DANIEL J. INMAN

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WITH CONTROL

SECOND EDITION

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Vibration with Control

Vibration with Control

Second Edition

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This book is dedicated to our grand children

Griffin Amherst Pitre
Benjamin Lafe Scamacca
Lauren Cathwren Scamacca
Conor James Scamacca
Jacob Carlin Scamacca

And our children

Daniel John Inman, II
Angela Wynne Pitre
Jennifer Wren Scamacca

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Preface

Advance level vibration topics are presented here, including lumped mass and distributed mass systems in the context of the appropriate mathematics along with topics from control that are useful in vibration analysis, testing and design. This text is intended for use in a second course in vibration, or a combined course in vibration and control. It is also intended as a reference for the field of structural control and could be used as a text in structural control. The control topics are introduced at the beginners' level with no prerequisite knowledge in controls needed to read the book.

The text is an attempt to place vibration and control on a firm mathematical basis and connect the disciplines of vibration, linear algebra, matrix computations, control and applied functional analysis. Each chapter ends with notes on further references and suggests where more detailed accounts can be found. In this way I hope to capture a "bigger picture" approach without producing an overly large book. The first chapter presents a quick introduction using single degree of freedom systems (second-order ordinary differential equations) to the following chapters, which extend these concepts to multiple degree of freedom systems (matrix theory and systems of ordinary differential equations) and distributed parameter systems (partial differential equations and boundary value problems). Numerical simulations and matrix computations are also presented through the use of MATLAB™.

New In This Edition – The book chapters have been reorganized (there are now 12 instead of 13 chapters) with the former chapter on design removed and combined with the former chapter on control to form a new chapter titled *Vibration Suppression*. Some older, no longer used material, has been deleted in an attempt to keep the book limited in size as new material has been added.

The new material consists of adding several modeling sections to the text, including corresponding problems and examples. Many figures have been redrawn throughout to add clarity with more descriptive captions. In addition, a number of new figures have been added. New problems and examples have been added and some old ones removed. In total, seven new sections have been added to introduce modeling, coupled systems, the use of piezoelectric materials, metastructures, and validation and verification.

Instructor Support – Power Point slides are available for presentation of the material, along with a complete solutions manual. These materials are available from the publisher for those who have adapted the book. The author is pleased to answer questions via the email listed below.

Student Support – The best place to get help is your instructor and others in your peer group through discussion of the material. There are also many excellent texts as referenced throughout the book and of course Internet searches can provide lots of help. In addition, feel free to email the author at the address below (but don't ask me to do your homework!).

Acknowledgements – I would like to thank two of my current PhD students, Katie Reichl and Brittany Essink, for checking some of the homework and providing some plots. I would like to thank all of my former and current PhD students for 36 years of wonderful research and discussions. Thanks are owed to the instructors and students of the previous edition who have sent suggestions and comments. Last, thanks to my lovely wife Catherine Ann Little for putting up with me.

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About the Companion Website

Vibration with Control, Second Edition is accompanied by a companion website:

www.wiley.com/go/inmanvibrationcontrol2e

The website includes:

- Powerpoint slides
- Solutions manual

1

Single Degree of Freedom Systems

1.1 Introduction

In this chapter, the vibration of a single degree of freedom system (SDOF) will be analyzed and reviewed. Analysis, measurement, design and control of SDOF systems are discussed. The concepts developed in this chapter constitute a review of introductory vibrations and serve as an introduction for extending these concepts to more complex systems in later chapters. In addition, basic ideas relating to measurement and control of vibrations are introduced that will later be extended to multiple degree of freedom systems and distributed parameter systems. This chapter is intended to be a review of vibration basics and an introduction to a more formal and general analysis for more complicated models in the following chapters.

Vibration technology has grown and taken on a more interdisciplinary nature. This has been caused by more demanding performance criteria and design specifications of all types of machines and structures. Hence, in addition to the standard material usually found in introductory chapters of vibration and structural dynamics texts, several topics from control theory are presented. This material is included not to train the reader in control methods (the interested student should study control and system theory texts), but rather to point out some useful connections between vibration and control as related disciplines. In addition, structural control has become an important discipline requiring the coalescence of vibration and control topics. A brief introduction to nonlinear SDOF systems and numerical simulation is also presented.

1.2 Spring-Mass System

Simple harmonic motion, or oscillation, is exhibited by structures that have elastic restoring forces. Such systems can be modeled, in some situations, by a spring-mass schematic (Figure 1.1). This constitutes the most basic vibration model of a structure and can be used successfully to describe a surprising number of devices, machines and structures. The methods presented here for solving such a simple mathematical model may seem to be more sophisticated than the problem

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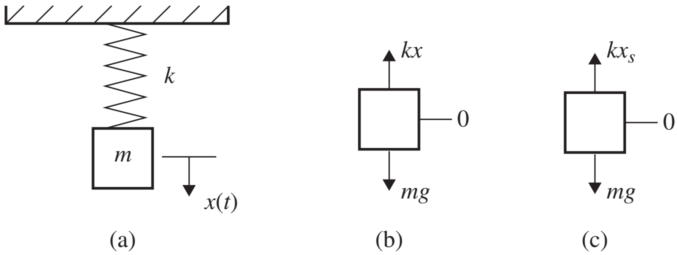


Figure 1.1 (a) A spring-mass schematic, (b) a free body diagram, and (c) a free body diagram of the static spring mass system.

requires. However, the purpose of this analysis is to lay the groundwork for solving more complex systems discussed in the following chapters.

If $x = x(t)$ denotes the displacement (in meters) of the mass m (in kg) from its equilibrium position as a function of time, t (in sec), the equation of motion for this system becomes (upon summing the forces in Figure 1.1b)

$$m\ddot{x} + k(x + x_s) - mg = 0$$

where k is the stiffness of the spring (N/m), x_s is the static deflection (m) of the spring under gravity load, g is the acceleration due to gravity (m/s^2) and the over dots denote differentiation with respect to time. A discussion of dimensions appears in Appendix A and it is assumed here that the reader understands the importance of using consistent units. From summing forces in the free body diagram for the static deflection of the spring (Figure 1.1c), $mg = kx_s$ and the above equation of motion becomes

$$m\ddot{x}(t) + kx(t) = 0 \tag{1.1}$$

This last expression is the equation of motion of an SDOF system and is a linear, second-order, ordinary differential equation with constant coefficients.

Figure 1.2 indicates a simple experiment for determining the spring stiffness by adding known amounts of mass to a spring and measuring the resulting static deflection, x_s. The results of this static experiment can be plotted as force (mass times acceleration) versus x_s, the slope yielding the value of k for the linear portion of the plot. This is illustrated in Figure 1.3.

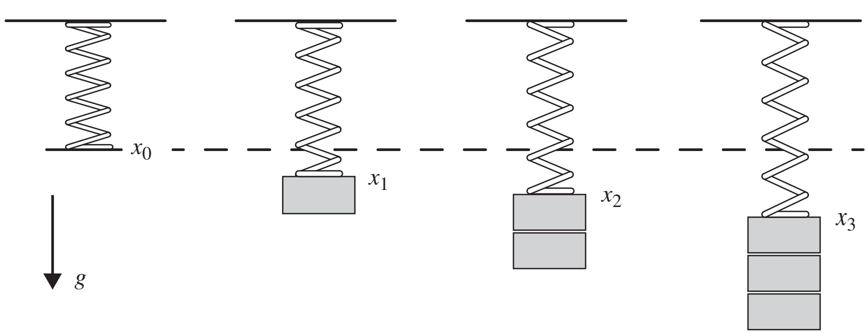


Figure 1.2 Measurement of spring constant using static deflection caused by added mass.

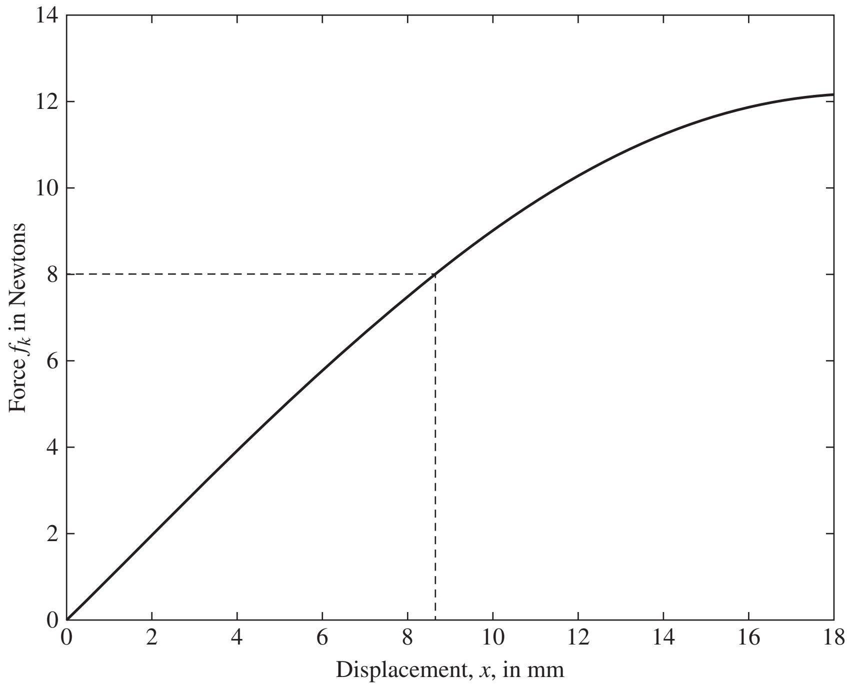


Figure 1.3 Determination of the spring constant. The dashed box indicates the linear range of the spring.

