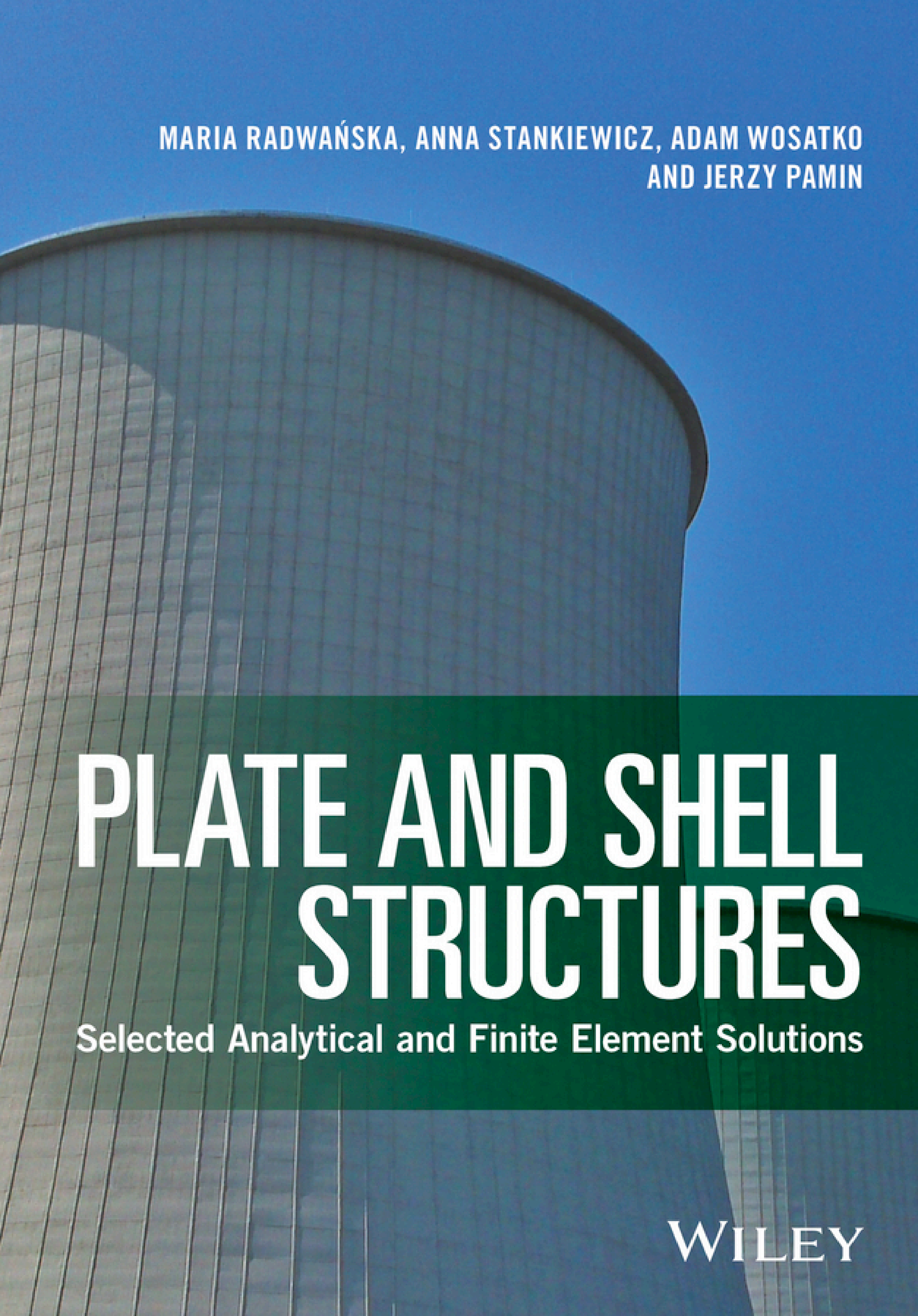

MARIA RADWAŃSKA, ANNA STANKIEWICZ, ADAM WOSATKO
AND JERZY PAMIN

PLATE AND SHELL STRUCTURES

Selected Analytical and Finite Element Solutions

WILEY

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To our families

Contents

Preface

This book deals with the mechanics and numerical simulations of plates and shells, which are flat and curved thin-walled structures, respectively (called shell structures for short in this book). They have very important applications as complete structures or structural elements in many branches of engineering. Examples of shell structures in civil and mechanical engineering include slabs, vaults, roofs, domes, chimneys, cooling towers, pipes, tanks, containers and pressure vessels; in shipbuilding – ship and submarine hulls, in the vehicle and aerospace industries – automobile bodies and tyres and the wings and fuselages of aeroplanes.

The scope of the book is limited to the presentation of the theory of elastic plates and shells undergoing small deformation (thus assuming linear constitutive and kinematic equations).

The book is aimed at the large international community of engineering students, university teachers, professional engineers and researchers interested in the mechanics of shell structures, as well as developers testing new simulation software. The book can be the basis of an intermediate-level course on (computational) mechanics of shell structures at the level of doctoral, graduate and undergraduate studies. The reader should have the basic knowledge of the strength of materials, theory of elasticity, structural mechanics and FEM technology; basic information in these areas is not repeated in the book.

The strength of the book results from the fact that it not only provides the theoretical formulation of fundamental problems of mechanics of plates and shells, but also several examples of analytical and numerical solutions for different types of shell structures. The book also contains some advanced aspects related to the stability analysis and a brief description of classical and modern finite element formulations for plates and shells, including the discussion of mixed/hybrid models and so-called locking phenomena.

The book contains a comprehensive presentation of the theory of elastic plates and shells, formulations and solutions of fundamental mechanical problems (statics, stability, free vibrations) for these structures using exact approaches and computational (approximate) methods, with emphasis on modern capabilities of the finite element (FE) technology. In the book we introduce a large number of examples that illustrate various physical phenomena associated with the behaviour of shell structures under external actions. Comparisons of analytical and numerical solutions are given for several benchmark tests. The book includes plenty of boxes and tables that contain sets of formulae or data and check values describing the examples. They help the reader to find and integrate the information provided and draw conclusions.

The authors are researchers and teachers from the Institute for Computational Civil Engineering of Cracow University of Technology. They have done research on structural mechanics for years, in particular on the theories and advanced computational methods for shell structures, and they also have a long history of teaching the subject to students and practitioners. The selection of the contents of the book is based on this experience. The motivation to write the present book has also come from the fact that there are no books that contain, in one volume, the foundation of the theory and solutions of selected problems using simultaneously analytical and numerical methods.

Following a sequence of subjects: mathematics, theoretical mechanics, strength of materials, structural mechanics, computer science, numerical methods and the finite element method – we have developed a comprehensive course on the mechanics of shell structures. This course contains: (i) discussion of the assumptions and limits of applicability of selected theories on which mathematical models are based, (ii) choice of a method to solve the problem efficiently, (iii) analytical and/or numerical calculations simulating physical phenomena or processes, (iv) confrontation of the results of theoretical and numerical analysis and (v) evaluation of the calculation methodology and results.

Maria Radwańska and Jerzy Pamin were members of Professor Zenon Waszczyszyn's research team, who implemented the finite element code ANKA for buckling and nonlinear analysis of structures at the end of the twentieth century. This resulted in the 1994 Elsevier book: Waszczyszyn, Z. and Cichoń, Cz. and Radwańska, M., *Stability of Structures by Finite Element Method*.

Next, we briefly describe the contents of the book, which is divided into five parts. Part 1 is the introductory part that gives a compact encyclopedic overview of the fundamentals of the theory and modelling of plates and shells in the linear elastic range. A description of static analysis of (plane) plates is contained in Part 2 and of (curved) shells in Part 3. Part 4 includes information on the selected problems of buckling and free vibrations of shell structures. In Part 5, the authors discuss the general aspects of finite element analysis, including the modelling process, evaluation of the quality of finite elements and accuracy of solutions, Part 5 also contains a brief presentation of advanced formulations of finite elements for plates and shells.

While working on the book, we felt special gratitude to two of our teachers: Professors Zenon Waszczyszyn and Michał Życzkowski, who we always thought of as scientific authorities in the field of structural mechanics. In particular, we are deeply indebted to Professor Zenon Waszczyszyn for his invaluable contribution to our knowledge, motivation to do research and to participate in high-level university education. Under his guidance we got to know the theory of plates and shells, computational mechanics applied in civil engineering and modern numerical methods; in particular, the finite element method.

The authors wish to express their appreciation to several colleagues from the Institute for Computational Civil Engineering for discussions and help during the preparation of the book, in particular to A. Matuszak, E. Pabisek, P. Pluciński, R. Putanowicz and T. Żebro. We also record our gratitude to our students who cooperated with us in the computation of numerous examples: M. Abramowicz, M. Bera, I. Bugaj, M. Florek, S. Janowiak, A. Kornaś and K. Kwinta.

