## The Power and Beauty of Electromagnetic Fields


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# THE POWER AND BEAUTY OF ELECTROMAGNETIC FIELDS 

F. R. Morgenthaler

The IEEE Press Series on Electromagnetic Wave Theory Andreas C. Cangellaris, Series Editor

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To my loving wife, Barbara, our splendid daughters, Ann and Janet, and their wonderful children, Sarah, Brian, Douglas, and Tessie

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## PREFACE

"The most beautiful thing we can experience is the mysterious.
It is the source of all true art and science."
—Albert Einstein

Anyone, who has taken the trouble to explore the rich legacy that James Clerk Maxwell left to the scientific world, cannot fail to be impressed. Because he made such outstanding and profound contributions in multiple disciplines within a life-span of only 48 years, his achievements are all the more remarkable. They include seminal work in thermodynamics, statistical mechanics, and electromagnetic theory, but it is the latter that we here consider.

Although this text was written to instruct advanced undergraduate and first-year graduate students in the basic concepts of classical macroscopic electromagnetic fields, it was done so with the hope of providing new insights into and appreciation for what is surely one of the supreme achievements in science. Certain topics (described below), that do not appear in traditional texts, are deserving of inclusion both in their own right and because they simplify the development of new material concerning electromagnetic power and energy. The title, The Power and Beauty of Electromagnetic Fields, was chosen because of its multiple meanings.

During a tour of the campus, more than one M.I.T. student has taken a non-scientist friend to the lobby of the Eastman Laboratories (Building 6) to stand before the marble wall on which mathematical symbols that represent Maxwell's Equations are inscribed in bronze. Usually, the student attempts to convey to his friend just how powerful and useful these compact equations are and how they at first predicted (rather than explained)
the existence of electromagnetic waves propagating at the speed of light, c. How that, in turn, led to their widespread application - to radio, microwaves, television, light, and x-rays; to electrical generators, motors, transducers, control systems, and power-grids; to integrated-circuits, computers, CD-ROMs, and the Internet; to electrocardiograms, pacemakers, magnetic resonance imaging (MRI), computer aided tomography (CAT), biosensors, and other emerging marvels of biomedical technology. And the list goes on and on .... No one can doubt the utility - the power - of electromagnetic theory.

Mathematicians find beauty and elegance in equations - especially if they have general applicability and can be expressed with brevity; many other scientists and engineers share these feelings as well. Surely, Maxwell's Equations qualify and may be considered beautiful as well as powerful. After all, Albert Einstein pondered their properties and shattered Newtonian concepts of space and time with his Theory of Special Relativity. That in turn led to the introduction (by Minkowski) of four-dimensional space-time (with $i$ ct the imaginary fourth-dimension). Recasting electrodynamics in four-dimensions made evident new symmetries that led to even higher levels of understanding and beauty. Finally, the pinnacle was reached when connections to the world of quantumphysics produced Quantum Electrodynamics (QED) - but the last is not a subject for this text.

But there are other meanings of power and beauty as well. In circuit theory, electrical power is voltage times current and flows into and out of network nodes; in electromagnetic theory, the power flux is commonly defined as the Poynting vector, $\mathbf{S}=\mathbf{E} \times \mathbf{H}$ (the vector cross-product of the electric and magnetic fields), which is largely exterior to all highly conducting pathways. Likewise, rather than being localized to the electric charges and currents, the Maxwell field energy density, $W$, with terms proportional to the scalar products $\mathbf{E} \cdot \mathbf{E}$ and $\mathbf{H} \cdot \mathbf{H}$, is distributed throughout space. These quantities, related to each other at every point by the Poynting Theorem, invite physical interpretation on a per unit area or per unit volume basis. Even at very low-frequencies, that interpretation is very different from the approximate, highly-localized circuit-theory representation.

Nevertheless, as recounted by Julius Stratton [1, pp. 134], the Poynting interpretation was criticized as early as 1902 by H. M. Macdonald [2], and later by George H. Livens [3], and Max Mason and Warren Weaver [4, pp. 264] among other writers. In their thought provoking analogy, Mason and Weaver, while excepting highly localized regions of space having little or no influence outside of them,
> "do not believe that 'Where?' is a fair or sensible question to ask concerning energy. Energy is a function of configuration, just as beauty of a certain black-and-white design [such as the 'Tiled-Photons' example] is a function of configuration. [They] see no more reason or excuse for speaking of a spatial energy density than they would for saying, in the case of a design, that its beauty was distributed over it with a certain density. Such a view would lead one to assign to a perfectly blank square inch in one portion of the design a certain amount of beauty, and to an equally blank square inch in another portion a certain different amount of beauty."

Many who have pondered that criticism applaud its cleverness - but consider it a false analogy that does little to advance one's understanding of the issue. After all, they had offered no alternate interpretation of (or replacement for) $\mathbf{S}$ and $W$. If not in the fields, where does radiating electromagnetic wave power and energy reside? Because circuit-power is confined to the wires that carry electrical currents, it cannot be used to explain radiation without the addition of a rather mysterious "radiation-resistance" to the circuit path. Even with that artifice, when the source-current is turned-off and a

'Tiled-Photons'
transmitting circuit stops radiating, one expects that the energy, that has been radiated, must be located somewhere in free-space prior to its eventual detection - possibly very far from the transmitter.

During his junior year as an electrical engineering student at M.I.T., this author recalls an instructor telling the class that "although sophomores believe that electrical power is confined to the wires that carry electrical currents in a circuit, we [who have learned about Poynting] know better." One skeptical student asked the instructor if his belief in the Poynting vector was strong enough that he would be willing to touch the wires of a high-voltage circuit. Without hesitation, the instructor responded, "of course, provided that you show me how I can do so without putting my hand in the field." We all laughed and became more than a little smug because of our new found superior knowledge - we really thought that we did know better! Nearly forty years later, I learned that we (or at least I) did not; that realization, which came near the end of a long academic career in electromagnetism, was the principal motivation for the writing of this text. The analysis that led to that realization forms an essential component of its content.

Although the "circuit" representation is very useful, conventional wisdom considers it only a low-frequency or quasistatic approximation. In this text, we show that this is not necessarily true and in the process find alternate representations of electromagnetic power and energy that differ from the familiar Poynting theorem values - yet are fully equivalent. The particular choice focussed on features highly-localized power and energy components and emphasizes the circuit rather than the wave nature of these quantities. Moreover, unlike the Poynting vector, this exact representation merges smoothly with well-known quasistatic approximations that have long been used to calculate power flows in both lumped and distributed circuits operating at low-frequencies. As required, the electromagnetic power-conversion density, $\mathbf{E} \cdot \mathbf{J}$ (the dot product of electric-field and
current-density), is the same in both the Poynting and all correct alternate representations of electromagnetic power. Maxwell's Equations and the fields they describe are, of course, left unchanged. It is also possible to alter the Maxwell stress-tensor, $\overline{\mathbf{T}}$, and the associated electromagnetic momentum-density, $\mathbf{G}$, in a similar manner without altering the electromagnetic force-density; when all four elements ( $\mathbf{S}, W, \overline{\mathbf{T}}$, and $\mathbf{G}$ ) are treated similarly, any of the resulting alternate-representations can replace the Maxwell-Poynting form without approximation. One specific choice is termed the Alternate-representation. For time-harmonic fields, the complex Poynting theorem, energy-theorem, and momentumtheorem all have alternate-counterparts. Electromagnetic angular-momentum also has an alternate counterpart that is shown to connect directly with the spin properties of photons.

For certain electromagnetic problems, the Alternate-representation leads to both conceptual and computational simplicity. It is especially useful when dealing with either antenna radiation or quasistatic fields. For example, the power radiated from a Hertzian electric-dipole and its radiation pattern both can be calculated without first finding the electric and/or magnetic field - not a single curl operation need be performed. For other problems, it is the Maxwell-Poynting representation that is mathematically simpler; having a choice adds both flexibility and insight to the process of problem solving. Some features are surprising; for steady-state single-frequency fields, the free-space Alternate power-flux and energy-density are time-independent. Consequently, Alternate reactive power-flows and energies are banished from free-space and restricted to the locations of the charges and currents.

The text is divided into four-parts: Basic Electromagnetic Theory; Four Dimensional Electromagnetism; Electromagnetic Examples; Backmatter. Part I is devoted to a fairly conventional presentation of the integral and differential forms of Maxwell's Equations in free-space containing electric charges and currents that are subject to Lorentz-forces. Conservation of charge is assumed. The equivalent representation in terms of the magnetic vector-potential and the electric scalar-potential is also given in both Coulomb and Lorenz gauges. Materials with polarization and magnetization that may be electrically-conducting are considered, as are the boundary conditions at material and source interfaces.

When wave-propagation effects are negligible, fields that are mainly electric or magnetic are classified as either electroquasistatic (EQS) or magnetoquasistatic (MQS). The properties of quasistatic fields and their analysis by approximate methods are developed.

Electromagnetic power, energy, stress, and momentum are presented in both the Maxwell-Poynting and Alternate representations. So too, are complex versions of power and energy theorems that apply to sinusoidal steady-state fields. For linear media, both homogeneous (source free) and inhomogeneous wave equations are studied in one, two, and three dimensions; extensive use is made of both symmetry and the principle of superposition The concepts of electromagnetic duality, equivalence, and Babinet's Principle for complementary structures are other important topics that are included; the uniquenesstheorem, induction-theorem, and reciprocity-theorem are also derived.

All derivations contained in Part I are carried out in conventional three-space coordinates with time as a parameter, but it is actually easier to formulate (and generalize) electrodynamics using four-dimensional representations of both fields and forces. It then follows that power, energy, stress, and momentum are unified in terms of an energymomentum tensor. In Part II, we introduce these concepts and emphasize the utility of expressing the various tensors in terms of four-vector electric and magnetic fields and the four-vector-potential. However, no prior knowledge of such representations is
assumed nor is the reader expected to be familiar with the electrodynamics of moving media which is also discussed. Although these topics must be considered advanced ${ }^{1}$, they depend upon straight-forward extensions of the standard vector-calculus - knowledge of which is a prerequisite for almost all texts (including this one) on electromagnetictheory. Using this approach, an infinity of energy-momentum tensors are found that are fully equivalent to the usual Maxwell-Poynting form; these change the representation of the electromagnetic power, energy, momentum and stress in free-space without altering the four-vector electromagnetic-force density. As noted above, emphasis is given to the particular choice that directly connects with circuit-theory representations of quasistaticfields. These results are generalized to include the presence of field interactions with matter; special emphasis is given to dielectric and magnetic materials that are linear. Because multiple representations lead to a deeper understanding, four-dimensional electromagnetics is presented in order to complement, illuminate, and (in many cases) simplify, the topics developed in Part I. In addition, the theory certainly possesses great "mathematical-beauty" and so is deserving of inclusion.

In Part III, classic field problems are considered that illustrate how solutions of Maxwell's Equations can be combined in order to satisfy boundary conditions for a wide variety of examples that, in the main, depend upon topics covered in Part I. Solving problems is the best (perhaps the only) way for a student of electromagnetics to master both physical concepts and mathematical techniques. Gradually, one develops physical intuition concerning fields; most find the process challenging, but the rewards great. It is reassuring that, once gained, mastery of the subject will not become obsolete. Certainly, its application to new materials and devices will continue to refresh electromagnetics, but, if history is any guide, the basic field-equations will not be supplanted; Maxwell seems to have got it very right. The specific examples were chosen, not only because the fields themselves are of interest, but because detailed comparisons can be made between the Alternate and Maxwell-Poynting representations. In many cases, the time-averaged distributions agree exactly. However, in others, there are significant differences and not only is Alternate power and energy highly-localized on currents and charges, but, astonishingly, also in regions of free-space. When compared with the usual Poynting energy-density, the propagation of such distributed and localized forms of Alternate-energy at first seems to create grave paradoxes. However, on closer inspection and analysis of the measurement process - by means of which power is detected, all of these are resolved. In other cases, Alternate-power exists in regions free of electric and magnetic fields or vice-versa. These strange results are also reconciled. In addition to the examples, which are worked through in detail, a chapter of over 120 practice problems is included so that the reader can test his/her understanding of the basic concepts and sharpen problem-solving skills. Many of the problems were created by the author for use in both M.I.T. undergraduate and graduate courses; however, new ones have been added as well.

