{2.8}$$

has been established. When $u(t)$ in equation (2.2) is a unit impulse, $x(t) = bF_q[-a,t]$ is seen to be the forced response of the fundamental fractional differential equation.

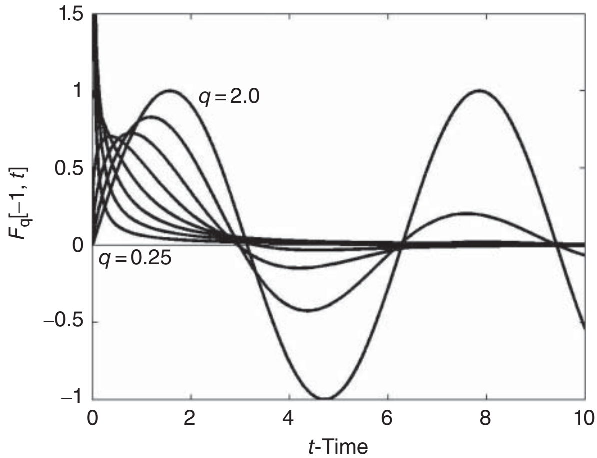


Figure 2.1 The $F_q[-1, t]$-function versus time as q varies from 0.25 to 2.0 in 0.25 increments.

This section has established the F-function as the impulse response of the fundamental linear fractional-order differential equation. The F-function generalizes the usual exponential function and is the fractional eigenfunction. The solution of equation (2.2), with $b=1$, the F-function, is shown for various values of q in Figure 2.1.

We note that $F_q[a, t]$ is a generalization of the exponential function, since for $q = 1$,

$$F_1[a, t] = \sum_{n=0}^{\infty} \frac{(at)^n}{\Gamma(n+1)} \equiv e^{at}, \quad t > 0. \tag{2.9}$$

This generalization, $F_q[a, t]$, is the basis for the solution of linear fractional-order differential equations composed of combinations of fractional poles of the type of equation (2.8).

2.3 Relationship of the *F*-function to the Mittag-Leffler Function

The F-function is closely related to the Mittag-Leffler function, $E_q[at^q]$ [96–98], which, at this time is commonly used in the fractional calculus. For this reason, we discuss the Mittag-Leffler function briefly; from Ref. [45], we have the following:

This function is defined as

$$E_q[x] \equiv \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(nq+1)}, \quad q > 0, \tag{2.10}$$

(Erdelyi et al. [34, 35]). Letting $x = -at^q$, this becomes

$$E_q\left[-at^q\right] \equiv \sum_{n=0}^{\infty} \frac{(-a)^n t^{nq}}{\Gamma(nq+1)}, \quad q > 0, \tag{2.11}$$

which is similar to, but not the same as equation (2.7). The Laplace transform of this equation (2.11) can also be obtained via term-by-term transformation, that is

$$L\left\{E_q\left[-at^q\right]\right\} = L\left\{\frac{1}{\Gamma(1)} - \frac{at^q}{\Gamma(1+q)} + \frac{a^2t^{2q}}{\Gamma(1+2q)} + \cdots\right\}$$

$$= \frac{1}{s} - \frac{a}{s^{q+1}} + \frac{a^2}{s^{2q+1}} + \cdots, \tag{2.12}$$

or equivalently

$$L\left\{E_q\left[-at^q\right]\right\} = \frac{1}{s}\left[1 - \frac{a}{s^q} + \frac{a^2}{s^{2q}} - \cdots\right] = \frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{-a}{s^q}\right)^n. \tag{2.13}$$

Using equation (2.5), equation (2.13) may be written as

$$L\left\{E_q\left[-at^q\right]\right\} = \frac{1}{s}\left[\frac{s^q}{s^q+a}\right] = \frac{1}{s}[s^q L\{F_q[-a,t]\}]. \tag{2.14}$$

Thus, the Laplace transform of the Mittag-Leffler function can be written as

$$L\left\{E_q\left[at^q\right]\right\} = \frac{s^{q-1}}{s^q - a}, \quad q > 0. \tag{2.15}$$

More importantly, from equation (2.14), the E-function and the F-function are related as follows:

$${}_0d_t^{q-1}F_q[a,t] = E_q[at^q]. \tag{2.16}$$

Functions similar to the F-function are mentioned by other authors as well. Oldham and Spanier [104], p. 122 mention it in passing in a footnote discussing eigenfunctions. Also, Robotnov [112, 113] studied a closely related function extensively with respect to hereditary integrals; he calls it the э-function. Also, other authors have used less direct methods to solve equation (2.2). Bagley and Calico [11] obtain a solution in terms of Mittag-Leffler functions. Miller and Ross's [95] solution is in terms of the fractional derivative of the exponential function. They use the function

$$E_t(\upsilon, a) \equiv {}_0d_t^{-\upsilon}e^{at}, \tag{2.17}$$

whose Laplace transform is

$$L\left\{E_t(\upsilon, a)\right\} = \frac{s^{-\upsilon}}{s-a}. \tag{2.18}$$

Also, Glockle and Nonnenmacher [38] obtain a solution in terms of the more complicated Fox Functions.