Once m and k are determined from static experiments, Equation (1.1) can be solved to yield the time history of the position of the mass m, given the initial position and velocity of the mass. The form of the solution of Equation (1.1) is found from substitution of an assumed periodic motion (from experience watching vibrating systems) of the form

$$x(t) = A \sin(\omega_n t + \phi) \tag{1.2}$$

where $\omega_n = \sqrt{k/m}$ is called the *natural frequency* in radians per second (rad/s). Here A, the *amplitude*, and ϕ, the *phase shift*, are constants of integration determined by the initial conditions.

The existence of a *unique* solution for Equation (1.1) with two specific initial conditions is well known and is given in Boyce and DiPrima (2012). Hence, if a solution of the form of Equation (1.2) is guessed and it works, then it is *the* solution. Fortunately, in this case, the mathematics, physics and observation all agree.

To proceed, if x_0 is the specified initial displacement from equilibrium of mass m, and v_0 is its specified initial velocity, simple substitution allows the constants of integration A and ϕ to be evaluated. The unique solution is

$$x(t) = \sqrt{\frac{\omega_n^2 x_0^2 + v_0^2}{\omega_n^2}} \sin\left[\omega_n t + \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right)\right] \tag{1.3}$$

Alternately, $x(t)$ can be written as

$$x(t) = \frac{v_0}{\omega_n} \sin \omega_n t + x_0 \cos \omega_n t \tag{1.4}$$

by using a simple trigonometric identity or by direct substitution of the initial conditions (Example 1.2.1).

A purely mathematical approach to the solution of Equation (1.1) is to assume a solution of the form $x(t) = Ae^{\lambda t}$ and solve for λ, i.e.

$$m\lambda^2 e^{\lambda t} + ke^{\lambda t} = 0$$

This implies that (because $e^{\lambda t} \neq 0$ and $A \neq 0$)

$$\lambda^2 + \left(\frac{k}{m}\right) = 0$$

or that

$$\lambda = \pm j \left(\frac{k}{m}\right)^{1/2} = \pm \omega_n j$$

where $j = (-1)^{1/2}$. Then the general solution becomes

$$x(t) = A_1 e^{-\omega_n jt} + A_2 e^{\omega_n jt} \tag{1.5}$$

where A_1 and A_2 are arbitrary complex conjugate constants of integration to be determined by the initial conditions. Use of Euler's formulas then yields Equations (1.2) and (1.4) (Inman, 2014). For more complicated systems, the exponential approach is often more appropriate than first guessing the form (sinusoid) of the solution from watching the motion.

Another mathematical comment is in order. Equation (1.1) and its solution are valid only as long as the spring is linear. If the spring is stretched too far or too much force is applied to it, the curve in Figure 1.3 will no longer be linear. Then Equation (1.1) will be nonlinear (Section 1.10). For now, it suffices to point out that initial conditions and springs should always be checked to make sure that they fall into the linear region, if linear analysis methods are going to be used.

Example 1.2.1

Assume a solution of Equation (1.1) of the form

$$x(t) = A_1 \sin \omega_n t + A_2 \cos \omega_n t$$

and calculate the values of the constants of integration A_1 and A_2 given arbitrary initial conditions x_0 and v_0, thus verifying Equation (1.4).

Solution: The displacement at time $t = 0$ is

$$x(0) = x_0 = A_1 \sin(0) + A_2 \cos(0)$$

or $A_2 = x_0$. The velocity at time $t = 0$ is

$$\dot{x}(0) = v_0 = \omega_n A_1 \cos(0) - \omega_n x_0 \sin(0)$$

Solving this last expression for A_1 yields $A_1 = v_0/x_0$, so that Equation (1.4) results in

$$x(t) = \frac{v_0}{x_0} \sin \omega_n t + x_0 \cos \omega_n t$$

Example 1.2.2
Compute and plot the time response of a linear spring-mass system to initial conditions of $x_0 = 0.5$ mm and $v_0 = 2\sqrt{2}$ mm/s, if the mass is 100 kg and the stiffness is 400 N/m.

Solution: The frequency is

$$\omega_n = \sqrt{k/m} = \sqrt{400/100} = 2 \text{ rad/s}$$

Next compute the amplitude from Equation (1.3):

$$A = \sqrt{\frac{\omega_n^2 x_0^2 + v_0^2}{\omega_n^2}} = \sqrt{\frac{2^2(0.5)^2 + (2\sqrt{2})^2}{2^2}} = 1.5 \text{ mm}$$

From Equation (1.3) the phase is

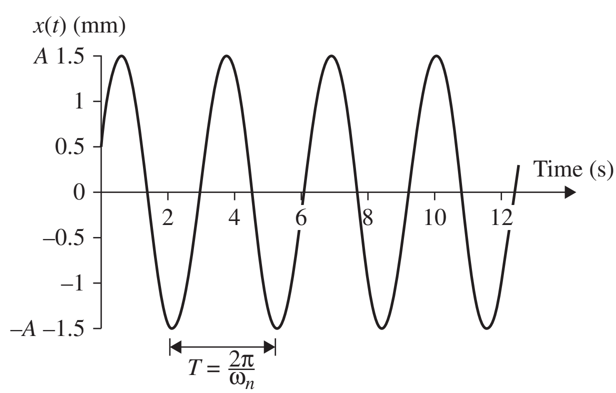
$$\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) = \tan^{-1}\left(\frac{2(0.5)}{2\sqrt{2}}\right) \approx 10 \text{ rad}$$

Thus the response has the form

$$x(t) = 1.5 \sin(2t + 10)$$

and this is plotted in Figure 1.4.

Figure 1.4 The response of a simple spring-mass system to an initial displacement of $x_0 =$ 0.5 mm and an initial velocity of $v_0 = 2\sqrt{2}$ mm/s. The period, defined as the time it takes to complete one cycle off oscillation, $T = 2\pi/\omega_n$, becomes $T = 2\pi/2 = \pi$s.



1.3 Spring-Mass-Damper System

Most systems will not oscillate indefinitely when disturbed, as indicated by the solution in Equation (1.3). Typically, the periodic motion dies down after some time. The easiest way to treat this mathematically is to introduce a velocity term, $c\dot{x}$, into Equation (1.1) and examine the equation

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{1.6}$$

This also happens physically with the addition of a *dashpot* or *damper* to dissipate energy, as illustrated in Figure 1.5.

Equation (1.6) agrees with summing forces in Figure 1.5 if the dashpot exerts a dissipative force proportional to velocity on the mass m. Unfortunately, the constant of proportionality, c, cannot be measured by static methods as m and k are. In addition, many structures dissipate energy in forms not proportional to velocity. The constant of proportionality c is given in Newton-second per meter (Ns/m) or kilograms per second (kg/s) in terms of fundamental units.

Again, the unique solution of Equation (1.6) can be found for specified initial conditions by assuming that $x(t)$ is of the form

$$x(t) = Ae^{\lambda t}$$

and substituting this into Equation (1.6) to yield

$$A\left(\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m}\right)e^{\lambda t} = 0 \tag{1.7}$$

Since a trivial solution is not desired, $A \neq 0$, and since $e^{\lambda t}$ is never zero, Equation (1.7) yields

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0 \tag{1.8}$$

Equation (1.8) is called the *characteristic equation* of Equation (1.6). Using simple algebra, the two solutions for λ are

$$\lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2}\sqrt{\frac{c^2}{m^2} - 4\frac{k}{m}} \tag{1.9}$$

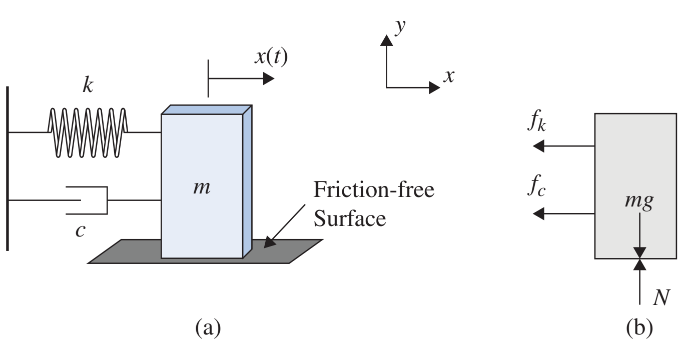


Figure 1.5 (a) Schematic of spring-mass-damper system. (b) A free-body diagram of the system in part (a).

The quantity under the radical is called the *discriminant* and together with the sign of m, c and k determines whether or not the roots are complex or real. Physically, m, c and k are all positive in this case, so the value of the discriminant determines the nature of the roots of Equation (1.8).

