Digital Image Interpolation in MATLAB®

Chi-Wah Kok • Wing-Shan Tam

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Chi-Wah Kok and Wing-Shan Tam
Canaan Semiconductor Limited
Hong Kong
China

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To my love Annie from Ted for putting up with it all once again
To mom Gloria Lee and the memory of dad, Simon Tam, dedicated by Wing-Shan

Contents

About the Authors

Chi-Wah Kok was born in Hong Kong. He was granted with a PhD degree from the University of Wisconsin–Madison. Since 1992, he has been working with various semiconductor companies, research institutions, and universities, which include AT&T Labs Research, Holmdel, SONY U.S. Research Labs, Stanford University, Hong Kong University of Science and Technology, Hong Kong Polytechnic University, City University of Hong Kong, Lattice Semiconductor, etc. In 2006, he founded Canaan Semiconductor Ltd., a fabless IC company with products in mixed-signal IC, high performance audio amplifier, and high-power MOSFETs and IGBTs. Dr. Kok embraces new technologies to meet the fast-changing market requirements. He has extensively applied signal processing techniques to improve the circuit topologies, designs, and fabrication technologies within Canaan. This includes the application of semidefinite programming to circuit design optimization, abstract algebra in switched capacitor circuit topologies, and nonlinear optimization method to optimize high voltage MOSFET layout and fabrication. He was MPEG (MPEG 4) and JPEG (JPEG 2000) standards committee member. He is the founding editor in chief of the journal *Solid State Electronics Letters* since 2017. He is also the author of three books by Prentice Hall and Wiley-IEEE and has written numerous papers on digital signal processing, multimedia signal processing, and CMOS circuits, devices, fabrication process, and reliability.

Wing-Shan Tam was born in Hong Kong. She received her PhD degree in electronic engineering from the City University of Hong Kong. She has been working in different telecommunication and semiconductor companies since 2004 and is currently the engineering manager of Canaan Semiconductor Ltd., where she works on both advanced CMOS sensor design and high-power device structure and process development. Dr. Tam has participated in professional services actively, in which she has been the researcher in different universities since 2007. She has been the invited speaker for different talks and seminars in numerous international conferences and renowned universities. She has served as guest editor in several journals published by IEEE and Elsevier. She is the founding editor of the journal *Solid State Electronics Letters* since 2017. She is the co-author of another Wiley-IEEE technology textbook and research papers with award quality. Her research interests include image interpolation algorithm, color enhancement algorithm and mixed-signal integrated circuit design for data conversion and power management, and device fabrication process and new device structure development.

Preface

The process of deriving real-world application from scientific knowledge is usually a very, very long process. However, with the advancement in complementary metal oxide semiconductor (CMOS) image sensor, and its application in handheld device, image interpolation has rapidly migrated from complex mathematics and academic publications to everyday applications in smartphones, laptops and tablets. Image interpolation has become a red-hot research topic in both academia and industry. One of the highly cited academic works in image interpolation is authored by Dr. Tam, which is an excerpt from her master thesis. Her work is also the origin of this book. However, this book is not intended to be a memoir of the work done by Dr. Tam and her research group; it is intended to be the course materials for senior- and graduate-level courses, training materials for engineers, and also a reference text for readers who are working in the field of digital imaging.

All the image interpolation algorithms discussed in this book will include both theories, where detailed analytic analysis are derived, and implementations through MATLAB into useful tools. Numerous algorithms are reviewed in this book together with detailed discussions on their origins, performances, and limitations. We are particularly happy with the numerical simulations presented for all the algorithms described in this book to clarify the observable but difficult to explain image interpolation artifacts, as the author shares the well-known Chinese saying that a picture is worth a thousand words. Furthermore, many of our unpublished works are included in this book, where new algorithms are developed to overcome various limitations.

This book is authored as much as it is collected. We have tried our best to cite references whenever we are aware of related works on the topics. However, we suspect that some topics may have been independently studied by many individuals, and thus we might have missed their citation. Over 30 years of research works are collected in one place, and we presented each selected topics in a self-contained format. If you are interested in further reading on any of these topics, you should look into the cited references and the Summary sections at the end of each chapter in this book. On a subject such as this one, which has been continuously investigated for over half a century, inevitably a number of valuable research results are not included in this book. It is nonetheless expected that the contents of this book will enable the careful readers to independently explore the more advanced image interpolation/processing technique.

Although much of the materials covered by this book are new to most students, our goal is to provide a working knowledge of various image interpolation algorithms without the need for additional course work besides freshman-level engineering mathematics and a junior-level matrix lab programming. To perform numerical simulation using computer, we must use a language that a computer can understand. This is why we choose to use MATLAB in this book, because MATLAB is not only a computer language. MATLAB, which is built with matrix data structure, is also a language of arithmetic. Once the MATLAB implementation of the algorithms have been learned, it will be fairly straightforward to implement them in other computer languages and VHDL for hardware synthesis. While almost all the MATLAB example codes presented in this book are co-developed from the basic and do not require any toolbox to run with, in Chapter 6, the author just cannot resist to make use of the wavelet toolbox developed by Prof. T.Q. Nguyen of UCSD who is also the PhD adviser of Dr. Kok back in the University of Wisconsin–Madison. The toolbox has made everything easy, which definitely helped the readers to understand the topics and ease their practical implementation tremendously.