2.4 Properties of the *F*-Function

In this section, the eigenfunction property of the F-function is derived. This essentially means that the qth-derivative of the function $F_q[a, t^q]$, returns the same function $F_q[a, t^q]$ for $t > 0$ (see equation 2.21). Then, from Ref. [45]:

Taking the uninitialized qth-derivative (${}_0d_t^q$) in the Laplace domain by multiplying by s^q gives

$$L^{-1}\left\{\frac{s^q}{s^q+a}\right\} = {}_0d_t^qF_q[-a,t]. \tag{2.19}$$

This equation can also be rewritten as

$$L^{-1}\left\{\frac{s^q}{s^q+a}\right\} = L^{-1}\left\{1 - \frac{a}{s^q+a}\right\} = \delta(t) - aF_q[-a,t], \tag{2.20}$$

where the delta function is recognized as the unit impulse function. Now comparing equations (2.19) and (2.20), it can be seen that

$$_0d_t^q F_q[-a,t] = \delta(t) - aF_q[-a,t]. \tag{2.21}$$

This equation demonstrates the eigenfunction property of returning the same function upon qth-order differentiation for $t > 0$. This is a generalization of the exponential function in integer-order calculus.

It is now easy to show that the F-function is the impulse response of the differential equation (2.2). Referring back to equation (2.2), and setting $u(t) = \delta(t)$ and $b = 1$, yields

$$_0d_t^q x(t) = -ax(t) + \delta(t). \tag{2.22}$$

For the F-function to be the impulse response, it must be the solution to equation (2.22), that is $x(t) = F_q[-a,t]$. Substituting this into equation (2.22) gives

$$_0d_t^q F_q[-a,t] = -aF_q[-a,t] + \delta(t). \tag{2.23}$$

This equation has been obtained by direct substitution into the differential equation. Referring back to equation (2.21), however, shows that the qth-derivative of the F-function on the left-hand side is in fact equal to the right-hand side of equation (2.23). Thus, it is shown by direct substitution that the F-function is indeed the impulse response of the differential equation (2.22).

These sections have developed the F-function and the properties that are key to the development of the fractional trigonometry. The following section develops further important properties of the F-function and illustrates its application to the solution of fractional differential equations.

2.5 Behavior of the *F*-Function as the Parameter *a* Varies

This section studies the F-function with complex values for the parameter a. A linear system theory approach provides further understanding of the properties of the F-function, when its Laplace transform is considered as a transfer function of a physical system. From Ref. [45]:

> Thus, we consider the Laplace transform of equation (2.22) as the transfer function of a related linear system
>
> $$L\{F_q[a,t]\} = \frac{1}{s^q - a}, \quad q > 0. \tag{2.24}$$
>
> In general, to understand the dynamics of any particular system, we often consider the nature of the s-domain singularities. We define $s = re^{i\theta}$ in what follows. For $a < 0$, the particular function of equation (2.24) does not have any poles on the primary Riemann sheet of the s-plane ($|\theta| < \pi$), as it is impossible to force the denominator of the right-hand side of equation (2.24) to zero. Note, however, that it is possible to force the denominator to zero if secondary Riemann sheets are considered. For example, the denominator of the Laplace transform
>
> $$L\{F_{1/2}[-1,t]\} = \frac{1}{s^{1/2} + 1}, \tag{2.25}$$
>
> does not go to zero anywhere on the primary sheet of the s-plane ($|\theta| < \pi$). It does go to zero on the secondary sheet, however. With $s = e^{\pm i2\pi}$, the denominator is indeed zero. Thus, this Laplace transform has a pole at $s = e^{\pm i2\pi}$, which is at $s = 1 + i0$ on the second