It is convenient to define the dimensionless *damping ratio*, ζ, as

$$\zeta = \frac{c}{2\sqrt{km}}$$

In addition, let the *damped natural frequency*, ω_d, be defined by (for $0 < \zeta < 1$)

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{1.10}$$

Then Equation (1.6) becomes

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \tag{1.11}$$

and Equation (1.9) becomes

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1} = -\zeta\omega_n \pm \omega_d j, \quad 0 < \zeta < 1 \tag{1.12}$$

Clearly the value of the damping ratio, ζ, determines the nature of the solution of Equation (1.6). There are three cases of interest. The derivation of each case is left as an exercise and can be found in almost any introductory text on vibrations (Inman, 2014; Meirovitch, 1986).

Underdamping occurs if the system's parameters are such that

$$0 < \zeta < 1$$

so that the discriminant in Equation (1.12) is negative and the roots form a complex conjugate pair of values. The solution of Equation (1.11) then becomes

$$x(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) \tag{1.13}$$

or

$$x(t) = Ce^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

where A, B, C and ϕ are constants determined by the specified initial velocity, v_0 and position, x_0

$$\begin{aligned} A &= x_0 & C &= \frac{\sqrt{(v_0+\zeta\omega_n x_0)^2 + (x_0\omega_d)^2}}{\omega_d} \\ B &= \frac{(v_0+\zeta\omega_n x_0)}{\omega_d} & \phi &= \tan^{-1}\left[\frac{x_0\omega_d}{(v_0+\zeta\omega_n x_0)}\right] \end{aligned} \tag{1.14}$$

The underdamped response has the form given in Figure 1.6 and consists of a decaying oscillation of frequency ω_d.

Overdamping occurs if the system's parameters are such that

$$\zeta > 1$$

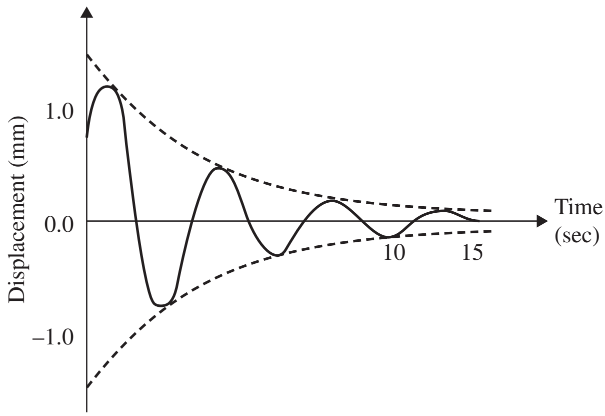


Figure 1.6 Response of an underdamped system illustrating oscillation with exponential decay.

so that the discriminant in Equation (1.12) is positive and the roots are a pair of negative real numbers. The solution of Equation (1.11) then becomes

$$x(t) = Ae^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + Be^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \tag{1.15}$$

where A and B are again constants determined by v_0 and x_0. They are

$$A = \frac{v_0 + \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}} \text{ and } B = -\frac{v_0 + \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}} \tag{1.16}$$

The overdamped response has the form given in Figure 1.7. An overdamped system does not oscillate, but rather returns to its rest position exponentially.

Critical Damping occurs if the system's parameters are such that $\zeta = 1$, so that the discriminant in Equation (1.12) is zero and the roots are a pair of negative real repeated numbers. The solution of Equation (1.11) then becomes

$$x(t) = e^{-\omega_n t}[(v_0 + \omega_n x_0)t + x_0] \tag{1.17}$$

The critically damped response is plotted in Figure 1.8 for values of the initial velocity v_0 of different signs and $x_0 = 0.25$ mm.

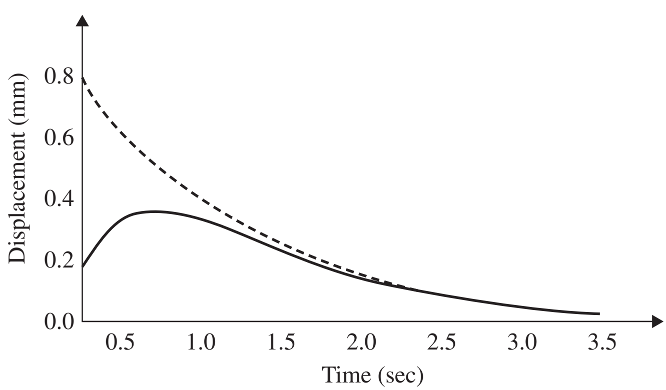


Figure 1.7 Response of an overdamped system illustrating exponential decay without oscillation.

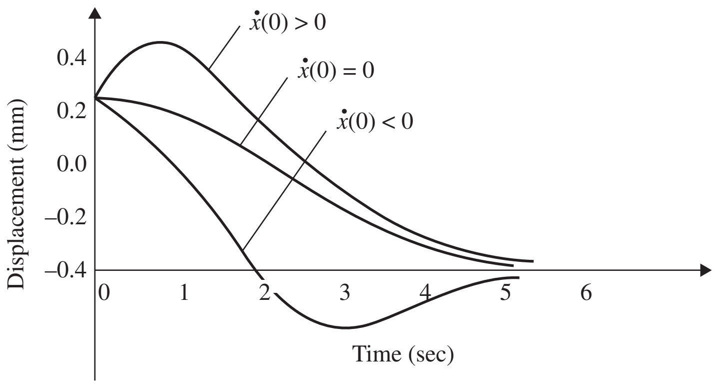


Figure 1.8 Response of critically damped system to an initial displacement and three different initial velocities indicating no oscillation.

It should be noted that critically damped systems can be thought of in several ways. First, they represent systems with the minimum value of damping rate that yields a non-oscillating system (Exercise 1.5). Critical damping can also be thought of as the case that separates non-oscillation from oscillation.

Example 1.3.1
Derive the constants A and B of integration for the overdamped case of Equation (1.15).

Solution: Substitution of $x(0) = x_0$ into Equation (1.15) yields

$$x(0) = Ae^0 + Be^0 \text{ or } x_0 = A + B \tag{1.18}$$

Differentiating Equation (1.15) and setting $t = 0$ in the result yields

$$\dot{x}(0) = A\lambda_1 e^0 + B\lambda_2 e^0 \text{ or } v_0 = \lambda_1 A + \lambda_1 B \tag{1.19}$$

where λ_1 and λ_2 are defined in Equation (1.12). These two initial conditions result in two independent equations in two unknowns, A and B, which can be solved in many ways. Writing Equations (1.17) and (1.18) as a single matrix equation yields

$$\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \text{ or } \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}^{-1} \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$$

Solving by computing matrix inverse (see Appendix B for details on computing a matrix inverse) yields

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$$

Expanding, substituting in the values for λ_1 and λ_2, recalling that they are real numbers (i.e. $\zeta^2 > 1$) and writing as two separate equations results in

$$A = \frac{-v_0 + (-\zeta - \sqrt{\zeta^2 - 1})\omega_n}{-2\omega_n\sqrt{\zeta^2 - 1}} \text{ and } B = \frac{v_0 + (\zeta - \sqrt{\zeta^2 - 1})\omega_n}{-2\omega_n\sqrt{\zeta^2 - 1}}$$

Factoring out the minus sign in the denominator results in Equations (1.16).

1.4 Forced Response

The preceding analysis considers the vibration of a device or structure due to some initial disturbance (nonzero v_0 and x_0). In this section, the vibration of a spring-mass-damper system subjected to an external force is considered. In particular, the response to harmonic excitations, impulses and step forcing functions is examined.

In many environments, rotating machinery, motors, etc., cause periodic motions of structures to induce vibrations into other mechanical devices and structures nearby. It is common to approximate the driving forces, $F(t)$, as periodic of the form

$$F(t) = F_0 \sin \omega t \tag{1.20}$$

where F_0 represents the amplitude of the applied force and ω denotes the frequency of the applied force, or the driving frequency, in rad/s. On summing forces, the equation for the forced vibration of the system in Figure 1.9 becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t \tag{1.21}$$

Recall from the discipline of differential equations (Boyce and DiPrima, 2012), that the solution of Equation (1.21) consists of the sum of the homogeneous solution Equation (1.5) and a particular solution. These are usually referred to as the *transient response* and the *steady-state response*, respectively. Physically, there is motivation to assume that the steady state response will follow the forcing function. Hence, it is tempting to assume that the particular solution has the form

$$x_p(t) = X \sin(\omega t - \theta) \tag{1.22}$$

where X is the steady-state amplitude and θ is the phase shift at steady state. Mathematically, the method is referred to as the method of undetermined coefficients. Substitution of Equation (1.22) into Equation (1.21) yields

$$X = \frac{F_0/k}{\sqrt{(1 - m\omega^2/k)^2 + (c\omega/k)^2}}$$

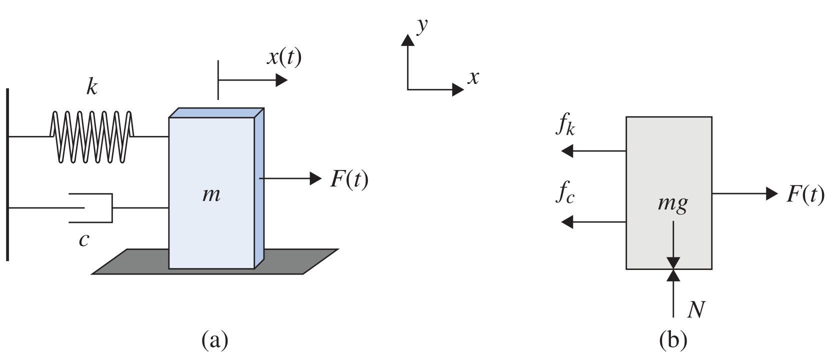


Figure 1.9 **(a)** The schematic of the forced spring-mass-damper system, assuming no friction on the surface. **(b)** The free-body diagram of the system of part (a).