The book is divided into nine chapters. Chapter 1 provides an account of basic signal processing and mathematical tools used in subsequent chapters. It also serves the purpose of getting the readers to be familiar with the mathematical notations adopted in the book. Chapter 2 introduces the important concepts of digital imaging and the operations that are useful to image interpolation algorithms. The quality and performance measures between the processed image and the original image are presented in Chapter 3. The human visual system that is first discussed in Chapter 2 will be extended here for the discussion of the *structural similarity* quality index. The nonparametric image interpolation algorithm developed around algebraic functions are presented in Chapter 4. This chapter ends with a discussion on the deficiency of nonadaptive interpolation methods. Chapter 5 discusses the interpolation by *Fourier* and other orthogonal series. We are particularly interested in interpolating image in the *discrete cosine transform* domain, which is motivated by current trends in international image compression and storage standards. The blocking noise resulted from transform domain zero padding interpolation with small block size is alleviated by variations of overlap and add interpolation techniques. An iterative algorithm is presented to improve the least squares solution of the conventional transform coefficients zero padding image interpolation algorithm. Note that iterative image interpolation algorithms are considered to be offline image interpolation algorithms. More about iterative interpolation algorithm that helps to maintain the original pixel values while improving the performance of the non-iterative image interpolation algorithms will be presented in subsequent chapters. Chapter 6 extends the block-based transform domain image interpolation to the wavelet domain. A number of the techniques presented in previous chapters are applicable to the wavelet domain image interpolation too, and various researchers have been given them different names in the literature. The performance of wavelet image interpolation can be improved by exploiting the scale-space relationships obtained by multi-resolution analysis through wavelet transform (a version of the human visual system). The explicit edge detection-based image interpolation methods discussed in Chapter 7 interpolate the image according to the edge-directed image perception property of human visual system. Various edge-directed interpolation methods will be discussed where edges are explicitly obtained by various edge detection methods discussed in Chapter 2, and implicit edge detection methods that the nature of the pixels to be interpolated is determined in the course of the estimation. The chapter concludes with discussions on the pros and cons of edge-directed image interpolation algorithm using explicit edge detection. Another type of edge-detected image interpolation method will be presented in Chapter 8, which is based on the edge geometric duality where a covariance-based implicit edge location and estimation method will interpolate the image along the edge to achieve good visual quality. Digital signal processing theory tells us that there is always room to improve the solutions of any estimation problem. Various improvements to the edge-directed interpolation problem will be discussed in this chapter to improve the preservation of edge geometric duality between the original image and the interpolated image, to reduce the interpolation error propagation by removing inter-processing dependence, and finally to improve the estimation solution through an iterative re-estimation algorithm. The book changes its course from linear statistical-based interpolation technique to fractal interpolation in Chapter 9.

It should be noticed that fractal is usually not considered to be a statistical-based interpolation algorithm. On the other hand, the generation of fractal map is based on similarity between image features, where the similarity is computed or classified via the statistics of the image or image blocks. Finally, an iterative algorithm is presented to improve the fractal image interpolation algorithm with the constraint that the original low-resolution image is the pivot of the interpolated image, i.e. the location and intensity invariance of the low-resolution image in the interpolation image is guaranteed. The advantage of such algorithmic constraint not only allows the preservation of the original low-resolution image pixel values in the interpolated image but also ensures the highest preservation of the structure property of the interpolated image. As a result, fractal image interpolation has been embedded in a number of successful image processing softwares. The book concludes with an appendix that lists all the MATLAB source codes discussed in the book.

Many people have contributed, directly or indirectly, over a long period of time, to the subjects presented in this book. Their contributions are cited appropriately in this book, and also in the *Summary* section at the end of each chapter. The Summary sections also aimed to detail the state-of-the-art development with respect to the topics discussed in each chapter. The exercises presented in the *Exercise* sections are essential parts of this text and often provide a discovery-like experience regarding the associated topics. It is our hope that the exercises will provide general guidelines to assist the readers to design new image interpolation algorithms for their own applications. The readers' effort spent on tackling the exercises will help them to develop a thorough consideration on the design of image processing algorithms for their future career in research and development in the field.