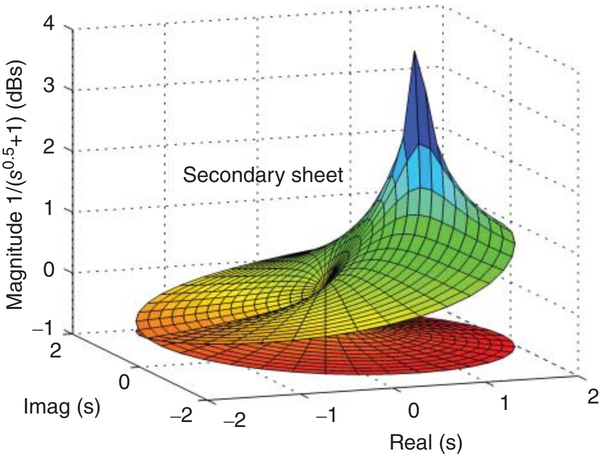


Figure 2.2 Both sheets of the Laplace transform of the F-function in the s-plane. Source: Hartley and Lorenzo 1998 [45]. Public domain. Please see www.wiley.com/go/Lorenzo/Fractional_Trigonometry for a color version of this figure.

Riemann sheet. This is shown in Figure 2.2, where $|1/s^{1/2} + 1|$ is plotted as a function of Real(s) and Imaginary(s).

As it is difficult to visualize multiple Riemann sheets, following LePage [65], it is useful to perform a conformal transformation into a new plane. Let

$$w = s^q. \tag{2.26}$$

The transform in equation (2.24) then becomes

$$L\{F_q[-a,t]\} = \frac{1}{s^q + a} \Leftrightarrow \frac{1}{w + a}. \tag{2.27}$$

With this transformation, we study the w-plane poles. Once we understand the time-domain responses that correspond to the w-plane pole locations, we will be able

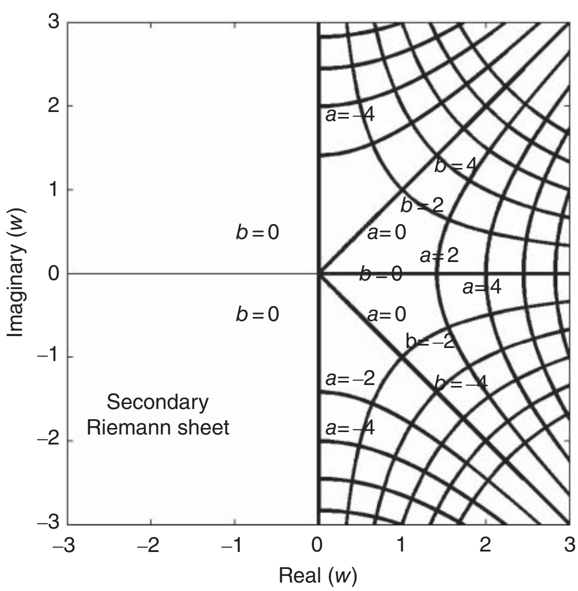


Figure 2.3 The w-plane for $q = 1/2$, with $w = s^{1/2}$ and $s = a + ib$. Source: Hartley and Lorenzo 1998 [45]. Public domain.

to clearly understand the implications of this new complex plane. To accomplish this, it is necessary to map the s-plane, along with the time-domain function properties associated with each point, into the new complex w-plane. To simplify the discussion, we limit the order of the fractional operator to $0 < q \leq 1$. Let

$$w = \rho e^{i\varphi} = \alpha + i\beta. \tag{2.28}$$

Then, referring to equation (2.26)

$$w = s^q = (re^{i\theta})^q = r^q e^{iq\theta} = \rho e^{i\varphi}. \tag{2.29}$$

Thus, $\rho = r^q$ and $\varphi = q\theta$. With this equation, it is possible to map either lines of constant radius or lines of constant angle from the s-plane into the w-plane. Of particular interest is the image of the line of s-plane stability (the imaginary axis), that is, $s = re^{\pm iq\pi/2}$. The image of this line in the w-plane is

$$w = r^q e^{\pm iq\pi/2}, \tag{2.30}$$

which is the pair of lines at $\varphi = \pm q\pi/2$. Thus, the right half of the s-plane maps into a wedge in the w-plane of angle less than $\pm 90q$ degrees, that is, the right half s-plane maps into

$$|\varphi| < q\pi/2. \tag{2.31}$$

For example, with $q = 1/2$, the right half of the s-plane maps into the wedge bounded by $-\pi/4 < \varphi < \pi/4$; see Figure 2.3.

