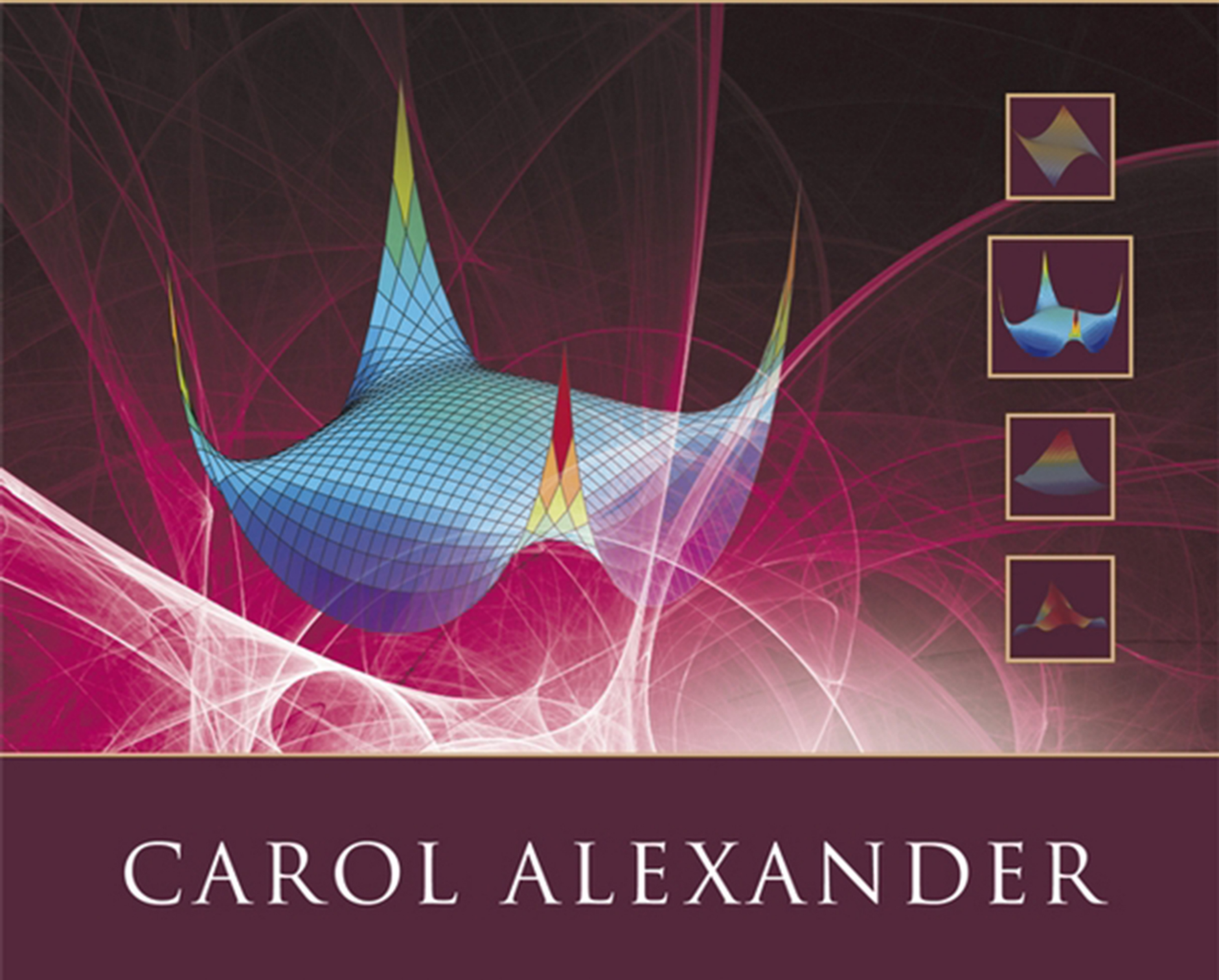
MARKET RISK ANALYSIS II



CAROL ALEXANDER

PRACTICAL FINANCIAL ECONOMETRICS

Market Risk Analysis
Volume II

Practical Financial Econometrics

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To Rick van der Ploeg

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Foreword

How many children dream of one day becoming risk managers? I very much doubt little Carol Jenkins, as she was called then, did. She dreamt about being a wild white horse, or a mermaid swimming with dolphins, as any normal little girl does. As I start crunching into two kilos of Toblerone that Carol Alexander-Pézier gave me for Valentine's day (perhaps to coax me into writing this foreword), I see the distinctive silhouette of the Matterhorn on the yellow package and I am reminded of my own dreams of climbing mountains and travelling to distant planets. Yes, adventure and danger! That is the stuff of happiness, especially when you daydream as a child with a warm cup of cocoa in your hands.