The book is definitely not meant to represent a comprehensive history about the development of image interpolation algorithms. On the other hand, it does provide a not so short review, which chronologically follows the evolution of some of the image interpolation algorithms that have direct implications on commercially available image processing softwares. In particular, we avoided with our best effort to provide a comprehensive survey of every image interpolation algorithms in literature and market. Instead, our selection of topics is on the importance of the algorithms with respect to their applications in image processing softwares in today's or near-future market. Our hope is that the book offers the readers a range of interesting topics and the current state-of-the-art image interpolation methods. In simple terms, image interpolation is an open problem that has no definite winner. Analyzing the design and performance trade-offs and proposing a range of attractive solutions to various image interpolation problems are the basic aims of this book. The book will underline the range of design considerations in an unbiased fashion, and the readers will be able to glean information from it in order to solve their own particular image interpolation problems. Most of all, we hope that the readers will find it an enjoyable and relatively effortless reading, providing them with intellectual stimulation.

Hong Kong, August 2018

Chi-Wah Kok
Wing-Shan Tam

Acknowledgments

Dr. Kok would like to thank his wife Dr. Annie Ko, an extraordinary woman with abiding faith in Christianity. He has acknowledged her in his previous book for her enormous contributions to his life – and still do. He thanks her for her encouragement, and she created enough time for him to write the book while being granted with *tenure* and awarded the *best teaching award* in her university. She has been his inspiration and motivation for continuing to improve his knowledge and move his career forward.

Dr. Kok would also like to thank Dr. Cindy Tam for allowing him to put up with far too many side projects while writing this book. He appreciates her belief in him to provide the leadership and knowledge to make this book a reality. She has provided research insights along the way, working with him to complete each chapter with the appropriate MATLAB sources and analytic details through revision and re-revision, pouncing on obscurities, decrying omissions, correcting misspelling, redrawing figures, and often making her life very much more difficult by his unrelenting insistence that the text and figures could be more literate, accurate, and intelligible. He is very pleased to see his illegible red marginalia have found their way into the text of this book. The last but not the least, he would like to thank her for contributing the beautiful photo of "BeBe" both as the designated simulation image source for all the examples and also the cover image of the book. This lovely cat is Dr. Tam's domestic cat, and the best model for image interpolations, because it contains all the necessary image features that can demonstrate the visual artifacts and performance of various image interpolation algorithms.

Dr. Tam is glad to write her second book with the topic on image interpolation, the same topic as her master thesis. This book project gives her precious opportunities to review the work done in her early years of research and a chance to refresh her knowledge with the ongoing technology development and to explore new research breakthroughs in the field. An interesting research topic always begins with some extraordinary idea. Dr. Tam would like to thank her best mentor and collaborator, Dr. Kok, who introduced and inspired her in this interesting topic.

Dr. Tam would not be able to finish her master thesis, and all other industrial and research projects, without the patience and guidance of Dr. Kok. Though sometimes the collaboration is challenging and bumpy, Dr. Tam believes all the experience and knowledge gained from their collaboration have laid the cornerstone for her future, both personally and professionally.

Dr. Tam would not be able to continue her research career without the love and support from her family. She would like to thank her mother, Gloria, for her love and support, offering her a warm shelter to rest her tired and frustrated body and mind for all these years, and her father, Simon, now in heaven watching and praying for her. Dr. Tam has inherited her father's spirit in striving for perfection, which keeps her moving and be a better researcher.

Her father would be happy to see the publication of her second book and all her research papers. Thanks also go to her sister Candy, brother-in-law Kelvin, niece Clarice, and nephew Kayven who have brought much happiness and laughter to her, the natural booster to keep her energetic year round.

We are in debt to many people, too numerous to mention. Our sincere gratitude is due to the numerous authors listed in the bibliography, as well as to those whose works were not cited due to space limitations. We are grateful for their contributions to the state of the art; without their contributions this book would not have materialized. In particular, we have to express exceptional and sincere gratitude to Dr. Min Li (of University of California, San Diego, and now Qualcomm) for her PhD research work contributed to the development of Markov random field-based edge-directed image interpolation. We are very sorry for the last minute decision to exclude the chapter about Dr. Li's work from the book. But our personal communications have made the book to be much better for the readers.

Despite the assistance, review, and editing by many people, both authors have no doubt that errors still lurk undetected. These are undoubtfully the authors' sin, and it is our hope that the readers of this book will discover them and bring them to our attention, so that they all may be eradicated. Finally, we acknowledge our thanks to God, who blessed this book project, through the words of the psalmist, "Give thanks to the Lord, for He is good; His love endures forever" (Psalms 107:1, NIV).

