DYNAMICS OF THE RIGID SOLID WITH GENERAL CONSTRAINTS BY A MULTIBODY APPROACH

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WILEY

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Preface

This book deals with both holonomic and non-holonomic constraints to study the mechanics of the constrained rigid body. The approach is completely matrix and we study all types of the general constraints that may appear at a rigid solid. The discussion is performed in the most general case, not in particular cases defined by certain types of mechanisms. Our approach is a multibody type one and the obtaining of the matrix of constraints is highlighted in each case discussed in the book. In addition, algorithms for the numerical calculations are given for each type of constraint. The theory is applied to numerical examples which are completely solved, the diagrams resulted being also presented.

The book contains eight chapters as follows. The first chapter is an introduction presenting the elements of mathematical calculation that will be used in the book. The second chapter treats the kinematics of the rigid solid and in this chapter we obtain the distribution of velocities and accelerations for a rigid body. The next chapter is dedicated to the general theorem in the dynamics of the rigid solid, that is, the theorem of momentum, the theorem of the moment of momentum, and the kinetic energy; all these theorems are developed in matrix form. In the fourth chapter are presented the matrix differential equations of motion in the general case of the rigid solid with constraints; the equations of motion are obtained using the general theorems and using the Lagrange equations; a completely new proof is given for the equivalence of these two approaches. In the fifth chapter we discuss the equilibrium of the rigid solid; we introduce the generalized forces and their expressions; as a particular case we study the equilibrium of a rigid solid hanged by springs. The next chapter deals with the motion of the rigid solid having constraints at given proper points; we discuss the rigid body with one fixed point, the rigid body in rotational motion, the rigid body with one or several points situated on given surfaces or curves. In the seventh chapter we discuss the motion of the rigid solid with constraints on given proper curves; the chapter is dedicated to the study of the rigid body at which given curves support on given curves or surfaces. The last chapter is dedicated to the motion of the rigid solid with constraints on the bounded surfaces; in this case the rigid body is supported at fixed points, or it rolls on curves or surfaces.

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This book is addressed to a large audience, to all those interested in using models and methods with holonomic and non-holonomic constraints in various fields like: mechanics, physics, civil and mechanical engineering, people involved in teaching, research or design, as well as students.

The book can be also used either as a stand-alone course for the master or PhD students, or as supplemental reading for the courses of computational mechanics, analytical mechanics, multibody mechanics etc. The prerequisites are the courses of elementary algebra and analysis, and mechanics.

Nicolae Pandrea and Nicolae-Doru Stănescu

1

Elements of Mathematical Calculation

This chapter is an introduction presenting the elements of mathematical calculation that will be used in the book.

1.1 Vectors: Vector Operations

A *vector* (denoted by $\mathbf{a}$) is defined by its numerical magnitude or modulus $|\mathbf{a}|$, by the direction Δ, and by sense. The vector is represented (Fig. 1.1) by an orientated segment of straight line.

The sum of two vectors $\mathbf{a}$, $\mathbf{b}$ is the vector $\mathbf{c}$ (Fig. 1.2) represented by the diagonal of the parallelogram constructed on the two vectors; it reads

$$\mathbf{c} = \mathbf{a} + \mathbf{b}. \tag{1.1}$$

The unit vector $\mathbf{u}$ of the vector $\mathbf{a}$ (or of the direction Δ) is defined by the relation

$$\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}. \tag{1.2}$$

If one denotes by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the unit vectors of the axes of dextrorsum orthogonal reference system $Oxyz$, and by a_x, a_y, a_z the projections of vector $\mathbf{a}$ onto the axes, then one may write the analytical expression

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}. \tag{1.3}$$

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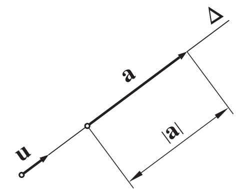

Figure 1.1 Representation of a vector.

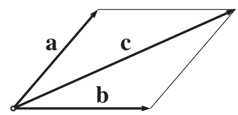

Figure 1.2 The sum of two vectors.

The scalar (dot) product of two vectors is defined by the expression

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\alpha, \tag{1.4}$$

where α is the angle between the two vectors.

We obtain the equalities

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0,\ \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1 \tag{1.5}$$

and, consequently, one deduces the analytical expressions

$$\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y + a_z b_z, \tag{1.6}$$

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2},\ |\mathbf{b}| = \sqrt{b_x^2 + b_y^2 + b_z^2}, \tag{1.7}$$

$$\cos\alpha = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}\sqrt{b_x^2 + b_y^2 + b_z^2}}. \tag{1.8}$$

The vector (cross) product of two vectors, denoted by **c**,

$$\mathbf{c} = \mathbf{a}\times\mathbf{b}, \tag{1.9}$$

is the vector perpendicular onto the plan of the vectors **a** and **b**, while the sense is given by the rule of the right screw when the vector **a** rotates over the vector **b** (making the smallest angle); the modulus has the expression

$$|\mathbf{c}| = |\mathbf{a}||\mathbf{b}|\sin\alpha, \tag{1.10}$$

α being the smallest angle between the vectors **a** and **b**.