As we grow up, dreams lose their naivety but not necessarily their power. Knowledge makes us discover new possibilities and raises new questions. We grow to understand better the consequences of our actions, yet the world remains full of surprises. We taste the sweetness of success and the bitterness of failure. We grow to be responsible members of society and to care for the welfare of others. We discover purpose, confidence and a role to fulfil; but we also find that we continuously have to deal with risks.

Leafing through the hundreds of pages of this four-volume series you will discover one of the goals that Carol gave herself in life: to set the standards for a new profession, that of market risk manager, and to provide the means of achieving those standards. Why is market risk management so important? Because in our modern economies, market prices balance the supply and demand of most goods and services that fulfil our needs and desires. We can hardly take a decision, such as buying a house or saving for a later day, without taking some market risks. Financial firms, be they in banking, insurance or asset management, manage these risks on a grand scale. Capital markets and derivative products offer endless ways to transfer these risks among economic agents.

But should market risk management be regarded as a professional activity? Sampling the material in these four volumes will convince you, if need be, of the vast amount of knowledge and skills required. A good market risk manager should master the basics of calculus, linear algebra, probability – including stochastic calculus – statistics and econometrics. He should be an astute student of the markets, familiar with the vast array of modern financial instruments and market mechanisms, and of the econometric properties of prices and returns in these markets. If he works in the financial industry, he should also be well versed in regulations and understand how they affect his firm. That sets the academic syllabus for the profession.

Carol takes the reader step by step through all these topics, from basic definitions and principles to advanced problems and solution methods. She uses a clear language, realistic illustrations with recent market data, consistent notation throughout all chapters, and provides a huge range of worked-out exercises on Excel spreadsheets, some of which demonstrate analytical tools only available in the best commercial software packages. Many chapters on advanced subjects such as GARCH models, copulas, quantile regressions, portfolio theory, options and volatility surfaces are as informative as and easier to understand than entire books devoted to these subjects. Indeed, this is the first series of books entirely dedicated to the discipline of market risk analysis written by one person, and a very good teacher at that.

A profession, however, is more than an academic discipline; it is an activity that fulfils some societal needs, that provides solutions in the face of evolving challenges, that calls for a special code of conduct; it is something one can aspire to. Does market risk management face such challenges? Can it achieve significant economic benefits?

As market economies grow, more ordinary people of all ages with different needs and risk appetites have financial assets to manage and borrowings to control. What kind of mortgages should they take? What provisions should they make for their pensions? The range of investment products offered to them has widened far beyond the traditional cash, bond and equity classes to include actively managed funds (traditional or hedge funds), private equity, real estate investment trusts, structured products and derivative products facilitating the trading of more exotic risks – commodities, credit risks, volatilities and correlations, weather, carbon emissions, etc. – and offering markedly different return characteristics from those of traditional asset classes. Managing personal finances is largely about managing market risks. How well educated are we to do that?

Corporates have also become more exposed to market risks. Beyond the traditional exposure to interest rate fluctuations, most corporates are now exposed to foreign exchange risks and commodity risks because of globalization. A company may produce and sell exclusively in its domestic market and yet be exposed to currency fluctuations because of foreign competition. Risks that can be hedged effectively by shareholders, if they wish, do not have to be hedged in-house. But hedging some risks in-house may bring benefits (e.g. reduction of tax burden, smoothing of returns, easier planning) that are not directly attainable by the shareholder.

Financial firms, of course, should be the experts at managing market risks; it is their métier. Indeed, over the last generation, there has been a marked increase in the size of market risks handled by banks in comparison to a reduction in the size of their credit risks. Since the 1980s, banks have provided products (e.g. interest rate swaps, currency protection, index linked loans, capital guaranteed investments) to facilitate the risk management of their customers. They have also built up arbitrage and proprietary trading books to profit from perceived market anomalies and take advantage of their market views. More recently, banks have started to manage credit risks actively by transferring them to the capital markets instead of warehousing them. Bonds are replacing loans, mortgages and other loans are securitized, and many of the remaining credit risks can now be covered with credit default swaps. Thus credit risks are being converted into market risks.