Chi-Wah Kok
Wing-Shan Tam

Nomenclature

$\lceil x \rceil$:	ceiling operator that returns the smallest integer larger than or equal to x
$\mathbb{Z}$:	the set of integers
$\mathbb{Z}^+$:	the set of positive integers (great than 0)
$\mathbb{R}$:	the set of real numbers
$\mathbb{C}$:	the set of complex numbers
$\mathbf{A}_{M,N}$:	arbitrary matrix of size $M \times N$ constructed by matrix entrance $a(m, n)$ with $\mathbf{A}_{M,N} = [a(m, n)]_{m,n}$ where $0 \leq m \leq M - 1$, and $0 \leq n \leq N - 1$
$\mathbf{I}_N$:	identity matrix of size $N \times N$
ℓ_2:	the space of all squares summable discrete functions/sequences
$\mathcal{L}^2$:	the space of all Lesbesgue squares integrable functions
$\mathcal{R}$:	real part of a number, matrix, or a function
$\mathcal{I}$:	imaginary part of a number, matrix, or a function
$\mathbf{sinc}(x)$:	Sinc function $\left(\frac{\sin(x)}{x}\right)$
δ:	Kronecker delta, or Dirac-delta function, or unit impulse with infinite size
j:	root of -1 and is equal to $\sqrt{-1}$
W_N:	Nth root of unity and equals to $e^{\frac{-j2\pi}{N}}$
$\mathcal{F}$:	discrete Fourier transform operator
$\mathcal{F}^{-1}$:	inverse discrete Fourier transform operator
$\mathbf{W}_N$:	discrete Fourier transform matrix of size $N \times N$; $\mathbf{W}_N = [W_N^{k,\ell}]_{k,\ell}$ with $0 \leq k, \ell \leq N - 1$. The Fourier matrix is of arbitrary size when N is missing
$\mathrm{C}_{M\times N}$:	discrete cosine transform matrix of size $M \times N$; the cosine matrix is of arbitrary size when $M \times N$ is missing
$\otimes$:	convolution operator
Δ_x:	interval in domain x; the interval domain is arbitrary when x is missing
ω:	angular frequency
ω_x:	spatial angular frequency in domain x
ω_{Δ_x}:	sampling angular frequency with sampling interval Δ_x in domain x $(= \frac{2\pi}{\Delta_x})$
$h_c(x, \Delta_x)$:	comb filter impulse response function in domain x with Δ_x being the separation between adjacent impulses in the comb filter; $h_c(x, \Delta_x) = \sum_{k=-\infty}^{\infty} \Delta_x \delta(x - k\Delta_x)$

$H_c(\omega, \Delta_x)$: frequency response of the comb filter $h_c(x, \Delta_x)$, i.e. $H_c(\omega, \Delta_x) = \mathcal{F}(h_c(x, \Delta_x)$

$f(x, \Delta)$: impulse train in analog domain x with Δ being the separation between adjacent indices $= \Delta \sum_{m=-\infty}^{\infty} \delta(x - m\Delta)$, with
$$\delta(k) = \begin{cases} 1/\Delta & \text{for } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$f[\mathrm{k}, N]$: discrete impulse sequence $= N \sum_{m=-\infty}^{\infty} \delta[k - mN]$, with
$$\delta[k] = \begin{cases} 1/N & \text{for } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

A word on notations

1. (Indices) We denote continuous variable (m) and discrete variable $[n]$ induced signals as $x(m)$ and $x[n]$, respectively.
2. (Vector-matrix) The blackboard bold $(\mathbf{A})$ is used to represent matrix-valued signal and function, and $(\mathbf{x})$ is used to represent the vector-valued signal and function. The normal characters (x) are used to represent signal in scalar form.
3. (Rows versus columns) For vector-matrix multiplication written as $\mathbf{xA}$, we may take vector $\mathbf{x}$ as a row vector.

Abbreviations

1D:	one-dimensional
2D:	two-dimensional
ADC:	analogue-to-digital converter
CFA:	color filter array
dB:	decibel
DCT:	discrete cosine transform
DFT:	discrete Fourier transform
DoG:	difference of Gaussian
DTFT:	discrete time Fourier transform
DWT:	discrete wavelet transform
FFT:	fast Fourier transform
FIR:	finite impulse response
FOH:	first-order hold
FRIQ:	full-reference image quality index
HR:	high-resolution
HVS:	human visual system
IDCT:	inverse discrete cosine transform
IDFT:	inverse discrete Fourier transform
IFS:	iterated function system
IIR:	infinite impulse response
JPEG:	joint photographic experts group
LoG:	Laplacian of Gaussian
LPF:	low-pass filter
LR:	low-resolution
MATLAB:	high-level technical computing language by MathWorks Inc.
MEDI:	modified edge-directed interpolation [59]
MOS:	mean opinion score
MRF:	Markov random field
MSE:	mean squares error
MSSIM:	mean structural similarity [63]
NEDI:	new edge-directed interpolation [40]
NRIQ:	no reference image quality index
PDF:	probability density function
PIFS:	partitioned iterated function system
PSNR:	peak signal-to-noise ratio

QMF: quadrature mirror filter
RGB: red, green, and blue color space
RMSE: root mean squares error
RRIQ: reduced reference image quality index
SNR: signal-to-noise ratio
SSIM: structural similarity [63]
YCbCr: luminance, blue chrominance, red chrominance color space
ZOH: zero-order hold

About the Companion Website

The companion website for this book is at:

www.wiley.com/go/ditmatlab

The website includes:

- MATLAB soruce code and figure inventory. Figure inventory includes certain figures from the book in PNG format for the convenience of the readers.
- PowerPoint file for lecturers[1]
- Solution manual[1]

Scan this QR code to visit the companion website.