Part IV contains the Backmatter including Summary, Appendices, Bibliography, and Index. A photo gallery of many (unfortunately not all) electromagnetic luminaries is included so that the reader can humanize the science of electromagnetics. As might be expected, many of the pioneers lived interesting and multi-faceted lives; fascinating

[^0]biographical information is readily available both in libraries and on the Internet. Eight appendices complete the text. Appendix-A includes elements of Special-Relativity including space-time coordinates and the Lorentz-transformation. Appendix-B is devoted to three-space vector identities and their four-space vector and tensor equivalents. AppendixC contains important properties of Laplacian-fields and tabulates solutions of Laplace's Equation. Appendix-D provides expressions for Alternate-power, energy, stress, and momentum in Cartesian, cylindrical, and spherical coordinate systems. Lorentz-torques and electromagnetic angular momentum formulations in both Maxwell-Poynting and Alternate formulations are also included. In Appendix-E, free-space fields associated with pre-specified electric charge and current distributions are considered. The method of images is included, as are formulas for the characteristic impedances of a variety of uniform TEM transmission-lines. Appendix-F reviews properties of Bessel functions and both Chebyshev and Hermite polynomials and provides useful recursion formulas and normalization integrals. Appendix-G discusses a very useful computer program: 4d-em.mac that has built-in knowledge of four-space electrodynamics; it runs under Macsyma and is included on the DVD that accompanies the text. For readers without access to Macsyma, the largely equivalent and freely available Maxima can be substituted and is included. Finally, Appendix-H contains a list of the avi movie files for several of the electromagnetic transients analyzed in the text. These animations are also included and can be viewed with standard media players.

The DVD also includes three electronic versions of the book. [Advanced Level] is the complete text (described above) that is the suitable for graduate courses, reference, and self study. [Introductory Level] is intended for undergraduates; [Intermediate Level] for graduate and well prepared undergraduate students. Both omit Part II and abridge Parts III and IV by including appropriate selections of the [Advanced Level] examples, practice problems, and appendicies. All three Levels are in PDF form suitable for on-line viewing using Adobe Reader; bookmarks and hyperlinks from the table of contents allow convenient browsing. The on-line versions of Appendix-H permit both the Macsyma Notebooks and the animations to be launched directly from icon hyperlinks.

The difference between the energy-momentum tensors in the Alternate and MaxwellPoynting representation is itself a four-tensor, $\mathrm{II}^{\mathrm{b}}$, that produces neither electromagneticforce nor $\mathbf{E} \cdot \mathbf{J}$ power-density. It is therefore an ephemeral quantity - yet one with components that can be calculated and presented graphically. In a bit of whimsy, that honors the insights of Mason and Weaver, this author has dubbed II' the "electromagneticbeauty;" this tensor joins electromagnetic-power to form still another set of meanings for the book title. The author believes that graphical rendering of the "beauty-power-flux" of an electromagnetic-field makes that final meaning plausible. Because selected examples are included in the Summary, the reader can either confirm or deny that judgement.

F. R. Morgenthaler

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It is a pleasure to acknowledge the influence of Julius A. Stratton whose seminal text: Electromagnetic Theory has inspired generations of students by revealing the true beauty of electromagnetism; that underlying the deceptively simple and very elegant Maxwell Equations.
It is an honor to acknowledge the debt owed to Lan Jen Chu who served for many years as mentor, advisor, and friend. He made significant contributions to electromagnetic theory in general, antenna theory in particular, and developed the elegant theory of electrodynamics that has come to be known as the Chu formulation. More than anyone, he encouraged his students to penetrate to the depths of a problem and taught, by example, that the solution of any electromagnetic problem was incomplete without a thorough understanding of the energies and power-flows.


Julius Adams Stratton (1901-1994)
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Lan Jen Chu (1913-1973)
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## PART I

## BASIC ELECTROMAGNETIC THEORY

## INTRODUCTION TO PART I

Michael Faraday, James Clerk Maxwell, and others ${ }^{1}$ postulated that the electromagnetic forces generated by electric charge and current densities ( $\rho$ and $\mathbf{J}$ ) are transmitted by electric and magnetic vector fields $(\mathbf{E}$ and $\mathbf{H})$ that exist in the surrounding free-space. This insight led to rapid progress in understanding the interconnections between timevarying sources and these fields and culminated in Maxwell's Equations as set forth in 1864. Their properties led to the prediction of electromagnetic waves traveling at the speed of light. A more complete understanding of the power and energy associated with these fields followed twenty years later.

[^1]The interaction of fields and macroscopic matter is complicated by the fact that (fortunately) materials have very diverse forms that include insulators, metals, and semiconductors. Almost all of these have dielectric properties; a few are strongly magneticsometimes even when no external fields are applied. The principal responses of electromagnetic character that are generated in materials are: conduction currents due to the motion of mobile unpaired charges, ${ }^{2}$ electric dipoles created by bound-charge pairs of opposite polarity, and magnetic dipoles generated by either tiny current loops or the coupling of the intrinsic magnetic moments of individual electrons (sometimes protons). Three vector fields are found to characterize a wide range of materials: They are the conduction-current density, $\mathbf{J}_{u}$, the electric-dipole density, $\mathbf{P}$, and the magnetic-dipole density, M. The constitutive laws that relate the material responses to $\mathbf{E}$ and $\mathbf{H}$ are usually nonlinear (especially in the case of ferromagnets), but are often approximated as linear-at least for weak fields.

Part I is devoted to a fairly conventional, but compact, presentation of both the integral and differential forms of Maxwell's Equations in free-space containing electric charges and currents. Rather than follow the historical development, we simply postulate the validity of conservation of charge, the Lorentz-force density, Faraday's Law, and Ampere's Law-as amended by Maxwell. All sources are assumed to be averaged over microscopic distances and times; therefore, the fields that they generate must also be considered macroscopic in nature. An equivalent representation of these fields, in terms of the magnetic vector potential and the electric scalar potential, is also given for both Coulomb and Lorenz gauges.

Next, inhomogeneous wave equations governing the electric and magnetic fields are derived-as are those governing the vector and scalar potentials in both the time and frequency domains. Solutions are found by direct integration of the currents and charges (assumed known). As an important example, the potentials associated with a line current of incremental length are derived; they provide the basis for understanding electromagnetic radiation. Materials with polarization and magnetization that may be electrically conducting are then considered as are the boundary conditions at material and source interfaces. When wave-propagation effects are negligible, fields that are mainly electric or magnetic are classified as either electroquasistatic (EQS) or magnetoquasistatic (MQS). The equations governing quasistatic fields are derived, and analysis by approximate methods is outlined; the approach taken follows that of earlier M.I.T. textbooks by Fano, Chu, and Adler [5] and Haus and Melcher [6].

As is customary, electromagnetic power, energy, stress, momentum, and angular momentum are defined within the Maxwell-Poynting representation that is based upon the Poynting vector and Maxwell stress tensor. However, the Alternate representation by Morgenthaler [8] is also presented that embraces localized circuit-theory concepts and merges with quasistatic representations, yet is exact; the unification of power, energy, stress, and momentum in these, and other representations, is delayed until Part II (where the appropriate mathematical tools are developed). Power-energy theorems for both representations are developed in the time domain. These are used to calculate Poynting power and energy of a uniform plane wave and a radiating electric dipole and compare them with the Alternate counterparts.

[^2]Next, homogeneous wave equations and free-space waves propagating in one, two, and three dimensions are studied in both the time and frequency domains. Waves of high symmetry are given special attention because their superposition can be used to represent inhomogeneous waves generated by arbitrary, but known, sources. Homogeneous waves in linear dielectric and magnetic materials (that may also be conductive) are considered; sinusoidal steady-state (harmonic) waves are given particular attention. In the limit of high conductivity, simple wave propagation is radically altered and replaced by the interrelated concepts of charge relaxation, magnetic diffusion, and skin depth. For lossfree dielectrics, power, energy, stress, and momentum associated with packets of uniform harmonic plane waves are considered from both wave and particle perspectives; the latter is shown to follow a Hamiltonian formulation.

The final chapter of Part I is devoted to important theorems and principles. Frequencydomain theorems involving electromagnetic power, stress, energy, and momentum are derived. These are followed by duality of fields and sources, the uniqueness theorem, the equivalence principle, the induction theorem, Babinet's Principle for complementary structures, and the reciprocity theorem.

When the Part III examples and explanatory texts are included, the overall coverage of the standard wave-propagation issues is similar to that of other texts but is built upon both the Maxwell-Poynting and Alternate representations. The most recent effort [9] (designed for the M.I.T. EECS undergraduate curriculum) is authored by D. H. Staelin, A. W. Morgenthaler, and J. A. Kong; older standards include Adler et al. [10] and Ramo et al. [11].

The text is designed to serve a variety of curriculums. Because much of Part III depends only upon Part I, an undergraduate course could omit the four-dimensional electrodynamics of Part II and include only basic examples; the focus of a particular graduate course might be restricted to advanced electrodynamics, antennas and diffraction, or transmission lines and microwave circuits.

## CHAPTER 1

## MAXWELL'S EQUATIONS

### 1.1 MATHEMATICAL NOTATION

- Scalar quantities, such as $\Phi$, are printed in normal type.
- Vectors, such as $\mathbf{E}$ and $\mathbf{H}$, are printed in bold type; their components are printed in normal type with subscripts that indicate coordinate directions.
- Unit vectors are printed in bold type, but with ${ }^{\text {- }}$ over the symbol. Thus, $\mathbf{E}=$ $\widehat{\mathbf{x}} E_{x}+\widehat{\mathbf{y}} E_{y}+\widehat{\mathbf{z}} E_{z}$ represents the electric field expressed in Cartesian coordinates as the sum of three orthogonal vectors. Numerical subscripts $1,2,3$ often substitute for $x, y, z$; for example, $E_{i}$ can be any one of the components.
- The scalar or dot product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is indicated generally by $\mathbf{A} \cdot \mathbf{B}$ or explicitly by either $A_{x} B_{x}+A_{y} B_{y}+A_{z} B_{z}$ or $\sum_{i=1}^{3} A_{i} B_{i}$. When (as here) indices are repeated, the summation symbol is often understood to exist and omitted in order to simplify the expression.
- The vector cross product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is indicated by $\mathbf{A} \times \mathbf{B}$.

[^3]- The del operator is defined by $\nabla=\widehat{\mathbf{x}} \frac{\partial}{\partial x}+\widehat{\mathbf{y}} \frac{\partial}{\partial y}+\widehat{\mathbf{z}} \frac{\partial}{\partial z}$ or in index notation by $\frac{\partial}{\partial x_{i}}$. It is used to express the gradient of a scalar and the divergence and curl of a vector. These operations are defined in Appendix B, Section B.2.
- Tensor quantities, such as $\overline{\mathbf{T}}$ are printed in bold type, but with a bar over the symbol; the components may be expressed in dyadic notation as two adjacent vectors $\mathbf{A B}$ or in component form using normal type. Examples are: $\overline{\mathbf{T}}=\mathbf{A B}$ $\left.=\left(\widehat{\mathbf{x}} A_{x}+\widehat{\mathbf{y}} A_{y}+\widehat{\mathbf{z}} A_{z}\right) \widehat{\mathbf{x}} B_{x}+\widehat{\mathbf{y}} B_{y}+\widehat{\mathbf{z}} B_{z}\right)$ or $T_{i j}=A_{i} B_{j}$. Notice that, in general, $T_{i j} \neq T_{j i}$.
- Four-dimensional vectors, such as $\mathcal{E}$ and $\mathcal{H}$, and dyads, like $\mathcal{A} \mathcal{B}$, are printed in calligraphic type.


### 1.2 FREE-SPACE FIELDS AND FORCES

In free-space regions that contain electric and magnetic field vectors, $\mathbf{E}$ and $\mathbf{H}$, the Lorentz force on a charge, $q_{i}$, moving with velocity, $\mathbf{v}_{i}$ (the subscript enumerates the charge and is not the index of a Cartesian component) is

$$
\begin{equation*}
\mathbf{F}_{i}=q_{i}\left(\mathbf{E}+\mathbf{v}_{i} \times \mu_{\mathrm{o}} \mathbf{H}\right) \tag{1.1}
\end{equation*}
$$

We employ SI units ${ }^{1}$ here and throughout the text.
If there are many charges within a small finite volume, $\Delta V$, the total force within it is

$$
\sum_{i} \mathbf{F}_{i}=\left(\sum_{i} q_{i}\right) \mathbf{E}+\left(\sum_{i} q_{i} \mathbf{v}_{i}\right) \times \mu_{0} \mathbf{H}
$$

When divided by $\Delta V$, Eq. (1.1) reduces to the Lorentz force density,

$$
\begin{equation*}
\mathbf{f}=\rho \mathbf{E}+\mathbf{J} \times \mu_{0} \mathbf{H} \tag{1.2}
\end{equation*}
$$

where $\rho=\lim _{\Delta V \rightarrow 0} \frac{\sum_{i} q_{i}}{\Delta V}$ is the macroscopic electric charge density and $\mathbf{J}=\lim _{\Delta V \rightarrow 0}$ $\frac{\sum_{i} q_{i} \mathbf{v}_{i}}{\Delta V}$ is the macroscopic electric-current density. In these averages, the limiting $\Delta V$ must remain large compared to microscopic dimensions. In the event that all of the velocities are equal to $\mathbf{v}$, the current, $\mathbf{J}=\rho \mathbf{v}$, is said to be convective. The $\mathbf{E}$ and $\mathbf{H}$ fields arise from all of the charges and currents and are themselves macroscopic averages. The equations that follow describe the interactions between these macroscopic quantities.