The rapid development of capital markets and, in particular, of derivative products bears witness to these changes. At the time of writing this foreword, the total notional size of all derivative products exceeds \$500 trillion whereas, in rough figures, the bond and money markets stand at about \$80 trillion, the equity markets half that and loans half that again. Credit derivatives by themselves are climbing through the \$30 trillion mark. These derivative markets are zero-sum games; they are all about market risk management – hedging, arbitrage and speculation.

This does not mean, however, that all market risk management problems have been resolved. We may have developed the means and the techniques, but we do not necessarily understand how to address the problems. Regulators and other experts setting standards and policies are particularly concerned with several fundamental issues. To name a few:

1. How do we decide what market risks should be assessed and over what time horizons? For example, should the loan books of banks or long-term liabilities of pension funds be marked to market, or should we not be concerned with pricing things that will not be traded in the near future? We think there is no general answer to this question about the most appropriate description of risks. The descriptions must be adapted to specific management problems.
2. In what contexts should market risks be assessed? Thus, what is more risky, fixed or floating rate financing? Answers to such questions are often dictated by accounting standards or other conventions that must be followed and therefore take on economic significance. But the adequacy of standards must be regularly reassessed. To wit, the development of International Accounting Standards favouring mark-to-market and hedge accounting where possible (whereby offsetting risks can be reported together).
3. To what extent should risk assessments be 'objective'? Modern regulations of financial firms (Basel II Amendment, 1996) have been a major driver in the development of risk assessment methods. Regulators naturally want a 'level playing field' and objective rules. This reinforces a natural tendency to assess risks purely on the basis of statistical evidence and to neglect personal, forward-looking views. Thus one speaks too often about risk 'measurements' as if risks were physical objects instead of risk 'assessments' indicating that risks are potentialities that can only be guessed by making a number of assumptions (i.e. by using models). Regulators try to compensate for this tendency by asking risk managers to draw scenarios and to stress-test their models.

There are many other fundamental issues to be debated, such as the natural tendency to focus on micro risk management – because it is easy – rather than to integrate all significant risks and to consider their global effect – because that is more difficult. In particular, the assessment and control of systemic risks by supervisory authorities is still in its infancy. But I would like to conclude by calling attention to a particular danger faced by a nascent market risk management profession, that of separating risks from returns and focusing on downside-risk limits.

It is central to the ethics of risk managers to be independent and to act with integrity. Thus risk managers should not be under the direct control of line managers of profit centres and they should be well remunerated independently of company results. But in some firms this is also understood as denying risk managers access to profit information. I remember a risk commission that had to approve or reject projects but, for internal political reasons, could not have any information about their expected profitability. For decades, credit officers in most banks operated under such constraints: they were supposed to accept or reject deals a priori, without knowledge of their pricing. Times have changed. We understand now, at least in principle, that the essence of risk management is not simply to reduce or control risks but to achieve an optimal balance between risks and returns.

Yet, whether for organizational reasons or out of ignorance, risk management is often confined to setting and enforcing risk limits. Most firms, especially financial firms, claim to have well-thought-out risk management policies, but few actually state trade-offs between risks and returns. Attention to risk limits may be unwittingly reinforced by regulators. Of course it is not the role of the supervisory authorities to suggest risk–return trade-offs; so supervisors impose risk limits, such as value at risk relative to capital, to ensure safety and fair competition in the financial industry. But a regulatory limit implies severe penalties if breached, and thus a probabilistic constraint acquires an economic value. Banks must therefore pay attention to the uncertainty in their value-at-risk estimates. The effect would be rather perverse if banks ended up paying more attention to the probability of a probability than to their entire return distribution.

With *Market Risk Analysis* readers will learn to understand these long-term problems in a realistic context. Carol is an academic with a strong applied interest. She has helped to design the curriculum for the Professional Risk Managers' International Association (PRMIA) qualifications, to set the standards for their professional qualifications, and she maintains numerous contacts with the financial industry through consulting and seminars. In *Market Risk Analysis* theoretical developments may be more rigorous and reach a more advanced level than in many other books, but they always lead to practical applications with numerous examples in interactive Excel spreadsheets. For example, unlike 90% of the finance literature on hedging that is of no use to practitioners, if not misleading at times, her concise expositions on this subject give solutions to real problems.

In summary, if there is any good reason for not treating market risk management as a separate discipline, it is that market risk management should be the business of *all* decision makers involved in finance, with primary responsibilities on the shoulders of the most senior managers and board members. However, there is so much to be learnt and so much to be further researched on this subject that it is proper for professional people to specialize in it. These four volumes will fulfil most of their needs. They only have to remember that, to be effective, they have to be good communicators and ensure that their assessments are properly integrated in their firm's decision-making process.