One obtains the equalities

$$\mathbf{i}\times\mathbf{j}=\mathbf{k},\ \mathbf{j}\times\mathbf{k}=\mathbf{i},\ \mathbf{k}\times\mathbf{i}=\mathbf{j}, \tag{1.11}$$

and the analytical expression

$$\mathbf{a}\times\mathbf{b}=\left(a_y b_z-a_z b_y\right)\mathbf{i}+\left(a_z b_x-a_x b_z\right)\mathbf{j}+\left(a_x b_y-a_y b_x\right)\mathbf{k}. \tag{1.12}$$

The mixed product of three vectors, defined by the relation $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ and denoted by $(\mathbf{a},\mathbf{b},\mathbf{c})$, leads to the successive equalities

$$(\mathbf{a},\mathbf{b},\mathbf{c})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \tag{1.13}$$

The mixed product $(\mathbf{a},\mathbf{b},\mathbf{c})$ is equal to the volume with sign of the parallelepiped constructed having the three vectors as edges (Fig. 1.3). It is equal to zero if and only if the three vectors are coplanar.

The double vector product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ satisfies the equality

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}. \tag{1.14}$$

The reciprocal vectors of the (non-coplanar) vectors **a**, **b**, **c** are defined by the expressions

$$\mathbf{a}^*=\frac{\mathbf{b}\times\mathbf{c}}{(\mathbf{a},\mathbf{b},\mathbf{c})},\ \mathbf{b}^*=\frac{\mathbf{c}\times\mathbf{a}}{(\mathbf{a},\mathbf{b},\mathbf{c})},\ \mathbf{c}^*=\frac{\mathbf{a}\times\mathbf{b}}{(\mathbf{a},\mathbf{b},\mathbf{c})}, \tag{1.15}$$

and satisfy the equality

$$(\mathbf{a}^*,\mathbf{b}^*,\mathbf{c}^*)=\frac{1}{(\mathbf{a},\mathbf{b},\mathbf{c})}. \tag{1.16}$$

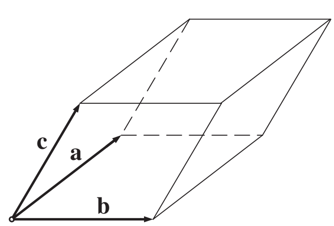

Figure 1.3 The geometric interpretation of the mixed product of three vectors.

An arbitrary vector **v** may be written in the form

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{a}^*)\mathbf{a} + (\mathbf{v} \cdot \mathbf{b}^*)\mathbf{b} + (\mathbf{v} \cdot \mathbf{c}^*)\mathbf{c}, \tag{1.17}$$

or as

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{a})\mathbf{a}^* + (\mathbf{v} \cdot \mathbf{b})\mathbf{b}^* + (\mathbf{v} \cdot \mathbf{c})\mathbf{c}^*. \tag{1.18}$$

1.2 Real Rectangular Matrix

By *real rectangular matrix* we understand a table with m rows and n columns ($m \neq n$)

$$[\mathbf{A}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \tag{1.19}$$

where the *elements* a_{ij} are real numbers.

Sometimes, we use the abridged notation

$$[\mathbf{A}] = \left(a_{ij}\right) \text{ or } [\mathbf{A}] = \left(a_{ij}\right)_{\substack{1 \le i \le m \\ 1 \le j \le n}}. \tag{1.20}$$

The multiplication between a matrix and a scalar $\lambda \in \mathbb{R}$ is defined by the relation

$$\lambda[\mathbf{A}] = \left(\lambda a_{ij}\right), \tag{1.21}$$

while *the sum of two matrices of the same type* (with the same number of rows and the same number of columns) is defined by

$$[\mathbf{A}] + [\mathbf{B}] = \left(a_{ij} + b_{ij}\right). \tag{1.22}$$

The zero matrix or the null matrix is the matrix denoted by $[\mathbf{0}]$, which has all its elements equal to zero.

The zero matrix verifies the relations

$$[\mathbf{A}] + [\mathbf{0}] = [\mathbf{0}] + [\mathbf{A}] = [\mathbf{A}]. \tag{1.23}$$

The transpose matrix $[\mathbf{A}]^{\mathrm{T}}$ is the matrix obtained transforming the rows of the matrix **[A]** into columns, that is

$$[\mathbf{A}]^{\mathrm{T}} = \left(a_{ji}\right). \tag{1.24}$$

The transposing operation has the following properties

$$\left[[\mathbf{A}]^{\mathrm{T}}\right]^{\mathrm{T}} = [\mathbf{A}], [[\mathbf{A}] + [\mathbf{B}]]^{\mathrm{T}} = [\mathbf{A}]^{\mathrm{T}} + [\mathbf{B}]^{\mathrm{T}}, \tag{1.25}$$

where we assumed that the sum can be performed.