## Integral form of Maxwell's Equations

For a region of free-space containing $\rho$ and $\mathbf{J}$, Maxwell's Equations, when expressed in integral form, are

$$
\begin{align*}
& \oint_{C_{0}} \mathbf{H} \cdot d \mathbf{s}-\frac{d}{d t} \int_{S} \varepsilon_{0} \mathbf{E} \cdot d \mathbf{a}=\int_{S} \mathbf{J} \cdot d \mathbf{a}  \tag{1.3a}\\
& \oint_{C_{0}} \mathbf{E} \cdot d \mathbf{s}+\frac{d}{d t} \int_{S} \mu_{0} \mathbf{H} \cdot d \mathbf{a}=0 \tag{1.3b}
\end{align*}
$$

[^4]

Figure 1.1 Surface spanning closed contour.
where $S$ is an arbitrary surface that spans the closed contour $C_{\mathrm{o}}$ that is depicted in Figure 1.1.

When $S$ is a minimal surface (that of a soap bubble film anchored by the contour), its positive side is defined as that seen from above as we circle the contour in a counterclockwise direction. The unit vector, $\widehat{\mathbf{n}}$, normal to that surface (or any other that can be formed by simple deformation), points outward from its positive side. The vector differential area of $S$ is defined by $d \mathbf{a}=\widehat{\mathbf{n}} d a$; the vector differential distance along the contour, $C_{\mathrm{o}}$, by $d \mathbf{s}=\widehat{\mathbf{t}} d s$, with $\widehat{\mathbf{t}}$ the unit vector tangential to the contour (and of positive polarity as we circle it). The first of these equations is Ampere's Law as amended by Maxwell to include the time-varying term (renowned as the Maxwell displacement current density); the second is Faraday's Law.

Because the electric charge is conserved, the total current flowing out of the closedsurface, $S_{0}$, that encloses an arbitrary volume, $V$, must equal the negative time-rate of change of the charge contained within it. Therefore,

$$
\begin{equation*}
\oint_{S_{\mathrm{o}}} \mathbf{J} \cdot d \mathbf{a}+\frac{d}{d t} \int_{V} \rho d V=0 \tag{1.4}
\end{equation*}
$$

is the integral form of the Law of Conservation of Charge. If the contour $C_{\mathrm{o}}$ is allowed to shrink to zero as $S \rightarrow S_{0}$, Eq. (1.3a) becomes

$$
-\frac{d}{d t} \oint_{S_{\mathrm{o}}} \varepsilon_{0} \mathbf{E} \cdot d \mathbf{a}=\oint_{S_{\mathrm{o}}} \mathbf{J} \cdot d \mathbf{a}
$$

provided that the magnetic-field tangential to $C_{\mathrm{o}}$ is finite and so cannot contribute to the limiting contour integral. When combined with Eq. (1.4) the result,

$$
\frac{d}{d t}\left(\oint_{S_{0}} \varepsilon_{0} \mathbf{E} \cdot d \mathbf{a}-\int_{V} \rho d V\right)=0
$$

can be integrated with respect to time to yield

$$
\oint_{S_{o}} \varepsilon_{0} \mathbf{E} \cdot d \mathbf{a}-\int_{V} \rho d V=\mathrm{constant}
$$

Consider a very small volume, $V$, centered upon some chosen point. Because $\rho$ and $\mathbf{E}$ can be forced to vanish in that region for at least one instant of time, the constant must be zero for that and (by extension) every region.

It follows that

$$
\begin{equation*}
\oint_{S_{\mathrm{o}}} \varepsilon_{0} \mathbf{E} \cdot d \mathbf{a}=\int_{V} \rho d V \tag{1.5a}
\end{equation*}
$$

applies to an arbitrary volume. Equation (1.5a) is known as Gauss' Law. Similar arguments applied to Eq. (1.3b) lead to

$$
\begin{equation*}
\oint_{S_{\mathrm{o}}} \mu_{0} \mathbf{H} \cdot d \mathbf{a}=0 \tag{1.5b}
\end{equation*}
$$

These equations are often included in the set of Maxwell's Equations, but are not independent. From Eqs. (1.3a) and (1.3b) and either Conservation of Charge or Gauss' Law the other equation follows.

## Units and fundamental constants

The units of the field vectors are $[E]=V m^{-1}$ and $[H]=A m^{-1}$, those of the electric charge and current densities are $[\rho]=\mathrm{Cm}^{-3}$, and $[\mathrm{J}]=\mathrm{Am}^{-2}$, and that of the Lorentz force density is $[f]=N \mathrm{~m}^{-3}$. The constants $\mu_{\mathrm{o}}$ and $\varepsilon_{\mathrm{o}}$ are, respectively, the permeability and permittivity of free-space. The former is legislated to have the value

$$
\mu_{\mathrm{o}}=4 \pi \times 10^{-7} \mathrm{Hm}^{-1}
$$

whereas the latter is a measured quantity,

$$
\varepsilon_{0}=8.854187817 \times 10^{-12} \mathrm{Fm}^{-1}
$$

Because

$$
\begin{equation*}
\mathrm{c}=\frac{1}{\sqrt{\mu_{\mathrm{o}} \varepsilon_{\mathrm{o}}}}=2.99792458 \times 10^{8} \mathrm{~ms}^{-1} \tag{1.6}
\end{equation*}
$$

(the velocity of light in vacuum) is so close to $3 \times 10^{8} \mathrm{~m} \mathrm{~s}^{-1}$, it is often convenient to approximate the permittivity by the exact value,

$$
\varepsilon_{0}=\frac{1}{36 \pi} \times 10^{-9} \mathrm{Fm}^{-1}
$$

We shall adhere to that practice. Another combination of these constants that is extremely useful is $c \mu_{\mathrm{o}}$ defined as the characteristic impedance of free-space. Given the symbol, $\eta_{\mathrm{o}}$, it has the value

$$
\eta_{\mathrm{o}}=\mathrm{c} \mu_{\mathrm{o}}=\sqrt{\frac{\mu_{\mathrm{o}}}{\varepsilon_{\mathrm{o}}}}=120 \pi \simeq 377 \Omega\left(S^{-1}\right)
$$

## Linearity and superposition

Because Maxwell's Equations are linear with respect to the fields and the charges and currents, superposition may be applied after dividing the sources into components with respect to either space or time (or some combination). In particular, if

$$
\begin{align*}
\rho & =\sum_{i} \rho_{(i)}  \tag{1.7a}\\
\mathbf{J} & =\sum_{i} \mathbf{J}_{(i)} \tag{1.7b}
\end{align*}
$$

and each $i$ th set of sources (not to be confused with a Cartesian index) is chosen so as to satisfy conservation of charge, then

$$
\begin{align*}
\mathbf{E} & =\sum_{i} \mathbf{E}_{(i)}  \tag{1.7c}\\
\mathbf{H} & =\sum_{i} \mathbf{H}_{(i)} \tag{1.7d}
\end{align*}
$$

where $\mathbf{E}_{(i)}$ and $\mathbf{H}_{(i)}$ are the Maxwellian fields generated when $\rho_{(i)}$ and $\mathbf{J}_{(i)}$ act alone. The components may be either discrete or differential; in the latter case the summations are replaced by integrals. When it is possible to decompose the sources into elements each providing a high degree of spatial symmetry, finding the field solutions is greatly facilitated. Fields generated by stationary spatially symmetric charge or current sources with either spherical, cylindrical, or planar symmetry are considered in Appendix C, Section C.1. Time-dependent sources are often considered to be the superposition of sinusoidal functions. This gives rise to representations employing Fourier series and Fourier integrals in the respective cases of periodic or aperiodic functions. As we shall learn in later sections of this chapter, representation in terms of complex exponential functions is also a standard technique-applicable to the fields as well as the sources.

## Symmetric field-transient example

When a highly symmetric source is suddenly switched on at $t=0$ and remains constant thereafter, it is sometimes possible to deduce the field solutions using only the integral form of the Maxwell Equations together with symmetry considerations. This method depends solely upon the material that has been presented up to this point and is employed in Part III, Chapter 15, Section 15.1. A more traditional solution of the same problem is presented in Section 15.2; it is based upon the symmetries that are made evident by the differential form of Maxwell's Equations. These are developed and built upon in the remainder of this chapter. Before continuing, the reader is encouraged to jump ahead to the first example in order to gain a preliminary understanding of both propagating electromagnetic transients and the quasistatic fields that follow in their wake. Both the speed of light and the ratio of electric to magnetic radiation field strengths (the characteristic impedance of free-space) emerge from that example.

## Differential form of Maxwell's Equations

If the contour $C_{\mathrm{o}}$ is chosen to lie in the $y-z$ plane and it and the surface $S$ spanning the contour are allowed to shrink to vanishingly small dimensions, the contour integral $\oint_{C_{o}} \mathbf{A} \cdot d \mathbf{s}$ (on a per unit area basis) equals the $x$ component of the curl $\mathbf{A}$. Suitable orientation of the contour (in the $z-x$ and $x-y$ planes) will produce the $y$ and $z$ components. Likewise, when the closed surface shrinks toward a point, the integral of the normal flux of a vector over that surface is (on a per unit volume basis) the divergence of that vector.

The set of integral equations is therefore equivalent to the differential forms

$$
\begin{align*}
\nabla \times \mathbf{H} & =\mathbf{J}+\varepsilon_{0} \frac{\partial \mathbf{E}}{\partial t}  \tag{1.8a}\\
\nabla \times \mathbf{E} & =-\mu_{0} \frac{\partial \mathbf{H}}{\partial t}  \tag{1.8b}\\
\nabla \cdot \varepsilon_{0} \mathbf{E} & =\rho  \tag{1.8c}\\
\nabla \cdot \mu_{0} \mathbf{H} & =0 \tag{1.8d}
\end{align*}
$$

where the curl $(\nabla \times)$ and divergence $(\nabla \cdot)$ operators are written in terms of the gradient operator, $\nabla$ (del). Alternatively, the Divergence Theorem and Stoke's Theorem (Appendix B, Eqs. (B.7) and (B.8)) can be applied to produce the same results. We prove in Chapter 6 that both the curl and divergence of a field vector are determining factors in specifying that vector, and so it is not surprising that Maxwell's Equations constrain these quantities for both $\mathbf{E}$ and $\mathbf{H}$.

Because $\nabla \cdot(\nabla \times \mathbf{H})=\nabla \cdot \mathbf{J}+\frac{\partial}{\partial t}\left(\nabla \cdot \varepsilon_{0} \mathbf{E}\right)=0$, the first and third equations imply conservation of electric charge,

$$
\begin{equation*}
\nabla \cdot \mathbf{J}+\frac{\partial \rho}{\partial t}=0 \tag{1.8e}
\end{equation*}
$$

This result also follows directly from Eq. (1.4).

### 1.3 VECTOR AND SCALAR POTENTIALS

From renewed use of the vector identity, $\nabla \cdot(\nabla \times \mathbf{A})=0$, it follows that Eq. (1.8d) is automatically satisfied by

$$
\begin{equation*}
\mu_{\mathrm{o}} \mathbf{H}=\nabla \times \mathbf{A} \tag{1.9}
\end{equation*}
$$

When substituted into Eq. (1.8b), the result is

$$
\nabla \times \mathbf{E}+\frac{\partial}{\partial t}(\nabla \times \mathbf{A})=\nabla \times\left(\mathbf{E}+\frac{\partial \mathbf{A}}{\partial t}\right)=0
$$

A second vector identity, $\nabla \times(\nabla \Phi)=0$, then permits one to express the electric field as

$$
\begin{equation*}
\mathbf{E}=-\frac{\partial \mathbf{A}}{\partial t}-\nabla \Phi \tag{1.10}
\end{equation*}
$$

The vector (A) and scalar ( $\Phi$ ) potentials (defined with the conventional polarities) are not unique because the set

$$
\begin{align*}
& \mathbf{A}^{\prime}=\mathbf{A}+\nabla \Psi  \tag{1.11a}\\
& \Phi^{\prime}=\Phi-\frac{\partial \Psi}{\partial t} \tag{1.11b}
\end{align*}
$$

produces the same values of both $\mathbf{E}$ and $\mu_{0} \mathbf{H}$. The primed and unprimed potentials are said to be related by a gauge transformation (set by the scalar function, $\Psi$ ). Evidently, the value of $\nabla \cdot \mathbf{A}$ is not unique and may be specified as desired; two possibilities are of special interest.

## Lorenz gauge

The Lorenz gauge ${ }^{2}$ is defined by

$$
\begin{equation*}
\nabla \cdot \mathbf{A}+\frac{1}{c^{2}} \frac{\partial \Phi}{\partial t}=0 \tag{1.12}
\end{equation*}
$$

with $c$ given by Eq. (1.6).
If that value is selected and Eqs. (1.9) and (1.10) are substituted into Eqs. (1.8e), the result is

$$
\begin{align*}
& \nabla^{2} \mathbf{A}-\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{A}}{\partial t^{2}}=-\mu_{\mathrm{o}} \mathbf{J}  \tag{1.13a}\\
& \nabla^{2} \Phi-\frac{1}{c^{2}} \frac{\partial^{2} \Phi}{\partial t^{2}}=-\frac{\rho}{\varepsilon_{0}} \tag{1.13b}
\end{align*}
$$

where $\mathbf{A}$ and $\mathbf{J}$ are assumed to be expressed in Cartesian components. All four components $\left(A_{x}, A_{y}, A_{z}, \Phi\right)$ are solutions of inhomogeneous wave equations; they are not independent because conservation of charge must be satisfied. The current and charge densities are those existing in free-space and must satisfy Eq. (1.8e); additional sources that model dielectric and magnetic materials are described in Section 1.7 and, more thoroughly, in Chapter 9.