It is also important to consider the mapping of the negative real s-plane axis, $s = re^{\pm i\pi}$. The image is

$$w = r^q e^{\pm iq\pi}. \tag{2.32}$$

Thus, the entire primary sheet of the s-plane maps into a w-plane wedge of angle less than $\pm 180q$ degrees. For example, if $q = 1/2$, then the negative real s-plane axis maps into the w-plane lines at ± 90 degrees; see Figure 2.3.

Continuing with the $q = 1/2$ example, and referring to Figure 2.3, it should now be clear that the right half of the w-plane corresponds to the primary sheet of the Laplace s-plane. The time responses we are familiar with from integer-order systems have poles that are in the right half of the w-plane. The left half of the w-plane, however, corresponds to the secondary Riemann sheet of the s-plane. A pole at $w = -1 + i0$ lies at $s = +1 + i0$, on the secondary Riemann sheet of the s-plane. This point in the s-plane is really not in the right half s-plane, corresponding to instability, but rather is "underneath" the primary s-plane Riemann sheet. As the corresponding time responses must then be even more than overdamped, we call any time response whose pole is on a secondary Riemann sheet, "hyperdamped." It should now be easy for the reader to extend this analysis to other values of q.

To summarize this, the shape of the F-function time response, $F_q[-a, t]$, depends upon both q and the parameter $-a$, which is the pole of equation (2.27). This is shown in Figure 2.4. For a fixed value of q, the angle φ of the parameter $-a$, as measured from the positive real w-axis, determines the type of response to expect. For small angles, $|\varphi| < q\pi/2$, the time response will be unstable and oscillatory, corresponding to poles in the right half s-plane. For larger angles, $q\pi/2 < |\varphi| < q\pi$, the time response will be stable and oscillatory, corresponding to poles in the left half s-plane. For even larger angles, $|\phi| > q\pi$, the time response will be hyperdamped, corresponding to poles on secondary Riemann sheets.

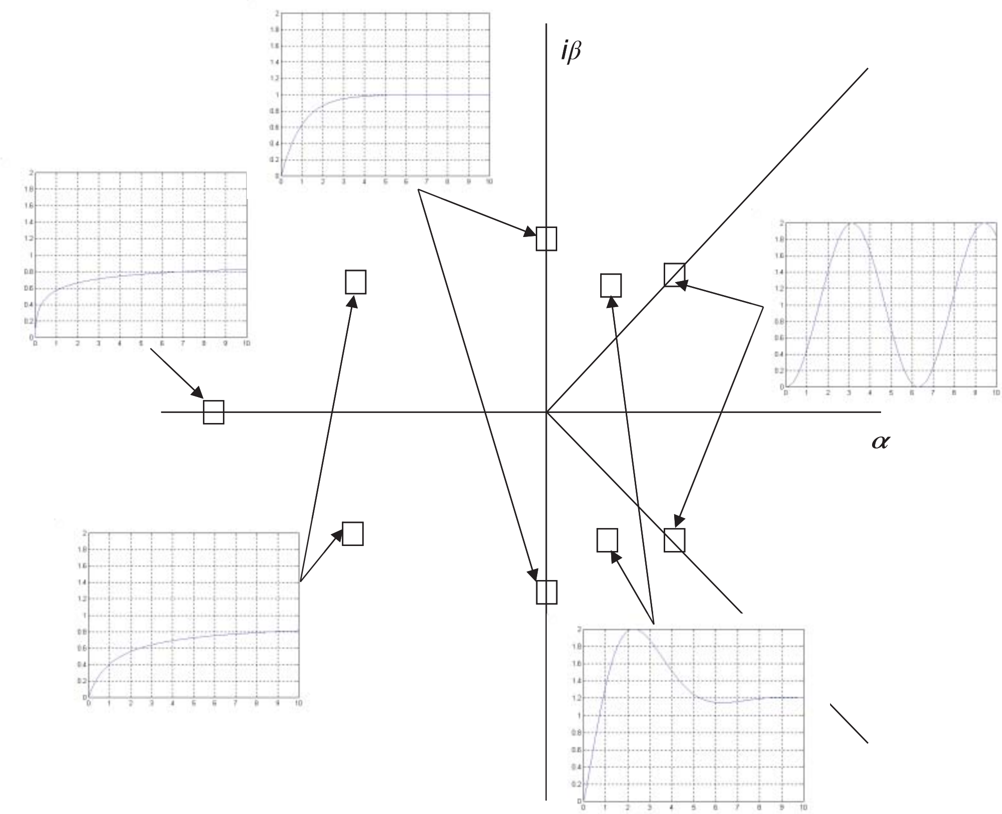


Figure 2.4 Step responses corresponding to various pole locations in the w-plane, for $q = 1/2$. Source: Hartley and Lorenzo 1998 [45]. Public domain.

