

The background features several mathematical diagrams. At the top right, a coordinate system with a vertical y-axis and a horizontal x-axis is shown. A point is marked on the y-axis with the label $\mu = 0.2$. Below this, a series of concentric circles are drawn, with an arrow indicating a clockwise direction. In the lower half of the image, a larger, more complex curve is plotted, labeled with Γ and C . This curve has several loops and arrows indicating a direction of flow. Vertical lines are drawn at $x = -1$ and $x = 1$, with labels $x = -1$ and $x = 1$ placed near the curves. The entire background has a warm, orange-to-yellow gradient with a subtle texture.

Ordinary Differential Equations

Michael D. Greenberg

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Contents

Preface	xiii
1 FIRST-ORDER DIFFERENTIAL EQUATIONS	1
1.1 MOTIVATION AND OVERVIEW	1
1.1.1 Introduction	1
1.1.2 Modeling	3
1.1.3 The order of a differential equation	4
1.1.4 Linear and nonlinear equations	5
1.1.5 Our plan	6
1.1.6 Direction field	7
1.1.7 Computer software	8
1.2 LINEAR FIRST-ORDER EQUATIONS	11
1.2.1 The simplest case	11
1.2.2 The homogeneous equation	12
1.2.3 Solving the full equation by the integrating factor method	14
1.2.4 Existence and uniqueness for the linear equation	17
1.3 APPLICATIONS OF LINEAR FIRST-ORDER EQUATIONS	24
1.3.1 Population dynamics; exponential model	24
1.3.2 Radioactive decay; carbon dating	26
1.3.3 Mixing problems; a one-compartment model	28
1.3.4 The phase line, equilibrium points, and stability	30
1.3.5 Electrical circuits	31
1.4 NONLINEAR FIRST-ORDER EQUATIONS THAT ARE SEPARABLE	43
1.5 EXISTENCE AND UNIQUENESS	50
1.5.1 An existence and uniqueness theorem	50
1.5.2 Illustrating the theorem	51
1.5.3 Application to free fall; physical significance of nonuniqueness	53
1.6 APPLICATIONS OF NONLINEAR FIRST-ORDER EQUATIONS	59

1.6.1	The logistic model of population dynamics	59
1.6.2	Stability of equilibrium points and linearized stability analysis	61
1.7	EXACT EQUATIONS AND EQUATIONS THAT CAN BE MADE EXACT	71
1.7.1	Exact differential equations	71
1.7.2	Making an equation exact; integrating factors	75
1.8	SOLUTION BY SUBSTITUTION	81
1.8.1	Bernoulli's equation	81
1.8.2	Homogeneous equations	83
1.9	NUMERICAL SOLUTION BY EULER'S METHOD	87
1.9.1	Euler's method	87
1.9.2	Convergence of Euler's method	90
1.9.3	Higher-order methods	92
	CHAPTER 1 REVIEW	95
2	HIGHER-ORDER LINEAR EQUATIONS	99
2.1	LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER	99
2.1.1	Introduction	99
2.1.2	Operator notation and linear differential operators	100
2.1.3	Superposition principle	101
2.2	CONSTANT-COEFFICIENT EQUATIONS	103
2.2.1	Constant coefficients	103
2.2.2	Seeking a general solution	104
2.2.3	Initial value problem	110
2.3	COMPLEX ROOTS	113
2.3.1	Complex exponential function	113
2.3.2	Complex characteristic roots	115
2.4	LINEAR INDEPENDENCE; EXISTENCE, UNIQUENESS, GENERAL SOLUTION	118
2.4.1	Linear dependence and linear independence	119
2.4.2	Existence, uniqueness, and general solution	121
2.4.3	Abel's formula and Wronskian test for linear independence	124
2.4.4	Building a solution method on these results	125
2.5	REDUCTION OF ORDER	128
2.5.1	Deriving the formula	128
2.5.2	The method rather than the formula	131
2.5.3	About the method of reduction of order	132
2.6	CAUCHY-EULER EQUATIONS	134
2.6.1	General solution	135
2.6.2	Repeated roots and reduction of order	136