## Coulomb gauge

The Coulomb gauge (sometimes called the radiation gauge) is defined by

$$
\begin{equation*}
\nabla \cdot \mathbf{A}=0 \tag{1.14}
\end{equation*}
$$

If that value is selected and Eqs. (1.9) and (1.10) are substituted into Eqs. (1.8e), the result is

$$
\begin{align*}
\nabla^{2} \Phi & =-\rho / \varepsilon_{\mathrm{o}}  \tag{1.15a}\\
\nabla^{2} \mathbf{A} & =-\mu_{\mathrm{o}}\left(\mathbf{J}+\varepsilon_{\mathrm{o}} \frac{\partial \mathbf{E}}{\partial t}\right)  \tag{1.15b}\\
\nabla^{2} \mathbf{A}-\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{A}}{\partial t^{2}} & =-\mu_{\mathrm{o}}\left(\mathbf{J}-\varepsilon_{\mathrm{o}} \nabla \frac{\partial \Phi}{\partial t}\right) \tag{1.15c}
\end{align*}
$$

Equation (1.15a) is Poisson's Equation, which does not contain time derivatives; consequently, the scalar potential at any position must respond instantaneously and simultaneously to all changes in the electric charge density, $\rho(\mathbf{r}, t)$. Consideration of Eq. (1.15b) reveals that the same is true of the vector potential, with respect to $\mathbf{J}(\mathbf{r}, t)+\varepsilon_{0} \frac{\partial \mathbf{E} \mathbf{( r}, t)}{\partial t}$ - the current density that includes the Maxwell displacement current. However, when $\mathbf{E}$ is expressed in terms of both the scalar and vector potentials, the wave equation,

[^5]Eq. (1.15c), emerges. Of course, the values of $\mathbf{E}$ and $\mathbf{H}$ calculated from the Coulomb gauge must (and do) agree with those calculated from the Lorenz (or any other) gauge.

Finally, when $\Phi=0$ (because, either there is no electric charge anywhere or else Eqs. (1.11) have been used), the Lorenz and Coulomb gauges can be identical, but only in charge-free regions of space. The reader should verify that the requisite $\Psi$ satisfies the homogeneous wave equation, except where $\partial(\nabla \cdot \mathbf{A}) / \partial t=-\rho / \varepsilon_{0}$ causes the Coulomb gauge to fail.

### 1.4 INHOMOGENEOUS WAVE EQUATIONS FOR E AND H

The electric and magnetic fields, interwoven by Maxwell's Equations, can be decoupled by use of the vector identities,

$$
\begin{aligned}
& \nabla \times(\nabla \times \mathbf{H})=\nabla(\nabla \cdot \mathbf{H})-\nabla^{2} \mathbf{H} \\
& \nabla \times(\nabla \times \mathbf{E})=\nabla(\nabla \cdot \mathbf{E})-\nabla^{2} \mathbf{E}
\end{aligned}
$$

and substitution of the curls and divergences of $\mathbf{H}$ and $\mathbf{E}$. The result (after reordering partial derivatives) is a pair of inhomogeneous wave equations (expressed in terms of Cartesian components),

$$
\begin{align*}
& \nabla^{2} \mathbf{H}-\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{H}}{\partial t^{2}}=-\nabla \times \mathbf{J}  \tag{1.16a}\\
& \nabla^{2} \mathbf{E}-\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{E}}{\partial t^{2}}=\frac{1}{\varepsilon_{0}} \nabla \rho+\mu_{\mathrm{o}} \frac{\partial \mathbf{J}}{\partial t} \tag{1.16b}
\end{align*}
$$

The solutions are not independent because, as noted previously, the current and charge densities must satisfy conservation of charge. The velocity of light, $c$, is given by Eq. (1.6).

An alternate, though less direct, derivation of Eqs. (1.16a) and (1.16b) follows (in the Lorenz gauge) from

$$
\begin{aligned}
\nabla \times\left(\nabla^{2} \mathbf{A}-\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{A}}{\partial t^{2}}+\mu_{\mathrm{o}} \mathbf{J}\right) & =0 \\
\frac{\partial}{\partial t}\left(\nabla^{2} \mathbf{A}-\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{A}}{\partial t^{2}}+\mu_{\mathrm{o}} \mathbf{J}\right)+\nabla\left(\nabla^{2} \Phi-\frac{1}{c^{2}} \frac{\partial^{2} \Phi}{\partial t^{2}}+\frac{\rho}{\varepsilon_{\mathrm{o}}}\right) & =0
\end{aligned}
$$

and (after again interchanging the order of the derivatives) substitution of Eqs. (1.9) and (1.10). Naturally, the final result is independent of the choice of gauge.

### 1.5 STATIC FIELDS

When there is no time variation in the charges and/or currents, the steady-state fields are also static and related by

$$
\begin{equation*}
\nabla^{2} \mathbf{E}=\frac{1}{\varepsilon_{0}} \nabla \rho \tag{1.17a}
\end{equation*}
$$

$$
\begin{align*}
\nabla^{2} \Phi & =-\frac{\rho}{\varepsilon_{0}}  \tag{1.17b}\\
\nabla^{2} \mathbf{A} & =-\mu_{\mathrm{o}} \mathbf{J} \tag{1.17c}
\end{align*}
$$

Static electric charge generates electrostatic fields $\Phi$ and $\mathbf{E}$; static electric current generates magnetostatic fields, $\mathbf{A}$ and $\mathbf{H}$. When the vectors are expressed in Cartesian coordinates, each of the scalar equations takes the form of Poisson's Equation:

$$
\begin{equation*}
\nabla^{2} \Psi=-s \tag{1.18}
\end{equation*}
$$

When $s=0$, the homogeneous equation is termed Laplace's Equation. Solutions in various coordinate systems are discussed in Appendix C, Section C.4; they are of great importance in solving both static and quasistatic boundary-value problems. For now, we assume that $s$ is a known function of position and attempt to integrate Poisson's Equation directly.

## Integration of Poisson's Equation

Because Eq. (1.18) is a linear partial differential equation, both the source density, $s$, and the response function, $\Psi$, can be expressed as the superposition of differential components that individually satisfy this equation.

When $s$ is a three-dimensional unit impulse (delta function) located at the origin of spherical coordinates, the spherically symmetric solution satisfies

$$
\begin{equation*}
\nabla^{2} \Psi(r)=\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} \frac{\partial \Psi}{\partial r}\right)=-\delta(r) \tag{1.19}
\end{equation*}
$$

with $\delta(r)=\frac{1}{4 \pi r^{2}} u_{0}(r)$. Here $u_{0}(r)$ is the derivative of the unit step function, $u_{-1}(r)$ (both are defined in Appendix B, Section B.1, but with origins shifted to $r=0^{+}$so that $\left.\int_{0}^{\infty} u_{0}(r) d r=1\right)$. It follows that $\Psi=1 /(4 \pi r)$ and (with a shift of origin) $d \Psi_{p}=$ $s_{q} d V_{q} /\left(4 \pi r_{q p}\right)$ where, as indicated in Figure 1.2,
$\mathbf{r}_{q p}=\mathbf{r}_{p}-\mathbf{r}_{q}$ is the vector between the differential source located at $\mathbf{r}_{q}$ and the response evaluated at $\mathbf{r}_{p}$ and $r_{q p}=\left|\mathbf{r}_{p}-\mathbf{r}_{q}\right|$. After integration, the general solution of Eq. (1.18) is found to be

$$
\begin{equation*}
\Psi_{p}\left(\mathbf{r}_{p}\right)=\int_{V} \frac{s_{q}\left(\mathbf{r}_{q}\right) d V_{q}}{4 \pi r_{q p}} \tag{1.20}
\end{equation*}
$$



Figure 1.2 Origin, source point, $q$, and field point, $p$.

When the field point is distant from all of the sources, $1 / r_{q p}$ can be expanded in a series of powers of $1 / r_{p}$. Together these constitute the multipole expansion that is developed in Appendix C, Section C.2; the $1 / r_{p}$ term is the monopole, $1 / r_{p}^{2}$ the dipole, $1 / r_{p}^{3}$ the quadrupole, and so on.

## Electrostatics

The equations of electrostatics in integral form are

$$
\begin{align*}
& \Phi_{p}\left(\mathbf{r}_{p}\right)=\int_{V} \frac{\rho_{q}\left(\mathbf{r}_{q}\right) d V_{q}}{4 \pi \varepsilon_{0} r_{q p}}  \tag{1.21a}\\
& \mathbf{E}_{p}\left(\mathbf{r}_{p}\right)=\int_{V} \widehat{\mathbf{r}}_{q p} \frac{\rho_{q}\left(\mathbf{r}_{q}\right) d V_{q}}{4 \pi \varepsilon_{0} r_{q p}^{2}} \tag{1.21b}
\end{align*}
$$

and together form Coulomb's Law.

## Sphere of uniform charge

As a simple example, consider a sphere of radius $R$ that contains a uniform volume charge density, $\rho_{0}$. Without loss of generality we may define spherical coordinates so that the point $p$ lies on the negative $z$ axis. Then Eq. (1.21a) becomes

$$
\Phi\left(r_{p}\right)=\int_{0}^{R} \int_{0}^{\pi} \frac{\rho_{0} 2 \pi r_{q}^{2} d r_{q} \sin \theta_{q} d \theta_{q}}{4 \pi \varepsilon_{0} \sqrt{r_{p}^{2}+r_{q}^{2}+2 r_{p} r_{q} \cos \theta_{q}}}
$$

where $r_{p}$ may be smaller or larger than $R$. The result of the two integrations (after dropping the subscript $p$ ) is

$$
\Phi(r)=\frac{\rho_{\mathrm{o}}}{\varepsilon_{\mathrm{o}}} \begin{cases}\frac{1}{2} R^{2}-\frac{1}{6} r^{2}, & r \leq R \\ \frac{R^{3}}{3 r}, & r \geq R\end{cases}
$$

From the gradient operator, or from Eq. (1.21b), the electrostatic electric field is

$$
\mathbf{E}(r)=\widehat{\mathbf{r}} \frac{\rho_{\mathrm{o}}}{3 \varepsilon_{\mathrm{o}}} \begin{cases}r, & r \leq R \\ \frac{R^{3}}{r^{2}}, & r \geq R\end{cases}
$$

This solution was carried out only to illustrate the procedure; the preferred method of solution is to use symmetry and Gauss' Law to immediately solve for the electric field.

## Magnetostatics

The equations of magnetostatics in integral form are (with $\nabla \cdot \mathbf{J}=0$ ):

$$
\begin{align*}
\mathbf{A}_{p}\left(\mathbf{r}_{p}\right) & =\int_{V} \frac{\mu_{\mathrm{o}} \mathbf{J}_{q}\left(\mathbf{r}_{q}\right) d V_{q}}{4 \pi r_{q p}}  \tag{1.22a}\\
\mu_{\mathrm{o}} \mathbf{H}_{p}\left(\mathbf{r}_{p}\right) & =\nabla_{p} \times \mathbf{A}_{p}=\frac{\mu_{\mathrm{o}}}{4 \pi} \int_{V} \nabla_{p} \times\left[\frac{\mathbf{J}_{q}\left(\mathbf{r}_{q}\right)}{r_{q p}}\right] d V_{q} \\
& =\frac{\mu_{\mathrm{o}}}{4 \pi} \int_{V} \nabla_{p}\left(\frac{1}{r_{q p}}\right) \times \mathbf{J}_{q}\left(\mathbf{r}_{q}\right) d V_{q} \tag{1.22b}
\end{align*}
$$

Because $\nabla_{p}\left(\frac{1}{r_{q p}}\right)=-\nabla_{q}\left(\frac{1}{r_{q p}}\right)=-\frac{\widehat{\mathbf{r}}_{q p}}{r_{q p}^{2}}$,

$$
\begin{equation*}
\mathbf{H}_{p}\left(\mathbf{r}_{p}\right)=\frac{1}{4 \pi} \int_{V} \frac{\mathbf{J}_{q} \times \mathbf{r}_{q p}}{r_{q p}^{3}} d V_{q} \tag{1.22c}
\end{equation*}
$$

Equation (1.22c) is known as the Biot-Savart Law.