or

$$\frac{Xk}{F_0} = \frac{1}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2 + [2\zeta(\omega/\omega_n)]^2}} \tag{1.23}$$

and

$$\tan\theta = \frac{(c\omega/k)}{1 - m\omega^2/k} = \frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2} \tag{1.24}$$

where $\omega_n = \sqrt{k/m}$ as before. Since the system is linear, the sum of two solutions is a solution, and the total time response for the system in Figure 1.9 for the case $0 < \zeta < 1$ becomes

$$x(t) = e^{-\zeta\omega_n t}(A\sin\omega_d t + B\cos\omega_d t) + X\sin(\omega t - \theta) \tag{1.25}$$

Here A and B are constants of integration determined by the initial conditions and the forcing function (and in general will be different than the values of A and B determined for the free response). See Examples 1.4.2 and 1.5.1 for the case where the driving force is a cosine function.

Examining Equation (1.25), two features are important and immediately obvious. First, as t gets larger, the transient response (the first term) becomes very small – hence the term steady-state response is assigned to the particular solution (the second term). The second observation is that the coefficient of the steady state response, or particular solution, becomes large when the excitation frequency is close to the undamped natural frequency, i.e. $\omega \approx \omega_n$. This phenomenon is known as *resonance* and is extremely important in design, vibration analysis and testing.

Example 1.4.1

Compute the response of the following system (assuming consistent units)

$$\ddot{x}(t) + 0.4\dot{x}(t) + 4x(t) = \frac{1}{\sqrt{2}}\sin 3t, \quad x(0) = \frac{-3}{\sqrt{2}}, \quad \dot{x}(0) = 0$$

Solution: First solve for the particular solution by using the more convenient form of

$$x_p(t) = X_1 \sin 3t + X_2 \cos 3t$$

rather than the magnitude and phase form, where X_1 and X_2 are the constants to be determined. Differentiating x_p yields

$$\dot{x}_p(t) = 3X_1\cos 3t - 3X_2 \sin 3t$$
$$\ddot{x}_p(t) = -9X_1 \sin 3t - 9X_2\cos 3t$$

Substitution of x_p and its derivatives into the equation of motion and collecting like terms yields

$$\left(-9X_1 - 1.2X_2 + 4X_1 - \frac{1}{\sqrt{2}}\right)\sin 3t + (-9X_2 + 1.2X_1 + 4X_2)\cos 3t = 0$$

Since the sine and cosine are independent, the two coefficients in parenthesis must vanish, resulting in two equations in the two unknowns, X_1 and X_2. This solution yields

$$x_p(t) = -0.134 \sin 3t - 0.032 \cos 3t$$

Next consider adding the free response to this. From the problem statement

$$\omega_n = 2 \text{ rad/s}, \quad \zeta = \frac{0.4}{2\omega_n} = 0.1 < 1, \quad \omega_d = \omega_n\sqrt{1-\zeta^2} = 1.99 \text{ rad/s}$$

Thus, the system is underdamped, and the total solution is of the form

$$x(t) = e^{-\zeta\omega_n t}(A \sin \omega_d t + B \cos \omega_d t) + X_1 \sin \omega t + X_2 \cos \omega t$$

Applying the initial conditions requires the derivative

$$\begin{aligned}\dot{x}(t) &= e^{-\zeta\omega_n t}(\omega_d A \cos \omega_d t - \omega_d B \sin \omega_d t) + \omega X_1 \cos \omega t \\ &\quad - \omega X_2 \sin \omega t - \zeta\omega_n e^{-\zeta\omega_n t}(A \sin \omega_d t + B \cos \omega_d t)\end{aligned}$$

The initial conditions yield the constants A and B

$$x(0) = B + X_2 = \frac{-3}{\sqrt{2}} \Rightarrow B = -X_2 - \frac{3}{\sqrt{2}} = -2.089$$

$$\dot{x}(0) = \omega_d A + \omega X_1 - \zeta\omega_n B = 0 \Rightarrow A = \frac{1}{\omega_d}(\zeta\omega_n B - \omega X_1) = -0.008$$

Thus the total solution is

$$x(t) = -e^{-0.2t}(0.008 \sin 1.99t + 2.089 \cos 1.99t) - 0.134 \sin 3t - 0.032 \cos 3t$$

Example 1.4.2
Calculate the form of the forced response if, instead of a sinusoidal driving force, the applied force is given by

$$F(t) = F_0 \cos \omega t.$$

Solution: In this case, assume that the response is also a cosine function out of phase or

$$x_p(t) = X \cos(\omega t - \theta)$$

To make the computations easy to follow, this is written in the equivalent form using a basic trig identity

$$x_p(t) = A_s \cos \omega t + B_s \sin \omega t$$

where the constants $A_s = X \cos \theta$ and $B_s = X \sin \theta$ satisfying

$$X = \sqrt{A_s^2 + B_s^2} \quad \text{and} \quad \theta = \tan^{-1} \frac{B_s}{A_s}$$

are undetermined constant coefficients. Taking derivatives of the assumed form of the solution and substitution of these into the equation of motion yields

$$\left(-\omega^2 A_s + 2\zeta\omega_n\omega B_s + \omega_n^2 A_s - f_0\right)\cos\omega t \\ + \left(-\omega^2 B_s - 2\zeta\omega_n\omega A_s + \omega_n^2 B_s\right)\sin\omega t = 0$$

This equation must hold for all time, in particular for $t = \pi/2\omega$, so that the coefficient of sin ωt must vanish. Similarly, for $t = 0$, the coefficient of cos ωt must vanish. This yields the two equations

$$\left(\omega_n^2 - \omega^2\right) A_s + \left(2\zeta\omega_n\omega\right) B_s = f_0$$

and

$$(-2\zeta\omega_n\omega)A_s + \left(\omega_n^2 - \omega^2\right) + B_s = 0$$

in the two undetermined coefficients A_s and B_s. Solving yields

$$A_s = \frac{(\omega_n^2 - \omega^2)f_0}{\left(\omega_n^2 - \omega^2\right)^2 + (2\zeta\omega_n\omega)^2}$$

$$B_s = \frac{2\zeta\omega_n\omega f_0}{\left(\omega_n^2 - \omega^2\right)^2 + (2\zeta\omega_n\omega)^2}$$

Substitution of these expressions into the equations for X and θ yields the particular solution

$$x_p(t) = \overbrace{\frac{f_0}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + (2\zeta\omega_n\omega)^2}}}^{X} \cos\left(\omega t - \overbrace{\tan^{-1}\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}}^{\theta}\right)$$

Resonance is generally to be avoided in designing structures, since it means large amplitude vibrations, which can cause fatigue failure, discomfort, loud noises, etc. Occasionally, the effects of resonance are catastrophic. However, the concept of resonance is also very useful in testing structures and in certain applications such as energy harvesting (Section 7.10). In fact, the process of modal testing (Chapter 12) is based on resonance. Figure 1.10 illustrates how ω_n and ζ affect the amplitude at resonance. The dimensionless quantity Xk/F_0 is called the *magnification factor* and Figure 1.10 is called a *magnification curve* or *magnitude plot*. The maximum value at resonance, called the *peak resonance*, and denoted by M_p, can be shown (Inman, 2014) to be related to the damping ratio by

$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{1.26}$$

Also, Figure 1.10 can be used to define the *bandwidth* of the structure, denoted by *BW*, as the value of the driving frequency at which the magnitude drops below 70.7% of its zero frequency value (also said to be the 3-dB down point from the zero

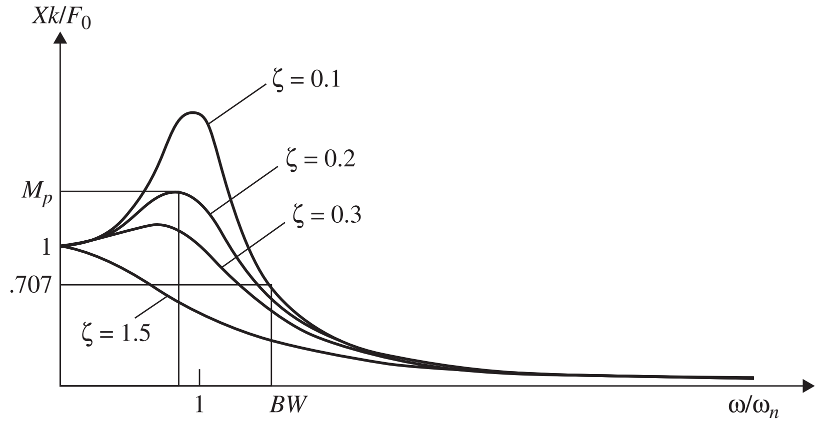


Figure 1.10 Magnification curves (dimensionless) for an SDOF system showing the normalized amplitude of vibration versus the ratio of driving frequency to natural frequency ($r = \omega/\omega_n$).