Jacques Pézier

Preface to Volume II

For well over a decade, econometrics has been one of the major routes into finance. I took this route myself several years ago. Starting an academic career as an algebraist, I then had a brief encounter with game theory before discovering that the skills of an econometrician were in greater demand. I would have found econometrics much more boring than algebra or game theory had it not been for the inspiration of some great teachers at the London School of Economics, and of Professor Robert Engle who introduced me to GARCH models some twenty years ago.

At that time finance was one of the newest areas of applied econometrics and it was relatively easy to find interesting problems that were also useful to practitioners. And this was how my reputation grew, such as it is. I was building GARCH models for banks well before they became standard procedures in statistical packages, applying cointegration to construct arbitrage strategies for fund managers and introducing models for forecasting very large covariance matrices. In the end the appreciation of this work was much greater than the appreciation I received as an academic so I moved, briefly, to the City. Then, almost a decade ago, I returned to academic life as a professor of financial risk management. In fact, I believe I was the first professor to have this title in the UK, financial risk management being such a new profession at that time. It was the late 1990s, and by then numerous econometricians were taking the same route into finance that I had. Some of the top finance journals were populating many of their pages with applied financial econometrics, and theoretical econometric journals were becoming increasingly focused on financial problems. Of course I wanted to read and learn all about this so that I could publish the academic papers that are so important to our profession. But I was disappointed and a little dismayed by what I read. Too few of the papers were written by authors who seemed to have a proper grasp of the important practical problems in finance. And too much journal space was devoted to topics that are at best marginal and at worst completely irrelevant to financial practitioners.

Econometrics has now become a veritable motorway into finance where, for many, prospects are presently more lucrative than those for standard macro- or micro-economists. The industry has enormous demand for properly trained financial econometricians, and this demand will increase. But few econometricians enter the industry with an adequate knowledge of how their skills can be employed to the best advantage of their firm and its clients, and many financial econometricians would benefit from improving their understanding of what constitutes an important problem.

AIMS AND SCOPE

This book introduces the econometric techniques that are commonly applied to finance, and particularly to resolve problems in market risk analysis. It aims to fill a gap in the market by offering a critical text on econometrics that discuss what is and what is not important to financial practitioners. The book covers material for a one-semester graduate course in applied financial econometrics in a very pedagogical fashion. Each time a concept is introduced, an empirical example is given, and whenever possible this is illustrated with an Excel spreadsheet.

In comparison with Greene (2007), which has become a standard graduate econometrics text and which contains more than enough material for a one-year course, I have been very selective in the topics covered. The main focus is on models that use time series data, and relatively few formal proofs are given. However, every chapter has numerous empirical examples that are implemented in Excel spreadsheets, many of which are interactive. And when the practical illustration of the model requires a more detailed exposition, case studies are included. More details are given in the section about the website below.

Econometrics is a broad discipline that draws on basic techniques in calculus, linear algebra, probability, statistics and numerical methods. Readers should also have a rudimentary knowledge of regression analysis and the first chapter, which is on factor model, refers to the capital asset pricing model and other models derived from the theory of asset pricing. All the prerequisite material is covered *Market Risk Analysis* Volume I: *Quantitative Methods in Finance*. However, there is only one chapter on basic regression in Volume I. A very comprehensive introductory text, written at a more elementary level than this but also aimed towards the finance student market, is Brooks (2008). For many years Professor Chris Brooks has been a close colleague at the ICMA Centre.

The other volumes in *Market Risk Analysis* are Volume III: *Pricing, Hedging and Trading Financial Instruments* and Volume IV: *Value at Risk Models*. Although the four volumes of *Market Risk Analysis* are very much interlinked, each book is self-contained. This book could easily be adopted as a stand-alone course text in applied financial econometrics, leaving students to follow up cross-references to other volumes only if they wish.

OUTLINE OF VOLUME II

Chapter 1, *Factor Models*, describes the models that are applied by portfolio managers to analyse the potential returns on a portfolio of risky assets, to determine the allocation of their funds to different assets and to measure portfolio risk. The chapter deals with models having fundamental factors and which are normally estimated by regression. We focus on the Barra model, giving a detailed description of its construction, and emphasizing the dangers of using tracking error as a risk metric for actively managed portfolios.

Chapter 2, *Principal Component Analysis*, covers statistical factor models, which are also used for portfolio management and risk management, but they are most successful when applied to a highly correlated system such as a term structure of interest rates, of futures prices or of volatility. Since it is not easy to find a complete treatment of principal component analysis in a finance-oriented text, we provide full details of the mathematics but, as usual, we focus on the applications. Empirical examples include bond portfolio immunization, asset–liability management and portfolio risk assessment.