Notation

Detailed notation for theoretical analysis

Indices

$\alpha, \beta = 1, 2$	Greek indices (for curvature lines and surface coordinates)
$i, j, k = 1, 2, 3$	Latin indices (for 3D space)
n, m, t	indices for membrane, bending, transverse shear states
$j = 0, 1, 2, \ldots$	number of a components of trigonometric series or number of circumferential wave (half-wave)
(i, j)	indices describing number of waves of deformation in two directions

Coefficients and variables

α_T	coefficient of thermal expansion of a material
A_α: A_1, A_2	Lame coefficients
$\hat{\mathbf{b}} = [\hat{b}_x, \hat{b}_y, \hat{b}_z]^{\mathrm{T}}$	prescribed body forces
$b_{\alpha\beta}$	components of II (second) metric tensor
$\beta = \sqrt{\dfrac{1}{R\,h}}\ \sqrt[4]{3(1-\nu^2)}$	coefficient in equation of local bending state in cylindrical shell
C^0, C^*	initial and current configuration of a body (shell)
$\mathbf{C}$, $\mathbf{E} = \mathbf{C}^{-1}$	matrices of local flexibility and stiffness in constitutive equations
$D^n = \dfrac{Eh}{1-\nu^2}$	cross-sectional stiffness in membrane state
$D^m = \dfrac{Eh^3}{12(1-\nu^2)}$	cross-sectional stiffness in bending state
$D^t = \dfrac{k\,Eh}{2(1+\nu)}, k = \dfrac{5}{6}$	cross-sectional stiffness in transverse shear state
$(\mathbf{e}_\alpha, \mathbf{n})$, $(\mathbf{e}_\alpha^{(z)}, \mathbf{n})$	local base versors on middle surface and on equidistant surface from the middle surface in initial configuration
$(\mathbf{e}_\alpha^*, \mathbf{n}^*)$	local base versors on middle surface in current configuration
$\mathbf{e}^{\mathrm{T}} = [\mathbf{e}^n, \mathbf{e}^m, \mathbf{e}^t]$	generalized strain vector (membrane, bending and transverse shear components)

$\mathbf{e}^n = [\varepsilon_{11}, \varepsilon_{22}, \gamma_{12}]^{\mathrm{T}}$	membrane strain vector
$\varepsilon_{11}, \varepsilon_{22}, \gamma_{12} = \varepsilon_{12} + \varepsilon_{21}$	membrane strains: normal and shear in middle surface
$\varepsilon_x, \varepsilon_\theta, \gamma_{x\theta}$	membrane strains in cylindrical system
$\varepsilon_\varphi, \varepsilon_\theta, \gamma_{\varphi\theta}$	membrane strains in spherical system
$\mathbf{e}^m = [\kappa_{11}, \kappa_{22}, \chi_{12}]^{\mathrm{T}}$	bending strain vector
$\kappa_{11}, \kappa_{22}, \chi_{12} = \kappa_{12} + \kappa_{21}$	bending strains: changes of curvature and warping of middle surface
$\kappa_x, \kappa_\theta, \chi_{x\theta}$	bending strains in cylindrical system
$\kappa_\varphi, \kappa_\theta, \chi_{\varphi\theta}$	bending strains in spherical system
$\mathbf{e}^t = [\gamma_{1z}, \gamma_{2z}]^{\mathrm{T}}$	transverse shear strain vector
$\gamma_{xz}, \gamma_{\theta z}$	transverse shear strains in cylindrical system
$\gamma_{\varphi z}, \gamma_{\theta z}$	transverse shear strains in spherical system
$E, \nu, G = E/(2 + 2\ \nu)$	material constants: Young's modulus, Poisson's ratio, Kirchhoff's modulus
f	rise of shallow shell
F	Airy's stress function
$\mathbf{i}_k$	base versors related to Cartesian coordinates x^k
$g_{\alpha\beta}$	components of I (first) metric tensor
h	thickness of shell
K	Gaussian curvature of surface
$\lambda = \pi/\beta$	length of half-wave for exponential-trigonometric function in local membrane-bending state of cylindrical shell
$m_{11}, m_{22}, m_{12} = m_{21}$	moments: bending and twisting in middle surface
$m_x, m_\theta, m_{x\theta}$	moments in cylindrical system
$m_\varphi, m_\theta, m_{\varphi\theta}$	moments in spherical system
$n_{11}, n_{22}, n_{12} = n_{21}$	membrane forces: normal and tangential in middle surface
$n_x, n_\theta, n_{x\theta}$	membrane forces in cylindrical system
$n_\varphi, n_\theta, n_{\varphi\theta}$	membrane forces in spherical system
$n_{\mathrm{I}}, n_{\mathrm{II}}, m_{\mathrm{I}}, m_{\mathrm{II}}$	principal membrane forces and bending moments
$\tilde{n}_{\nu s}, \tilde{t}_\nu$	effective boundary forces (tangential membrane and transverse shear)
$n_\nu, \tilde{n}_{\nu s}, \tilde{t}_\nu, m_\nu$	generalized boundary forces
$\hat{n}_\nu, \hat{n}_{\nu s}, \hat{t}_\nu, \hat{m}_\nu$	prescribed generalized boundary loads
ν, s, n	directions of boundary base vectors
$\hat{\mathbf{p}} = [\hat{p}_1, \hat{p}_2, \hat{p}_n]^{\mathrm{T}}$	vector of prescribed surface loads
$\hat{\mathbf{p}}_b, \hat{\mathbf{u}}_b$	vectors of prescribed generalized boundary loads and displacements
$\hat{P}_i$	prescribed concentrated force in corner i
$\Pi, \Pi^{(z)}$	middle and equidistant surfaces in initial configuration
$\Pi^*, \Pi^{*\,(z)}$	middle and equidistant surfaces in current configuration

Π_ν, Π_s	cross-sectional planes: normal and tangent to middle surface
Π_α: Π_1, Π_2	two transverse cross-sectional planes normal to middle surface
Π, U, W	total potential energy, internal energy, external load work
$r = f(x)$, $r = f(\varphi)$	meridian equation for axisymmetric shell
R_α: R_1, R_2	principal curvature radii for middle surface of a shell
s	arch coordinate for a line on surface
$\mathbf{s}^{\mathrm{T}} = [\mathbf{s}^n, \mathbf{s}^m, \mathbf{s}^t]$	vector of generalized resultant forces for membrane, bending and transverse shear states
$\mathbf{s}_b = [n_\nu, \tilde{n}_{\nu s}, \tilde{t}_\nu, m_\nu]^{\mathrm{T}}$	vector of generalized boundary forces
$\hat{\mathbf{s}}_b = [\hat{n}_\nu, \hat{n}_{\nu s}, \hat{t}_\nu, \hat{m}_\nu]^{\mathrm{T}}$	vector of presribed boundary forces
$\mathbf{s}^n = \mathbf{n} = [n_{11}, n_{22}, n_{12}]^{\mathrm{T}}$	vector of membrane forces
$\mathbf{s}^m = \mathbf{m} = [m_{11}, m_{22}, m_{12}]^{\mathrm{T}}$	vector of bending and twisting moments
$\mathbf{s}^t = \mathbf{t} = [t_1, t_2]^{\mathrm{T}}$	vector of transverse shear forces
$\varsigma = \sqrt{\frac{R}{h}}\,\sqrt[4]{3(1-\nu^2)}$	coefficient in equation of local bending state in spherical shell
T_i	effective force in a corner used in static boundary conditions
$\vartheta = [\vartheta_1, \vartheta_2, \vartheta_n]^{\mathrm{T}}$	vector of rotations
ϑ_α: ϑ_1, ϑ_2	two rotations of normal to middle surface
ϑ_n	rotation around normal to middle surface
$\vartheta_x = \varphi_y$, $\vartheta_y = -\varphi_x$	two rotations of normal to middle plane of plate under bending in Cartesian system (two alternative notations)
σ_{nn}, σ_{ns}, σ_{nz}	stresses: in-plane normal, in-plane tangential, transverse shear
$u = u_x$, $v = u_y$, w	translations with respect to local system (x, y, z)
U, V, W	translations with respect to global system (X, Y, Z)
$\mathbf{u} = [u_1, u_2, w, \vartheta_1, \vartheta_2, \vartheta_n]^{\mathrm{T}}$	generalized displacement vector
$\mathbf{u} = [u_1, u_2, w]^{\mathrm{T}}$	translation vector in three-parameter thin shell theory
$\mathbf{u} = [w, \vartheta_1, \vartheta_2]^{\mathrm{T}}$	generalized displacement vector in three-parameter moderately thick plate theory
$\mathbf{u} = [u_1, u_2, w, \vartheta_1, \vartheta_2]^{\mathrm{T}}$	generalized displacement vector in five-parameter moderately thick shell theory
$\mathbf{u}_b = [u_\nu, u_s, w, \vartheta_\nu]$	vector of generalized boundary displacements
$\hat{\mathbf{u}}_b = [\hat{u}_\nu, \hat{u}_s, \hat{w}, \hat{\vartheta}_\nu]$	vector of prescribed generalized boundary displacements
U^n, U^m, U^t	strain energy in membrane, bending and transverse shear states
ξ_α: ξ_1, ξ_2	curvilinear surface coordinates on middle surface $z = 0$
$\xi_\alpha = \text{const.}$	coordinate lines on middle surface
$\xi_1 = x$, $\xi_2 = y$	Cartesian coordinates
$\xi_1 = \varphi$, $\xi_2 = \theta$	spherical coordinates