frequency point). The bandwidth can be calculated (Kuo and Golnaraghi, 2009: p. 359) in terms of the damping ratio by

$$BW = \omega_n \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \tag{1.27}$$

Two other quantities are used in discussing the vibration of underdamped structures. They are the *loss factor* defined at resonance (only) to be

$$\eta = 2\zeta \tag{1.28}$$

and the *Q value*, or *resonance sharpness* factor, given by

$$Q = \frac{1}{2\zeta} = \frac{1}{\eta} \tag{1.29}$$

Another common situation focuses on the transient nature of the response, namely, the response of Equation (1.6) to an impulse, to a step function, or to initial conditions. Many mechanical systems are excited by loads, which act for a very brief time. Such situations are usually modeled by introducing a fictitious function called the *unit impulse function*, or the *Dirac delta function*. This delta function, denoted δ, is defined by the two properties

$$\begin{aligned} &\delta(t - a) = 0 \qquad t \neq a \\ &\int_{-\infty}^{\infty} \delta(t - a)\, dt = 1 \end{aligned} \tag{1.30}$$

where a is the instant of time at which the impulse is applied. Strictly speaking, the quantity $\delta(t)$ is not a function; however, it is very useful in quantifying important physical phenomena of an impulse.

The response of the system of Figure 1.9 for the underdamped case (with $a = x_0 = v_0 = 0$) can be given by

$$x(t) = \begin{Bmatrix} 0 & t < a \\ \dfrac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t & t \geq a \end{Bmatrix} \tag{1.31}$$

Note from Equation (1.13) that this corresponds to the transient response of the system to the initial conditions $x_0 = 0$ and $v_0 = 1/m$. Hence, the impulse response is equivalent to giving a system at rest an initial velocity of $(1/m)$. This makes the impulse response, $x(t)$, important in discussing the transient response of more complicated systems. The impulse is also very useful in making vibration measurements, as described in Chapter 12.

A physical impact applied to a structure can be modeled by using the Dirac delta function with a magnitude representing the size of the impact. In this case, the impulse applied to the structure is modeled as having a magnitude F applied over a short time period Δt so that the effective change in momentum is $mv_0 - 0 = F\,\Delta t$, assuming the structure is initially at rest. This is equivalent to imparting an initial velocity of $v_0 = F\,\Delta t/m$. Thus, for an impulse of magnitude F applied over time Δt, the response becomes

$$x(t) = \begin{Bmatrix} 0 & t < a \\ \dfrac{F\Delta t}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t & t \geq a \end{Bmatrix} \tag{1.32}$$

Often design problems are stated in terms of certain specifications based on the response of the system to step function excitation. The response of the system in Figure 1.9 to a step function (of magnitude $m\omega_n^2$ for convenience), with initial conditions both set to zero, is calculated for underdamped systems from

$$m\ddot{x} + c\dot{x} + kx = m\omega_n^2 \mu(t), \qquad \mu(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \tag{1.33}$$

to be

$$x(t) = 1 - \frac{e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)}{\sqrt{1 - \zeta^2}} \tag{1.34}$$

where

$$\phi = \arctan\left[\frac{\sqrt{1 - \zeta^2}}{\zeta}\right] \tag{1.35}$$

A sketch of the response is given in Figure 1.11, along with the labeling of several significant specifications for the case $m = 1$, $\omega_n = 2$ and $\zeta = 0.2$.

In some situations, the steady-state response of a structure may be at an acceptable level, but the transient response may exceed acceptable limits. Hence, one important measure is the *overshoot*, labeled O.S. in Figure 1.11 and defined to be

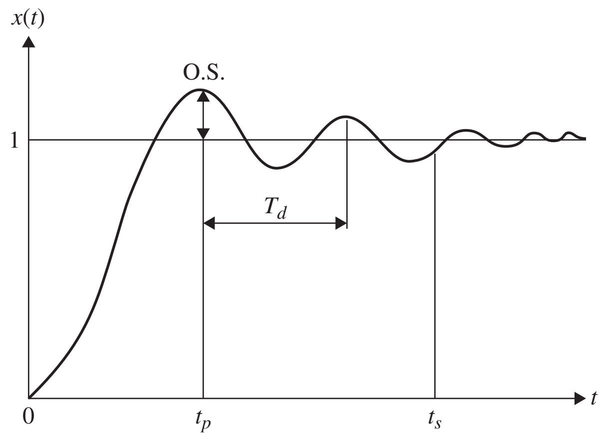


Figure 1.11 Step response of an SDOF system.

the maximum value of the response minus the steady-state value of the response. From Equation (1.34) it can be shown that

$$\text{overshoot} = \text{O.S.} = x_{\max}(t) - 1 = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \tag{1.36}$$

This occurs at the *peak time*, t_p, which can be shown to be

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{1.37}$$

In addition, the period of oscillation, T_d, is given by

$$T_d = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = 2t_p \tag{1.38}$$

Another useful quantity, which indicates the behavior of the transient response, is the *settling time*, t_s. This is the time it takes the response to get within ±5% of the steady-state response and remain within ±5%. One approximation of t_s is given by Kuo and Golnaraghi (2009: p. 263)

$$t_s = \frac{3.2}{\omega_n\zeta} \tag{1.39}$$

The preceding definitions allow designers and vibration analysts to specify and classify precisely the nature of the transient response of an underdamped system. These definitions also give some indication of how to adjust the physical parameters of the system so that the response has a desired shape.

The response of a system to an impulse may be used to determine the response of an underdamped system to any input $F(t)$ by defining the *impulse response function* by

$$h(t) = \frac{1}{m\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t \tag{1.40}$$

Then the solution of

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

can be shown to be

$$x(t) = \int_0^t F(\tau)h(t-\tau)d\tau = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t F(\tau)e^{\zeta\omega_n \tau} \sin\omega_d(t-\tau)d\tau \qquad (1.41)$$

for the case of zero initial conditions. This last expression gives an analytical representation for the response to any driving force that has an integral.

Example 1.4.3
Consider a spring-mass-damper system with $m = 1$ kg, $c = 2$ kg/s and $k = 2000$ N/m, with an impulsive force applied to it of 10,000 N for 0.01 s. Compute the resulting response.

Solution: A 10,000 N force acting over 0.01 s provides (area under the curve) a value of $F\Delta t = 10000 \times 0.01 = 100$ N · s Using the values given, the equation of motion is

$$\ddot{x}(t) + 2\dot{x}(t) + 2000x(t) = 100\delta(t)$$

Thus the natural frequency, damping ratio and damped natural frequency are

$$\omega_n = \sqrt{\frac{2000}{1}} = 44.721 \text{ rad/s}, \ \zeta = \frac{2}{2\sqrt{1 \times 2000}} = 0.022,$$

$$\omega_d = 44.721\sqrt{1 - 0.022^2} = 44.71 \text{ rad/s}$$

Using Equation (1.32), the response becomes

$$x(t) = \frac{\hat{F}e^{-\zeta\omega_n t}}{m\omega_d} \sin\omega_d t = 2.237e^{-0.1t}\sin(44.71t)$$

1.5 Transfer Functions and Frequency Methods

The preceding analysis of the response was carried out in the time domain. Current vibration measurement methodology (Ewins, 2000), as well as much control analysis (Kuo and Golnaraghi, 2009), often takes place in the frequency domain. Hence, it is worth the effort to reexamine these calculations using frequency domain methods (a phrase usually associated with linear control theory). The frequency domain approach arises naturally from mathematics (ordinary differential equations) via an alternative method of solving differential equations, such as Equations (1.21) and (1.33), using the Laplace transform (Boyce and DiPrima, 2012; Chapter 6).

Taking the Laplace transform of Equation (1.33), assuming both initial conditions to be zero, yields

$$X(s) = \left[\frac{1}{ms^2 + cs + k}\right] \mu(s) \qquad (1.42)$$

where $X(s)$ denotes the Laplace transform of $x(t)$, and $\mu(s)$ is the Laplace transform on the right-hand side of Equation (1.33). If the same procedure is applied to Equation (1.21), the result is

$$X(s) = \left[\frac{1}{ms^2 + cs + k}\right] F_0(s) \tag{1.43}$$

where $F_0(s)$ denotes the Laplace transform of $F_0 \sin \omega t$. Note that

$$G(s) = \frac{X(s)}{\mu(s)} = \frac{X(s)}{F_0(s)} = \frac{1}{ms^2 + cs + k} \tag{1.44}$$

Thus, it appears that the quantity $G(s) = [1/(ms^2 + cs + k)]$, the ratio of the Laplace transform of the output (response) to the Laplace transform of the input (applied force) to the system characterizes the system (structure) under consideration. This characterization is independent of the input or driving function. This ratio, $G(s)$, is defined as the *transfer function* of this system in control analysis (or of this structure in vibration analysis). The transfer function can be used to provide analysis of the vibrational properties of the structure, as well as to provide a means of measuring the structure's dynamic response.