## Circular current loop

As an important example, we calculate the magnetic field on the axis of a planar circular loop of wire carrying current $\mathrm{I}_{0}$; the wire radius is assumed to be very small compared to the loop radius, $R$. Integrating over the area of the wire results in

$$
\begin{equation*}
\mathbf{H}_{p}\left(\mathbf{r}_{p}\right)=\frac{\mathrm{I}_{\mathrm{o}}}{4 \pi} \oint \frac{\mathbf{d s}_{q} \times \mathbf{r}_{q p}}{r_{q p}^{3}} \tag{1.23a}
\end{equation*}
$$

where $\mathbf{d s}_{q}$ is the differential vector length of the current element. If the current loop lies in the plane $z=0$ with its center at the origin, the on-axis field (which by symmetry is $z$-directed) is easily evaluated and found to be

$$
\begin{equation*}
\mathbf{H}(r=0, z)=\widehat{\mathbf{z}} \frac{\mathrm{I}_{0} R^{2}}{2\left(R^{2}+z^{2}\right)^{3 / 2}} \tag{1.23b}
\end{equation*}
$$

The current direction and $\mathbf{H}$ obey the right-hand rule (with fingers curled in the direction of the current, the thumb points in the direction of the field).

### 1.6 INTEGRATION OF THE INHOMOGENEOUS WAVE EQUATION

The inhomogeneous scalar wave equation is defined by

$$
\begin{equation*}
\nabla^{2} \Psi-\frac{1}{c^{2}} \frac{\partial^{2} \Psi}{\partial t^{2}}=-s(\mathbf{r}, t) \tag{1.24}
\end{equation*}
$$

If the spatial derivatives dominate over those with respect to time, the inhomogeneous wave equation reduces to Poisson's Equation, and an approximation to the solution is

$$
\begin{equation*}
\Psi_{p}\left(\mathbf{r}_{p}, t\right) \simeq \int_{V} \frac{s_{q}\left(\mathbf{r}_{q}, t\right) d V_{q}}{4 \pi r_{q p}} \tag{1.25}
\end{equation*}
$$

Such quasistatic solutions are often useful approximations to the exact solution.
Because there is a time delay of $r_{q p} / c$ before $\Psi_{p}$ can respond to changes in the source located at $\mathbf{r}_{q}$, the simplest correction to Eq. (1.25) that might be expected to improve the approximation is to replace $t$ with the retarded time, $t-r_{q p} / c$, where

$$
\begin{equation*}
r_{q p}=\sqrt{r_{q}^{2}+r_{p}^{2}-2 \mathbf{r}_{q} \cdot \mathbf{r}_{p}} \tag{1.26}
\end{equation*}
$$

Remarkably,

$$
\begin{equation*}
\Psi_{p}\left(\mathbf{r}_{p}, t\right)=\int \frac{s_{q}\left(\mathbf{r}_{q}, t-r_{q p} / c\right) d V_{q}}{4 \pi r_{q p}} \tag{1.27}
\end{equation*}
$$

is an exact solution of the wave equation. Although this result can be verified by direct substitution, we postpone proofs until Chapter 4, Section 4.5 and Chapter 7, Section 7.7.

In cases where the source term, $s_{q}\left(\mathbf{r}_{q}, t\right)$ has a sinusoidal time dependence at frequency $\omega$, it is advantageous to introduce complex functions $\underline{s}_{q}\left(\mathbf{r}_{q}, j \omega\right)$ and $\underline{\Psi}_{p}\left(\mathbf{r}_{p}, j \omega\right)$ such that

$$
\begin{aligned}
s_{q}\left(\mathbf{r}_{q}, t\right) & =\operatorname{Re}\left\{\underline{s}_{q}\left(\mathbf{r}_{q}, j \omega\right) \exp (j \omega t)\right\} \\
\Psi_{p}\left(\mathbf{r}_{p}, t\right) & =\operatorname{Re}\left\{\underline{\Psi}_{p}\left(\mathbf{r}_{p}, j \omega\right) \exp (j \omega t)\right\}
\end{aligned}
$$

where $\operatorname{Re}\}$ is the operator that extracts the real part of the expression contained within the curly brackets.

Substitution into Eq. (1.24) produces the complex form of the inhomogeneous wave equation,

$$
\begin{equation*}
\nabla^{2} \underline{\Psi}+k^{2} \underline{\Psi}=-\underline{s}_{q}\left(\mathbf{r}_{q}\right) \tag{1.28}
\end{equation*}
$$

where the wavenumber $k$ is defined by

$$
\begin{equation*}
k=\omega \sqrt{\mu_{\mathrm{o}} \varepsilon_{\mathrm{o}}} \tag{1.29}
\end{equation*}
$$

Because it is customary to define $\omega=2 \pi f$ and $k=2 \pi / \lambda_{0}$, the frequency $f$ and free-space wavelength $\lambda_{o}$ are related by

$$
\begin{equation*}
f \lambda_{\mathrm{o}}=c \tag{1.30}
\end{equation*}
$$

From Eq. (1.27), the integral form of the complex solution of Eq. (1.28) is

$$
\begin{equation*}
\Psi_{p}\left(\mathbf{r}_{p}\right)=\int \frac{s_{q}\left(\mathbf{r}_{q}\right) \exp \left(-j k r_{q p}\right) d V_{q}}{4 \pi r_{q p}} \tag{1.31}
\end{equation*}
$$

The complex exponential accounts for time retardation through the phase factor $k r_{q p}$. Integrals of this form are commonly encountered in diffraction and antenna theory; they can often be approximated with simpler forms when the observation point, $\mathbf{r}_{p}$, is very distant from all source locations, $\mathbf{r}_{q}$. In such cases, Eq. (1.26) is well-approximated by

$$
r_{q p} \simeq r_{p}-\frac{\mathbf{r}_{q} \cdot \mathbf{r}_{p}}{r_{p}}
$$

and the so-called "far-field" solution is

$$
\begin{equation*}
\Psi_{p}\left(\mathbf{r}_{p}\right) \simeq \frac{\exp \left(-j k r_{p}\right)}{4 \pi r_{p}} \int_{V} \underline{s_{q}\left(\mathbf{r}_{q}\right) \exp \left(j k \frac{\mathbf{r}_{q} \cdot \mathbf{r}_{p}}{r_{p}}\right) d V_{q} .} \tag{1.32}
\end{equation*}
$$

where the amplitude has been safely approximated using, $r_{q p} \simeq r_{p}$.
We note that in either the time or frequency domain, $\Psi$ can be replaced by the scalar $\Phi$ or any Cartesian component of $\mathbf{A}, \mathbf{E}$, or $\mathbf{H}$ with the corresponding $s$ evaluated from $\rho$ and $\mathbf{J}$. Equation (1.32) will be made use of repeatedly in the examples analyzed in Part III, Chapter 22.

## Current element (Hertzian electric dipole)

As an important example, we calculate the vector potential of a line current, of length $d$ and magnitude $\mathrm{I}_{0} \sin \omega t$, that is parallel to the $z$ axis and is "electrically short," that is $d \ll \lambda_{0}(k d \ll 1)$. Because we are assuming a steady-state current, it is convenient to employ complex vectors and scalars. With $\underline{s}_{q}\left(\mathbf{r}_{q}\right)=\underline{J}_{z}\left(x_{q}, y_{q}, z_{q}\right)$ and $\underline{\Psi}_{p}\left(\mathbf{r}_{p}\right)=\underline{A}_{z}\left(\mathbf{r}_{p}\right)$, it follows that for all $r_{p} \gg d$, Eq. (1.32),

$$
\underline{A}_{z}\left(\mathbf{r}_{p}\right)=\mu_{\mathrm{o}} \frac{\exp \left(-j k r_{p}\right)}{4 \pi r_{p}} \iiint \underline{J}_{z}\left(x_{q}, y_{q}, z_{q}\right) \exp \left(j k \frac{\mathbf{r}_{q} \cdot \mathbf{r}_{p}}{r_{p}}\right) d x_{q} d y_{q} d z_{q}
$$

can be used. After integrating over the cross section of the current and realizing that $\exp \left(j k \frac{\mathbf{r}_{q} \cdot \mathbf{r}_{p}}{r_{p}}\right) \simeq 1$ (because the argument of the exponential never exceeds $k \mathrm{~d}$ ), the result (after dropping the subscript $p$ ) is

$$
\underline{A}_{z}(\mathbf{r})=\mu_{0} \frac{\exp (-j k r)}{4 \pi r} \int_{0}^{\mathrm{d}} \underline{\mathrm{I}}\left(z_{q}\right) d z_{q}
$$

If the current is uniform, the integral is simply $\underline{I}_{0} d$; if not, it can be written as $\underline{I}_{0} d_{\text {eff }}$ and

$$
\begin{equation*}
\underline{A}_{z}(\mathbf{r})=\mu_{\mathrm{o}} \frac{\exp (-j k r)}{4 \pi r} \underline{\mathrm{I}}_{0} d_{\mathrm{eff}} \tag{1.33}
\end{equation*}
$$

Because of charge conservation, the current cannot go to zero at the ends of the line element without electric charges being created at the ends and/or along the length according to

$$
\frac{\partial \underline{\mathrm{I}}(z)}{\partial z}+j \omega Q^{\prime}(z)=0
$$

If $\frac{\partial \underline{I}(z)}{\partial z}=0$ (except at the ends), point charges of equal magnitude,
$\underline{\mathrm{Q}}_{\mathrm{o}}=\underline{\mathrm{I}}_{0} / \omega$, but opposite polarity, will exist at the ends and together create an electric dipole of moment, $\underline{\mathrm{Q}}_{0} d$; otherwise a line-charge density, $Q^{\prime}(z)$, will exist along the current and form a distributed dipole. In all cases, bipolar charges are the source of a complex scalar potential that satisfies

$$
\begin{equation*}
\Phi\left(\mathbf{r}_{p}\right)=\frac{\exp \left(-j k r_{p}\right)}{-j \omega 4 \pi \varepsilon_{0} r_{p}} \int_{0}^{\mathrm{d}} \frac{\partial \underline{\mathrm{I}}\left(z_{q}\right)}{\partial z_{q}} \exp \left(j k \frac{\mathbf{r}_{q} \cdot \mathbf{r}_{p}}{r_{p}}\right) d z_{q} \tag{1.34}
\end{equation*}
$$

This integral is a little tricky to evaluate correctly, so we choose the alternate approach of simply invoking the Lorenz gauge, $\nabla \cdot \underline{\mathbf{A}}+j \omega \mu_{0} \varepsilon_{0} \underline{\Phi}=0$ (which by itself imposes conservation of charge). The result is

$$
\begin{aligned}
\Phi & =\frac{-1}{j \omega \mu_{\mathrm{o}} \varepsilon_{0}} \frac{\partial A_{z}(\mathbf{r})}{\partial z}=\frac{-\mathrm{I}_{0} d_{\mathrm{eff}}}{j \omega 4 \pi \varepsilon_{0}} \frac{\partial}{\partial z}\left[\frac{\exp (-j k r)}{r}\right] \\
& =\frac{\mathrm{I}_{0} d_{\mathrm{eff}}}{j \omega 4 \pi \varepsilon_{0}}\left(j k+\frac{1}{r}\right) \frac{\exp (-j k r)}{r} \cos \theta
\end{aligned}
$$

The reader should verify that the same result is obtained from Eq. (1.34).
For the $\sin \omega t$ dependence specified at the outset, $\underline{I}_{0}=-j \mathrm{I}_{0}$; therefore the timedependent versions of the potentials (evaluated by multiplying by $\exp (j \omega t)$ and taking the real part) are

$$
\begin{align*}
& A_{z}(\mathbf{r}, t)=\mu_{0} I_{\mathrm{o}} d_{\mathrm{eff}} \frac{\sin (\omega t-k r)}{4 \pi r}  \tag{1.35a}\\
& \Phi(\mathbf{r}, t)=\sqrt{\frac{\mu_{\mathrm{o}}}{\varepsilon_{0}}} I_{0} d_{\mathrm{eff}} \frac{\sin (\omega t-k r)-\frac{1}{k r} \cos (\omega t-k r)}{4 \pi r} \cos \theta \tag{1.35b}
\end{align*}
$$

The Hertzian electric dipole is revisited in Chapter 3, Section 3.5 and Part III, Chapter 13, Section 13.1.

## Current loop (Hertzian magnetic dipole)

As a second example, consider a small circular loop of radius $R$ that lies in the $x-y$ plane, centered at $z=0$. A uniform current, $\underline{\mathrm{I}}(t)=\underline{\mathrm{I}}_{0} \cos \omega t$, circulates around the loop.