The matrix with one column bears the name *column matrix* or *column vector* and it is denoted by $\{\mathbf{A}\}$, that is

$$\{\mathbf{A}\} = [a_{11} \quad a_{21} \quad \dots \quad a_{m1}]^{\mathrm{T}}, \tag{1.26}$$

while the matrix with one row is called *row matrix* or *row vector* and is denoted as

$$[\mathbf{A}] = [a_{11} \quad a_{12} \quad \dots \quad a_{1n}], \tag{1.27}$$

or

$$[\mathbf{A}] = \{\mathbf{A}\}^{\mathrm{T}}, \tag{1.28}$$

where

$$\{\mathbf{A}\} = [a_{11} \quad a_{12} \quad \dots \quad a_{1n}]^{\mathrm{T}}. \tag{1.29}$$

If the matrix $[\mathbf{A}]$ has m rows and n columns, and the matrix $[\mathbf{B}]$ has n rows and p columns, *then* the two matrices *can be multiplied* and the result is a matrix $[\mathbf{C}]$ with m rows and p columns

$$[\mathbf{C}] = [\mathbf{A}][\mathbf{B}], \tag{1.30}$$

where the elements c_{ij}, $1 \leq i \leq m$, $1 \leq j \leq p$, of the matrix $[\mathbf{C}]$ satisfy the equality

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \tag{1.31}$$

that is, the elements of the product matrix are obtained by multiplying the rows of matrix $[\mathbf{A}]$ by the columns of matrix $[\mathbf{B}]$.

The transpose of the product matrix is given by the relation

$$[[\mathbf{A}][\mathbf{B}]]^{\mathrm{T}} = [\mathbf{B}]^{\mathrm{T}}[\mathbf{A}]^{\mathrm{T}}. \tag{1.32}$$

In some cases, there may exist *matrices of matrices* and the multiplication is performed as in the following example

$$\begin{bmatrix} [\mathbf{A}_1] & [\mathbf{A}_2] \\ [\mathbf{A}_3] & [\mathbf{A}_4] \\ [\mathbf{A}_5] & [\mathbf{A}_6] \end{bmatrix} \begin{bmatrix} [\mathbf{B}_1] & [\mathbf{B}_2] \\ [\mathbf{B}_3] & [\mathbf{B}_4] \end{bmatrix} = \begin{bmatrix} [\mathbf{A}_1][\mathbf{B}_1] + [\mathbf{A}_2][\mathbf{B}_3] & [\mathbf{A}_1][\mathbf{B}_2] + [\mathbf{A}_2][\mathbf{B}_4] \\ [\mathbf{A}_3][\mathbf{B}_1] + [\mathbf{A}_4][\mathbf{B}_3] & [\mathbf{A}_3][\mathbf{B}_2] + [\mathbf{A}_4][\mathbf{B}_4] \\ [\mathbf{A}_5][\mathbf{B}_1] + [\mathbf{A}_6][\mathbf{B}_3] & [\mathbf{A}_5][\mathbf{B}_2] + [\mathbf{A}_6][\mathbf{B}_4] \end{bmatrix}, \tag{1.33}$$

where we assumed that the operations of multiplication and addition of matrices can be performed for each separate case.

1.3 Square Matrix

The matrix $[\mathbf{A}]$ is a *square matrix* if the number of rows is equal to the number of columns; hence

$$[\mathbf{A}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \tag{1.34}$$

where the number n is *the dimension* or *the order of the matrix*.

The determinant associated to the matrix $[\mathbf{A}]$ is denoted by $\det[\mathbf{A}]$.

If $[\mathbf{A}_{ij}]$ is the matrix obtained from the matrix $[\mathbf{A}]$ by the suppression of the row i and the column j, then *the algebraic complement* a_{ij}^* is given by the expression

$$a_{ij}^* = (-1)^{i+j} \det[\mathbf{A}_{ij}], 1 \leq i, j \leq n, \tag{1.35}$$

and the following relation holds true

$$\sum_{k=1}^{n} a_{ik} a_{jk}^* = \sum_{k=1}^{n} a_{kj} a_{ki}^* = \begin{cases} 0 & \text{for } i \neq j \\ \det[\mathbf{A}] & \text{for } i = j \end{cases}. \tag{1.36}$$

The determinants of the matrices satisfy the equalities

$$\det[\mathbf{A}] = \det[\mathbf{A}]^{\mathrm{T}}, \tag{1.37}$$

$$\det[[\mathbf{A}][\mathbf{B}]] = \det[\mathbf{A}] \cdot \det[\mathbf{B}], \tag{1.38}$$

where we assumed that the matrices [**A**] and [**B**] have the same order.

In general, *the multiplication of matrices* is not commutative,

$$[\mathbf{A}][\mathbf{B}] \neq [\mathbf{B}][\mathbf{A}], \tag{1.39}$$

but it is associative and distributive, that is

$$[\mathbf{A}][[\mathbf{B}][\mathbf{C}]] = [[\mathbf{A}][\mathbf{B}]][\mathbf{C}] = [\mathbf{A}][\mathbf{B}][\mathbf{C}], \tag{1.40}$$

$$[\mathbf{A}][[\mathbf{B}] + [\mathbf{C}]] = [\mathbf{A}][\mathbf{B}] + [\mathbf{A}][\mathbf{C}], \tag{1.41}$$

where the matrices [**A**], [**B**] and [**C**] have the same order.

The trace of a matrix, denoted by Tr[**A**] is equal to the sum of the elements situated on the principal diagonal

$$\mathrm{Tr}[\mathbf{A}] = \sum_{i=1}^{n} a_{ii}. \tag{1.42}$$

The diagonal matrix is the matrix with all the elements equal to zero, except *some elements* situated on the principal diagonal.