2.6.3	Complex roots	138
2.7	THE GENERAL THEORY FOR HIGHER-ORDER EQUATIONS	142
2.7.1	Theorems for n th-order linear equations	143
2.7.2	Constant-coefficient equations	144
2.7.3	Cauchy–Euler equations	146
2.8	NONHOMOGENEOUS EQUATIONS	149
2.8.1	General solution	149
2.8.2	The scaling and superposition of forcing functions	151
2.9	PARTICULAR SOLUTION BY UNDETERMINED COEFFICIENTS	155
2.9.1	Undetermined coefficients	155
2.9.2	A special case; the complex exponential method	160
2.10	PARTICULAR SOLUTION BY VARIATION OF PARAMETERS	163
2.10.1	First-order equations	163
2.10.2	Second-order equations	165
	CHAPTER 2 REVIEW	170
3	APPLICATIONS OF HIGHER-ORDER EQUATIONS	173
3.1	INTRODUCTION	173
3.2	LINEAR HARMONIC OSCILLATOR; FREE OSCILLATION	174
3.2.1	Mass–spring oscillator	174
3.2.2	Undamped free oscillation	176
3.2.3	Pendulum	179
3.3	FREE OSCILLATION WITH DAMPING	186
3.3.1	Underdamped	187
3.3.2	Critically damped	188
3.3.3	Overdamped	188
3.4	FORCED OSCILLATION	193
3.4.1	Undamped, $c = 0$	193
3.4.2	Damped, $c > 0$	196
3.5	STEADY-STATE DIFFUSION; A BOUNDARY VALUE PROBLEM	202
3.5.1	Boundary value problems; existence and uniqueness	202
3.5.2	Steady-state heat conduction in a rod	203
3.6	INTRODUCTION TO THE EIGENVALUE PROBLEM; COLUMN BUCKLING	211
3.6.1	An eigenvalue problem	211
3.6.2	Application to column buckling	213
	CHAPTER 3 REVIEW	218

4	SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS	219
4.1	INTRODUCTION, AND SOLUTION BY ELIMINATION . . .	219
4.1.1	Introduction	219
4.1.2	Physical examples	220
4.1.3	Solutions, existence, and uniqueness	221
4.1.4	Solution by elimination	222
4.1.5	Auxiliary variables	225
4.2	APPLICATION TO COUPLED OSCILLATORS	230
4.2.1	Coupled oscillators	230
4.2.2	Reduction to first-order system by auxiliary variables . .	231
4.2.3	The free vibration	231
4.2.4	The forced vibration	234
4.3	N -SPACE AND MATRICES	238
4.3.1	Passage from 2-space to n -space	238
4.3.2	Matrix operators on vectors in n -space	240
4.3.3	Identity matrix and zero matrix	242
4.3.4	Relevance to systems of linear algebraic equations	242
4.3.5	Vector and matrix functions	244
4.4	LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS	247
4.4.1	Linear dependence of a set of constant vectors in n -space	247
4.4.2	Linear dependence of vector functions in n -space	250
4.5	EXISTENCE, UNIQUENESS, AND GENERAL SOLUTION . .	253
4.5.1	The key theorems	253
4.5.2	Illustrating the theorems	257
4.6	MATRIX EIGENVALUE PROBLEM	261
4.6.1	The eigenvalue problem	261
4.6.2	Solving an eigenvalue problem	262
4.6.3	Complex eigenvalues and eigenvectors	266
4.7	HOMOGENEOUS SYSTEMS WITH CONSTANT COEFFICIENTS	270
4.7.1	Solution by the method of assumed exponential form . .	270
4.7.2	Application to the two-mass oscillator	274
4.7.3	The case of repeated eigenvalues	276
4.7.4	Modifying the method if there are defective eigenvalues .	278
4.7.5	Complex eigenvalues	279
4.8	DOT PRODUCT AND ADDITIONAL MATRIX ALGEBRA . .	283
4.8.1	More about n -space: dot product, norm, and angle	283
4.8.2	Algebra of matrix operators	285
4.8.3	Inverse matrix	289
4.9	EXPLICIT SOLUTION OF $\mathbf{x}' = \mathbf{A}\mathbf{x}$ AND THE MATRIX EXPONENTIAL FUNCTION	297

4.9.1	Matrix exponential solution	297
4.9.2	Getting the exponential matrix series into closed form . .	300
4.10	NONHOMOGENEOUS SYSTEMS	307
4.10.1	Solution by variation of parameters	307
4.10.2	Constant coefficient matrix	310
4.10.3	Particular solution by undetermined coefficients	311
	CHAPTER 4 REVIEW	314
5	LAPLACE TRANSFORM	317
5.1	INTRODUCTION	317
5.2	THE TRANSFORM AND ITS INVERSE	318
5.2.1	Laplace transform	318
5.2.2	Linearity property of the transform	321
5.2.3	Exponential order, piecewise continuity, and conditions for existence of the transform	323
5.2.4	Inverse transform	326
5.2.5	Introduction to the determination of inverse transforms .	327
5.3	APPLICATION TO THE SOLUTION OF DIFFERENTIAL EQUATIONS	334
5.3.1	First-order equations	334
5.3.2	Higher-order equations	336
5.3.3	Systems	339
5.3.4	Application to a nonconstant-coefficient equation; Bessel's equation	341
5.4	DISCONTINUOUS FORCING FUNCTIONS; HEAVISIDE STEP FUNCTION	347
5.4.1	Motivation	347
5.4.2	Heaviside step function and piecewise-defined functions	347
5.4.3	Transforms of Heaviside and time-delayed functions . .	349
5.4.4	Differential equations with piecewise-defined forcing func- tions	351
5.4.5	Periodic forcing functions	353
5.5	CONVOLUTION	358
5.5.1	Definition of Laplace convolution	358
5.5.2	Convolution theorem	359
5.5.3	Applications	360
5.5.4	Integro-differential equations and integral equations . . .	362
5.6	IMPULSIVE FORCING FUNCTIONS; DIRAC DELTA FUNCTION	366
5.6.1	Impulsive forces	366
5.6.2	Dirac delta function	368
5.6.3	The jump caused by the delta function	370