In control theory, the transfer function of a system is defined in terms of an output to input ratio, but the use of a transfer function in structural dynamics and vibration testing implies certain physical properties, depending on whether position, velocity or acceleration is considered as the response (output). It is common, for instance, to measure the response of a structure by using an accelerometer. The transfer function resulting is then $s^2X(s)/U(s)$, where $U(s)$ is the Laplace transform of the input and $s^2X(s)$ is the Laplace transform of the acceleration. This transfer function is called the *inertance* and its reciprocal is referred to as the *apparent mass*. Table 1.1 lists the nomenclature of various transfer functions. The physical basis for these names can be seen from their graphical representation.

The transfer function representation of a structure is very useful in control theory as well as in vibration testing. It also forms the basis of impedance methods discussed in the next section. The variable s in the Laplace transform is a complex variable, which can be further denoted by

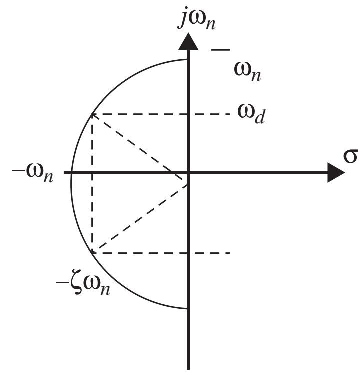
$$s = \sigma + j\omega_d$$

where the real numbers σ and ω_d denote the real and imaginary parts of s, respectively ($j = \sqrt{-1}$). Thus, the various transfer functions are also complex-valued.

Table 1.1 Various transfer functions.

Response Measurement	**Transfer Function**	**Inverse Transfer Function**
Acceleration	Inertance	Apparent mass
Velocity	Mobility	Impedance
Displacement	Compliance	Dynamic stiffness

Figure 1.12 Complex *s*-plane of the poles (roots of the characteristics of Equation (1.39).



In control theory, the values of s where the denominator of the transfer function $G(s)$ vanishes are called the *poles* of the transfer function. A plot of the poles of the compliance (also called receptance) transfer function for Equation (1.44) in the complex s-plane is given in Figure 1.12. The points on the semi-circle occur where the denominator of the transfer function is zero. These values of s ($s = -\zeta\omega_n \pm \omega_d j$) are exactly the roots of the characteristic equation for the structure. The values of the physical parameters m, c and k determine the two quantities ζ and ω_n , which in turn determine the position of the poles in Figure 1.12.

Another graphical representation of a transfer function useful in control is the *block diagram* illustrated in Figure 1.13a. This diagram is an icon for the definition of a transfer function. The control terminology for the physical device represented by the transfer function is the *plant*, whereas in vibration analysis the plant is usually referred to as the structure. The block diagram of Figure 1.13b is meant to imply the formula

$$\frac{X(s)}{U(s)} = \frac{1}{(ms^2 + cs + k)} \tag{1.45}$$

exactly.

The response of Equation (1.21) to a sinusoidal input (forcing function) motivates a second description of a structure's transfer function called the *frequency response function* (often denoted by FRF). The FRF is defined as the transfer function evaluated at $s = j\omega$, i.e. $G(j\omega)$. The significance of the FRF follows from Equation (1.22), namely, that the steady-state response of a system driven sinusoidally is a sinusoid of the same frequency with different amplitude and phase. In fact,

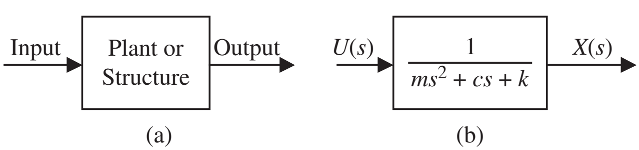


Figure 1.13 Block diagram representation of an SDOF system.

substitution of $j\omega$ into Equation (1.45) yields exactly Equations (1.23) and (1.24) from

$$\frac{X}{F_0} = |G(j\omega)| = \sqrt{x^2(\omega) + y^2(\omega)} \tag{1.46}$$

where $|G(j\omega)|$ indicates the magnitude of the complex FRF

$$\phi = \tan^{-1} G(j\omega) = \tan^{-1}\left[\frac{y(\omega)}{x(\omega)}\right] \tag{1.47}$$

indicates the phase of the FRF, and

$$G(j\omega) = x(\omega) + y(\omega)j \tag{1.48}$$

This mathematically expresses two ways to represent a complex function, as the sum of its real part (Re $G(j\omega) = x(\omega)$) and its imaginary part (Im $(G(j\omega)) = y(\omega)$), or by its magnitude ($|G(j\omega)|$) and phase (ϕ). In more physical terms, the FRF of a structure represents the magnitude and phase shift of its steady-state response under sinusoidal excitation. While Equations (1.23), (1.24), (1.46) and (1.47) verify this for an SDOF viscously damped structure, it can be shown in general for any linear time invariant plant (Melsa and Schultz, 1969: p. 187)).

It should also be noted that the FRF of a linear system can be obtained from the transfer function of the system and vice versa. Hence, the FRF uniquely determines the time response of the structure to any known input.

Graphical representations of the FRF form an extensive part of control analysis and also form the backbone of vibration measurement analysis. Next, three sets of FRF plots that are useful in testing vibrating structures are examined. The first set of plots consists simply of plotting the imaginary part of the FRF versus the driving frequency and the real part of the FRF versus the driving frequency. These are shown for the damped SDOF system in Figure 1.14 (the compliance FRF for $\zeta = 0.01$ and $\omega_n = 20$ rad/s).

The second representation consists of a single plot of the imaginary part of the FRF versus the real part of the FRF. This type of plot is called a *Nyquist plot* (also called an *Argand plane plot*) and is used for measuring the natural frequency and damping in testing methods and for stability analysis in control system design. The

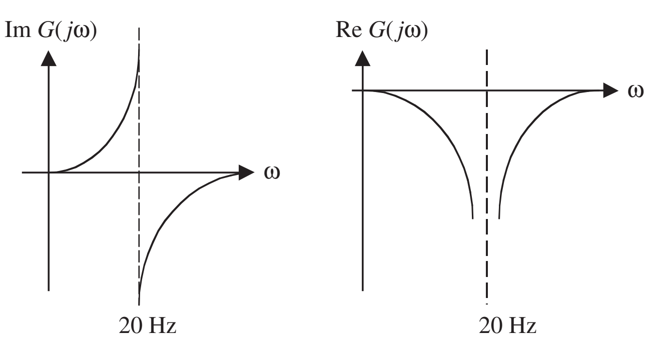


Figure 1.14 Plots of the real part and the imaginary part of the FRF.

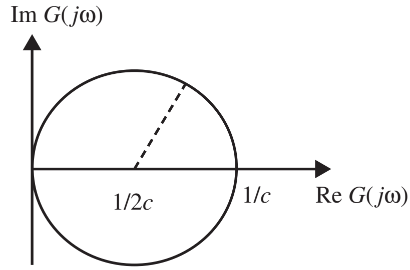


Figure 1.15 Nyquist plot for Equation 1.44.

Nyquist plot of the mobility FRF of a structure modeled by Equation (1.44) is given in Figure 1.15.

The last plots considered for representing the FRF are called *Bode plots* and consist of a plot of the magnitude of the FRF versus the driving frequency and the phase of the FRF versus the driving frequency (a complex number requires *two* real numbers to describe it completely). Bode plots have long been used in control system design and analysis as well as for determining the plant transfer function of a system. More recently, Bode plots have been used in analyzing vibration test results and in determining the physical parameters of the structure.

In order to represent the complete Bode plots in a reasonable space, $\log_{10}$ scales are often used to plot $|G(j\omega)|$. This has given rise to the use of the decibel and decades in discussing the magnitude response in the frequency domain. The magnitude and phase plots (for the compliance transfer function) for the system in Equation (1.21) are shown in Figures 1.16 and 1.17 for different values of ζ. Note the phase change at resonance (90°), as this is important in interpreting measurement data.

Note that Figures 1.10 and 1.17 show the same physical phenomenon and are both plots of the compliance transfer function. However, the magnitude in Figure 1.10 is dimensionless versus dimensionless frequency, while Figure 1.17 is usually the magnitude in decibels versus frequency on a semi-log scale.