1 PowerPoint file and Solution manual are available under subscription for professors/lecturers who intent to use this book in their courses.

1

Signal Sampling

We are living in an analog world that makes it fairly easy to overwhelm our computation system to process the vast information carried by the analog signal. To process the analog signal, it will have to be sampled in a way that the sampled signal can be handled by our computation system. The sampled signal should be able to faithfully represent the analog signal. With this, it is natural to ask: "Is it possible to reconstruct the analog signal from the samples?" Such an important question has been answered by the *sampling theorem* [56]. The sampling theorem considers the signal sequence $f[k]$ obtained by uniformly sampling an analog function $f(x)$ with a sampling interval Δ_x, such that

$$f[k] = f(x)\delta(x - k\Delta_x) = f(k\Delta_x), \quad \forall k \in \mathbb{Z}, \tag{1.1}$$

where $\delta(\cdot)$ is a Dirac delta function and $\mathbb{Z}$ is the set of integers. The sampling theorem tells us when and how to reconstruct the analog signal $f(x)$ from the sampled signal sequence $f[k]$. At the same time, the signal sequence $f[k]$ to be handled by the computation system is not only a sampled version of $f(x)$ along x; the amplitude of the signal is also "*sampled*" by a process known as *quantization*. We shall discuss the x domain (also known as the time domain) sampling process in the next section and the quantization process in Section 1.3. Following the presentation of the sampling theorem, the signal reconstruction problem is alleviated by means of interpolation and/or approximation. Other problems that affect the signal reconstruction accuracy, including quantization, will be discussed in Section 1.3. The quantization problem is an important problem because the quantization process is lossy, which poses tremendous difficulties in the recovery of the analog signal. A number of reconstruction methods for *imperfect signal* will be discussed subsequently.

1.1 Sampling and Bandlimited Signal

The readers should have studied Engineering Mathematics in their freshman year; therefore, we shall not discuss the Fourier theorem in detail. Nevertheless, the discrete Fourier transform (DFT) of sampled signal sequence will be introduced in Section 1.2.1 to familiarize the readers with the mathematical notations used in this book. This book also assumes the readers have already acquired the basic knowledge about spectral domain signal processing, and, therefore, this section starts with a formal definition

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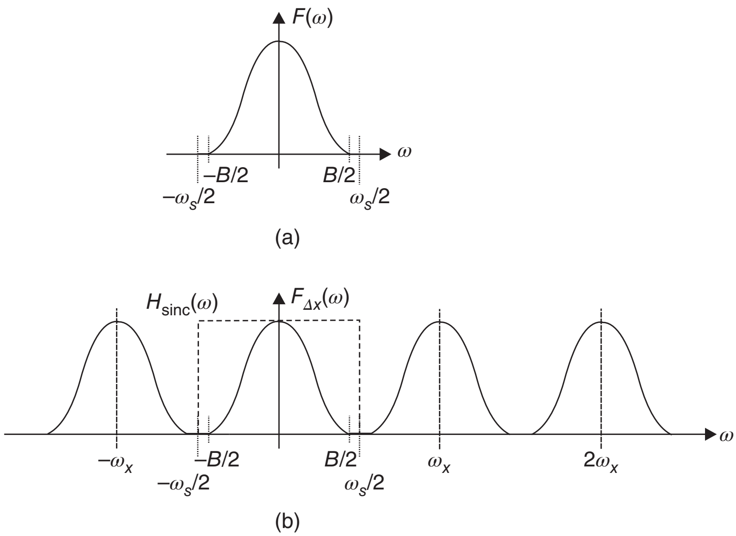

Figure 1.1 (a) Spectrum of a bandlimited signal $f(x)$ with bandwidth B; (b) sampled with rate $\Delta_x = \frac{2\pi}{\omega_s}$ with $B \leq \omega_x$ can be recovered with a sinc filter with bandwidth ω_s.

of bandlimited signal. A signal $f(x)$ is said to be bandlimited with bandwidth B if and only if it does not contain any frequency components outside the spectral range of $-B/2 \leqslant \omega \leqslant B/2$, where ω is the angular frequency. An example of bandlimited signal is shown in Figure 1.1, where the B bandlimited signal $f(x)$ has its Fourier transform $F(\omega)$ equal 0 with $|\omega| > B/2$.