Chapter 3, *Classical Models of Volatility and Correlation*, provides a critical review of the time series models that became popular in the industry during the 1990s, making readers aware of the pitfalls of using simple moving averages for estimating and forecasting portfolio risk. These are based on the assumption that returns are independent and identically distributed so the volatility and correlation forecasts from these models are equal to the current estimates. The sample estimates vary over time, but this is only due to sampling error. There is nothing in the model to capture the volatility and correlation clustering that is commonly observed in financial asset returns.

Chapter 4, *Introduction to GARCH Models*, provides a complete and up-to-date treatment of the generalized autoregressive conditional heteroscedasticity models that were introduced by Engle (1982) and Bollerslev (1986). We explain how to: estimate the model parameters by maximizing a likelihood function; use the model to forecast term structures for volatility and correlation; target the long term volatility or correlation and use the GARCH model to forecast volatility and correlation over the short and medium term; and extend the model to capture non-normal conditional returns distributions and regime-switching volatility behaviour. There are so many approaches to modelling multivariate distributions with time varying volatility and correlation that I have been very prescriptive in my treatment of multivariate GARCH models, recommending specific approaches for different financial problems. Throughout this long chapter we illustrate the GARCH model optimization with simple Excel spreadsheets, employing the Excel Solver whenever possible. Excel parameter estimates for GARCH are not recommended, so the estimates are compared with those obtained using GARCH procedures in the Matlab and EViews software. The section on simulation is enlightening, since it demonstrates that only regime-switching GARCH models can properly capture the observed behaviour of financial asset returns. The final section covers the numerous applications of GARCH models to finance, including option pricing, risk measurement and portfolio optimization.

Chapter 5 is on *Time Series Models and Cointegration*. Building on the introduction to stochastic processes given in Chapter I.3, this begins with a mathematical introduction to stationary and integrated processes, multivariate vector autoregressions and unit root tests. Then we provide an intuitive definition of cointegration and review the huge literature on applications of cointegration in financial markets. A case study focuses on the benchmark tracking and statistical arbitrage applications that I developed more than a decade ago, and which are now used by major fund managers. The final section provides a didactic approach to modelling short term dynamics using error correction models, focusing on the response of cointegrated asset prices to market shocks and the time taken for a spread to mean-revert. Another case study examines pairs trading volatility indices.

Chapter 6, *Introduction to Copulas*, took much longer to write than the other chapters. I was less familiar with copulas than with the other topics in this book, and found the available literature a little obscure and off-putting. However, copulas are of crucial importance to the development of our subject and no reputable financial econometrician can afford to ignore them. So it became quite a challenge to present this material in the pedagogical style of the rest of the book. I have programmed several copulas, including the normal, normal mixture, Student's t, Clayton and Gumbel copulas, in interactive Excel spreadsheets, so that you can see how the shape of the copula alters on changing its parameters. The quantile curves of conditional copulas play a crucial role in financial applications – for instance, in quantile regression – so these have been derived mathematically and also encoded into Excel. Many other applications such as value-at-risk measurement, portfolio optimization and risk aggregation, which are discussed in the last section of the chapter, are based on simulation with copulas. Two simulation algorithms are described and spreadsheets generate simulations based on different copulas.

Chapter 7 covers the *Advanced Econometric Models* that have important applications to finance. A significant portion of this chapter provides a tutorial on quantile regression, and contains two case studies in Excel. The first implements linear and non-linear quantile regressions to examine the relationship between an equity index and its volatility, and the second demonstrates how non-linear quantile regression using copulas can be applied to hedge a portfolio with futures. A relatively brief treatment of other non-linear models is restricted to polynomial regression and discrete choice models, the latter being illustrated with an application to credit scoring models. What I hope is an accessible specification of Markov switching models is followed with a short review of their applications and the software that can be used for estimation, and the chapter concludes by describing the main high frequency data sets and two of the most important financial problems in high frequency data analysis. First, for capturing the clustering of the times between trades we describe the autoregressive conditional duration model. Then we review the large and growing literature on using high frequency data to forecast realized variance and covariance, this being important for pricing the variance swaps and covariance swaps that are actively traded in over-the-counter markets.