The unity matrix, generally denoted by [**I**], is *the diagonal matrix* that has all the elements of the principal diagonal equal to unity,

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \tag{1.43}$$

The unity matrix verifies the relations

$$[\mathbf{A}][\mathbf{I}] = [\mathbf{I}][\mathbf{A}] = [\mathbf{A}]. \tag{1.44}$$

The adjunct matrix $\mathbf{A}^*$ is defined by the relation

$$[\mathbf{A}^*] = \left(a_{ij}^*\right). \tag{1.45}$$

The matrix [**A**] is called *singular* if $\det[\mathbf{A}] = 0$; it is called a *non-singular* one if $\det[\mathbf{A}] \neq 0$.

The non-singular matrices [**A**] admit *inverse* matrices $[\mathbf{A}]^{-1}$; the inverse matrices fulfill the conditions

$$[\mathbf{A}]^{-1} = \frac{1}{\det[\mathbf{A}]}[\mathbf{A}^*], \tag{1.46}$$

$$[\mathbf{A}][\mathbf{A}]^{-1} = [\mathbf{A}]^{-1}[\mathbf{A}] = [\mathbf{I}], \tag{1.47}$$

$$\left[[\mathbf{A}]^{\mathrm{T}}\right]^{-1} = \left[[\mathbf{A}]^{-1}\right]^{\mathrm{T}}. \tag{1.48}$$

The matrix $[\mathbf{A}]$ is called *symmetric* if

$$[\mathbf{A}] = [\mathbf{A}]^{\mathrm{T}}; \tag{1.49}$$

it is called *anti-symmetric* or *skew* if

$$[\mathbf{A}] = -[\mathbf{A}]^{\mathrm{T}}. \tag{1.50}$$

The matrix $[\mathbf{A}]$ is called *orthogonal* if it fulfills the condition

$$[\mathbf{A}][\mathbf{A}]^{\mathrm{T}} = [\mathbf{I}]. \tag{1.51}$$

The orthogonal matrix $[\mathbf{A}]$ satisfies the equalities

$$[\mathbf{A}]^{\mathrm{T}} = [\mathbf{A}]^{-1}, \det[\mathbf{A}] = \pm 1. \tag{1.52}$$

The equation of nth degree

$$\det[\lambda[\mathbf{I}] - [\mathbf{A}]] = 0 \tag{1.53}$$

is *the characteristic equation* of the matrix $[\mathbf{A}]$; its roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called *the eigenvalues of the matrix* $[\mathbf{A}]$.

The vectors $\left\{\mathbf{v}^{(m_i)}\right\}$ which are obtained from the equality

$$[\mathbf{A}]\left\{\mathbf{v}^{(m_i)}\right\} = \lambda_m \left\{\mathbf{v}^{(m_i)}\right\}, 1 \le m \le k, \tag{1.54}$$

are called *eigenvectors* and, if the matrix $[\mathbf{A}]$ is a symmetric one, then its eigenvectors are orthogonal

$$\left\{\mathbf{v}^{(r)}\right\}^{\mathrm{T}}\left\{\mathbf{v}^{(s)}\right\} = 0, \text{ if } s \ne r. \tag{1.55}$$

Using the notation

$$b_j = \mathrm{Tr}\left[[\mathbf{A}]^j\right], \tag{1.56}$$

one obtains the characteristic equation

$$\sum_{j=0}^{n} c_{n-j}\lambda^j = 0, \tag{1.57}$$

where the coefficients c_j are given by the iterative relations

$$c_0 = 0,\ c_j = -\frac{1}{j}\sum_{k=0}^{j-1} c_k b_{j-k}. \tag{1.58}$$

Observation 1.3.1.

i. The eigenvalues of the matrix [**A**] of order n can be real or complex, distinct or not.
ii. One or more eigenvectors correspond to an eigenvalue λ_m, depending on the order of multiplicity for that eigenvalue.
iii. No matter if the eigenvalue is real or not, keeping into account that the matrix [**A**] has real components, the eigenvectors associated to that eigenvalue are matrices with n rows and one column, with real elements.

Observation 1.3.2. Let us consider that the matrix [**A**] is a square one, of order 3.

i. If the eigenvalues are real and distinct $\lambda_i \in \mathbb{R}$, $\lambda_i \neq \lambda_j$, $i, j \in \{1, 2, 3\}$, $i \neq j$, then the eigenvalues are obtained by solving three matrix equations of the form

$$[\mathbf{A}]\{\mathbf{v}_i\} = \lambda_i\{\mathbf{v}_i\}, i = 1, 2, 3. \tag{1.59}$$

ii. If the eigenvalues are real, but two of them are equal, $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$, $\lambda_1 = \lambda_2$, $\lambda_3 \neq \lambda_1$, then the eigenvalues result by solving the matrix equations

$$[\mathbf{A}]\{\mathbf{v}_1\} = \lambda_1\{\mathbf{v}_1\}, ([\mathbf{A}] - \lambda_1[\mathbf{I}])\{\mathbf{v}_2\} = \{\mathbf{v}_1\}, [\mathbf{A}]\{\mathbf{v}_3\} = \lambda_3\{\mathbf{v}_3\}. \tag{1.60}$$

iii. If the eigenvalues are real and equal, $\lambda_i = \lambda$, $i = 1, 2, 3$, then the eigenvector are obtained by solving the matrix equations