$\xi_1 = r, \xi_2 = \theta$	polar coordinates
$\xi_1 = x, \xi_2 = \theta$	cylindrical coordinates
z	coordinate in direction normal to the middle surface Π (distance of equidistant surface $\Pi^{(z)}$ from middle surface Π, $z = 0$ corresponds to the middle surface Π)
$x^k: x^1, x^2, x^3$	Cartesian coordinates with respect to versors $\mathbf{i}_k$
(X, Y, Z)	Cartesian coordinate system
Ω	problem domain
$\partial\Omega$	boundary of domain
$\partial\Omega_\sigma, \partial\Omega_u$	boundary with prescribed loads and displacements, respectively
T_r	reference temperature
T_0	temperature on middle surface
$\Delta T_0 = T_0 - T_\mathrm{r}$	temperature change (independent of z) with respect to reference temperature
$\Delta T_h = \Delta T(h/2) - \Delta T(-h/2)$	temperature difference between limiting shell surfaces $z = \pm h/2$

Detailed notation for numerical analysis

Indices

e	index of finite element (FE)
(ef)	index of interelement boundary
n, $node$	index of FE node

Abbreviations

NNDOF, *NEDOF*, *NSDOF*	number of degrees of freedom (dofs) for node, element and structure
NSE	number of FEs in a structure
NEN, *NSN*	number of FE nodes and of structure nodes
NGP	number of Gauss points

Coefficients and variables

$\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_\sigma, \boldsymbol{\alpha}_\varepsilon$	mathematical dofs for interpolation of displacement, stress, strain fields
$\mathbf{B}^n, \mathbf{B}^m, \mathbf{B}^t$	matrices in kinematic relations for membrane, bending and transverse shear states
$\mathbf{D}^n, \mathbf{D}^m, \mathbf{D}^t$	matrices in constitutive equations for membrane, bending and transverse shear states

$\mathbf{f}^e, \mathbf{f}_b^e$	vector of substitute nodal forces which represent loads in FE and on FE boundary
$\mathbf{F}$	global vector of substitute nodal forces after assembly process
$I_{\mathrm{p}}[\mathbf{u}], I_{\mathrm{c}}[\boldsymbol{\sigma}]$	potential and complementary energy functionals
$I_{\mathrm{p,m}}, I_{\mathrm{c,m}}$	modified potential and complementary energy functionals
$I_{\mathrm{H\text{-}R}}[\mathbf{u}, \boldsymbol{\sigma}]$	two-field Hellinger–Reissner functional
$I_{\mathrm{H\text{-}W}}[\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\epsilon}]$	three-field Hu–Washizu functional
$G^{(ef)}[\boldsymbol{\sigma}, \mathbf{u}^{(ef)}]$	component added to functional and associated with the equilibrium of tractions on interelement boundary
$H^{(ef)}[\mathbf{u}, \mathbf{t}^{(ef)}]$	component added to functional and associated with the continuity of displacements on interelement boundary
J, $\det \mathbf{J}$	jacobian, determinant of Jacobi matrix
$\mathbf{k}^{en}, \mathbf{k}^{em}, \mathbf{k}^{et}$	element stiffness matrix for membrane, bending and transverse shear states
$\mathbf{k}_\sigma^e$	stress stiffness matrix in initial and linearized buckling analysis
$\mathbf{k}_u^e$	displacement stiffness matrix for FE in linearized buckling analysis
$\mathbf{L}$	matrix of differential operators in kinematic strain-displacement equations $\boldsymbol{\varepsilon} = \mathbf{L}\,\mathbf{u}$
$\mathbf{N}$	matrix of shape functions used for displacement field approximation
$\mathbf{N}_u, \mathbf{N}_\sigma, \mathbf{N}_\varepsilon$	matrices for approximation of displacement, stress, strain fields in two- or three-field formulation in mixed FEs
$\mathbf{P}^*, \mathbf{Q}^*$	vectors of reference loads and displacements for one-parameter loading process
$\mathbf{q}^e = \mathbf{q}_u^e$	element generalized displacement vector for displacement-based FE model
$\mathbf{q}_{node}$	nodal generalized displacement (dof) vector for displacement-based FE model
$\mathbf{q}_u^e, \mathbf{q}_\sigma^e, \mathbf{q}_\varepsilon^e$	vectors of element generalized displacement, stress and strain dofs, respectively, for different FE models
$\mathbf{q}_u^{(ef)}, \mathbf{q}_t^{(ef)}$	vectors of generalized displacement or, respectively, traction dofs on interelement boundary
$\mathbf{Q}$	vector of generalized displacements for structure
$\mathbf{u}(\mathbf{x}), \boldsymbol{\sigma}(\mathbf{x}), \boldsymbol{\varepsilon}(\mathbf{x})$	displacement, stress, strain fields approximated within FE domain
$\mathbf{u}^{(ef)}(s), \mathbf{t}^{(ef)}(s)$	displacement and traction function approximated along interelement boundary
$\mathbf{R}_{\mathrm{supp}}$	support reaction vector for structure
ξ, η, ζ	natural normalized dimensionless coordinates
$\Omega^e, \partial\Omega^e$	area and boundary of FE
$\partial\Omega^{(ef)}$	interelement boundary

Conversions between imperial and metric system units

Quantity	Imperial units	International System of Units (SI)	
length	1 in.	= 2.54 cm	= 0.0254 m
	1 ft.	= 30.48 cm	= 0.3048 m
area	1 in.^{2}	= 6.45 cm^2	= 0.000645 m^2
	1 ft^2	= 929 cm^2	= 0.0929 m^2
force	1 lb-f = 1 lbf	= 4.45 N	= 0.00445 kN
moment	1 lbf-in.	= 11.31 Ncm	= 0.0001131 kNm
intensity of membrane force	1 lbf/in.	= 1.751 N/cm	= 0.175 kN/m
intensity of moment	1 lbf-in./in.	= 4.45 N/cm/cm	= 0.00445 kNm/m
pressure	1 psi = 1 lbf/in.^{2}	= 0.690 N/cm^2	= 6.90 kN/m^2

Part 1

Fundamentals: Theory and Modelling

1

General Information

1.1 Introduction

In the classification of mechanical structures, somewhere between one-dimensional (1D) bar structures and three-dimensional (3D) solid structures, a class of two-dimensional (2D) plates and shells (thin-walled flat and curved structures) can be distinguished. The attention is focused on a deformable solid body, which is limited by two surfaces (top and bottom) and lateral surfaces, see Figure 1.1. The distance between the top and bottom surfaces, identified as the thickness, is small compared to the other dimensions of the body (e.g. radius of curvature or span), measured referring to the so-called primary surface (2D physical model), most often taken as the middle surface defined as equidistant from the top and bottom surfaces.

The following, generally accepted nomenclature is going to be used throughout the book:

- shells = thin-walled curved shells
- curved membranes = special shells that have no bending rigidity
- plates = thin plane structures that have some subclasses:
  - flat membranes = plates with load in the middle plane, sometimes also called panels
  - plates under bending = plates with transverse load (normal to the middle plane), sometimes also called slabs

In the general description, for all these classes we will use the name ‘shell structures’ or, in brief, ‘shells’. In other words, we understand that shell structures can be flat.

Scientists, teachers, students, engineers and even the authors of software are interested in the mechanics of plates and shells. Due to the variety of potential users, the following variants of the mechanical theory have evolved:

- general advanced tensorial shell theory
- technical (engineering) shell theory

The scope of this book is limited to the case of linear constitutive and kinematic equations.

The theory is the basis for the construction of appropriate mathematical models (sets of differential and algebraic equations) and is associated with the calculation method that can be used to solve general or particular mechanical problems.

Plate and Shell Structures: Selected Analytical and Finite Element Solutions, First Edition.
Maria Radwańska, Anna Stankiewicz, Adam Wosatko and Jerzy Pamin.