We continue to define $k=\omega / c$ and employ spherical coordinates with $\widehat{\phi}$ the unit vector in the azimuthal direction. For $k R \ll 1$, the complex vector and scalar retarded potentials are

$$
\begin{aligned}
\underline{\mathbf{A}}\left(\mathbf{r}_{p}\right) & =\mu_{\mathrm{o}} \frac{\exp \left(-j k r_{p}\right)}{4 \pi r_{p}} \oint_{\text {loop }} \mathrm{I}_{o} \widehat{\phi}_{q} \exp \left(j k \frac{\mathbf{r}_{q} \cdot \mathbf{r}_{p}}{r_{p}}\right) d s_{q} \\
\underline{\Phi} & =0
\end{aligned}
$$

The reader is cautioned to remember that although the current is $\mathrm{I}_{0} \widehat{\phi}$, the unit vector cannot simply be moved from inside to outside the integral. Instead, it must be converted to its Cartesian components before the integrations that determine $\mathbf{A}$ are attempted. Afterwards, the components can be converted to spherical coordinates; of course, only the $\phi$ component will survive. The electric potential is zero because the time-varying current is uniform and does not produce electric charges. The complex exponential when expanded in a Taylor series becomes

$$
\exp \left(j k \frac{\mathbf{r}_{q} \cdot \mathbf{r}_{p}}{r_{p}}\right) \simeq 1+j k \frac{\mathbf{r}_{q} \cdot \mathbf{r}_{p}}{r_{p}}+\cdots
$$

Because the leading term vanishes when integrated around the closed loop, the first-order term must be retained (unlike the case of the electric dipole). The far-field complex vector potential evaluates to

$$
\underline{\mathbf{A}}(\mathbf{r})=j k\left(1+\frac{1}{j k r}\right) R \frac{\mu_{0} \underline{I}_{0} R}{4} \frac{\exp (-j k r)}{r} \sin \theta \widehat{\phi}
$$

The time-dependent form is

$$
\begin{equation*}
\mathbf{A}(\mathbf{r})=\widehat{\phi} \frac{\mu_{\mathrm{o}}}{4 \pi} \frac{m_{0}[-k r \sin (\omega t-k r)+\cos (\omega t-k r)]}{r^{2}} \sin \theta \tag{1.36}
\end{equation*}
$$

where $m_{0}=\underline{\mathrm{I}}_{0} \pi R^{2}$ is defined as the magnetic dipole moment.
The Hertzian magnetic dipole is revisited in Part III, Chapter 13, Section 13.2. We postpone further discussion of complex fields or the introduction of the complex Maxwell's Equations until after the time-domain version has been generalized to include dielectric and magnetic materials.

### 1.7 POLARIZABLE, MAGNETIZABLE, AND CONDUCTING MEDIA

In the free-space formulation considered so far, only free charges and their associated electric currents are present. When dielectric and magnetic materials are considered, the response of electric and/or magnetic dipoles must be considered. Commonly, these are considered to be paired electric charges (bound to each other), in the case of dielectrics, and small electric-current loops (or their equivalent), in the case of magnetics. Other models (permitted by the equivalence principle) are possible; these include using fictitious magnetic charges of opposing polarities to model magnetic dipoles and/or small current loops carrying fictitious magnetic current to model electric dipoles. Moreover, various superpositions of all four types can be employed. In all models, the individual dipoles are of microscopic or mesoscopic dimensions; this permits one to characterize the material in terms of its polarization $(\mathbf{P})$ and magnetization $(\mathbf{M})$ vectors. Each is defined as the
appropriate macroscopically averaged dipole-moment density. There may be spontaneous values of $\mathbf{M}$ or $\mathbf{P}$; more generally, they arise in response to applied electric or magnetic fields. The response of dielectric materials can often be approximated as linear with respect to the excitation. That of magnetic materials is generally nonlinear and hysteretic, yet linear operation with respect to some operating point is sometimes possible. When the material is deforming and/or accelerating, additional complexity and subtlety is involved.

Because, at present, we wish to continue using a vector-potential formulation, it is convenient to employ a model for materials that is based solely on electric charge and current. The previous formulation can then be generalized by adding additional electric currents to those generated by the free (unpaired) charges. The form of Maxwell's Equations remains unchanged. Other ways of incorporating magnetization into electrodynamic theories are reviewed in Part II; the Chu formulation is of special interest because electric and magnetic dipoles are modelled symmetrically.

## Polarization and Amperian electric currents

The dielectric polarization can be modeled as identical charge distributions of opposite polarity, $\pm \rho_{0}$, displaced from one another by a small vector distance $\mathbf{d}$. This is depicted in Figure 1.3 with the separation greatly exaggerated.


Figure 1.3 Separated bipolar charge densities.
The electric-dipole moment of an incremental volume of the dielectric is the increment $\Delta \mathbf{p}=\rho_{0} \mathbf{d} \Delta V$ [refer to Appendix C, Eq. (C.9)]; therefore

$$
\begin{equation*}
\mathbf{p}=\int \mathbf{P}^{\mathrm{o}} d V \tag{1.37}
\end{equation*}
$$

where the polarization vector is defined as $\mathbf{P}^{0}=\rho_{\mathrm{o}} \mathbf{d}{ }^{3}$
The time rate of change of the polarization produces a current density,

$$
\begin{equation*}
\mathbf{J}_{\text {polarization }}=\rho_{0} \frac{\partial \mathbf{d}}{\partial t}=\frac{\partial \mathbf{P}^{\circ}}{\partial t} \tag{1.38}
\end{equation*}
$$

and an associated charge density, $\rho_{\text {polarization }}$. Because the latter is conserved independently of the free charge, we obtain

$$
\nabla \cdot \frac{\partial \mathbf{P}^{\mathrm{o}}}{\partial t}=\frac{\partial}{\partial t}\left(\nabla \cdot \mathbf{P}^{\mathrm{o}}\right)=-\frac{\partial \rho_{\text {polarization }}}{\partial t}
$$

[^6]and
\[

$$
\begin{equation*}
\rho_{\text {polarization }}=-\nabla \cdot \mathbf{P}^{\mathrm{o}} \tag{1.39}
\end{equation*}
$$

\]

Notice that this charge density is not $\rho_{0}$. Indeed, if $\mathbf{P}^{0}$ is spatially uniform, $\rho_{\text {polarization }}=0$, except on the boundaries of the dielectric. There, surface polarization charges reside wherever $\mathbf{P}^{0} \cdot \widehat{\mathbf{n}}$ is nonzero ( $\widehat{\mathbf{n}}$ is the normal to the surface).

For magnetic materials, the magnetization is assumed to arise from tiny Amperiancurrent loops (shown schematically in Figure 1.4) that produce magnetic-dipole moments.


Figure 1.4 Amperian current loops.

If the current circulating around each loop is uniform, no electric charge will be present. Because conservation of Amperian electric charge requires that

$$
\nabla \cdot \mathbf{J}_{\text {amperian }}=-\frac{\partial \rho_{\text {amperian }}}{\partial t}=0
$$

it follows that

$$
\begin{align*}
& \mathbf{J}_{\text {amperian }}=\nabla \times \mathbf{M}^{\mathrm{o}}  \tag{1.40a}\\
& \rho_{\text {amperian }}=0 \tag{1.40b}
\end{align*}
$$

where $\mathbf{M}^{0}$ is, as yet, an undefined vector.
However, the net magnetic moment, $\mathbf{m}$, of an electric-current distribution is defined as $\int \frac{1}{2} \mathbf{r}^{\prime} \times \mathbf{J} d V \quad$ [refer to Appendix C, Eq. (C.14)], which in this case is

$$
\mathbf{m}=\int \frac{1}{2} \mathbf{r}^{\prime} \times\left(\nabla \times \mathbf{M}^{\mathrm{o}}\right) d V
$$

and where the choice of the origin of $\mathbf{r}^{\prime}$ is arbitrary. The material volume is the sum of increments $\Delta V$; each contributes an incremental magnetic moment $\Delta \mathbf{m}$. As we consider each increment in turn, the origin of $\mathbf{r}^{\prime}=\mathbf{r}$ can be redefined so as to be centered upon it. Each incremental moment is therefore of the form $\Delta \mathbf{m}=\left[\lim _{\mathbf{r} \rightarrow 0} \frac{1}{2} \mathbf{r} \times\left(\nabla \times \mathbf{M}^{\circ}\right)\right] \Delta V$.

But

$$
\mathbf{r} \times\left(\nabla \times \mathbf{M}^{\mathrm{o}}\right)=(\nabla \cdot \mathbf{r}-1) \mathbf{M}^{\mathrm{o}}+\nabla\left(\mathbf{M}^{\mathrm{o}} \cdot \mathbf{r}\right)-\nabla \cdot\left(\mathbf{M}^{\mathrm{o}} \mathbf{r}\right)
$$

Therefore, because $\nabla \cdot \mathbf{r}=3$ and the other terms vanish in the limit, we obtain $\Delta \mathbf{m}=$ $\mathbf{M}^{\mathrm{o}} \Delta V$. Consequently, the total magnetic moment of the material volume can also be written as

$$
\begin{equation*}
\mathbf{m}=\int \mathbf{M}^{\mathrm{o}} d V \tag{1.41}
\end{equation*}
$$

with $\mathbf{M}^{0}$ now identified as the magnetization vector (the magnetic-dipole moment per unit volume). If $\mathbf{M}^{\circ}$ is spatially uniform, $\mathbf{J}_{\text {amperian }}=0$, except on the boundaries of the magnetic material. There, surface magnetization currents reside wherever $\mathbf{M}^{0} \times \widehat{\mathbf{n}}$ is nonzero ( $\widehat{\mathbf{n}}$ is the normal to the surface).

When polarization currents and Amperian currents are added to $\mathbf{J}_{\mathrm{u}}$, the current associated with the free (unpaired) charge, it follows that

$$
\begin{align*}
& \mathbf{J}_{\text {total }}=\mathbf{J}_{\mathrm{u}}+\mathbf{J}_{\text {polarization }}+\mathbf{J}_{\text {amperian }}=\mathbf{J}_{\mathrm{u}}+\frac{\partial \mathbf{P}^{\mathrm{o}}}{\partial t}+\nabla \times \mathbf{M}^{\mathrm{o}}  \tag{1.42a}\\
& \rho_{\text {total }}=\rho_{\mathrm{u}}+\rho_{\text {polarization }}+\rho_{\text {amperian }}=\rho_{\mathrm{u}}-\nabla \cdot \mathbf{P}^{\mathrm{o}}+0 \tag{1.42b}
\end{align*}
$$

With these field sources, it is customary to express Maxwell's Equations in terms of the field vector $\mathbf{B} / \mu_{0}$ rather than $\mathbf{H}$. With $\mathbf{E}^{0}$ substituted for $\mathbf{E}$ to distinguish the electric field from that used in the Chu formulation, the result is

$$
\begin{align*}
\frac{1}{\mu_{\mathrm{o}}} \nabla \times \mathbf{B}-\varepsilon_{\mathrm{o}} \frac{\partial \mathbf{E}^{\mathrm{o}}}{\partial t} & =\mathbf{J}_{\text {total }}=\mathbf{J}_{\mathrm{u}}+\frac{\partial \mathbf{P}^{\mathrm{o}}}{\partial t}+\nabla \times \mathbf{M}^{\mathrm{o}}  \tag{1.43a}\\
\varepsilon_{\mathrm{o}} \nabla \cdot \mathbf{E}^{\mathrm{o}} & =\rho_{\text {total }}=\rho_{\mathrm{u}}-\nabla \cdot \mathbf{P}^{\mathrm{o}}  \tag{1.43b}\\
\nabla \times \mathbf{E}^{\mathrm{o}}+\frac{\partial \mathbf{B}}{\partial t} & =0  \tag{1.43c}\\
\nabla \cdot \mathbf{B} & =0 \tag{1.43d}
\end{align*}
$$

where

$$
\begin{align*}
\mathbf{B} & =\nabla \times \mathbf{A}  \tag{1.44a}\\
\mathbf{E}^{\mathrm{o}} & =-\frac{\partial \mathbf{A}}{\partial t}-\nabla \Phi \tag{1.44b}
\end{align*}
$$

In this Amperian formulation, the Lorentz force density is

$$
\begin{equation*}
\mathbf{f}=\rho_{\text {total }} \mathbf{E}^{\mathrm{o}}+\mathbf{J}_{\text {total }} \times \mathbf{B} \tag{1.45}
\end{equation*}
$$

It is customary to define

$$
\begin{align*}
& \mathbf{D}=\varepsilon_{0} \mathbf{E}^{\mathrm{o}}+\mathbf{P}^{\mathrm{o}}  \tag{1.46a}\\
& \mathbf{B}=\mu_{\mathrm{o}}\left(\mathbf{H}^{\mathrm{o}}+\mathbf{M}^{\mathrm{o}}\right) \tag{1.46b}
\end{align*}
$$

because, with their use, Maxwell's Equations simplify to the Minkowski form,

$$
\begin{align*}
\nabla \times \mathbf{H}^{\mathrm{o}} & =\mathbf{J}_{\mathrm{u}}+\frac{\partial \mathbf{D}}{\partial t}  \tag{1.47a}\\
\nabla \cdot \mathbf{D} & =\rho_{\mathrm{u}}  \tag{1.47b}\\
\nabla \times \mathbf{E}^{\mathrm{o}} & =-\frac{\partial \mathbf{B}}{\partial t}  \tag{1.47c}\\
\nabla \cdot \mathbf{B} & =0 \tag{1.47d}
\end{align*}
$$