The sampling theorem tells us the sufficient conditions for the reconstructed signal $g(x)$ obtained from

$$g(x) = f[k] \otimes h(x) = \sum_{k=-\infty}^{\infty} f(k\Delta_x)h(x - k\Delta_x), \tag{1.2}$$

where $h(x)$ is the reconstruction function and the sample sequence $f[k] = f(k\Delta_x)$ with $k \in \mathbb{Z}$ and $\Delta_x > 0$ (as discussed in Eq. (1.1)) is lossless, such that $g(x) = f(x)$, with $f(x)$ being bandlimited by B with sampling frequency $\omega_x = \frac{2\pi}{\Delta_x} \geqslant B$. A formal and also one of the oldest definition of the sampling theorem is given by the following

Theorem 1.1 ***Sampling theorem:*** *Consider a sampled signal $f[k]$ with samples taken at a B-bandlimited function $f(x)$ at sampling period Δ_x. The reconstructed signal,*

$$g(x) = \sum_{k=-\infty}^{\infty} f[k]\text{sinc}\left(\frac{\pi(x - k\Delta_x)}{\Delta_x}\right) = \sum_{k=-\infty}^{\infty} f[k]\text{sinc}\left(\frac{\omega_x}{2}(x - k\Delta_x)\right), \tag{1.3}$$

with $\omega_x = \frac{2\pi}{\Delta_x}$ being the sampling frequency and sinc$(a) = \sin(a)/a$ *being a sinc function, is an exact reconstruction of $f(x)$ when $\omega_s \geqslant B$. It should be noted that both ω_x and B are in radian and $\omega_x = B$ is known as the* Nyquist frequency *or* Nyquist rate.

To understand Eq. (1.3) of the sampling theorem, we can make use of the *discrete time Fourier transform* (DTFT) to examine the reconstructed signal $g(x)$.

$$
\begin{aligned}
G(\omega) &= \sum_k f[k] e^{-2j\omega k} \times \mathcal{F}\left(\text{sinc}\left(\frac{\omega_s}{2}(x - k\Delta_x)\right)\right), \\
&= \frac{H_{\text{sinc},\Delta x}(\omega)}{\Delta_x} \sum_{k=-\infty}^{\infty} F(\omega - k\omega_s),
\end{aligned}
\tag{1.4}
$$

where $H_{\text{sinc},\Delta x}(\omega)$ is the DTFT of sinc$(\cdot)$ that is a box function of height Δ_x in the spectral domain from $[-\omega_x/2, \omega_x/2]$, and zero everywhere else, and $\mathcal{F}$ is the Fourier transform operator. It is vivid from Eq. (1.4) that the spectrum of the sampled signal is a series of duplications of the original analog signal spectrum of $F(\omega)$ located at spectral locations $k\omega_x$ with $k \in \mathbb{Z}$ as shown in Figure 1.1b. Therefore, when the bandwidth of $f[k]$ is smaller than ω_s, the contributions of the duplicated spectral components $F(\omega - k\omega_x)$ at different k will not overlap (also known as *aliasing*-free). Otherwise, as shown in Figure 1.2b, when the signal spectrum of $f(x)$ has a bandwidth wider than ω_s as shown in Figure 1.2a, the spectral contributions of the sampled signal spectra at different k will overlap. As a result, the reconstructed signal obtained by filtering with $H_{\text{sinc}}(\omega)$ will be a distorted signal $\hat{F}(\omega)$ (not the same as $F(\omega)$). Such kind of distortion is known as the *aliasing* distortion. This helps to illustrate the *Nyquist frequency* ($\omega_x = B$) as a *sufficient* condition to perfectly reconstruct the analog function $f(x)$ from its sample sequence $f[k]$ at a sampling rate $\Delta_x = \frac{2\pi}{B}$.

The sampling theorem (Theorem 1.1) stated that a bandlimited signal $f(x)$ can be sampled at a rate equal to or higher than the Nyquist rate and then reconstructed from its sample sequence without loss by passing the sample sequence $f[k]$ through a noncausal

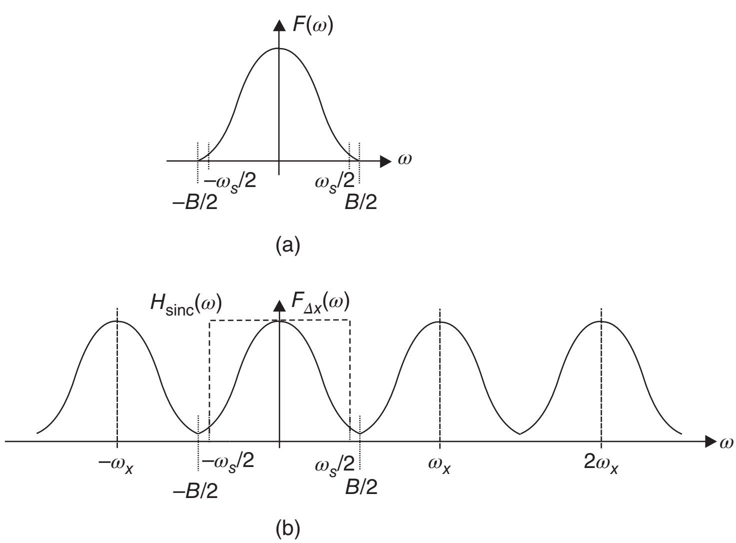

Figure 1.2 (a) Spectrum of a bandlimited signal $f(x)$ with bandwidth B; (b) sampled with rate $\Delta_x = \frac{2\pi}{\omega_x}$ with $B > \omega_s$ will suffer from spectrum overlap error, also known as aliasing noise, which makes it difficult to be recovered by a sinc filter with bandwidth ω_s.