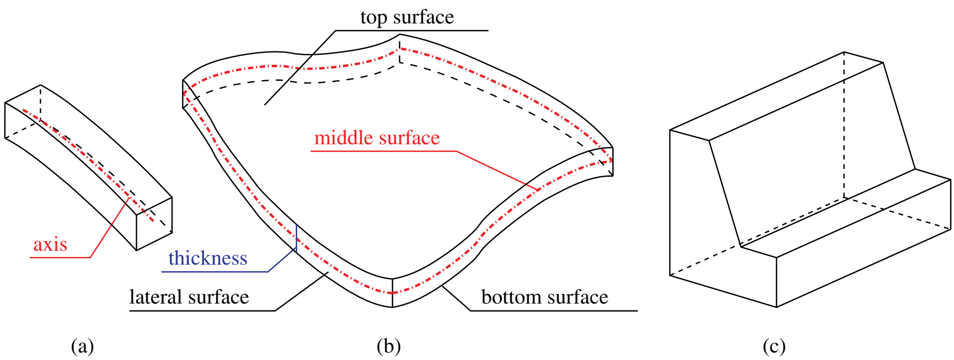


Figure 1.1 Structures: (a) bar (1D), (b) surface (2D) and (c) solid (3D)

In Section 1.2, encyclopedic information on the development of theories describing elastic plates and shells is included.

The description of shell structures, which makes them different from bar (1D) and solid (3D) bodies, must contain the following aspects:

- information on the coordinate systems and geometry of representative surfaces
- specification of kinematic constraints related to the mode of deformation
- definitions of so-called generalized strains with respect to the middle surface
- definitions of resultant forces and moments on the middle surface
- characteristics of fundamental stress and strain states

Detailed discussion is given in Sections 1.3 and 1.4.

A classification of plates and shells can be performed taking into account the slenderness (thickness to span ratio), the shape of the middle surface, the definitions and assumptions presented further in Section 1.4 and the character of stress distribution along the thickness, related to the stress state. In Section 1.5 and in Box 1.1, we present the classification of surface structures according to these aspects.

Thin-walled shell structures of various types are very important structural elements. Examples of shell structures can be encountered in civil and mechanical engineering (slabs, vaults, roofs, domes, chimneys, cooling towers, pipes, tanks, containers, pressure vessels), shipbuilding (ship hulls, submarine hulls) and in the vehicle and aerospace industries (car bodies and tyres, wings and fuselages of aeroplanes).

From an engineering point of view, it is necessary to predict different modes of behaviour of plates and shells under applied loading. In the case of a (flat) plate subjected to a transverse load, static equilibrium is preserved by the action of bending and twisting moments and transverse shear forces. On the other hand, the (curved) shell structure is able to carry the load inducing membrane tension or compression, distributed uniformly throughout the thickness (it is an optimal case from the viewpoint the material strength). This feature of shell structures makes them more economical and stiffer in comparison to plates.

Familiarity with the technical shell theory is necessary for engineers who are responsible for the safety of structures and are supposed to take into account various safety factors using computer-aided design.

Box 1.1 Summary of classification of shell structures

Thin plates for $\frac{h}{L_{\min}} < \frac{1}{10}$

Moderately thick plates for $\frac{1}{10} \leq \frac{h}{L_{\min}}$

Thin shells for $\frac{h}{R_{\min}} \leq \frac{1}{20}$

Moderately thick shells for $\frac{1}{20} < \frac{h}{R_{\min}} \leq \frac{1}{6}$

Thick shells for $\frac{1}{6} < \frac{h}{R_{\min}}$

Shallow shell for $\frac{f}{L} < \frac{1}{5}$

h – thickness of plate or shell

L – characteristic dimension of plate or shell

$L_{\min}$ – the smallest dimension in the middle plane of a plate

$R_{\min}$ – smaller of two principal radii of curvature

f – distance of shell from the horizontal plane, of its projection, that is rise

w – representative deflection

Geometrically linear theory of plates with small deflections for $|w| < \frac{h}{5}$

von Kármán theory of plates with moderately large deflections for $|w| \approx h$

Geometrically nonlinear theory of plates with large deflections for $|w| > 5h$

As emphasized in Ramm and Wall (2004), shell structures exhibit the strong influence of initial geometry, slenderness, type of loading and boundary conditions on the deformation and load carrying capacity. Small variations or even imperfections of these parameters can change the structural response significantly and, in particular, cause loss of stability.

Shells are characterized by an advantageous ratio of stiffness to weight, which makes them suitable for lightweight and long-spanned structures. Moreover, optimal shells are designed to carry predominantly membrane forces with minimum bending effects. It is therefore extremely important to understand the principal mechanical features of plates and shells before using computer-aided design involving numerical simulation.

1.2 Review of Theories Describing Elastic Plates and Shells

The general description of the historical development of plate and shell theory, as well as details of specific theories are referred to in a lot of books and monographs. Here, the authors do not try to present the developmental trends of this branch of mechanics, even limiting interest to the theory of elastic plates and shells undergoing small deformations.

The beginnings of the linear theory of plates and shells date back to the nineteenth century, however, the vibration problem of bells was considered by Leonhard Euler in 1764. The name of Sophie Germain is associated with the theory of plates: in 1811 she submitted work on plates for a contest announced by the French Academy of Sciences.

Following the two encyclopaedic elaborations:

- Mechanics of Elastic Plates and Shells, vol. 8 in *Technical Mechanics* (Borkowski et al. 2001)
- Models and Finite Elements for Thin-walled Structures, Chapter 3 by Bischoff et al. in vol. 2 of *The Encyclopedia of Computational Mechanics* (Stein et al. 2004)

the authors of this book would like to mention the names of researchers associated with the theories of plates and shells from three different, consecutive periods (listing names in alphabetical order):

- nineteenth century:
  A. Cauchy, S. Germain, A.E. Green, G. Kirchhoff, A.H. Love, S.D. Poisson and L. Rayleigh
- first half of the twentieth century:
  E. Cosserat and F. Cosserat, A.L. Gol'denveizer, Th. von Kármán, S. Lévy, A.I. Lur'e and E. Reissner
- second half of the twentieth century:
  Y. Başar, B. Budiansky, L.H. Donnell, J.L. Ericksen, W. Flügge, J.M. Gere, K. Girkmann, K.Z. Golimov, R. Harte, Z. Kączkowski, W.T. Koiter, W.B. Krätzig, H. Kraus, R.D. Mindlin, K.M. Mushtari, P.M. Naghdi, F.I. Niordson, W. Nowacki, V.V. Novozhilov, W. Pietraszkiewicz, E. Ramm, J.L. Sanders, J.G. Simmonds, I. Szabó, S.P. Timoshenko, C. Truesdell, V.Z. Vlasov, W. Wunderlich, S. Woinowski-Krieger, Cz. Woźniak and W. Zerna

We also mention some previous works relevant to the subject of this book, dividing them into:

- books dealing with the basis of mechanics: Timoshenko and Goodier (1951), Fung (1965), Washizu (1975), Reddy (1986), Borkowski et al. (2001) and Stein et al. (2004)
- monographs related to the theories of plates and shells: Girkmann (1956), Timoshenko and Woinowsky-Krieger (1959), Kolkunov (1972), Nowacki (1980), Niordson (1985), Noor et al. (1989), Waszczyszyn and Radwańska (1995), Reddy (1999), Başar and Krätzig (2001), Borkowski et al. (2001), Reddy (2007), Radwańska (2009), Wiśniewski (2010) and Oñate (2013)

The general formulation of the theory of thin-walled structures is determined by their specific geometry with one dimension (thickness) much smaller in comparison to the other two dimensions. There are two essential concepts that can be used to formulate the mathematical description of the problem.

One possibility is to start from the equations of three-dimensional continuum, describing a body with a specified geometry. Applying a power series representation of certain quantities as a function of coordinate z (measured in the direction of a thickness) the reduction to a two-dimensional theory is performed. Using a specified number of terms of this representation a 2D problem with varying accuracy of approximation is obtained.

Alternatively, one can adopt suitable kinematic assumptions and treat a thin-walled structure as a two-dimensional continuum representation of a substitute problem, (see Borkowski et al. 2001). This option is associated with direct methods of formulating two-dimensional models of plates and shells, based on appropriate static and kinematic hypotheses. The approximation in this theory is that the deformed state of the shell is determined entirely by the configuration of its middle surface.

Beside the two approaches based on three-dimensional continuum mechanics or two-dimensional surface-based theories we mention a so-called Cosserat surface concept, see for instance Chapter 3 in vol. 2 of Stein et al. (2004). This approach is an extension of classical continuum formulation by adding information about the orientation of a material point equipped with rotational degrees of freedom.