The last chapter, Chapter 8 on *Forecasting and Model Evaluation*, describes how to select the best model when several models are available. The model specification and evaluation criteria and tests described here include goodness-of-fit criteria and tests, which measure the success of a model to capture the empirical characteristics of the estimation sample, and post-sample prediction criteria and tests, which judge the ability of the model to provide accurate forecasts. Models for the conditional expectation, volatility and correlation of financial asset returns that were introduced in earlier chapters are considered here, and we explain how to apply both statistical and operational criteria and tests to these models. Amongst the statistical tests, we emphasize the Kolmogorov–Smirnoff and related tests for the proximity of two distributions and the coverage tests that are applied to evaluate models for predicting quantiles of conditional distributions. We also explain how to simulate the critical values of non-standard test statistics. A long section on operational evaluation first outlines the model backtesting procedure in general terms, and then explains how backtests are applied in specific contexts, including tests of: factor models used in portfolio management; covariance matrices used for portfolio optimization and value-at-risk estimation; and models that are used for short term hedging with futures, trading implied volatility, trading variance swaps and hedging options

ABOUT THE WEBSITE

Whenever possible the econometric models, tests and criteria that are introduced in this book are illustrated in an Excel spreadsheet. The Excel workbooks for each chapter may be found on the accompanying website. Simply search for the book on wiley.com and select 'Related Resources' to access the material. Many of the spreadsheets are interactive, so readers may change any parameters of the problem (the parameters are indicated in *red*) and see the new solution (the output is indicated in *blue*). Rather than using VBA code, which will be obscure to many readers, I have encoded the formulae directly into the spreadsheet. Thus the reader need only click on a cell to read the formula. Whenever a data analysis tool such as regression or a numerical tool such as Solver is used, clear instructions are given in the text, and/or using comments and screenshots in the spreadsheet. Hence, the spreadsheets are designed to offer tutors the possibility to set, as exercises for their courses, an unlimited number of variations on the examples in the text.

Excel is not always an adequate program for estimating econometric models, and I have been particularly emphatic on this point for the spreadsheets that estimate GARCH model parameters. Excel has its limits in other respects, too, and so references to and recommendations of proper econometric programs are given where necessary. For instance, the website includes the EViews code for Markov switching models that was written by my PhD student Andreas Kaeck.

Several case studies, based on complete and up-to-date financial data, and all graphs and tables in the text are also contained in the Excel workbooks on the website. The case study data can be used by tutors or researchers since they were obtained from free internet sources, and references for updating the data are provided. Also the graphs and tables can be modified if required, and copied and pasted as enhanced metafiles into lecture notes based on this book.

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But most of all, thanks to Ronnie Barnes and Philippe Derome. We first 'met' when they, independently, contacted me regarding errors in the first printing of Volume I. Ever since they have been working tirelessly to help find typos, and to improve clarity, in all four books in this series. Ronnie is a highly trained mathematician with an eagle's eye and Philippe accepts nothing without rigorous examination. Entirely thanks to Ronnie and Philippe, the reprints of Volume II now have an extremely high standard of presentation.

Discussion forums and other resources for the Market Risk Analysis series are available at **www.marketriskanalysis.com**

II.1 Factor Models

II.1.1 INTRODUCTION

This chapter describes the factor models that are applied by portfolio managers to analyse the potential returns on a portfolio of risky assets, to choose the optimal allocation of their funds to different assets and to measure portfolio risk. The theory of linear regression-based factor models applies to most portfolios of risky assets, excluding options portfolios but including alternative investments such as real estate, hedge funds and volatility, as well as traditional assets such as commodities, stocks and bonds. Stocks and bonds are the major categories of risky assets, and whilst bond portfolios could be analysed using regression-based factor models a much more powerful factor analysis for bond portfolios is based on principal component analysis (see Chapter II.2).

An understanding of both multiple linear regression and matrix algebra is necessary for the analysis of multi-factor models. Therefore, we assume that readers are already familiar with matrix theory from Chapter I.2 and the theory of linear regression from Chapter I.4. We also assume that readers are familiar with the theory of asset pricing and the optimal capital allocation techniques that were introduced in Chapter I.6.

Regression-based factor models are used to forecast the expected return and the risk of a portfolio. The expected return on each asset in the portfolio is approximated as a weighted sum of the expected returns to several market risk factors. The weights are called *factor sensitivities* or, more specifically, *factor betas* and are estimated by regression. If the portfolio only has cash positions on securities in the same country then market risk factors could include broad market indices, industry factors, style factors (e.g. value, growth, momentum, size), economic factors (e.g. interest rates, inflation) or statistical factors (e.g. principal components).[1] By inputting scenarios and stress tests on the expected returns and the volatilities and correlations of these risk factors, the factor model representation allows the portfolio manager to examine expected returns under different market scenarios.