filter with the impulse response equal to a sinc function. In reality, Eq. (1.3) is of theoretical interest only because the equation is numerically ill conditioned (the range of $f[k]$ includes both causal and noncausal components). However, it is intuitively clear that the analog function could be closely reconstructed from the sampled sequence using practical reconstruction function (provided that the signal does not change too rapidly and hence bandlimited), and the sampling frequency is relatively high when compared with that of the signal (in that case the sampling frequency is higher than that of the signal bandwidth).

1.2 Unitary Transform

The DTFT can be applied to signal sequence with infinite length to represent the signal in frequency domain. For finite length signals, the concept of spectral (Fourier) domain representation is generalized to transform domain representation with unitary transforms. Let us consider a length N finite duration sequence

$$\mathbf{f} = [\; f[0] \quad f[1] \quad \cdots \quad f[N-1] \;]^T, \tag{1.5}$$

where $\mathbf{f}$ can be a vector in either $\mathbb{R}^{N\times 1}$ or $\mathbb{C}^{N\times 1}$. Similarly, consider an invertible matrix $\mathbf{U}$ that is in either $\mathbb{R}^{N\times N}$ or $\mathbb{C}^{N\times N}$, which is known as the *basis matrix* or *kernel matrix*. A linear transform and the associated inverse transform of $\mathbf{f}$ by $\mathbf{U}$ are defined to be

$$\mathbf{F} = \mathbf{U} \cdot \mathbf{f}, \tag{1.6}$$

$$\mathbf{f} = \mathbf{U}^{-1} \cdot \mathbf{F}, \tag{1.7}$$

with $\mathbf{F} \in \mathbb{R}^{N\times 1}$ or $\mathbb{C}^{N\times 1}$ being the transform coefficient vector of $\mathbf{f}$. In other words, the signal vector $\mathbf{f}$ is represented by $\mathbf{F}$ in a domain described by the basis matrix $\mathbf{U}$. The transform defined by the set of Eqs. (1.6) and (1.7) is said to be a *unitary transform* pair when $\mathbf{U} \in \mathbb{R}^{N\times N}$ and

$$\mathbf{U}^{-1} = \mathbf{U}^T \quad \rightleftharpoons \quad \mathbf{U}\mathbf{U}^T = \mathbf{I}. \tag{1.8}$$

In the case of $\mathbf{U} \in \mathbb{C}^{N\times N}$, the basis matrix $\mathbf{U}$ is a unitary transform when it satisfies

$$\mathbf{U}^{-1} = \mathbf{U}^{\dagger} \quad \rightleftharpoons \quad \mathbf{U}\mathbf{U}^{\dagger} = \mathbf{I}, \tag{1.9}$$

where the superscript † denotes the complex conjugate transpose operation and the resulting matrix is known as the Hermitian matrix. The following will present an example of the complex unitary transform, the DFT.

1.2.1 Discrete Fourier Transform

The DFT is derived from the DTFT by assuming $f[n]$ is periodic, which implicitly defines a mapping from $\mathbb{C}^N$ to $\mathbb{C}^N$ between $f[n]$ and $F[k]$ as

$$f[n] \xrightarrow{\mathcal{F}} F[k] = \sum_{n=0}^{N-1} e^{\frac{-j2\pi kn}{N}} f[n], \qquad \forall k = 0, \ldots, N-1, \tag{1.10}$$

with $j = \sqrt{-1}$. The inverse discrete Fourier transform (IDFT) of the sequence $F[k]$ is given by

$$F[k] \xrightarrow{\mathcal{F}^{-1}} f[n] = \sum_{k=0}^{N-1} e^{\frac{j2\pi kn}{N}} F[k], \qquad \forall n = 0, \ldots, N-1. \tag{1.11}$$

In the form of unitary transform, the transform kernel of the DFT is given by the $N \times N$ DFT (Fourier) matrix $\mathbf{W}_N$, where the subscript N indicates the kernel size.

$$\mathbf{W}_N = \left[e^{\frac{-j2\pi kn}{N}}\right]_{0 \leqslant k,n < N}. \tag{1.12}$$

If we denote $W_N^k = e^{\frac{-j2\pi k}{N}}$, the Nth root of unity, then the Fourier matrix can be expressed as a Vandermonde matrix in W. As an example, the 3×3 Fourier matrix is given by

$$\mathbf{W}_3 = \begin{bmatrix} W_3^0 & W_3^0 & W_3^0 \\ W_3^0 & W_3^1 & W_3^2 \\ W_3^0 & W_3^2 & W_3^4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^1 & W_3^2 \\ 1 & W_3^2 & W_3^1 \end{bmatrix}. \tag{1.13}$$