Among the developed theories for shells a few specific approaches can be distinguished:

- general theory applying any parametrization of the curved middle surface
- theory that uses the orthogonal parameterization of the middle surface based on principal curvature coordinates
- general membrane-bending shell theory with or without the consideration of transverse shear deformation
- theories for particular cases of shells (e.g. for cylindrical or spherical shells of revolution)
- theory of plates
- theory of flat membranes

The full set of equations of the linear theory of shells, which contains Kirchhoff plate equations as a special case, are given in pages 173–174 of Love (1944). This theory is called the Kirchhoff–Love (K–L) theory of first approximation or order. In theory based on assumptions of K–L the effects of transverse shear and normal strains in the thickness direction are neglected. The weakening of these assumptions leads to enhanced variants of the equations, the so-called second and third approximations. This involves more complex forms of measurement of deformation and construction of constitutive equations. In fact, the first approximation theory is mathematically and physically incorrect. When the kinematic equations and constitutive equations, used in this approach, are substituted into the sixth equilibrium equation (expressing equilibrium of moments around the normal to the middle surface), the equation is not satisfied. The sixth equilibrium equation guarantees that all strains vanish for small rigid-body rotations of the shell.

The inconsistency of Love's first order theory was removed in the improved theory for thin shells by Sanders (1959), formulating the equations in principal curvature coordinates. For this new improved theory modified equilibrium equations, strain-displacement relations and boundary conditions were derived using the principle of virtual work. The detailed information about the basics of theory of Sanders is presented in Chapter 3.

Koiter checked and corrected Love's theory (see Koiter 1960). An assessment of the order of magnitude of the terms in Love's strain-energy expression was carried out. Appropriate consistent stress-strain relations for stress resultants and equilibrium equations in tensorial form were presented. In the theory the sixth equation of equilibrium is satisfied identically.

In work of Budiansky and Sanders (1963) the equations of the 'best' first-order linear elastic shell theory were formulated for shells of arbitrary shape in a coordinate system related to the middle surface using general tensor notation.

In the broad literature a variety of kinematic and constitutive equations can be found, because different simplifications were used in their derivation. The summary of various descriptions of the strain state and kinematic relations (even for linear analysis) is also presented in the work by Lewiński (1980). The following four essential features of the improved first approximation shell theory are cited here from this paper:

- matrices of generalized strains (membrane strains and changes of curvature) and stress resultants (forces and moments) are symmetric
- constitutive equations are decoupled
- the sixth equation of equilibrium is identically satisfied
- a rigid motion of the shell does not cause strains or stresses

Now, a little information about the classical three-parameter Sanders thin shell theory is given, because this formulation is applied in our book. The equations are considered to be the most suitable with respect to both theoretical and numerical applications. In the geometry description the orthogonality of the coordinate lines implies that the first metric tensor is diagonal, and the surface is described by only two Lame parameters and two radii of curvature (or curvatures themselves), see Subsection 1.3.2. The following fields are used in the shell problem description: translation(s), rotation(s), generalized strain(s) and stress resultants, all defined with respect to the two-dimensional middle surface. In this three-parameter thin shell theory three translations u_1, u_2, w are adopted as independent variables in the description of the deformation (see Subsection 1.4.1).

The five-parameter theory is used to describe moderately thick shells with five independent generalized displacements: three translations u_1, u_2, w and two rotations ϑ_1, ϑ_2 (see Subsection 1.4.2).

At this point the assumptions adopted in this book are specified:

- translations, rotations and strains are assumed to be small enough for nonlinear components in the kinematic and equilibrium equations to be omitted (thus taking into account only the first order terms)
- the initial undeformed configuration of the structure is the reference configuration
- the material is treated as isotropic linearly elastic, described by Hooke's constitutive equations, that is to define the material only two parameters are used: Young's modulus and Poisson's ratio

A more advanced tensorial formulation of the theory of shell structures can be found, for instance, in Başar and Krätzig (2001). The theoretical foundations there are coupled with:

- local formulation using differential and algebraic equations
- global formulations employing energy theorems and variational principles for plates and shells

In Chapter 3 of Vol. 2 of Stein et al. (2004), entitled 'Models and Finite Elements for Thin-walled Structures', both the mathematical and mechanical foundations of the theory of plates and shells and the description of FE formulations are presented. The chapter includes an extensive derivation of kinematic equations and strains, constitutive equations and stresses as well as the parametrization of displacements and rotations, both in linear and nonlinear range. The long list of references contains 211 items from 1833 to 2003.

Most recent efforts of scientists are aimed at the analysis of:

- anisotropic, composite (in particular layered) shells
- shells undergoing large deformations (with varying magnitude of displacements, rotations and strains)
- shell in inelastic (in particular plastic) states

However, these issues are beyond the scope of this book. The reader is referred to the following works on nonlinear theories of plates and shells: Woźniak (1966), Pietraszkiewicz (1977, 1979, 2001), Crisfield (1982), Hinton et al. (1982), Kleiber (1985), Borkowski et al. (2001), Wiśniewski (2010), de Borst et al. (2012).

1.3 Description of Geometry for 2D Formulation

The description of the geometry of 2D surfaces is based on the works by Waszczyszyn and Radwańska (1995) and Radwańska (2009).

1.3.1 Coordinate Systems, Middle Surface, Cross Section, Principal Coordinate Lines

The analysis of thin and moderately thick shell structures is most often performed with respect to the middle surface, that is to a geometrically two-dimensional object; only thick shells are treated as three-dimensional bodies.

The geometry of a shell structure is defined when the shape of the middle surface, the boundary contour and the thickness distribution have been specified. In the theoretical consideration we assume for simplicity that the thickness is constant.

Two families of curves are introduced. They are parametrized with so-called curvilinear coordinates ξ_1, ξ_2, see Figure 1.2a, used for an explicit definition of the position of a point on the surface, In most cases a general curvilinear coordinate system will be employed, and further a discussion of particular cases will be provided, for instance the cylindrical (x, θ), spherical (φ, θ) or Cartesian (x, y) coordinate systems will be applied. In the two-dimensional description of shells analogous pairs of variables (e.g. R_α) or pairs of formulae (e.g. $\mathrm{d}s_\alpha = A_\alpha \mathrm{d}\xi_\alpha$) will be used, where the Greek index α represents numbers 1 or 2.

On the middle surface the so-called principal curvature lines related to principal curvature radii are specified. Many equations formulated for particular shells refer to these principal (extreme) curvature lines.

At any point P on the middle surface a cross section can be defined. We consider two normal section planes Π_1 and Π_2, see Figure 1.2b. These planes are perpendicular to each other and their intersections with the middle surface generate arc segments of unit length $ds_\alpha = 1$. We emphasize that the intersection of the planes Π_α is a

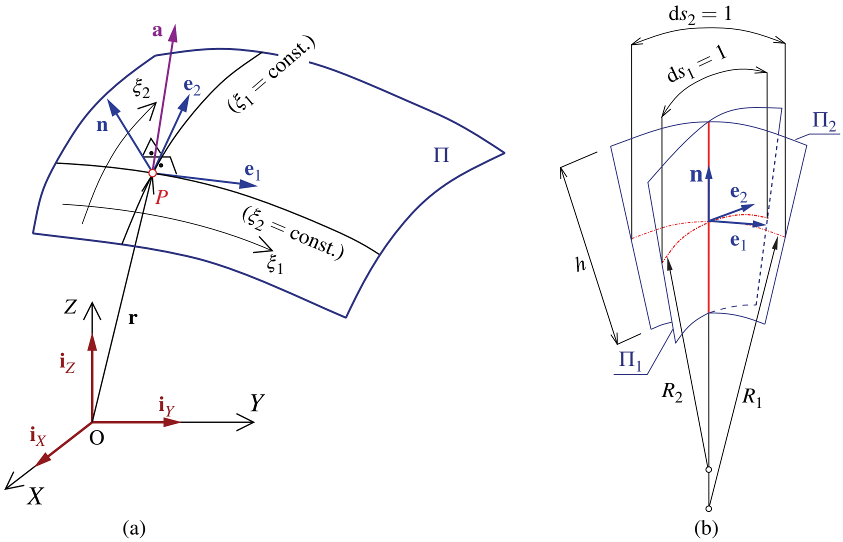


Figure 1.2 (a) A middle surface with curvilinear coordinates ξ_α and local base vectors $\mathbf{e}_\alpha$, $\mathbf{n}$ at point P, (b) straight fibre – intersection of planes Π_α, $\alpha = 1, 2$. Source: Waszczyszyn and Radwańska (1995). Reproduced with permission of Waszczyszyn.

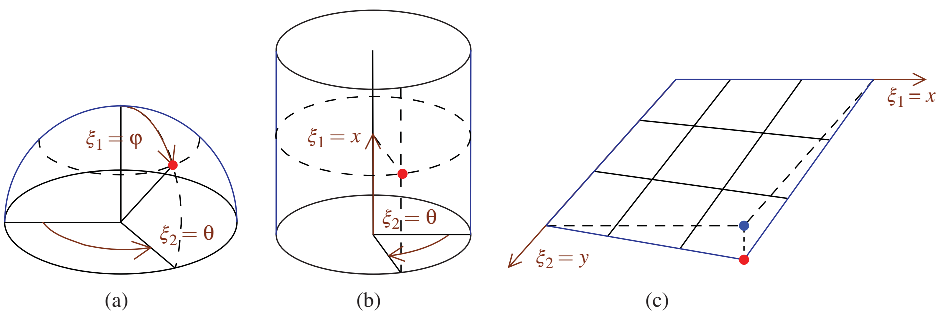


Figure 1.3 Three surfaces: (a) spherical, (b) cylindrical and (c) shallow hyperbolic, corresponding to appropriate coordinate systems

straight fibre (the so-called director), see Figure 1.2b. Its behaviour during deformation is precisely described according to the so-called kinematic hypothesis of Kirchhoff–Love (see Subsection 1.4.1) or Mindlin–Reissner (see Subsection 1.4.2).