Example 1.5.1

Solve the following system using the Laplace Transform method and using a Table of Laplace Transform pairs (from the Internet)

$$m\ddot{x}(t) + kx(t) = F_0 \cos \omega(t),\ x(0) = x_0,\ \dot{x}(0) = v_0$$

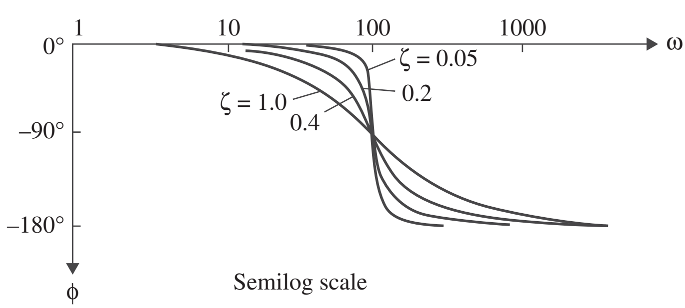


Figure 1.16 Bode phase plot for Equation (1.39) showing resonance at –90°.

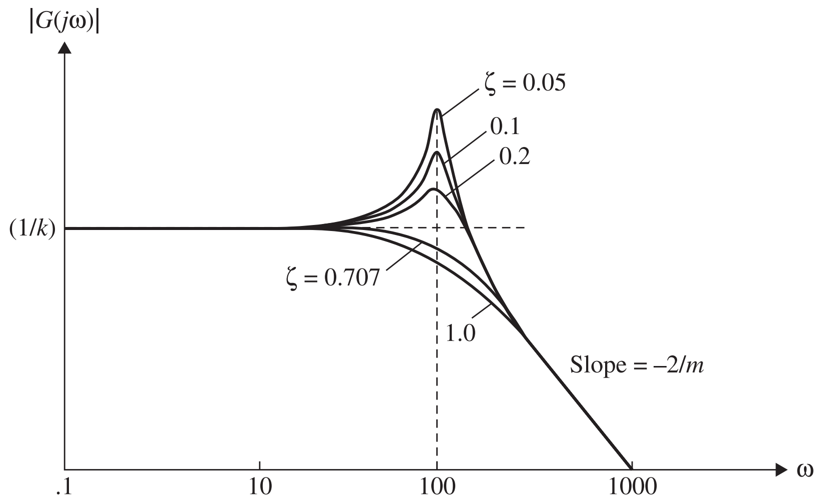


Figure 1.17 Bode magnitude plot for Equation (1.39) showing resonance and values of mass and stiffness.

Solution: First divide through by the mass to get

$$\ddot{x}(t) + \omega_n^2 x(t) = f_0 \cos \omega t, \; x(0) = x_0, \; \dot{x}(0) = v_0$$

Here $f_0 = F_0/m$. Taking the Laplace Transform (see the Table of Laplace Transforms: from the Internet) of the equation of motion considering the initial conditions yields

$$s^2 X(s) - s x_0 - v_0 + \omega_n^2 X(s) = \frac{s f_0}{s^2 + \omega^2}$$

$$\Rightarrow \left(s^2 + \omega_n^2\right) X(s) = s x_0 + v_0 + \frac{s f_0}{s^2 + \omega^2}$$

Solving this for $X(s)$ yields

$$X(s) = \frac{s x_0 + v_0}{s^2 + \omega_n^2} + \frac{s f_0}{\left(s^2 + \omega_n^2\right)\left(s^2 + \omega^2\right)}$$

$$= (x_0) \frac{s}{s^2 + \omega_n^2} + \left(\frac{v_0}{\omega_n}\right) \frac{\omega_n}{s^2 + \omega_n^2} + \frac{s f_0}{\left(s^2 + \omega_n^2\right)\left(s^2 + \omega^2\right)}$$

Taking the Inverse Laplace Transform using an online table of each term yields

$$x(t) = x_0 \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2}(\cos \omega t - \cos \omega_n t)$$

$$= \frac{v_0}{\omega_n} \sin \omega_n t + \left(x_0 - \frac{f_0}{\omega_n^2 - \omega^2}\right) \cos \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \cos \omega t$$

In comparing this with the solution given in Equation (1.25) for zero damping, note that Equation (1.25) is the solution for the case where the driving force is a sine function instead of a cosine as solved here.

1.6 Complex Representation and Impedance

Table 1.1 formally defines impedance as the ratio of a sinusoidal driving force, F, acting on the system to the resulting velocity, v, of the system. Impedance is usually denoted by the symbol Z and is a measure of a structure's resistance to motion. In working with impedance methods it is common to use the complex exponential notation to represent harmonic quantities. Using the exponential notation, the sinusoidal force in Equation (1.21) can be written as

$$F(t) = F_0 e^{j\omega t} \tag{1.49}$$

Here, ω is the driving frequency as before. The impedance approach offers an alternative way to examine systems vibrating harmonically based on using complex functions to represent the response.

A useful way to visualize harmonic motion is to think of the response $x(t)$ as a vector rotating in the complex plane, as illustrated in Figure 1.18. Here the vector has magnitude A and rotates an angle ωt in the complex plane. From Euler's formula for the complex exponential function

$$x(t) = Ae^{j\omega t} = A\cos\omega t + Aj\sin\omega t \tag{1.50}$$

which agrees with representation in Figure 1.18. Differentiation of the complex exponential yields simply

$$\begin{aligned} \frac{d}{dt}(Ae^{j\omega t}) &= j\omega Ae^{j\omega t} = j\omega x(t) \\ \frac{d^2}{dt^2}(Ae^{j\omega t}) &= j^2\omega^2 Ae^{j\omega t} = -\omega^2 x(t) \end{aligned} \tag{1.51}$$

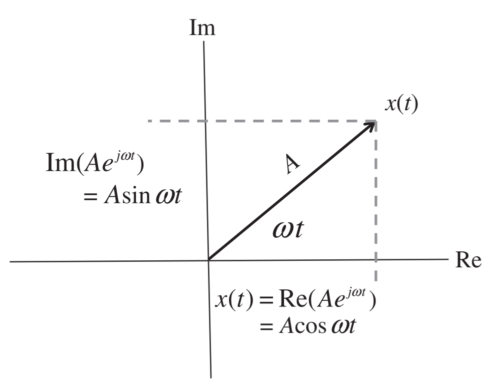


Figure 1.18 Graphic illustration of Euler's formula of the complex exponential.

Thus, each differentiation of the complex exponential results in simply multiplying by $j\omega$, similar to multiplying by s in the Laplace domain.

From the Figure 1.18, the physical displacement is interpreted from the complex exponential as just the real part of Equation (1.50). Thus the velocity becomes the real part of the derivative of the complex exponential and the acceleration is the real part of the derivative of that or

$$\begin{aligned} x(t) &= \mathrm{Re}(Ae^{j\omega t}) = A\cos(\omega t) \\ \dot{x}(t) &= \mathrm{Re}(j\omega Ae^{j\omega t}) = -\omega A\sin(\omega t) \\ \ddot{x}(t) &= \mathrm{Re}(j^2\omega^2 Ae^{j\omega t}) = -\omega^2 A\cos(\omega t) \end{aligned} \tag{1.52}$$

If the displacement is thought to be a sine function, then the physical motion variables become the imaginary parts of the complex exponential. Using the complex notation equation for the forced response of an SDOF system becomes

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_0 e^{jwt} \tag{1.53}$$

Assuming the resulting displacement is of the form

$$x(t) = A\sin(\omega t - \theta)$$

its complex form is the corresponding velocity as

$$v(t) = Aj\omega e^{j(\omega t+\theta)} \tag{1.54}$$

Here ω and θ are the driving frequency and phase shift between the applied force and the resulting response respectively. Substituting the complex form of $x(t)$ into Equation (1.48) yields

$$[-\omega^2 m + j\omega c + k]Ae^{j\omega - j\theta} = F(t) \tag{1.55}$$

Solving for the complex value A yields

$$A = \frac{F_0 e^{j\theta}}{[-\omega^2 m + j\omega c + k]} \tag{1.56}$$

which has magnitude and phase given by

$$|A| = \frac{F}{\sqrt{(k-\omega^2 m)^2 + (\omega c)^2}} \quad \text{and} \quad \theta = \tan\frac{\omega c}{k - \omega^2 m} \tag{1.57}$$

These values are of course the same as those derived in the previous section in Equations (1.23 and 1.24).

Examination of the force/velocity expressions for each element reveals the impedance of each, and these are given in Table 1.2.

Table 1.2 Impedance values for mass, damping and stiffness.

Mass	$Z = j\omega m$
Damping	$Z = c$
Stiffness	$Z = {}^{-jk}/_{\omega}$

Example 1.6.1
Compute the mechanical impedance of the spring-mass-damper system of Figure 1.9.

Solution: Dividing Equation (1.55) by (1.54) and simplifying yields that directly the mechanical impedance of the spring-mass-damper system becomes

$$\begin{aligned} Z &= \frac{F}{v} = \frac{[k - \omega^2 m + j\omega c]Ae^{j\omega t - j\theta}}{Aj\omega e^{j\omega t - j\theta}} = \frac{1}{j\omega}(k - \omega^2 m + j\omega c) \\ &= \omega j m + c - \frac{k}{j\omega} \end{aligned} \tag{1.58}$$

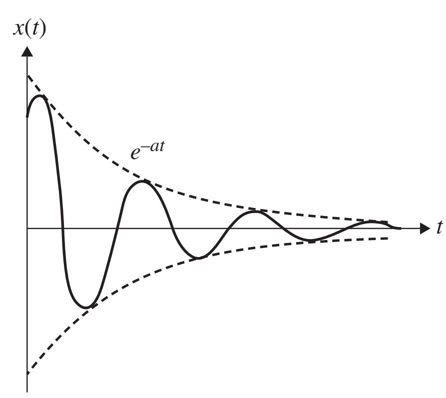
Comparing this expression to the terms in Table 1.2 reveals that the mechanical impedance of the system is just the sum of the impedance expressions for each element. The use of the impedance method is essentially the existence of following rules developed in electrical engineering for combining deferent circuit elements by adding their impedances (e.g. series and parallel combinations) and making the analogy to electrical components of capacitance (reciprocal of stiffness), inductance (mass) and resistance (damping). The units of mechanical impedance are kg/s, the same as the viscous damping coefficient.