Factor models also allow the market risk manager to quantify the systematic and specific risk of the portfolio:

- The market risk management of portfolios has traditionally focused only on the *undiversifiable risk* of a portfolio. This is the risk that cannot be reduced to zero by holding a large and diversified portfolio. In the context of a factor model, which aims to relate the distribution of a portfolio's return to the distributions of its risk factor returns, we also call the undiversifiable risk the *systematic risk*. A multi-factor model, i.e. a factor model with more than one risk factor, would normally be estimated using a multiple linear regression where the dependent variable is the return on an individual asset and the

[1] But for international portfolios exchange rates also affect the returns, with a beta of one. And if the portfolio contains futures then zero coupon rates should also be included in the market risk factors.

independent variables are the returns on different risk factors. Then the systematic risk is identified with the risk of the factor returns and the net portfolio sensitivities to each risk factor.

- The *specific risk*, also called the *idiosyncratic risk* or *residual risk*, is the risk that is not associated with the risk factor returns. In a linear regression model of the asset return on risk factor returns, it is the risk arising from the variance of the residuals. The specific risk on an individual asset may be high, especially when the model has only a few factors to explain the asset's returns. But in a sufficiently large and diversified portfolio the specific risk may be reduced to almost zero, since the specific risks on a large number of assets in different sectors of the economy, or in different countries, tend to cancel each other out.

The outline of the chapter is as follows. Section II.1.2 explains how a single-factor model is estimated. We compare two methods for estimating factor betas and show how the total risk of the portfolio can be decomposed into the systematic risk due to risk of the factors, and the specific risk that may be diversified away by holding a sufficiently large portfolio. Section II.1.3 describes the general theory of multi-factor models and explains how they are used in style attribution analysis. We explain how multi-factor models may be applied to different types of portfolios and to decompose the total risk into components related to broad classes of risk factors. Then in Section II.1.4 we present an empirical example which shows how to estimate a fundamental factor model using time series data on the portfolio returns and the risk factor returns. We suggest a remedy for the problem of multicollinearity that arises here and indeed plagues the estimation of most fundamental factor models in practice.

Then Section II.1.5 analyses the Barra model, which is a specific multi-factor model that is widely used in portfolio management. Following on from the Barra model, we analyse the way some portfolio managers use factor models to quantify *active risk,* i.e. the risk of a fund relative to its benchmark. The focus here is to explain why it is a mistake to use *tracking error*, i.e. the volatility of the active returns, as a measure of active risk. Tracking error is a metric for active risk only when the portfolio is tracking the benchmark. Otherwise, an increase in tracking error does not indicate that active risk is increased and a decrease in tracking error does not indicate that active risk has been reduced. The active risk of actively managed funds which by design do not track a benchmark cannot be measured by tracking error. However, we show how it is possible to adjust the tracking error into a correct, but basic active risk metric. Section II.1.6 summarizes and concludes.

II.1.2 SINGLE FACTOR MODELS

This section describes how single factor models are applied to analyse the expected return on an asset, to find a portfolio of assets to suit the investor's requirements, and to measure the risk of an existing portfolio. We also interpret the meaning of a factor beta and derive a fundamental result on portfolio risk decomposition.

II.1.2.1 Single Index Model

The capital asset pricing model (CAPM) was introduced in Section I.6.4. It hypothesizes the following relationship between the expected excess return on any single risky asset and the expected excess return on the market portfolio:

$$E(R_i) - R_f = \beta_i(E(R_M) - R_f),$$

where R_i is the return on the ith risky asset, R_f is the return on the *risk free* asset, R_M is the return on the *market portfolio* and β_i is the beta of the ith risky asset. The CAPM implies the following linear model for the relationship between ordinary returns rather than excess returns:

$$E(R_i) = \alpha_i + \beta_i E(R_M), \tag{II.1.1}$$

where $\alpha_i \neq 0$ unless $\beta_i = 1$.

The single index model is based on the expected return relationship (II.1.1) where the return X on a factor such as a broad market index is used as a proxy for the market portfolio return R_M. Thus the single index model allows one to investigate the risk and return characteristics of assets *relative* to the broad market index. More generally, if the performance of a portfolio is measured relative to a benchmark other than a broad market index, then the benchmark return is used for the factor return X.