It is now possible to do fractional system analysis and design, for commensurate-order fractional systems, directly in the w-plane. To do this, it is necessary to first choose the greatest common fraction (q) of a particular system (clearly nonrationally related powers are an important problem although a close approximation of the irrational number will be sufficient for practical application). Once this is done, all powers of s^q are replaced by powers of w. Then the standard pole-zero analysis procedures can be done with the w-variable, being careful to recognize the different areas of the particular w-plane. This analysis includes root finding, partial fractions (note that complex conjugate w-plane poles still occur in pairs), root locus, compensation, and so on. We have now characterized the possible behaviors for fractional commensurate-order systems in a new complex w-plane; that is, given a set of w-plane poles, the corresponding time-domain functions are known both quantitatively and qualitatively. Although most of the discussion has actually been for $0 < q \leq 1/2$, it is reasonably applicable to larger values of q with the appropriate modifications for many-to-many mappings.

2.6 Example

In this example, we consider the impulse response of the inductor–supercapacitor pair shown in Figure 2.5. In the study of Hartley et al. [49], it was shown that a particular

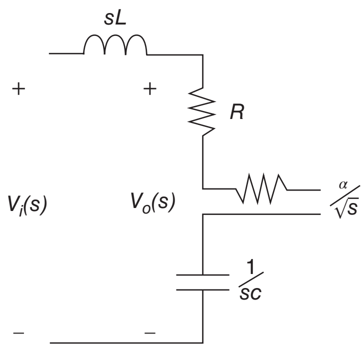


Figure 2.5 Supercapacitor circuit example.

commercial supercapacitor is accurately modeled with the impedance transfer function $Z(s) = R + \frac{\alpha}{\sqrt{s}} + \frac{1}{sC}$. The voltage across this device is chosen as the output voltage for this example. Then, in Figure 2.5, the voltage transfer function from the input terminals to the supercapacitor terminals is found to be

$$\frac{V_o(s)}{V_i(s)} = \frac{R + \frac{\alpha}{\sqrt{s}} + \frac{1}{sC}}{sL + R + \frac{\alpha}{\sqrt{s}} + \frac{1}{sC}} = \frac{RCs + \alpha C\sqrt{s} + 1}{LCs^2 + RCs + \alpha C\sqrt{s} + 1}. \tag{2.33}$$

For this example, we let $RC = 1, \quad \alpha C = 1, \quad$ and $\quad LC = 1$. Then,

$$\frac{V_o(s)}{V_i(s)} = \frac{s + \sqrt{s} + 1}{s^2 + s + \sqrt{s} + 1}. \tag{2.34}$$

The poles and zeros of this transfer function are plotted on the w-plane in Figure 2.6, which also shows the 45° stability lines for $q = 0.5$. Clearly, there are two poles in the right half of the w-plane, but to the left of the stability boundary. These pole locations correspond to complex stable poles in the s-plane and imply a damped oscillatory impulse response. We obtain the impulse response of equation (2.34) using F-functions. First, it is necessary to define the new

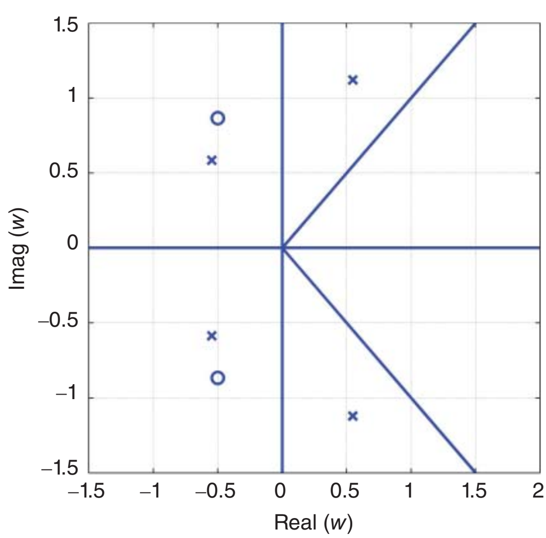


Figure 2.6 The w-plane stability diagram for supercapacitor.