$$[\mathbf{A}]\{\mathbf{v}_1\} = \lambda\{\mathbf{v}_1\}, ([\mathbf{A}] - \lambda[\mathbf{I}])\{\mathbf{v}_2\} = \{\mathbf{v}_1\}, ([\mathbf{A}] - \lambda[\mathbf{I}])\{\mathbf{v}_3\} = \{\mathbf{v}_2\}. \tag{1.61}$$

iv. If the eigenvalues are one real, $\lambda_1 \in \mathbb{R}$, and two complex conjugate, $\lambda_2 = \alpha + \mathrm{i}\beta$, $\lambda_3 = \alpha - \mathrm{i}\beta$, $\alpha, \beta \in \mathbb{R}$, $\mathrm{i}^2 = -1$, then the eigenvectors result by solving the matrix equations

$$[\mathbf{A}](\{\mathbf{v}_2\} + \mathrm{i}\{\mathbf{v}_3\}) = (\alpha + \mathrm{i}\beta)(\{\mathbf{v}_2\} + \mathrm{i}\{\mathbf{v}_3\}); \tag{1.62}$$

1.4 Skew Matrix of Third Order

Starting from the relation of definition (1.49), it results that a third order skew matrix may be written in the form

$$[\mathbf{B}] = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}. \tag{1.63}$$

One associates to the skew matrix $[\mathbf{B}]$ *the column matrix (vector)*

$$\{\mathbf{b}\} = [b_1 \quad b_2 \quad b_3]^{\mathrm{T}} \tag{1.64}$$

and the vector

$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}. \tag{1.65}$$

It results the equality

$$[\mathbf{B}]\{\mathbf{b}\} = \{\mathbf{0}\}. \tag{1.66}$$

Being given the skew matrices $[\mathbf{A}]$, $[\mathbf{B}]$, and the eigenvectors associated to these matrices, then the vector equality

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \tag{1.67}$$

may be put in the matrix expression

$$[\mathbf{A}]\{\mathbf{b}\} = -[\mathbf{B}]\{\mathbf{a}\}. \tag{1.68}$$

For *the skew matrix* $[\mathbf{B}]$ one may write the following relations (obtained by elementary calculation)

$$\det[\mathbf{B}] = 0, \tag{1.69}$$

$$[\mathbf{B}]^2 = -\left(b_1^2 + b_2^2 + b_3^2\right)[\mathbf{I}] + \{\mathbf{b}\}\{\mathbf{b}\}^{\mathrm{T}}, \tag{1.70}$$

$$[\mathbf{B}]^3 = -\left(b_1^2 + b_2^2 + b_3^2\right)[\mathbf{B}]. \tag{1.71}$$

For the *skew* matrices $[\mathbf{A}]$, $[\mathbf{B}]$ and the associated vectors $\mathbf{a}$, $\mathbf{b}$, denoting the vector product by $\mathbf{c}$, $\mathbf{c} = \mathbf{a} \times \mathbf{b}$, and by $[\mathbf{C}]$ the associated skew matrix, one obtains the relations

$$[\mathbf{A}][\mathbf{B}] = -(a_1b_1 + a_2b_2 + a_3b_3)[\mathbf{I}] + \{\mathbf{b}\}\{\mathbf{a}\}^{\mathrm{T}}, \tag{1.72}$$

$$[\mathbf{B}][\mathbf{A}] = -(a_1 b_1 + a_2 b_2 + a_3 b_3)[\mathbf{I}] + \{\mathbf{a}\}\{\mathbf{b}\}^{\mathrm{T}}, \tag{1.73}$$

$$[\mathbf{C}] = [\mathbf{A}][\mathbf{B}] - [\mathbf{B}][\mathbf{A}] = \{\mathbf{b}\}\{\mathbf{a}\}^{\mathrm{T}} - \{\mathbf{a}\}\{\mathbf{b}\}^{\mathrm{T}}, \tag{1.74}$$

$$\begin{gathered}[\mathbf{C}]^2 = (a_1 b_1 + a_2 b_2 + a_3 b_3)\left[\{\mathbf{a}\}\{\mathbf{b}\}^{\mathrm{T}} - \{\mathbf{b}\}\{\mathbf{a}\}^{\mathrm{T}}\right] - \\ \left(b_1^2 + b_2^2 + b_3^2\right)\{\mathbf{a}\}\{\mathbf{a}\}^{\mathrm{T}} - \left(a_1^2 + a_2^2 + a_3^2\right)\{\mathbf{b}\}\{\mathbf{b}\}^{\mathrm{T}}.\end{gathered} \tag{1.75}$$

If the matrix [**A**] is an *arbitrary* third order one, and the matrices [**B**], [**C**] are *skew* ones, then the matrix

$$[\mathbf{D}] = [\mathbf{A}]^{\mathrm{T}}[\mathbf{B}][\mathbf{A}] \tag{1.76}$$

is a *skew* matrix, and the associated column matrices {**b**}, {**c**}, {**d**} satisfy the equalities

$$\{\mathbf{d}\} = [\mathbf{A}^*]\{\mathbf{b}\}, \tag{1.77}$$

$$[\mathbf{A}]^{\mathrm{T}}[\mathbf{B}][\mathbf{A}]\{\mathbf{c}\} = -[\mathbf{C}][\mathbf{A}^*]\{\mathbf{b}\}, \tag{1.78}$$

where $[\mathbf{A}^*]$ is the adjunct matrix of the matrix [**A**].