5.6.4	Caution	371
5.6.5	Impulse response function	372
CHAPTER 5 REVIEW		376
6	SERIES SOLUTIONS	379
6.1	INTRODUCTION	379
6.2	POWER SERIES AND TAYLOR SERIES	380
6.2.1	Power series	380
6.2.2	Manipulation of power series	382
6.2.3	Taylor series	383
6.3	POWER SERIES SOLUTION ABOUT A REGULAR POINT	387
6.3.1	Power series solution theorem	387
6.3.2	Applications	389
6.4	LEGENDRE AND BESSEL EQUATIONS	395
6.4.1	Introduction	395
6.4.2	Legendre's equation	395
6.4.3	Bessel's equation	399
6.5	THE METHOD OF FROBENIUS	408
6.5.1	Motivation	408
6.5.2	Regular and irregular singular points	409
6.5.3	The method of Frobenius	410
CHAPTER 6 REVIEW		420
7	SYSTEMS OF NONLINEAR DIFFERENTIAL EQUATIONS	423
7.1	INTRODUCTION	423
7.2	THE PHASE PLANE	424
7.2.1	Phase plane method	424
7.2.2	Application to nonlinear pendulum	427
7.2.3	Singular points and their stability	431
7.3	LINEAR SYSTEMS	435
7.3.1	Introduction	435
7.3.2	Purely imaginary eigenvalues (CENTER)	437
7.3.3	Complex conjugate eigenvalues (SPIRAL)	438
7.3.4	Real eigenvalues of the same sign (NODE)	439
7.3.5	Real eigenvalues of opposite sign (SADDLE)	443
7.4	NONLINEAR SYSTEMS	447
7.4.1	Local linearization	448
7.4.2	Predator-prey population dynamics	450
7.4.3	Competing species	453
7.5	LIMIT CYCLES	463
7.6	NUMERICAL SOLUTION OF SYSTEMS BY EULER'S METHOD	468

7.6.1	Initial value problems	468
7.6.2	Existence and uniqueness for nonlinear systems	471
7.6.3	Linear boundary value problems	472
CHAPTER 7 REVIEW		476
APPENDIX A: REVIEW OF PARTIAL FRACTION EXPANSIONS		479
APPENDIX B: REVIEW OF DETERMINANTS		483
APPENDIX C: REVIEW OF GAUSS ELIMINATION		491
APPENDIX D: REVIEW OF COMPLEX NUMBERS AND THE COMPLEX PLANE		497
ANSWERS TO EXERCISES		501
INDEX		521

Preface

PURPOSE AND PREREQUISITES

This book is intended for students of science, engineering, and mathematics, as a textbook for a first course in differential equations, typically in the student's third or fourth semester. It is expected that the student has completed a differential and integral calculus sequence, but prior knowledge of linear algebra is not a prerequisite, and that material is provided here when it is needed.

TO THE INSTRUCTOR

The following points are to describe and explain some of the pedagogical decisions and approaches adopted in this text.

- 1. Transition to higher-order equations.** The passage from first-order equations to equations of higher order is, we believe, often made more difficult for the student than necessary. Typically, the discussion of higher-order equations begins with the citing of an existence-and-uniqueness theorem and the introduction of linear independence and Wronskians, none of which is needed for the important case of equations with constant coefficients, which is indeed the first case to be studied. Consequently, second-order equations tend to look like a “new subject” to the student. Instead, we focus immediately on second-order equations with constant coefficients and derive their general solution in only a few pages (Theorem 2.2.1 of Section 2.2), using only results obtained in Chapter 1 for first-order equations. Proof of Theorem 2.2.1 is elementary, relying only on the factoring of the differential operator and the known solution of first-order equations with constant coefficients. The latter is not put forward as a solution method, but only to prove the theorem, and we are careful to caution the student that factorization cannot be expected to be useful for nonconstant-coefficient equations.

The advantage of this approach is that the general solution of $y'' + p_1y' + p_2y = 0$ (in which the p_j 's are constants) is obtained quickly and easily, without first introducing an existence-and-uniqueness theorem, linear independence, or Wronskians. The remainder of Sections 2.2 and 2.3 is devoted to familiarizing the student with the various solution forms: (a) the real exponentials and hyperbolic functions and (b) the complex exponentials and the circular functions. With that done, linear independence, Wronskians, existence, uniqueness, and general solution are introduced next, in Section 2.4, at which point the discussion can then be more readily grasped by the student, by virtue of the already completed discussion of the constant-coefficient case in Sections 2.2 and 2.3.