Therefore, one can view the computation of $F[k]$ from $f[n]$ as a matrix vector product of

$$\mathcal{F}(f) = \mathbf{W}_N \mathbf{f} = \mathbf{F}. \tag{1.14}$$

The IDFT can be easily obtained by multiplication of $\mathbf{W}_N^{-1}$ to Eq. (1.14). Since the matrix $\mathbf{W}_N$ is an orthogonal matrix, therefore, $\mathbf{W}_N^{-1} = \mathbf{W}_N^{\dagger}$ as given by Eq. (1.9). In image interpolation, N is usually very large, and an efficient method to compute the DFT is required. In MATLAB, an efficient computation of the DFT is available by means of the *fast Fourier transform* (FFT) command `fft`.

It is vivid that the kernel of the Fourier matrix $\mathbf{W}_N$ is a function of $j = \sqrt{-1}$, which makes this kernel complex. As a result the power of the signal in frequency domain (Fourier domain) given by the power spectrum $P[u]$ is obtained as the sum of squares of the real and imaginary part of the DFT

$$P[u] = |F[u]|^2 = (R^2(\mathcal{F}[u]) + \mathcal{I}^2(F[u])), \tag{1.15}$$

which measures the power of individual sinusoidal components contained in the signal.

1.3 Quantization

The time domain (x domain) sampled signal has a continuum of values, as can be observed from the solid line in Figure 1.3. However, the sampled analog signal must be representable in digital form for storage or transmission. Since the number of bits (binary digits) for representing each signal sample is limited, the analog samples must be *quantized* to a finite number of levels before it can be coded in the form of binary numbers. As a result, the quantization process compresses the continuum of analog values to a finite number of discrete values. It is vivid that the quantization process will introduce distortion into the quantized signal when compared with the original

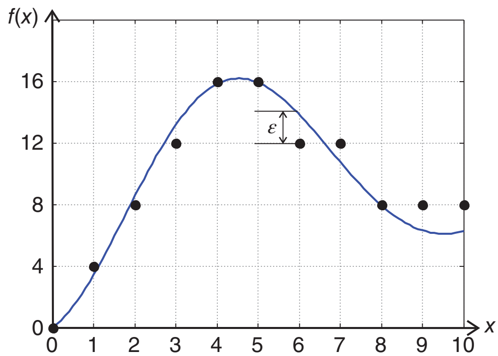

Figure 1.3 Sampling and quantization of a one-dimensional continuous signal.

analog signal. This kind of distortion is known as *quantization noise*. In simple terms, a scalar quantizer for real signal is a mapping from $\mathbb{R}$ to a finite set of discrete values on the real number line. The quantized value is chosen to be the closest approximation to the amplitude of the input signal within the finite set. Formally, a scalar quantizer $Q(\cdot)$ defines the mapping of the input *decision intervals* $(d_k : k = 0, 1, \dots, L)$ to output or *reconstruction levels* $(r_k : k = 0, \dots, L-1)$. The quantized signal is given by

$$f_Q(x) = Q(f(x)) = r_k \quad \text{with } d_k \leqslant f(x) < d_{k+1} \qquad \text{for } k = 0, \dots, L-1. \tag{1.16}$$

Without loss of generality, the decision levels are chosen such that

$$d_0 < d_1 < \cdots < d_L. \tag{1.17}$$

Furthermore, d_0 and d_L are selected to be the minimum and maximum possible input signals. It should be noted that $d_0 = -\infty$ and $d_L = \infty$ are valid and are being chosen for most of the quantizers applied in practice. As a result, the number of bits required to address any one of the output levels is $\lceil \log_2 L \rceil$ bits with $\lceil \cdot \rceil$ being the ceiling operator that returns the smallest integer equal to or larger than $\log_2 L$. There exist a lot of quantizers (a particular choice of d_k and r_k) that are optimal for different applications. Without loss of generality and limitation in our discussions, we shall focus on uniform quantizer in this book, where the difference between decision levels of the quantizer equals to a constant step size Δ_Q.

$$\Delta_Q = d_k - d_{k-1}, \qquad \forall k \in \mathbb{Z}^+. \tag{1.18}$$

An example of an analog signal being sampled and quantized is shown in Figure 1.3, where the analog signal plotted in the figure is a damped cosine function.

$$f(x) = 10e^{-x/10} \cos\left(\frac{x}{10}\omega - \theta\right) - \gamma, \tag{1.19}$$

with $\omega = 2\pi$, $\theta = 3$, and $\gamma = -9.9$. The sampled and quantized signal samples are plotted in Figure 1.3 by black dots together with the analog signal $f(x)$ by solid line. It can be observed that the sampled signal can faithfully represent the analog signal with quantization error $\epsilon(x)$ (also known as quantization noise as marked in Figure 1.3 for the case of $x = 6$). The quantization error is highly correlated with the number of bits applied to quantize the signal. Shown in Figure 1.4 is the same signal being sampled with the same