When the matrix [**A**] is orthogonal, one obtains the equalities

$$\{\mathbf{d}\} = [\mathbf{A}]^{\mathrm{T}}\{\mathbf{b}\}, [\mathbf{A}]^{\mathrm{T}}[\mathbf{B}][\mathbf{A}]\{\mathbf{c}\} = -[\mathbf{C}][\mathbf{A}]^{\mathrm{T}}\{\mathbf{b}\}. \tag{1.79}$$

More general, if the matrix [**A**] has k rows and 3 columns, then it results that the kth order square matrix

$$[\mathbf{D}] = [\mathbf{A}]^{\mathrm{T}}[\mathbf{B}][\mathbf{A}] \tag{1.80}$$

is a skew matrix; moreover, it results that if $k = 1$, then the matrix [**D**] is the zero matrix with only one element.

Sometimes, in the analytical calculations, it is useful to use the skew matrices associated to the unit vectors **i**, **j**, **k**,

$$[\mathbf{U}_1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, [\mathbf{U}_2] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, [\mathbf{U}_3] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.81}$$

and the column matrices

$$\{\mathbf{u}_1\} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \{\mathbf{u}_2\} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \{\mathbf{u}_3\} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{1.82}$$

respectively.

One obtains the expressions

$$[\mathbf{B}] = \sum_{i=1}^{3} b_i [\mathbf{U}_i], \{\mathbf{b}\} = \sum_{i=1}^{3} b_i \{\mathbf{u}_i\}, \tag{1.83}$$

$$[\mathbf{U}_i]\left[\mathbf{U}_j\right][\mathbf{U}_i] = [\mathbf{0}], (\forall)\ i \neq j, \tag{1.84}$$

$$[\mathbf{U}_1] = [\mathbf{U}_2][\mathbf{U}_3] - [\mathbf{U}_3][\mathbf{U}_2] \tag{1.85}$$

and the analogous,

$$[\mathbf{U}_1][\mathbf{U}_2][\mathbf{U}_3] + [\mathbf{U}_3][\mathbf{U}_2][\mathbf{U}_1] = [\mathbf{0}]. \tag{1.86}$$

and the analogous.

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2

Kinematics of the Rigid Solid

The chapter treats the kinematics of the rigid solid. Here we obtain the distribution of velocities and accelerations for a rigid body.

2.1 Finite Displacements of the Points of Rigid Solid

The motion of a rigid solid relative to a tri-orthogonal dextrorsum reference system O_0XYZ is completely determined by the motion of a tri-orthogonal dextrorsum reference system $Oxyz$ jointed to the rigid solid, relative to the reference system O_0XYZ (Fig. 2.1).

One considers that at the initial moment the system $Oxyz$ coincides to the system O_0XYZ. In these conditions, the finite displacement of the point O is $\mathbf{O}_0\mathbf{O} = \mathbf{s}_O$, while the finite displacement of an arbitrary point P (Fig. 2.1) is $\mathbf{P}_0\mathbf{P} = \mathbf{s}$.

We denote by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the unit vectors of the mobile axes, by $\mathbf{i}_0$, $\mathbf{j}_0$, $\mathbf{k}_0$ the unit vectors of the fixed axes, by a_{1i}, a_{2i}, a_{3i}, $i = 1, 2, 3$, the director cosines of the axes Ox, Oy, Oz, by X, Y, Z, x, y, z the coordinates of the point P relative to the two reference systems, and by X_O, Y_O, Z_O the coordinates of the point O relative to the fixed reference system. Keeping into account that the point P_0 has the same position relative to the system O_0XYZ as the point P relative to the system $Oxyz$, one may write the vectors

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},\ \mathbf{r}_0 = x\mathbf{i}_0 + y\mathbf{j}_0 + z\mathbf{k}_0,\ \mathbf{s}_O = X_O\mathbf{i}_0 + Y_O\mathbf{j}_0 + Z_O\mathbf{k}_0,\ \mathbf{R} = X\mathbf{i}_0 + Y\mathbf{j}_0 + Z\mathbf{k}_0, \tag{2.1}$$

Dynamics of the Rigid Solid with General Constraints by a Multibody Approach, First Edition.
Nicolae Pandrea and Nicolae-Doru Stănescu.

Companion website: www.wiley.com/go/pandrea

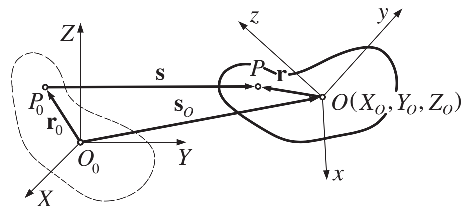

Figure 2.1 Finite displacements of the rigid solid.