## Chu formulation

An alternate formulation, due to Lan Jen Chu, uses electric charges to model the polarization and magnetic charges to model the magnetization. In this case, the sources are duals of one another (as discussed in Chapter 6, Section 6.5) and Maxwell's Equations become

$$
\begin{align*}
& \nabla \times \mathbf{H}-\varepsilon_{0} \frac{\partial \mathbf{E}}{\partial t}=\mathbf{J}_{\mathrm{e}}=\mathbf{J}_{\mathrm{u}}+\frac{\partial \mathbf{P}}{\partial t}+\nabla \times(\mathbf{P} \times \mathbf{v})  \tag{1.48a}\\
& \varepsilon_{\mathrm{o}} \nabla \cdot \mathbf{E}=\rho_{\mathrm{e}}=\rho_{\mathrm{u}}-\nabla \cdot \mathbf{P}  \tag{1.48b}\\
& \nabla \times \mathbf{E}+\mu_{\mathrm{o}} \frac{\partial \mathbf{H}}{\partial t}=-\mathbf{J}_{\mathrm{m}}=-\mu_{\mathrm{o}}\left[\frac{\partial \mathbf{M}}{\partial t}+\nabla \times(\mathbf{M} \times \mathbf{v})\right]  \tag{1.48c}\\
& \mu_{\mathrm{o}} \nabla \cdot \mathbf{H}=\rho_{\mathrm{m}}=-\mu_{\mathrm{o}} \nabla \cdot \mathbf{M} \tag{1.48d}
\end{align*}
$$

These are consistent with Eqs. (1.47) provided

$$
\begin{align*}
\mathbf{E}^{\mathrm{o}} & =\mathbf{E}+\mu_{\mathrm{o}} \mathbf{M} \times \mathbf{v}  \tag{1.49a}\\
\mathbf{H}^{\mathrm{o}} & =\mathbf{H}-\mathbf{P} \times \mathbf{v}  \tag{1.49b}\\
\mathbf{P}^{\mathrm{o}} & =\mathbf{P}-\mu_{\mathrm{o}} \varepsilon_{\mathrm{o}} \mathbf{M} \times \mathbf{v}  \tag{1.49c}\\
\mathbf{M}^{\mathrm{o}} & =\mathbf{M}+\mathbf{P} \times \mathbf{v} \tag{1.49d}
\end{align*}
$$

Notice that neither $\mathbf{D}$ nor $\mathbf{B}$ requires superscript labeling because

$$
\begin{align*}
& \mathbf{D}=\varepsilon_{0} \mathbf{E}+\mathbf{P}=\varepsilon_{\mathrm{o}} \mathbf{E}^{\mathrm{o}}+\mathbf{P}^{\mathrm{o}}  \tag{1.50a}\\
& \mathbf{B}=\mu_{\mathrm{o}}(\mathbf{H}+\mathbf{M})=\mu_{\mathrm{o}}\left(\mathbf{H}^{\mathrm{o}}+\mathbf{M}^{\mathrm{o}}\right) \tag{1.50b}
\end{align*}
$$

In the Chu formulation, there are separate Lorentz-force densities acting upon both the electric and magnetic charges:

$$
\begin{align*}
\mathbf{f}_{\mathrm{e}} & =\rho_{\mathrm{e}} \mathbf{E}+\mathbf{J}_{\mathrm{e}} \times \mu_{\mathrm{o}} \mathbf{H}  \tag{1.51a}\\
\mathbf{f}_{\mathrm{m}} & =\rho_{\mathrm{m}} \mathbf{E}-\mathbf{J}_{\mathrm{m}} \times \varepsilon_{0} \mathbf{E} \tag{1.51b}
\end{align*}
$$

The total force density is their sum which is different from Eq. (1.45) because magnetic charges have been employed; the Minkowski formulation leads to yet another force density. These are all forces of "electromagnetic origin" that are dependent on how the sources are modelled; they must be augmented by mechanical and/or other forces. When done so properly, the total force density is independent of the electromagnetic formulation chosen. Despite the explicit velocity dependence, the Chu formulation is often (but not always) simpler to use. However, because we wish to employ the vector-potential, we continue at present with the Amperian formulation (expressed in terms of the Minkowski electric and magnetic fields).

For regions of free-space it follows that the constitutive laws that relate the four field vectors are (with $\mathbf{E}^{0}=\mathbf{E}, \mathbf{H}^{0}=\mathbf{H}$ ):

$$
\begin{align*}
& \mathbf{D}=\varepsilon_{0} \mathbf{E}  \tag{1.52a}\\
& \mathbf{B}=\mu_{0} \mathbf{H} \tag{1.52b}
\end{align*}
$$

## Electrically conducting materials

For linear isotropic (stationary) conductors that obey Ohm's Law,

$$
\begin{equation*}
\mathbf{J}_{\mathrm{u}}=\sigma \mathbf{E} \tag{1.53a}
\end{equation*}
$$

where $\sigma$ is the electrical-conductivity with units of $S m^{-1}=A V^{-1} m^{-1}$.
For such materials, conservation of charge becomes

$$
\begin{equation*}
\nabla \cdot(\sigma \mathbf{E})+\frac{\partial \rho_{\mathrm{u}}}{\partial t}=0 \tag{1.53b}
\end{equation*}
$$

These materials may also have dielectric/magnetic properties.

## Perfect conductors

A perfect conductor is defined by $\sigma \rightarrow \infty$ and therefore unless $\mathbf{J}_{u}$ is infinite, the electric field inside it must vanish. It follows that current must flow only in surface layers of thickness $\delta \rightarrow 0$ and $\mathbf{J}_{\mathrm{u}} \rightarrow \infty$ such that there is a finite surface current density, $\mathbf{K}_{\mathrm{u}}$ $=\lim _{\delta \rightarrow 0} \mathbf{J}_{\mathbf{u}} \delta$. Because the time rate of change of any magnetic field inside the conductor would produce electric fields, only static magnetic fields are tenable. Nevertheless, we define a perfect conductor as one that is completely field-free.

## Dielectric and magnetic materials

For linear isotropic (stationary) materials, the constitutive laws that relate the four field vectors are

$$
\begin{align*}
& \mathbf{D}=\varepsilon \mathbf{E}  \tag{1.54a}\\
& \mathbf{B}=\mu \mathbf{H} \tag{1.54b}
\end{align*}
$$

or equivalently

$$
\begin{align*}
\mathbf{P} & =\left(\varepsilon-\varepsilon_{\mathrm{o}}\right) \mathbf{E}=\chi_{\mathrm{e}} \varepsilon_{\mathrm{o}} \mathbf{E}  \tag{1.55a}\\
\mathbf{M} & =\left(\mu / \mu_{\mathrm{o}}-1\right) \mathbf{H}=\chi_{\mathrm{m}} \mathbf{H} \tag{1.55b}
\end{align*}
$$

where $\varepsilon$ and $\mu$ are, respectively, the dielectric permittivity and the magnetic permeability. The ratios, $\varepsilon / \varepsilon_{0}$ and $\mu / \mu_{\mathrm{o}}$ are defined as the relative permittivity (dielectric constant) and relative permeability. The electric and magnetic susceptibilities are the dimensionless quantities defined by

$$
\begin{align*}
\chi_{\mathrm{e}} & =\varepsilon / \varepsilon_{\mathrm{o}}-1  \tag{1.56a}\\
\chi_{\mathrm{m}} & =\mu / \mu_{\mathrm{o}}-1 \tag{1.56b}
\end{align*}
$$

The reader should again be cautioned that when the material is moving and/or deforming, the values of $\mathbf{E}$ and $\mathbf{H}$ in this formulation are not identical to those in the free-space equations. These complications are taken up in Part II of the text.

### 1.8 BOUNDARY CONDITIONS

The form of Maxwell's Equations reveals that in regions where the charges and currents are continuous functions of position and time, the divergence, curl, and time-derivative operators must all produce continuous functions. This fact requires the electric and magnetic fields to be continuous functions of space and time. However, there may be locations (such as the surface of a conductor or the interface between different materials) where there is an accumulation of charge and/or current that can be modeled as a finite surface density that within zero thickness has infinite volume density. At such locations $\nabla \cdot \mathbf{E}$ and/or $\nabla \times \mathbf{H}$ are infinite; these singularities lead to discontinuities in the respective field. We wish to determine exactly how large these will be.


Figure 1.5 Closed surface (normal fields).

## Electric surface charges

Assume that an unpaired electric surface charge of density $\sigma_{\mathrm{u}}^{s}$ resides on the surface shown in Figure 1.5. A circular pillbox is constructed with equal top and bottom areas, $\Delta A$, that are located on opposite sides of the surface; the height of the box is $\Delta h$. We apply Eqs. (1.5a) and (1.47b) to the closed surface defined by the Gaussian pillbox

$$
\begin{aligned}
\oint_{\text {top+bottom+sides }} \mathbf{D} \cdot \widehat{\mathbf{n}} d a & =\int_{\text {area }} \sigma_{\mathrm{u}}^{s} d a \\
{\left[\mathbf{D}^{(1)} \cdot \widehat{\mathbf{n}}_{1}+\mathbf{D}^{(2)} \cdot \widehat{\mathbf{n}}_{2}\right] \Delta A+\mathbf{D}^{(3)} \cdot \widehat{\mathbf{n}}_{3} 2 \sqrt{\pi \Delta A} \Delta h } & =\sigma_{\mathrm{u}}^{s} \Delta A
\end{aligned}
$$

If $\Delta h \ll \sqrt{\Delta A} \rightarrow 0$ and $\mathbf{E}$ is finite, it follows that because $\widehat{\mathbf{n}}_{1}=-\widehat{\mathbf{n}}_{2}=\widehat{\mathbf{n}}$, we obtain

$$
\begin{equation*}
\widehat{\mathbf{n}} \cdot\left[\mathbf{D}^{(1)}-\mathbf{D}^{(2)}\right]=\sigma_{\mathrm{u}}^{s} \tag{1.57}
\end{equation*}
$$

## Electric surface currents

Assume that an unpaired electric surface current of density $\mathbf{K}_{\mathrm{u}}^{s}$ resides on the horizontal surface shown in Figure 1.6.


Figure 1.6 Closed contour (tangential fields).

A rectangular area is constructed with equal top and bottom lengths, $\Delta s$, that are located on opposite sides of the surface; the height of the rectangle is $\Delta h$. We apply Eqs. (1.3a) and (1.47a) to the surface that is the plane of the rectangle:

$$
\begin{aligned}
\oint_{\text {perimeter }} \mathbf{H} \cdot \widehat{\mathbf{t}} d s=\int_{\text {area }} \mathbf{J}_{\mathrm{u}} \cdot(\widehat{\mathbf{n}} \times \widehat{\mathbf{t}}) d a & =\int_{\text {length }} \mathbf{K}_{\mathrm{u}}^{s} \cdot(\widehat{\mathbf{n}} \times \widehat{\mathbf{t}}) d s \\
{\left[\mathbf{H}^{(1)} \cdot \widehat{\mathbf{t}}_{1}+\mathbf{H}^{(2)} \cdot \widehat{\mathbf{t}_{2}}\right] \Delta s+\left[\mathbf{H}^{(3)} \cdot \widehat{\mathbf{t}}_{3}+\mathbf{H}^{(4)} \cdot \widehat{\mathbf{t}}_{4}\right] \Delta h } & =\mathbf{K}_{\mathrm{u}}^{s} \cdot(\widehat{\mathbf{n}} \times \widehat{\mathbf{t}}) \Delta s
\end{aligned}
$$

If $\Delta h \ll \Delta s \rightarrow 0$ and $\mathbf{H}$ is finite, it follows that because $\widehat{\mathbf{t}}_{1}=-\widehat{\mathbf{t}} 2=\widehat{\mathbf{t}}$ and $\widehat{\mathbf{n}} \cdot \widehat{\mathbf{t}}=0$, we have

$$
\begin{align*}
{\left[\mathbf{H}^{(1)}-\mathbf{H}^{(2)}\right] \cdot \widehat{\mathbf{t}} } & =\mathbf{K}_{\mathrm{u}}^{s} \cdot(\widehat{\mathbf{n}} \times \widehat{\mathbf{t}})=\left(\mathbf{K}_{\mathrm{u}}^{s} \times \widehat{\mathbf{n}}\right) \cdot \widehat{\mathbf{t}}  \tag{1.58a}\\
\mathbf{H}^{(1)}-\mathbf{H}^{(2)} & =\mathbf{K}_{\mathrm{u}}^{s} \times \widehat{\mathbf{n}}  \tag{1.58b}\\
\widehat{\mathbf{n}} \times\left[\mathbf{H}^{(1)}-\mathbf{H}^{(2)}\right] & =\mathbf{K}_{\mathrm{u}}^{s} \tag{1.58c}
\end{align*}
$$