complex variable $w = \sqrt{s}$. Then, assuming an impulse input, the output of the transfer function becomes

$$\begin{aligned} V_o(s) &= \frac{w^2 + w + 1}{w^4 + w^2 + w + 1} \\ &= \frac{-0.0570 - 0.4334i}{w - 0.5474 - 1.1209i} + \frac{-0.0570 + 0.4334i}{w - 0.5474 + 1.1209i} \\ &\quad + \frac{0.0570 - 0.1308i}{w + 0.5474 - 1.1209i} + \frac{0.0570 + 0.1308i}{w + 0.5474 + 1.1209i}. \end{aligned} \tag{2.35}$$

Inverse transforming the partial fractions with $w = \sqrt{s}$ yields

$$\begin{aligned} v_o(t) &= (-0.0570 - 0.4334i)F_{0.5}[0.5474 + 1.1209i, t] \\ &\quad + (-0.0570 + 0.4334i)F_{0.5}[0.5474 - 1.1209i, t] \\ &\quad + (0.0570 - 0.4334i)F_{0.5}[-0.5474 + 1.1209i, t] \\ &\quad + (0.0570 + 0.4334i)F_{0.5}[-0.5474 - 1.1209i, t]. \end{aligned} \tag{2.36}$$

This impulse response is plotted in Figure 2.7. Some damped oscillations are observed as expected from the pole-zero plot.

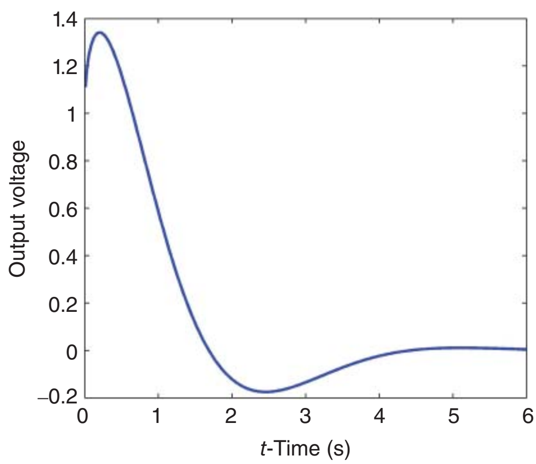


Figure 2.7 Supercapacitor impulse response.

This simple example is presented to demonstrate the use of the F-function to obtain the solution of a physical fractional-order system.

In this chapter, the fundamental linear fractional-order differential equation has been considered and its impulse response has been obtained as the F-function. This function most directly generalizes the exponential function for application to fractional differential equations. It is at the heart of our development of the fractional trigonometry. Also, several properties of this function have been presented and discussed. In particular, the Laplace transform properties of the F-function have been discussed using multiple Riemann sheets and a conformal mapping into a more readily useful complex w-plane.

It is felt that this generalization of the exponential function, the F-function, is the most easily understood and most readily implemented of the several other generalizations presented in the literature.

3

The Generalized Fractional Exponential Function: The *R*-Function and Other Functions for the Fractional Calculus

3.1 Introduction

This chapter generalizes the F-function, that is, the eigenfunction solution of the fundamental fractional differential equation, to the R-function, which carries in it the derivatives and integrals of the F-function. The R-function then becomes the basis for the development of the fractional trigonometric and fractional hyperbolic functions. This chapter also compares the properties of these functions with other important functions that have been associated with the fractional calculus.

The previous chapter developed the F-function, $F_q[a, t]$, for the solution of fractional differential equations. This function provided direct solution and important understanding for the fundamental linear fractional-order differential equation and for the related initial value problem (Hartley and Lorenzo [46]).

This chapter presents functions commonly used in the fractional calculus, and their Laplace transforms. A new function called the R-function and two related generalized functions, the G-function and the H-function, are presented for consideration. In the sequel, we will see that these functions are important in the development and application of the fractional trigonometries and are, therefore, useful in the solution of fractional differential equations. The R-function, $R_{q,\nu}(a, t)$, contains the νth-order derivatives and integrals of the F-function. With $\nu = 0$, the R-function becomes the F-function and thus the $R_{q,0}$-function also returns itself on qth-order differintegration. The G- and H-functions are needed for the analysis of repeated and partially repeated fractional poles. An example application of the R-function is provided. Sections 3.10–3.16 present some preliminaries to the development of the fractional trigonometries. The derivations of Sections 3.2–3.9 are adapted from Lorenzo and Hartley [69].

3.2 Functions for the Fractional Calculus

This section summarizes a number of functions that have been found useful in the solution of problems of the fractional calculus and more particularly in the solution of fractional differential equations.

The Fractional Trigonometry: With Applications to Fractional Differential Equations and Science, First Edition.
Carl F. Lorenzo and Tom T. Hartley.