the column matrices

$$\{\mathbf{r}\} = [x \ \ y \ \ z]^{\mathrm{T}}, \{\mathbf{r}_0\} = [x \ \ y \ \ z]^{\mathrm{T}}, \{\mathbf{s}_O\} = [X_O \ \ Y_O \ \ Z_O]^{\mathrm{T}}, \{\mathbf{R}\} = [X \ \ Y \ \ Z]^{\mathrm{T}}, \tag{2.2}$$

the rotational matrix

$$[\mathbf{A}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \tag{2.3}$$

and the matrix relation of transformation

$$\{\mathbf{R}\} = \{\mathbf{s}_O\} + [\mathbf{A}]\{\mathbf{r}\}. \tag{2.4}$$

At the initial moment, the point P being situated at the point P_0 (Fig. 2.1), the displacement $\mathbf{s} = \mathbf{P}_0\mathbf{P}$ of this point may be written in the form $\mathbf{s} = \mathbf{R} - \mathbf{r}_0$; keeping into account the notations (2.2), one obtains the matrix expression

$$\{\mathbf{s}\} = \{\mathbf{s}_O\} + [\mathbf{A}]\{\mathbf{r}\} - \{\mathbf{r}\} \tag{2.5}$$

or

$$\{\mathbf{s}\} = \{\mathbf{s}_O\} + [\mathbf{A}]\{\mathbf{r}_0\} - \{\mathbf{r}_0\}, \tag{2.6}$$

where $\{\mathbf{s}\}$ is the column matrix of the projection of vector $\mathbf{s}$ onto the axes of the fixed system O_0XYZ.

2.2 Matrix of Rotation: Properties

2.2.1 General Properties

From the vector relations $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$, $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1$ it results the scalar relations

$$\sum_{i=1}^{3} a_{ij} a_{ik} = \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \neq k, \end{cases} \quad j, k = 1, 2, 3, \tag{2.7}$$

which show that the matrix [**A**] is and orthogonal one; hence

$$[\mathbf{A}]^{-1} = [\mathbf{A}]^{\mathrm{T}}. \tag{2.8}$$

From the vector relation $\mathbf{i} = \mathbf{j} \times \mathbf{k}$ one obtains the scalar relations

$$a_{11} = a_{22}a_{33} - a_{32}a_{23},\ a_{21} = a_{23}a_{31} - a_{21}a_{33},\ a_{31} = a_{21}a_{32} - a_{22}a_{31}; \tag{2.9}$$

developing det[**A**] after the first column, we get

$$\det[\mathbf{A}] = 1. \tag{2.10}$$

Considering the vector defined by the elements of the column matrix $[\mathbf{A}]\{\mathbf{r}\}$ and denoting this vector by $\overline{[\mathbf{A}]\{\mathbf{r}\}}$, from the conditions of distances and angles preservation, it results the expressions

$$\overline{[\mathbf{A}]\{\mathbf{r}\}} = |\mathbf{r}|, \tag{2.11}$$

$$\overline{[\mathbf{A}]\{\mathbf{u}\}} \cdot \overline{[\mathbf{A}]\{\mathbf{v}\}} = \mathbf{u} \cdot \mathbf{v}, \tag{2.12}$$

$$\overline{[\mathbf{A}]\{\mathbf{u}\}} \times \overline{[\mathbf{A}]\{v\}} = \overline{[\mathbf{A}]\{\mathbf{q}\}},\ \text{where}\ \mathbf{q} = \mathbf{u} \times \mathbf{v}. \tag{2.13}$$

2.2.2 *Successive Displacements*

Let us consider two positions of the rigid solid and two jointed reference systems $O_1x_1y_1z_1$ and $O_2x_2y_2z_2$, respectively. Denoting by $[\mathbf{A}_{10}]$, $[\mathbf{A}_{20}]$ the rotational matrices relative to the fixed reference system O_0XYZ, one obtains the following relations for the column matrices $\{\mathbf{v}^{(1)}\}$, $\{\mathbf{v}^{(2)}\}$, $\{\mathbf{v}^{(0)}\}$ of the projections of an arbitrary vector **v** in the three reference systems

$$\left\{\mathbf{v}^{(0)}\right\} = [\mathbf{A}_{10}]\left\{\mathbf{v}^{(1)}\right\}, \left\{\mathbf{v}^{(0)}\right\} = [\mathbf{A}_{20}]\left\{\mathbf{v}^{(2)}\right\}. \tag{2.14}$$

It results

$$\left\{\mathbf{v}^{(1)}\right\} = [\mathbf{A}_{10}]^{\mathrm{T}}[\mathbf{A}_{20}]\left\{\mathbf{v}^{(2)}\right\}; \tag{2.15}$$

hence, the matrix of rotation of the reference system $O_2x_2y_2z_2$ relative to the reference system $O_1x_1y_1z_1$ reads

$$[\mathbf{A}_{21}] = [\mathbf{A}_{10}]^{\mathrm{T}}[\mathbf{A}_{20}]. \tag{2.16}$$

From the expression (2.16) it also results

$$[\mathbf{A}_{20}] = [\mathbf{A}_{10}][\mathbf{A}_{21}], \tag{2.17}$$

and therefore, in general, for n positions of the solid rigid, one obtains the matrix relation

$$[\mathbf{A}_{n0}] = [\mathbf{A}_{10}][\mathbf{A}_{21}]\ldots[\mathbf{A}_{n,n-1}]. \tag{2.18}$$