1.7 Measurement and Testing

One can also use the quantities defined in the previous sections to measure the physical properties of a structure. As mentioned before, resonance can be used to determine a system's natural frequency. Methods based on resonance are referred to as resonance testing (or modal analysis techniques) (Bishop and Gladwell, 1963) and are briefly introduced here and discussed in more detail in Chapter 8.

As mentioned earlier, the mass and stiffness of a structure can often be determined by making simple static measurements. However, damping rates require a dynamic measurement and hence are more difficult to determine. For underdamped systems one approach is to realize, from Figure 1.6, that the decay envelope is the function $e^{-\zeta\omega_n t}$. The points on the envelope illustrated in Figure 1.19

Figure 1.19 Free decay measurement method.



can be used to curve-fit the function e^{-at}, where a is the constant determined by the curve fit. The relation $a = \zeta\omega_n$ can next be used to calculate ζ and hence the damping rate c (assuming that m and k or ω_n are known).

A second approach is to use the concept of logarithmic decrement, denoted by δ (delta) and defined by

$$\delta = \ln \frac{x(t)}{x(t + T_d)} \tag{1.59}$$

where T_d is the period of oscillation. Using Equation (1.13) in the form

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) \tag{1.60}$$

the value for δ becomes

$$\delta = \ln\left[\frac{e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}{e^{-\zeta\omega_n(t+T_d)}\sin(\omega_d t + \omega_d T_d + \phi)}\right] = \ln e^{\zeta\omega_n T_d} = \zeta\omega_n T_d \tag{1.61}$$

where the sine functions cancel because $\omega_d T_d$ is a one period shift by definition. Further evaluating δ yields

$$\delta = \zeta\omega_n T_d = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{1.62}$$

Equation (1.62) can be manipulated to yield the damping ratio in terms of the decrement, i.e.

$$\zeta = \frac{\delta}{\sqrt{4\pi + \delta^2}} \tag{1.63}$$

Hence, if the decrement is measured, Equation (1.63) yields the damping ratio.

The various plots of the previous section can also be used to measure ω_n, ζ, m, c and k. For instance, the Bode diagram of Figure 1.17 can be used to determine the natural frequency, stiffness and damping ratio. The stiffness is determined from the intercept of the FRF and the magnitude axis, since the value of the magnitude of the FRF for small ω is $\log(1/k)$. This can be seen by examining the function $\log_{10}|G(j\omega)|$ for small ω. Note that

$$\log|G(j\omega)| = \log\frac{1}{k} - \frac{1}{2}\log\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2\right] = \log\left(\frac{1}{k}\right) \tag{1.64}$$

for very small values of ω. Also note that $|G(j\omega)|$ evaluated at ω_n yields

$$k|G(j\omega_n)| = \frac{1}{2\zeta} \tag{1.65}$$

which provides a measure of the damping ratio from the magnitude plot of the FRF.

Note that Equations (1.65) and (1.26) appear to contradict each other, since

$$\frac{1}{2\zeta\sqrt{1-\zeta^2}} = k\max|G(j\omega)| = M_p \neq k|G(j\omega_n)| = \frac{1}{2\zeta}$$

except in the case of very small ζ (i.e. the difference between M_p and $|G(j\omega_n)|$ goes to zero as ζ goes to zero). This indicates a subtle difference between using the damping ratio obtained by using resonance as the value of ω, where $|G(j\omega_n)|$ is a maximum, and using the point, where $\omega = \omega_n$, the undamped natural frequency. This point is also illustrated by noting that the damped natural frequency, Equation (1.8), is $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and ω_p, the frequency at which $|G(j\omega_n)|$ is maximum, is

$$\omega_p = \omega_n\sqrt{1-2\zeta^2} \tag{1.66}$$

Also note that Equation (1.66) is valid only if $0 < \zeta < 0.707$.

Finally, the mass can be related to the slope of the magnitude plot for the inertance transfer function, denoted by $G_I(s)$, by noting that

$$G_I(s) = \frac{s^2}{(ms^2+cs+k)} \tag{1.67}$$

and for large ω (i.e. $\omega_n \ll \omega$), the value of $|G_I(j\omega)|$ is

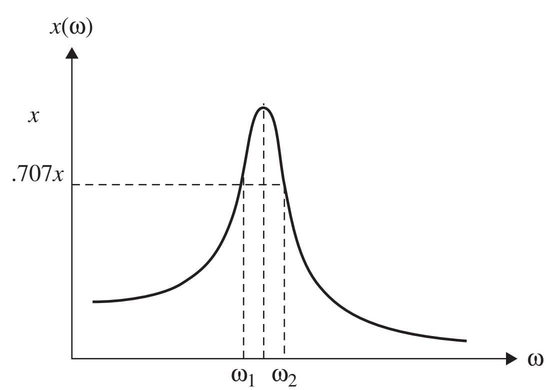
$$|G_I(j\omega)| \approx (1/m) \tag{1.68}$$

Plots of these values are referred to as straight-line approximations to the actual magnitude plot (Bode, 1945).

The preceding formulas relating the physical properties of the structure to the magnitude Bode diagrams suggest an experimental way to determine a structure's parameters: namely, if the structure can be driven by a sinusoid of varying frequency and if the magnitude and phase (needed to locate resonance) of the resulting response are measured, then the Bode plots and the preceding formulas can be used to obtain the desired physical parameters. This process is referred to as plant identification in the controls literature and can be extended to systems with more degrees of freedom (see Melsa and Schultz (1969), for a more complete account).

There are several other formulas for measuring the damping ratio and natural frequency from the results of such experiments, sine sweeps. For instance, if the Nyquist plot of the mobility transfer function is used, a circle of diameter $1/c$ results (Figure 1.15). Another approach is to plot the magnitude of the FRF on a linear scale near the region of resonance (Figure 1.20). If the damping is small enough so that the peak at resonance is sharp, the damping ratio can be determined by measuring the frequencies at 0.707 at the maximum value (also called

Figure 1.20 Quadrature peak picking method.



the 3-dB down point or half-power points), denoted by ω_1 and ω_2, respectively. Then, using the formula (Ewins, 2000)

$$\zeta = \frac{1}{2}\left[\frac{\omega_2 - \omega_1}{\omega_d}\right] \tag{1.69}$$

to compute the damping ratio. This method is referred to as *quadrature peak picking* and is illustrated in Figure 1.20.

1.8 Stability

In all the preceding analysis, the physical parameters *m, c* and *k* are, of course, positive quantities. There are physical situations, however, in which equations of the form of Equations (1.1) and (1.6) result but have one or more negative coefficients. Such systems are not well behaved and require some additional analysis.

Recalling that the solution to Equation (1.1) is of the form $A\sin(\omega t + \phi)$, where A is a constant, it is easy to see that the response, in this case $x(t)$, is bounded. That is to say that

$$|x(t)| \leq A \tag{1.70}$$

for all t where A is some finite constant and $|x(t)|$ denotes the absolute value of $x(t)$. In this case, the system is well behaved or *stable* (called marginally stable in the control's literature). In addition, note that the roots (also called *characteristic values* or eigenvalues) of

$$\lambda^2 m + k = 0$$

are purely complex numbers $\pm j\omega_n$ as long as m and k are positive (or have the same sign). If k happens to be negative and m is positive, the solution becomes

$$x(t) = A\sinh\omega_n t + B\cosh\omega_n t \tag{1.71}$$

which increases without bound as t does. Such solutions are called *divergent* or *unstable*.

If the solution of the damped system of Equation (1.6) with positive coefficients is examined, it is clear that $x(t)$ approaches zero as t becomes large, because of the exponential term. Such systems are considered to be *asymptotically stable* (called stable in the controls literature). Again, if one or two of the coefficients are negative, the motion grows without bound and becomes unstable as before. In this case, however, the motion may become unstable in one of two ways. Similar to overdamping and underdamping, the motion may grow without bound and not oscillate, or it may grow without bound and oscillate. The first case is referred to as *divergent instability* and the second case as *flutter instability*; together they fall under the topic of self-excited vibrations.

Apparently, the sign of the coefficient determines the stability behavior of the system. This concept is pursued in Chapter 4, where these stability concepts are formally defined. Figures 1.21 to 1.24 illustrate each of these concepts.

These stability definitions can also be stated in terms of the roots of the characteristic Equation (1.8) or in terms of the poles of the transfer function of the