Surface coordinates ξ_α are used to identify three common types of surface (see Figure 1.3):

(a) spherical surface described in a spherical coordinate system (φ, θ)
(b) cylindrical surface in a cylindrical system (x, θ)
(c) shallow ruled hyperbolic surface in a Cartesian system (x, y)

1.3.2 Geometry of Middle Surface

For point P on the middle surface Π the connection between global Cartesian coordinates X, Y, Z and local curvilinear coordinates ξ_1, ξ_2 is expressed by the relation

$$\mathbf{r} = X\,\mathbf{i}_X + Y\,\mathbf{i}_Y + Z\,\mathbf{i}_Z \tag{1.1}$$

where:

$$X = f_1(\xi_1, \xi_2), \qquad Y = f_2(\xi_1, \xi_2), \qquad Z = f_3(\xi_1, \xi_2) \tag{1.2}$$

On the middle surface, a two-dimensional segment $P - P_1 - M - P_2$ is identified, resulting from the intersection of four lines $\xi_1 = \text{const.}$, $\xi_1 + d\xi_1 = \text{const.}$, $\xi_2 = \text{const}$, $\xi_2 + d\xi_2 = \text{const.}$ (see Figure 1.4a). Next, curve l is considered. The curve, parametrized by coordinate λ, passes through points P and M that are located on the elementary surface subdomain, with lengths of sides ds_α, $\alpha = 1, 2$ measured by so-called Lame parameters A_α, which are magnitudes of the tangential vectors $\mathbf{r}_{,\alpha}$:

$$ds_\alpha = A_\alpha d\xi_\alpha, \qquad A_\alpha = |\mathbf{r}_{,\alpha}| = |\mathbf{g}_\alpha|, \qquad (\;)_{,\alpha} = \frac{\partial(\;)}{\partial \xi_\alpha}, \qquad \alpha = 1, 2 \tag{1.3}$$

$$\mathbf{r} = \mathbf{r}[\xi_1(\lambda), \xi_2(\lambda)], \qquad d\mathbf{r} = \left(\frac{\partial \mathbf{r}}{\partial \xi_1} \frac{d\xi_1}{d\lambda} + \frac{\partial \mathbf{r}}{\partial \xi_2} \frac{d\xi_2}{d\lambda} \right) d\lambda = \mathbf{r}_{,1}\, d\xi_1 + \mathbf{r}_{,2}\, d\xi_2 \tag{1.4}$$

The length of the arch between points P and M on line l is calculated using the formula

$$\begin{aligned}(ds)^2 &= \mathbf{r}_{,1} \cdot \mathbf{r}_{,1}\, (d\xi_1)^2 + 2\mathbf{r}_{,1} \cdot \mathbf{r}_{,2}\, d\xi_1 d\xi_2 + \mathbf{r}_{,2} \cdot \mathbf{r}_{,2}\, (d\xi_2)^2 \\ &= (A_1)^2\, (d\xi_1)^2 + 2A_1 A_2 \cos(\mathbf{g}_1, \mathbf{g}_2)\, d\xi_1 d\xi_2 + (A_2)^2\, (d\xi_2)^2 \end{aligned} \tag{1.5}$$

The product of tangential vectors $\mathbf{g}_\alpha$ defines the first (I) metric tensor $g_{\alpha\beta}$

$$g_{\alpha\beta} = \mathbf{g}_\alpha \cdot \mathbf{g}_\beta = \mathbf{r}_{,\alpha} \cdot \mathbf{r}_{,\beta} \tag{1.6}$$

Moreover, the I fundamental quadratic form of the surface is derived

$$(ds)^2 = g_{11}\, (d\xi_1)^2 + 2g_{12}\, d\xi_1 d\xi_2 + g_{22}\, (d\xi_2)^2 \tag{1.7}$$

Next, base vectors with unit length (versors) $\mathbf{e}_\alpha$, $\mathbf{n}$ are obtained:

$$\mathbf{e}_\alpha = \frac{\mathbf{r}_{,\alpha}}{A_\alpha} = \frac{\mathbf{g}_\alpha}{A_\alpha}, \qquad \mathbf{n} = \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 \tag{1.8}$$

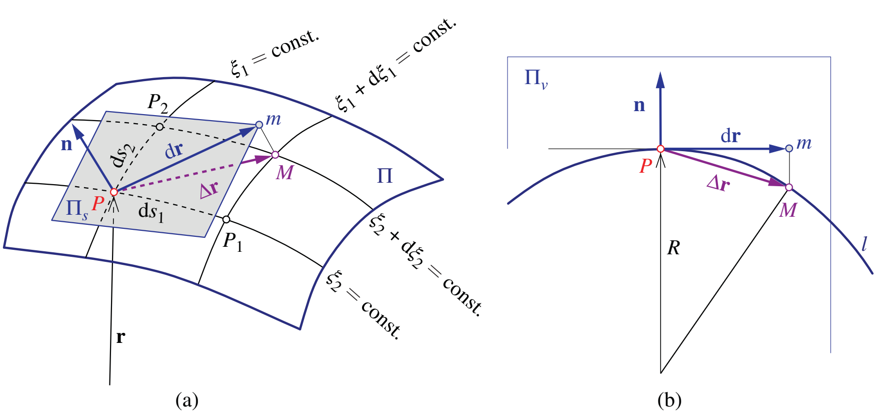


Figure 1.4 Description of objects: (a) on middle surface Π and in plane Π_s, (b) in plane Π_v. Source: Waszczyszyn and Radwańska (1995). Reproduced with permission of Waszczyszyn.

where $\times$ denotes the vector product of two vectors. The components of load $\hat{\mathbf{p}}$ and displacement vectors $\mathbf{u}$ can be defined using the local base versors ($\mathbf{e}_\alpha$, $\mathbf{n}$):

$$\hat{\mathbf{p}} = \hat{p}_1\mathbf{e}_1 + \hat{p}_2\mathbf{e}_2 + \hat{p}_n\mathbf{n} \tag{1.9}$$

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + w\mathbf{n} \tag{1.10}$$

The measure of the middle surface curvature for a shell, denoted by m, can be calculated as the length of projection of vector $\Delta\mathbf{r}$ on direction $\mathbf{n}$, see Figure 1.4b

$$m = \mathbf{n}\cdot\Delta\mathbf{r} = \mathbf{n}\cdot\left(\mathrm{d}\mathbf{r} + \frac{1}{2}\mathrm{d}^2\mathbf{r} + \dots\right) = \frac{1}{2}\mathbf{n}\cdot\mathrm{d}^2\mathbf{r} + \dots\,. \tag{1.11}$$

To this end the second (II) metric tensor $b_{\alpha\beta}$

$$b_{\alpha\beta} = \mathbf{r}_{,\alpha\beta}\cdot\mathbf{n} = -\mathbf{r}_{,\alpha}\cdot\mathbf{n}_{,\beta} \tag{1.12}$$

and the II fundamental form of the surface

$$2m = b_{11}\,(\mathrm{d}\xi_1)^2 + 2b_{12}\,\mathrm{d}\xi_1\mathrm{d}\xi_2 + b_{22}\,(\mathrm{d}\xi_2)^2 \tag{1.13}$$

are defined. For line l its curvature radius R and curvature k are calculated as

$$\frac{1}{R} \equiv k = \lim_{|\Delta\mathbf{r}|\to 0}\frac{2m}{|\Delta\mathbf{r}|^2} = \mathbf{n}\cdot\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2} \tag{1.14}$$

In relation to the so-called principal coordinate lines, for which $g_{12} = b_{12} = 0$, two extreme principal curvature radii $R_{\alpha\alpha}$, as well as two characteristic parameters, mean curvature H and so-called Gaussian curvature K, are calculated using the formulae:

$$k_{\alpha\alpha} = -\frac{1}{R_{\alpha\alpha}} = \frac{b_{\alpha\alpha}}{g_{\alpha\alpha}} = \frac{b_{\alpha\alpha}}{(A_\alpha)^2} \tag{1.15}$$

$$k^2 - 2Hk + K = 0, \qquad H = \frac{1}{2}(k_1 + k_2), \qquad K = k_1k_2 \tag{1.16}$$