Companion Website: www.wiley.com/go/Lorenzo/Fractional_Trigonometry

3.2.1 Mittag-Leffler's Function

The Mittag-Leffler function [96, 97, 98] is given by the following equation:

$$E_q[t] = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(nq+1)}, \quad q > 0. \tag{3.1}$$

This function often appears with the argument $-at^q$, and its Laplace transform then is given as

$$L\{E_q[-at^q]\} = L\left\{ \sum_{n=0}^{\infty} \frac{(-a)^n t^{nq}}{\Gamma(nq+1)} \right\} = \frac{s^q}{s(s^q+a)}, \quad q > 0. \tag{3.2}$$

3.2.2 Agarwal's Function

The Mittag-Leffler function is generalized by Agarwal [5, 6] as follows:

$$E_{\alpha,\beta}(t) = \sum_{m=0}^{\infty} \frac{t^{\left(m+\frac{\beta-1}{\alpha}\right)}}{\Gamma(\alpha m+\beta)}. \tag{3.3}$$

This function is particularly interesting to the fractional-order system theory due to its Laplace transform, given by Agarwal as

$$L\{E_{\alpha,\beta}[t^\alpha]\} = \frac{s^{\alpha-\beta}}{s^\alpha - 1}. \tag{3.4}$$

This function is the $(\alpha - \beta)$-order fractional derivative of the F-function with argument $a = 1$.

3.2.3 Erdelyi's Function

Erdelyi et al. [34] has studied the following related generalization of the Mittag-Leffler function:

$$E_{\alpha,\beta}(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\alpha m+\beta)}, \quad \alpha, \beta > 0, \tag{3.5}$$

where the powers of t are integer. The Laplace transform of this function is given by

$$L\{E_{\alpha,\beta}(t)\} = \sum_{m=0}^{\infty} \frac{\Gamma(m+1)}{\Gamma(\alpha m+\beta)s^{m+1}}, \qquad \alpha > 1,\ \beta > 0. \tag{3.6}$$

As this function cannot be easily generalized, it is not considered further.

3.2.4 Oldham and Spanier's, Hartley's, and Matignon's Function

To effect the direct solution of the fundamental linear fractional-order differential equation (Chapter 2), the following function was used (Hartley and Lorenzo [45]):

$$F_q[-a,t] = t^{q-1} \sum_{n=0}^{\infty} \frac{(-a)^n t^{nq}}{\Gamma(nq+q)}, \qquad q > 0. \tag{3.7}$$

This function had been mentioned earlier in a footnote by Oldham and Spanier [104], p. 122. The important feature of this function is the power and simplicity of its Laplace transform, namely

$$L\{F_q[a,t]\} = \frac{1}{s^q - a}, \qquad q > 0. \tag{3.8}$$

This function, with $a = 1$, is the fractional eigenfunction in that it returns itself on qth-order differintegration. We note that Matignon [92] has also recognized this function as the fractional eigenfunction.

3.2.5 Robotnov's Function

Robotnov [112, 113] used the function

$$\xi_q(a,t) = \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)(q+1)-1}}{\Gamma((n+1)(q+1))} \tag{3.9}$$

in his study of hereditary integrals. The Laplace transform of this function is

$$L\{\xi_q(a,t)\} = \frac{1}{s^{q+1} - a}. \tag{3.10}$$

Using Robotnov's function, $\xi_{q-1}(1,t)$ is the eigenfunction. Continuing from [69]:

3.2.6 Miller and Ross's Function

Miller and Ross [95], pp. 80 and 309–351 introduce another function as the basis of the solution of the fractional-order initial value problem. It is defined as the νth integral of the exponential function, that is,

$$E_t(\nu,a) = \frac{d^{-\nu}}{dt^{-\nu}} e^{at} = t^{\nu} e^{at} \gamma^*(\nu, at) = t^{\nu} \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(\nu + k + 1)}, \tag{3.11}$$

where $\gamma^*(\nu, at)$ is the incomplete gamma function. The Laplace transform of equation (3.11) follows directly as

$$L\{E_t(\nu,a)\} = \frac{s^{-\nu}}{s - a}, \qquad \mathrm{Re}(\nu) > 1\,. \tag{3.12}$$

Miller and Ross then show that

$$L\left\{ \sum_{j=1}^{q} a^{j-1} E_t(j\nu - 1, a^q) \right\} = \frac{1}{s^{\nu} - a}, \quad q = 1, 2, 3, \dots, \quad \nu = \frac{1}{q} = 1, \frac{1}{2}, \frac{1}{3}, \dots, \tag{3.13}$$

which is a special case of the F-function.