2.2.3 Eigenvalues: Eigenvectors

The eigenvalues λ_1, λ_2, λ_3 of the matrix [**A**] are obtained by solving the equation

$$\det[[\mathbf{A}] - \lambda[\mathbf{I}]] = 0, \tag{2.19}$$

where [**I**] is the unity matrix.
It results the equation

$$(\lambda - 1)\left[\lambda^2 - (\delta - 1)\cdot\lambda + 1\right] = 0, \tag{2.20}$$

where

$$\delta = \mathrm{Tr}[\mathbf{A}] = a_{11} + a_{22} + a_{33}. \tag{2.21}$$

Keeping into account the relations (2.1), it results $(\delta - 1)^2 \le 4$ and, consequently, one obtains a real solution

$$\lambda_1 = 1 \tag{2.22}$$

and two complex solutions

$$\lambda_{2,3} = \frac{\delta - 1}{2} \pm \mathrm{i}\sqrt{1 - \left(\frac{\delta - 1}{2}\right)^2}. \tag{2.23}$$

To determine the real unitary vector $\{\mathbf{u}\}$ it is necessary to solve the system of equations

$$[\mathbf{A}]\{\mathbf{u}\} = \{\mathbf{u}\}, \tag{2.24}$$

$$\{\mathbf{u}\}^{\mathrm{T}}\{\mathbf{u}\} = 1. \tag{2.25}$$

From the matrix equation (2.24) one obtains the scalar relations

$$u_x(a_{32}+a_{23}) = u_y(a_{13}+a_{31}) = u_z(a_{21}+a_{12}); \tag{2.26}$$

keeping into account the equalities

$$a_{13}^2 - a_{31}^2 = a_{21}^2 - a_{12}^2 = a_{32}^2 - a_{23}^2, \tag{2.27}$$

the relations (2.26) become

$$\frac{u_x}{a_{32}-a_{23}} = \frac{u_y}{a_{13}-a_{31}} = \frac{u_z}{a_{21}-a_{12}}. \tag{2.28}$$

The relations (2.28) and (2.27) show that the vectors

$$\mathbf{w} = \frac{1}{2}[(a_{32}-a_{23})\mathbf{i}_0 + (a_{13}-a_{31})\mathbf{j}_0 + (a_{21}-a_{12})\mathbf{k}_0], \tag{2.29}$$

$$\mathbf{w}^* = \frac{1}{a_{32}+a_{23}}\mathbf{i}_0 + \frac{1}{a_{13}+a_{31}}\mathbf{j}_0 + \frac{1}{a_{21}+a_{12}}\mathbf{k}_0, \tag{2.30}$$

(which are not unitary ones) verify the matrix equation (2.24), that is, they are eigenvectors of the matrix [**A**].

If the vector **w** is not equal to zero, then the unitary eigenvector is calculated using the relation

$$\mathbf{u} = \frac{\mathbf{w}}{|\mathbf{w}|}; \tag{2.31}$$

if the vector **w** is a null one, then we have to use the relations (2.24) and (2.25).

Example 2.2.1 Determine the real unitary eigenvectors for the rotational matrices

$$[\mathbf{A}_1] = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, [\mathbf{A}_2] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, [\mathbf{A}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{2.32}$$

Solution: Since

$$\det[\mathbf{A}_1] = \det[\mathbf{A}_2] = \det[\mathbf{A}_3] = 1, \tag{2.33}$$

it results that the given matrices correspond to dextrorsum tri-orthogonal systems.

For the matrix $[\mathbf{A}_1]$ we get

$$\mathbf{w}_1 = \frac{1}{2}(\mathbf{i}_0 - \mathbf{j}_0 + \mathbf{k}_0),\ \mathbf{u}_1 = \frac{\mathbf{w}_1}{|\mathbf{w}_1|} = \frac{1}{\sqrt{3}}(\mathbf{i}_0 - \mathbf{j}_0 - \mathbf{k}_0). \tag{2.34}$$

For the matrices $[\mathbf{A}_2]$, $[\mathbf{A}_3]$ one obtains the vectors

$$\mathbf{w}_2 = \mathbf{w}_3 = \mathbf{0} \tag{2.35}$$

and then, from the equations (2.24) and (2.25) it results

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}}(\mathbf{i}_0 + \mathbf{k}_0),\ \mathbf{u}_3 = \mathbf{i}_0. \tag{2.36}$$

2.2.4 The Expression of the Matrix of Rotation with the Aid of the Unitary Eigenvector and the Angle of Rotation

One considers the case in which a rigid solid rotates with the angle ξ (Fig. 2.2) about an axis Δ of unit vector $\mathbf{u}$.

By this motion, an arbitrary point P_0 transforms in the point P, the vector $\mathbf{O_0P_0}$ becomes the vector $\mathbf{O_0P}$, the vector $\mathbf{CP_0}$ (C being the projection of the point P_0 onto the axis Δ) transforms in the vector $\mathbf{CP}$, while the reference system O_0XYZ becomes the system $Oxyz$, $O \equiv O_0$.

It is obvious that the rotation angle ξ is the angle between the vectors $\mathbf{CP_0}$ and $\mathbf{CP}$.

The Rodrigues relation reads

$$\mathbf{O_0P} = \mathbf{O_0P_0} + \mathbf{u} \times \mathbf{O_0P_0} \sin\xi + \mathbf{u} \times (\mathbf{u} \times \mathbf{O_0P_0})(1 - \cos\xi), \tag{2.37}$$

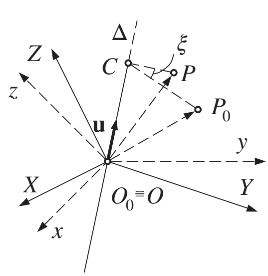

Figure 2.2 The rotation about the axis Δ.