1.3.3 Geometry of Surface Equidistant from Middle Surface

Similar to point P on the middle surface Π (see Figure 1.5), we consider point $P^{(z)}$ on surface $\Pi^{(z)}$, equidistant from the middle surface. The position vector $\mathbf{r}^{(z)}$ of point $P^{(z)}$ is the sum of position vector $\mathbf{r}$ of point P and vector $z\mathbf{n}$:

$$\mathbf{r}^{(z)} = \mathbf{r} + z\mathbf{n}, \qquad -\frac{h}{2} \le z \le \frac{h}{2} \tag{1.17}$$

The following objects $\mathrm{d}s_\alpha^{(z)}$, $\mathbf{e}_\alpha^{(z)}$, $A_\alpha^{(z)}$, $R_\alpha^{(z)}$, $\alpha = 1, 2$, are introduced for the equidistant surface. They are associated with analogous objects for the middle surface by linear functions of coordinate z:

$$\mathrm{d}s_\alpha^{(z)} = A_\alpha^{(z)}\mathrm{d}\xi_\alpha, \qquad \mathbf{e}_\alpha^{(z)} = \frac{1}{A_\alpha^{(z)}}\mathbf{r}_{,\alpha}^{(z)}, \qquad \mathbf{n}^{(z)} \equiv \mathbf{n} \tag{1.18}$$

$$A_\alpha^{(z)} = A_\alpha\left(1 + \frac{z}{R_\alpha}\right), \qquad R_\alpha^{(z)} = R_\alpha\left(1 + \frac{z}{R_\alpha}\right) \tag{1.19}$$

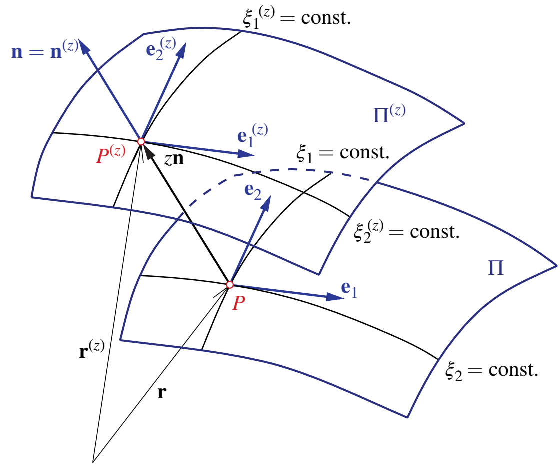


Figure 1.5 Middle surface Π and equidistant surface $\Pi^{(z)}$. Source: Waszczyszyn and Radwańska (1995). Reproduced with permission of Waszczyszyn.

1.3.4 Geometry of Selected Surfaces

We will now present three typical coordinate systems and three selected surfaces as well as scalar, vector and tensor quantities, useful in the description of a surface identified with the middle surface of a shell structure. We will specify base vectors and the first metric tensor. Omitting detailed derivations, we will provide formulae used for the description of geometry of these surface. For more information on the subject, the reader is referred, for instance, to Başar and Krätzig (2001).

1.3.4.1 Spherical Surface

A spherical surface is located in a 3D space with a Cartesian coordinate system ($x^1 = x, x^2 = y, x^3 = z$). On this surface point P is considered, whose position is defined using two spherical surface coordinates $\xi_1 = \varphi, \xi_2 = \theta$ and radius $R_1 = R_2 = R$ (see Figure 1.6).

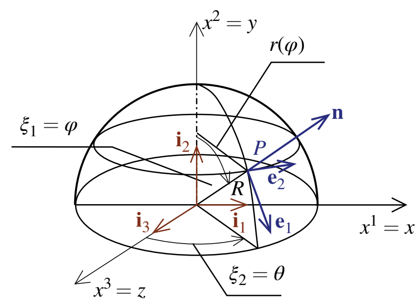
In the global system of axes x^i, $i = 1, 2, 3$, the position vector $\mathbf{r}$ of point P is written first with Cartesian coordinates x^i, and next using two spherical coordinates ξ_α

$$\mathbf{r} = x^i\ \mathbf{i}_i = R\ \sin\varphi \sin\theta\ \mathbf{i}_1 + R\ \cos\varphi\ \mathbf{i}_2 + R\ \sin\varphi\cos\theta\ \mathbf{i}_3 \tag{1.20}$$

Base vectors ($\mathbf{e}_\alpha, \mathbf{n}$) are derived from the formulae:

$$\begin{gathered}\begin{bmatrix}\mathbf{g}_1\\ \mathbf{g}_2\end{bmatrix} = R\begin{bmatrix}\cos\varphi\sin\theta\ \mathbf{i}_1 - \sin\varphi\ \mathbf{i}_2 + \cos\varphi\cos\theta\ \mathbf{i}_3\\ \sin\varphi\cos\theta\ \mathbf{i}_1 + 0\ \mathbf{i}_2 - \sin\varphi\sin\theta\ \mathbf{i}_3\end{bmatrix}\\ \mathbf{e}_\alpha = \frac{1}{R}\,\mathbf{g}_\alpha, \qquad \mathbf{n} = \sin\varphi\sin\theta\ \mathbf{i}_1 + \cos\varphi\ \mathbf{i}_2 + \sin\varphi\cos\theta\ \mathbf{i}_3\end{gathered} \tag{1.21}$$

Figure 1.6 Spherical surface



The following formulae are used in the description of a sphere:

- Lame parameters:

$$A_1 = |\mathbf{g}_1| = R, \qquad A_2 = |\mathbf{g}_2| = R \ \sin\varphi \tag{1.22}$$

- first metric tensor

$$g_{\alpha\beta} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\varphi \end{bmatrix} \tag{1.23}$$

- principal curvature radii

$$R_1 = R_2 = R \tag{1.24}$$

- Gaussian and mean curvatures:

$$K = \frac{1}{R^2}, \qquad H = \frac{1}{R} \tag{1.25}$$

1.3.4.2 Cylindrical Surface

A cylindrical surface, for which the symmetry axis is identical to axis $x^1 = x$ of the Cartesian coordinate system (x, y, z), is shown in Figure 1.7. The position of point P from the cylindrical surface is specified using three Cartesian coordinates x^i, which are related to two cylindrical surface coordinates $\xi_1 = x$ and $\xi_2 = \theta$ and radius R.

The main formulae for the calculation of characteristic parameters, vectors and tensor are (the names are as for the previous surface):

$$\mathbf{r} = x^i \ \mathbf{i}_i = x \, \mathbf{i}_1 + R \ \sin\theta \, \mathbf{i}_2 + R \ \cos\theta \, \mathbf{i}_3 \tag{1.26}$$

$$\begin{gathered}
\begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix} = \begin{bmatrix} 1 \, \mathbf{i}_1 \\ R \ \cos\theta \, \mathbf{i}_2 - R \ \sin\theta \, \mathbf{i}_3 \end{bmatrix} \\
\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} 1 \, \mathbf{i}_1 \\ \cos\theta \, \mathbf{i}_2 - \sin\theta \, \mathbf{i}_3 \end{bmatrix} \\
\mathbf{n} = \sin\theta \, \mathbf{i}_2 + \cos\theta \, \mathbf{i}_3
\end{gathered} \tag{1.27}$$

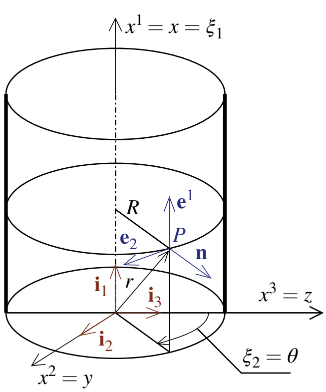


Figure 1.7 Cylindrical surface

$$A_1 = |\mathbf{g}_1| = 1, \qquad A_2 = |\mathbf{g}_2| = R \tag{1.28}$$

$$g_{\alpha\beta} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & R^2 \end{bmatrix} \tag{1.29}$$

$$R_1 = \infty, \qquad R_2 = R \tag{1.30}$$

$$K = 0, \qquad H = \frac{1}{2R} \tag{1.31}$$

1.3.4.3 Hyperbolic Paraboloid

The surface called the hyperbolic paraboloid is defined over a rectangle with dimensions $2a \times 2b$ on plane $x^3 = z = 0$ with two Cartesian coordinates $\xi_1 = x$, $\xi_2 = y$. The surface (see Figure 1.8) is defined by the equation:

$$z(x,y) = kxy, \qquad k = \frac{f}{ab}, \quad m = z_{,y} = kx, \quad n = z_{,x} = ky \tag{1.32}$$

The characteristic formulae used to describe the surface in question are:

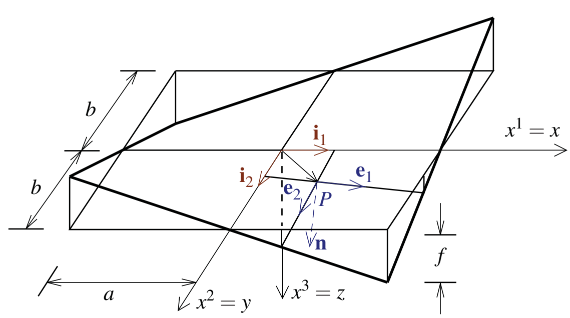
$$\mathbf{r} = x^i\,\mathbf{i}_i = x\,\mathbf{i}_1 + y\,\mathbf{i}_2 + kxy\,\mathbf{i}_3 \tag{1.33}$$

$$\begin{gathered} \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix} = \begin{bmatrix} 1\,\mathbf{i}_1 + n\,\mathbf{i}_3 \\ 1\,\mathbf{i}_2 + m\,\mathbf{i}_3 \end{bmatrix} \\ \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = \frac{1}{\sqrt{1+m^2+n^2}} \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix} \\ \mathbf{n} = \frac{1}{\sqrt{1+m^2+n^2}} (-n\,\mathbf{i}_1 - m\,\mathbf{i}_2 + \mathbf{i}_3) \end{gathered} \tag{1.34}$$

$$A_1 = |\mathbf{g}_1| \approx 1, \qquad A_2 = |\mathbf{g}_2| \approx 1 \tag{1.35}$$

$$g_{\alpha\beta} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} 1+n^2 & mn \\ mn & 1+m^2 \end{bmatrix} \tag{1.36}$$

Figure 1.8 Hyperbolic paraboloid



1.4 Definitions and Assumptions for 2D Formulation

1.4.1 Generalized Displacements and Strains Consistent with the Kinematic Hypothesis of Three-Parameter Kirchhoff–Love Shell Theory

The Kirchhoff–Love (K–L) kinematic hypothesis, adopted for thin shell structures, can be formulated in the following manner:

> *A straight fibre, located at the intersection of two cross-sectional planes, normal to the undeformed (initial) middle surface of a shell, after application of external actions remains straight and normal to the deformed (current) middle surface and has an unchanged length.*

To describe the fields of generalized displacements and strains it is necessary to use two surfaces Π, $\Pi^{(z)}$ in the initial configuration, as well as two analogous surfaces Π^*, $\Pi^{*(z)}$, marked by $*$ and related to the current configuration (after deformation).

In the description of kinematics two middle surfaces Π and Π^* (in initial and current configurations, respectively) are used (see Figure 1.9).

In the analysis of the current configuration the following vectors are distinguished: position vector $\mathbf{r}$, displacement (translation) vector $\mathbf{u}$ and rotation vector $\boldsymbol{\vartheta}$, whose components are related to the local base $(\mathbf{e}_\alpha, \mathbf{n})$ from the initial middle surface Π:

$$\mathbf{r}^* = \mathbf{r} + \mathbf{u} \tag{1.37}$$

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + w\mathbf{n} \tag{1.38}$$

$$\boldsymbol{\vartheta} = -\vartheta_2\mathbf{e}_1 + \vartheta_1\mathbf{e}_2 + \vartheta_n\mathbf{n} = \varphi_1\mathbf{e}_1 + \varphi_2\mathbf{e}_2 + \varphi_n\mathbf{n} \tag{1.39}$$

For the rotation vector we can use two types of components: ϑ_α, ϑ_n or φ_α, φ_n (see Figure 1.10).

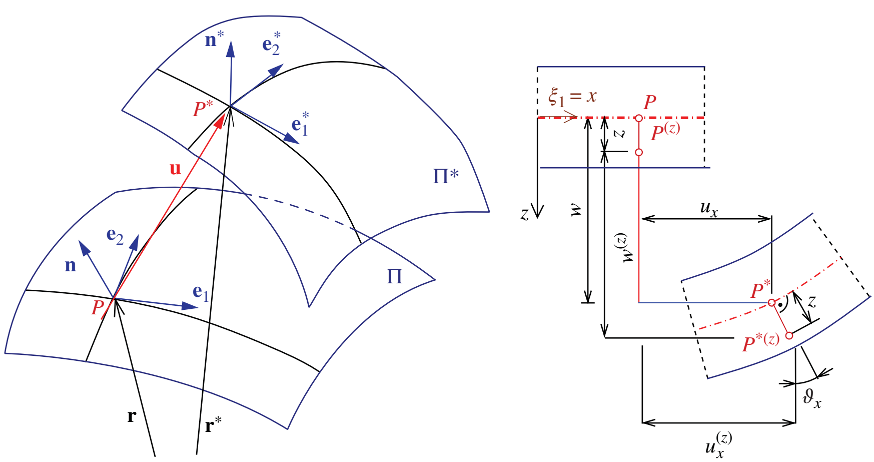


Figure 1.9 (a) Middle surfaces Π and Π^* (before and after deformation), (b) graphical interpretation of kinematic K–L hypothesis for the special case of a flat shell (plate) on plane (ξ_1, z); analogical section can be shown for plane (ξ_2, z). Source: Waszczyszyn and Radwańska (1995). Reproduced with permission of Waszczyszyn.

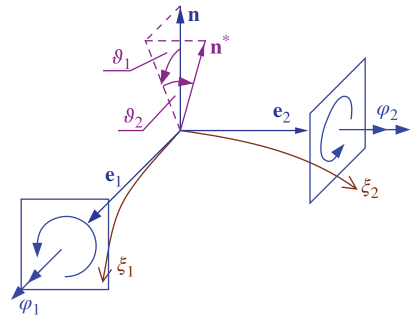


Figure 1.10 Description of rotations of vector **n** normal to middle surface. Source: Waszczyszyn and Radwańska (1995). Reproduced with permission of Waszczyszyn.

During deformation of the middle surface the orthogonal unit base $(\mathbf{e}_\alpha, \mathbf{n})$ changes into a different base $(\mathbf{e}^*_\alpha, \mathbf{n}^*)$, in general nonorthogonal:

$$\begin{aligned}\mathbf{e}^*_\alpha &= \frac{1}{A^*_\alpha}\mathbf{r}^*_{,\alpha} \approx \mathbf{e}_\alpha + \delta\mathbf{e}_\alpha \\ &= \mathbf{e}_\alpha + \frac{1}{A_\alpha}\left(u_{\beta,\alpha} - \frac{A_{\alpha,\beta}}{A_\beta}u_\alpha\right)\mathbf{e}_\beta + \frac{1}{A_\alpha}\left(w_{,\alpha} - \frac{A_\alpha}{R_\alpha}u_\alpha\right)\mathbf{n}\end{aligned} \tag{1.40}$$

$$\mathbf{n}^* = \mathbf{e}^*_1 \times \mathbf{e}^*_2 = \mathbf{n} + \delta\mathbf{n} = \mathbf{n} - \vartheta_2\mathbf{e}_1 + \vartheta_1\mathbf{e}_2 \tag{1.41}$$

The formula (1.41) expresses the change of normal vector **n** into new vector $\mathbf{n}^*$ by means of two rotations ϑ_α (see Figure 1.10).

The displacements at point $P^{(z)}$ on surface $\Pi^{(z)}$, equidistant from the middle surface Π, are calculated on the basis of translations u_α, w and rotations ϑ_α, defined at point P on the middle surface Π:

$$u^{(z)}_1 = u_1 + z\,\vartheta_1, \qquad u^{(z)}_2 = u_2 + z\,\vartheta_2, \qquad w^{(z)} = w \tag{1.42}$$

The K–L kinematic constraints imply the following relations between two rotations ϑ_α and three translations u_α, w:

$$\begin{aligned}&\vartheta_1 = -\frac{1}{A_1}\frac{\partial w}{\partial \xi_1} + \frac{u_1}{R_1}, \qquad \vartheta_2 = -\frac{1}{A_2}\frac{\partial w}{\partial \xi_2} + \frac{u_2}{R_2} \\ &\vartheta_\alpha = -\frac{1}{A_\alpha}\frac{\partial w}{\partial \xi_\alpha} + \frac{u_\alpha}{R_\alpha}, \qquad \alpha = 1, 2\end{aligned} \tag{1.43}$$

Equations (1.43) show the possibility of using a shortened notation of two analogous formulae to describe two-dimensional shell structures.

The third rotation ϑ_n, around the normal, is related to the translations by the following equation

$$\vartheta_n = \frac{1}{2}\left[\left(\frac{1}{A_2}\frac{\partial u_1}{\partial \xi_2} - \frac{1}{A_1}\frac{\partial u_2}{\partial \xi_1}\right) - \left(\frac{A_{1,2}u_1}{A_1A_2} - \frac{A_{2,1}u_2}{A_1A_2}\right)\right] \tag{1.44}$$

The name three-parameter theory of shell structures originates from the fact that only three translations of points from the middle surface, written in a vector

$$\mathbf{u} = [u_1, u_2, w]^{\mathrm{T}} \tag{1.45}$$

and treated as independent components, suffice to describe generalized displacements of the shell (three translations and three rotations).