## Conservation of charge

If a surface of discontinuity supports $\sigma_{\mathrm{u}}^{s}$ and/or $\mathbf{K}_{\mathrm{u}}^{s}$ and the material on one or both sides supports volume currents, $\mathbf{J}_{\mathrm{u}}$, the boundary conditions must maintain conservation of unpaired electric charge. A Gaussian pillbox is again erected to provide a closed surface over which

$$
\nabla \cdot \mathbf{J}_{u}+\frac{\partial \rho_{\mathrm{u}}}{\partial t}=0
$$

can be integrated. Comparison with Eq. (1.47b) suggests that

$$
\mathbf{n} \cdot\left[\mathbf{J}_{\mathrm{u}}^{(1)}-\mathbf{J}_{\mathrm{u}}^{(2)}\right]=-\frac{\partial \sigma_{\mathrm{u}}^{s}}{\partial t}
$$

However, in this case, $\mathbf{J}_{\mathrm{u}} \rightarrow \infty$, and so contributions from the sides cannot be ignored in the limit $\Delta h \rightarrow 0$. The correct boundary condition is

$$
\begin{equation*}
\mathbf{n} \cdot\left[\mathbf{J}_{\mathrm{u}}^{(1)}-\mathbf{J}_{\mathrm{u}}^{(2)}\right]+\nabla_{\Sigma} \cdot \mathbf{K}_{\mathrm{u}}^{s}+\frac{\partial \sigma_{\mathrm{u}}^{s}}{\partial t}=0 \tag{1.59}
\end{equation*}
$$

where $\nabla_{\Sigma} \cdot \mathbf{K}_{\mathrm{u}}^{s}$ is the two-dimensional surface divergence that accounts for the net outward flow along the surface. It is defined by

$$
\begin{equation*}
\nabla_{\Sigma} \cdot \mathbf{K}_{\mathrm{u}}^{s}=\lim _{\Delta A \rightarrow 0} \frac{\oint_{C_{\mathrm{o}}} \mathbf{K}_{\mathrm{u}}^{s} \cdot(\mathbf{t} \times \mathbf{n}) d s}{\Delta A} \tag{1.60}
\end{equation*}
$$

Because polarization charges are separately conserved, this form of boundary condition applies to them as well as to the total current and charge densities. There is no electric charge density associated with the Amperian current density, $\nabla \times \mathbf{M}$; but a surface density,

$$
\mathbf{K}_{M}^{s}=\widehat{\mathbf{n}} \times\left[\mathbf{M}^{(1)}-\mathbf{M}^{(2)}\right]
$$

exists wherever there are discontinuities in the tangential magnetization vector.
Because there are no magnetic charges or magnetic currents, the boundary conditions for $\mathbf{E}$ and $\mathbf{B}$ are

$$
\begin{align*}
\mathbf{n} \times\left[\mathbf{E}^{(1)}-\mathbf{E}^{(2)}\right] & =0  \tag{1.61a}\\
\mathbf{n} \cdot\left[\mathbf{B}^{(1)}-\mathbf{B}^{(2)}\right] & =0 \tag{1.61b}
\end{align*}
$$

It also follows that

$$
\begin{align*}
\mathbf{n} \cdot \varepsilon_{0}\left[\mathbf{E}^{(1)}-\mathbf{E}^{(2)}\right] & =\sigma_{\text {total }}^{s}=\sigma_{\mathrm{u}}^{s}+\sigma_{\text {polarization }}^{s}  \tag{1.62a}\\
\mathbf{n} \cdot\left[\mathbf{P}^{(1)}-\mathbf{P}^{(2)}\right] & =-\sigma_{\text {polarization }}^{s}  \tag{1.62b}\\
\mathbf{n} \cdot\left[\mathbf{H}^{(1)}-\mathbf{H}^{(2)}\right] & =-\mathbf{n} \cdot\left[\mathbf{M}^{(1)}-\mathbf{M}^{(2)}\right] \tag{1.62c}
\end{align*}
$$

### 1.9 THE COMPLEX MAXWELL EQUATIONS

In the sinusoidal steady state, time-harmonic vectors and scalars can be conveniently expressed as

$$
\begin{aligned}
\mathbf{F}(\mathbf{r}, t) & =\operatorname{Re}\{\underline{\mathbf{F}}(\mathbf{r}) \exp (j \omega t)\} \\
\Psi & =\operatorname{Re}\{\underline{\Psi}(\mathbf{r}) \exp (j \omega t)\}
\end{aligned}
$$

where the symbol $R e$ stands for the real part of the expression in the curly brackets $\}$, and the vector or scalar that is underscored is a complex quantity that may be a constant or a function of position. Because spatial and temporal derivatives are operations that commute with taking the real part, one may initially suppress the Re until the very last stage of analysis and simply insert the bracketed terms into Maxwell's Equations. When this is done, each time derivative will cause multiplication by the factor $j \omega$, but $\exp (j \omega t)$ will remain a common factor in every term; these are cumbersome and can be ignored if we remember to reinsert it before taking the real part.

The complex forms of Eqs. (1.47a)-(1.47d) (with the time dependence suppressed) are therefore

$$
\begin{align*}
\nabla \times \underline{\mathbf{H}}^{\mathrm{o}}-j \omega \underline{\mathbf{D}} & =\underline{\mathbf{J}_{\mathbf{u}}}  \tag{1.63a}\\
\nabla \cdot \underline{\mathbf{D}} & =\underline{\rho_{\mathbf{u}}}  \tag{1.63b}\\
\nabla \times \underline{\mathbf{E}}^{\mathrm{o}}+j \omega \underline{\mathbf{B}} & =0  \tag{1.63c}\\
\nabla \cdot \underline{\mathbf{B}} & =0 \tag{1.63d}
\end{align*}
$$

The complex forms of Eqs. (1.44a) and (1.44b) are

$$
\begin{align*}
\underline{\mathbf{B}} & =\nabla \times \underline{\mathbf{A}}  \tag{1.64a}\\
\underline{\mathbf{E}}^{\mathrm{o}} & =-j \omega \underline{\mathbf{A}}-\nabla \underline{\Phi} \tag{1.64b}
\end{align*}
$$

In most cases, we assume that the material is stationary and nondeforming; then, the superscripts can be removed from the electric and magnetic fields.

## CHAPTER 2

## QUASISTATIC APPROXIMATIONS

There are many electromagnetic-field configurations in which propagation effects are negligible. For example, if the time required for the field to travel through a structure is very small compared to the time variation of the sources that generate the fields, then the wave velocity can be considered to be infinite. In such cases, the electromagnetic wave equation is well approximated by Laplace's Equation, and the superposition of Laplacian fields ${ }^{1}$ that match the appropriate boundary conditions provides a satisfactory solution. In the limit of slow time variation (low frequency for sinusoidal steady-state fields), it often occurs that the electric field predominates over the magnetic field or vice versa. This leads to the classification of quasistatic fields that are either electroquasistatic (EQS) or magnetoquasistatic (MQS); references 5 [Chapter 6, p. 213] and 6 both contain treatments of quasistatic analysis, especially for sinusoidal steady-state fields. In some cases involving electrical conduction, the fields may be comparable yet both quasistatic in character. These three types of fields will be considered separately. In all cases the

[^7][^8]electric and magnetic fields are expressed as
\[

$$
\begin{aligned}
\mathbf{E} & =\mathbf{E}_{(0)}+\mathbf{E}_{(1)}+\cdots \\
\mathbf{H} & =\mathbf{H}_{(0)}+\mathbf{H}_{(1)}+\cdots
\end{aligned}
$$
\]

The main advantage of quasistatic analysis is that for many configurations that model practical devices, in which propagation effects are secondary, very complicated field problems can often be broken into comparatively simple pieces that nevertheless provide a basic understanding of the device operation. In numerous cases, the theoretical analysis required can be carried out by "back of the envelope calculations." When this is possible, the enhanced physical intuition that is gained is of incalculable value. Even when computer simulations are required (or desired), quasistatic analysis is often a valuable tool and guide.

### 2.1 QUASISTATIC EXPANSIONS OF A STANDING WAVE

To make clear the nature of quasistatic fields, we consider the exact solution of a standing electromagnetic wave of frequency $\omega$ and its quasistatic expansion. Linearly polarized fields formed by a pair of equal and opposite harmonic plane waves directed along the $z$ axis are

$$
\begin{aligned}
\mathbf{E} & =\widehat{\mathbf{x}} E_{\mathrm{o}} \cos (k z) \cos (\omega t) \\
\mathbf{H} & =\widehat{\mathbf{y}} \frac{1}{c \mu_{\mathrm{o}}} E_{\mathrm{o}} \sin (k z) \sin (\omega t) \\
k & =\omega \sqrt{\mu_{\mathrm{o}} \varepsilon_{\mathrm{o}}}
\end{aligned}
$$

Near $z=0$, the field is principally electric in character and so there the expansion is $E Q S$. Taylor series expansions of the $z$ variations yield

$$
\begin{aligned}
& \mathbf{E}=\mathbf{E}_{(0)}+\mathbf{E}_{(2)}+\cdots=\widehat{\mathbf{x}} E_{0} \cos (\omega t)\left[1-\frac{1}{2!}(k z)^{2}+\frac{1}{4!}(k z)^{4}-\cdots\right] \\
& \mathbf{H}=\mathbf{H}_{(1)}+\mathbf{H}_{(3)}+\cdots=\widehat{\mathbf{y}} \frac{1}{c \mu_{0}} E_{0} \sin (\omega t)\left[k z-\frac{1}{3!}(k z)^{3}+\cdots\right]
\end{aligned}
$$

where

$$
\begin{aligned}
& \mathbf{E}_{(n)}=\widehat{\mathbf{x}} E_{\mathrm{o}}(-1)^{n / 2} \frac{(k z)^{n}}{n!} \cos (\omega t) \\
& \mathbf{H}_{(n)}=\widehat{\mathbf{y}} \frac{1}{c \mu_{\mathrm{o}}} E_{\mathrm{o}}(-1)^{(n-1) / 2} \frac{(k z)^{n}}{n!} \sin (\omega t) \quad(n \text { even })
\end{aligned}
$$

Near $k z=\pi / 2$, the dominant field is magnetic and so there the expansion is $M Q S$. Although a shift of the $z$-coordinate origin is convenient, a simpler method is to consider the dual field:

$$
\begin{aligned}
\mathbf{H} & =\widehat{\mathbf{x}} H_{\mathrm{o}} \cos (k z) \cos (\omega t) \\
\mathbf{E} & =-\widehat{\mathbf{y}} \mathrm{c} \mu_{\mathrm{o}} H_{\mathrm{o}} \sin (k z) \sin (\omega t) \\
k & =\omega \sqrt{\mu_{\mathrm{o}} \varepsilon_{0}}
\end{aligned}
$$


[^0]:    ${ }^{1}$ If considered too demanding (or when there is insufficient time), Part II may be delayed (or bypassed altogether) until selected chapters of Part III have been studied. These contain topics and illustrative examples that depend mainly on knowledge of Part I. Used in this way, the text can serve both undergraduate and graduate students.

[^1]:    ${ }^{1}$ Originally, electricity and magnetism were separate fields of inquiry, each with its own nomenclature and units. That changed in 1819 when the Danish physicist Hans Christian Oersted (1777-1851) discovered the interaction of electric currents on a compass needle and coined the word "electromagnetism." Following up on this revelation, the French physicist Andre Ampere (1775-1836) studied the forces between electric currents.

[^2]:    ${ }^{2}$ In most metals, diffusion currents are negligible because conduction and charge neutrality combine to remove concentration gradients that drive diffusion, In semiconductors, diffusion can be made to dominate; the bipolar transistor depends upon it.

[^3]:    The Power and Beauty of Electromagnetic Fields, First Edition. F. R. Morgenthaler.
    (c) 2011 John Wiley \& Sons, Inc. Published 2011 by John Wiley \& Sons, Inc.

[^4]:    ${ }^{1}$ In this system of units, the abbreviations used include: $m$ (meter), $s$ (second), $k g$ (kilogram), $C$ (coulomb), $V$ (volt), $A$ (ampere), $F$ (farad), $H$ (henry), $S$ (siemens), $S^{-1}=\Omega$ (ohms).

[^5]:    ${ }^{2}$ The Lorenz gauge formulated by the Danish mathematician and physicist, Ludwig V. Lorenz (1829-1891), is often incorrectly attributed to Hendrick Lorentz; the latter is noted for the Lorentz transformation, the Lorentz contraction, and his theory of the electron.

[^6]:    ${ }^{3}$ The superscript o is added because we reserve unscripted variables for field quantities in the Chu formulation of electrodynamics. When the material is stationary and nondeformable, there is no difference between corresponding quantities with and without a superscript.

[^7]:    ${ }^{1}$ In some cases, such as those involving piezoelectric or magnetoelastic materials, hybrid waves can propagate with predominatly mechanical energies. Although the weak electric or magnetic fields (associated with what are basically elastic waves) are subject to quasistatic analysis, they are non-Laplacian in character and statements made in this text about Laplacian $E Q S$ and $M Q S$ fields do not apply.

[^8]:    The Power and Beauty of Electromagnetic Fields, First Edition. F. R. Morgenthaler.
    (C) 2011 John Wiley \& Sons, Inc. Published 2011 by John Wiley \& Sons, Inc.