3.2.7 Gorenflo and Mainardi's, and Podlubny's Function

Gorenflo and Mainardi [40] and Podlubny [109] use the function

$$\xi_q(t,a,q,\nu) = \sum_{n=0}^{\infty} \frac{a^n t^{qn+\nu-1}}{\Gamma(nq+\nu)}. \tag{3.14}$$

This convenient function has the Laplace transform

$$L\{\xi_q(t,a,q,\nu)\} = \frac{s^{q-\nu}}{s^q - a}. \tag{3.15}$$

Table 3.1 presents a summary of the defining series and respective Laplace transforms for these important functions discussed here and shows the relation of their Laplace transforms to

Table 3.1 Special fractional calculus functions.

Function	Time expression	Laplace transform	Remarks
Mittag-Leffler (1903) [96, 97]	$E_q[at^q] = \sum_{n=0}^{\infty} \frac{a^n t^{nq}}{\Gamma(nq+1)}$	$\frac{s^q}{s(s^q - a)}$	$(q-1)$ differintegral of eigenfunction
Agarwal (1953) [5, 6]	$E_{q,\beta}[t^q] = \sum_{n=0}^{\infty} \frac{t^{(n+(\beta-1)/q)q}}{\Gamma(nq+\beta)}$	$\frac{s^{q-\beta}}{(s^q-1)}$	
Erdelyi et al. (1954) [34]	$E_{q,\beta}[t^q] = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(nq+\beta)}$	$\sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(qn+\beta)s^{n+1}}$	
Oldham and Spanier (1974) [104], Hartley and Lorenzo (1998) [44], Matignon (1998) [92]	$F_q[a,t] = \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)q-1}}{\Gamma((n+1)q)}$	$\frac{1}{s^q - a}$	eigenfunction
Miller and Ross (1993)[95]	$E_t[\nu,a] = \sum_{n=0}^{\infty} \frac{a^n t^{n+\nu}}{\Gamma(\nu+n+1)}$	$\frac{s^{-\nu}}{(s-a)}$	
Lorenzo and Hartley (1999) [69]	$R_{q,\nu}[a,t] = \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)q-1-\nu}}{\Gamma((n+1)q-\nu)}$	$\frac{s^{\nu}}{(s^q-a)}$	ν^{th} differintegral of eigenfunction
Robotnov (1969) [112]	$\xi_q(a,t) = \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)(1+q)-1}}{\Gamma((n+1)(1+q))}$	$\frac{1}{s^{q+1}-a}$	$\xi_{q-1}(a,t)$ is eigenfunction
Gorenflo and Mainardi (1997) [40], Podlubny (1999) [109, 110]	$\xi(t,a,q,\nu) = \sum_{n=0}^{\infty} \frac{a^n t^{qn+\nu-1}}{\Gamma(nq+\nu)}$	$\frac{s^{q-\nu}}{(s^q-a)}$	$(q-\nu)$ differintegral of eigenfunction

The G- and H-functions may be found in Sections 3.9.1 and 3.9.2.
Source: Adapted from Lorenzo and Hartley 1999 [69].

that of the F-function. Clearly, many of these functions are useful for the solution of various fractional differential equations, but the F-function presented in the previous chapter appears to most properly generalize the exponential function.

3.3 The *R*-Function: A Generalized Function

It is of great interest to develop a generalized function which, when fractionally differintegrated by any order, returns itself (with a new parameter). A function of this type would be useful for the solution of fractional-order differential equations. The following form is proposed [69, 80, 81]. Consider the function

$$R_{q,\nu}[a,t] = \sum_{n=0}^{\infty} \frac{(a)^n t^{(n+1)q-1-\nu}}{\Gamma((n+1)q-\nu)} . \tag{3.16}$$

For $t < 0$, R will be complex except for the cases when the exponent $((n+1)q-1-\nu)$ is integer. Clearly, when $\nu = 0$ in equation (3.16), the R-function becomes the F-function (equation (2.7)); that is, $R_{q,0}[a,t] = F_q[a,t]$. The Laplace transform of such a function will facilitate the solution of fractional-order differential equations.

The Laplace transform of the R-function is determined as follows:

$$L\{R_{q,\nu}[a,t]\} = L\sum_{n=0}^{\infty} \left\{ \frac{(a)^n t^{(n+1)q-1-\nu}}{\Gamma((n+1)q-\nu)} \right\}, \quad t > 0. \tag{3.17}$$