We can express the single index model in the form

$$R_{it} = \alpha_i + \beta_i X_t + \varepsilon_{it}, \quad \varepsilon_{it} \sim \text{i.i.d.}(0, \sigma_i^2). \tag{II.1.2}$$

Here α_i measures the asset's expected return relative to the benchmark or index (a positive value indicates an expected outperformance and a negative value indicates an expected underperformance); β_i is the *risk factor sensitivity* of the asset; $\beta_i \sigma_X$ is the *systematic volatility* of the asset, σ_X being the volatility of the index returns; and σ_i is the *specific volatility* of the asset.

Consider a portfolio containing m risky assets with portfolio weights $\mathbf{w} = (w_1, w_2, \ldots, w_m)'$, and suppose that each asset has a returns representation (II.1.2). Then the portfolio return may be written

$$Y_t = \alpha + \beta X_t + \varepsilon_t \quad t = 1, \ldots, T, \tag{II.1.3}$$

where each characteristic of the portfolio (i.e. its alpha and beta and its specific return) is a weighted sum of the individual assets' characteristics, i.e.

$$\alpha = \sum_{i=1}^{m} w_i \alpha_i, \quad \beta = \sum_{i=1}^{m} w_i \beta_i, \quad \varepsilon_t = \sum_{i=1}^{m} w_i \varepsilon_{it}. \tag{II.1.4}$$

Now the portfolio's characteristics can be estimated in two different ways:

- Assume some portfolio weights $\mathbf{w}$ and use estimates of the alpha, beta and residuals for each asset in (II.1.4) to infer the characteristics of this hypothetical portfolio. This way an *asset manager* can compare many different portfolios for recommendation to his investors.
- A *risk manager*, on the other hand, will apply the weights $\mathbf{w}$ of an existing portfolio that is held by an investor to construct a constant weighted artificial returns history for the portfolio. This series is used for Y_t in (II.1.3) to assess the relative performance, the systematic risk and the specific risk of an existing portfolio.[2]

Thus risk managers and asset managers apply the same factor model in different ways, because they have different objectives. Asset managers need estimates of (II.1.2) for every

[2] The reconstructed 'constant weight' series for the portfolio returns will not be the same as the actual historical returns series for the portfolio, unless the portfolio was rebalanced continually so as to maintain the weights constant. The reason for using current weights is that the risk manager needs to represent the portfolio as it is now, not as it was last week or last year, and to use this representation to forecast its risk over a future risk horizon of a few days, weeks or months.

asset in the investor's universe in order to forecast the performance of many different portfolios and hence construct an optimal portfolio; by contrast, a risk manager takes an existing portfolio and uses (II.1.3) to forecast its risk characteristics. The next section explains how risk managers and asset managers also use different data and different statistical techniques to estimate the factor models that they use.

II.1.2.2 Estimating Portfolio Characteristics using OLS

The main lesson to learn from this section is that risk managers and asset managers require quite different techniques to estimate the parameters of factor models because they have different objectives:

- When asset managers employ a factor model of the form (II.1.2) they commonly use long histories of asset prices and benchmark values, measuring returns at a weekly or monthly frequency and assuming that the true parameters are constant. In this case, the *ordinary least squares* (OLS) estimation technique is appropriate and the more data used to estimate them the better, as the sampling error will be smaller. Three to five years of monthly or weekly data is typical.
- When risk managers employ a factor model of the form (II.1.3) they commonly use shorter histories of portfolio and benchmark values than the asset manager, measuring returns daily and not assuming that the true values of the parameters are constant. In this case, a time varying estimation technique such as exponentially weighted moving averages or generalized autoregressive conditional heteroscedasticity is appropriate.

We shall now describe how to estimate (II.1.2) and (II.1.3) using the techniques that are appropriate for their different applications. For model (II.1.2) the OLS parameter estimates based on a sample of size T are given by the formulae[3]

$$\hat{\beta}_i = \frac{\sum_{t=1}^{T}\left(X_t - \overline{X}\right)\left(R_{it} - \overline{R}_i\right)}{\sum_{t=1}^{T}\left(X_t - \overline{X}\right)^2} \quad \text{and} \quad \hat{\alpha}_i = \overline{R}_i - \hat{\beta}_i \overline{X}, \tag{II.1.5}$$

where $\overline{X}$ denotes the sample mean of the factor returns and $\overline{R}_i$ denotes the sample mean of the ith asset returns. The OLS estimate of the specific risk of the ith asset is the estimated standard error of the model, given by

$$s_i = \sqrt{\frac{RSS_i}{T-2}}, \tag{II.1.6}$$

where RSS_i is the residual sum of squares in the ith regression. See Section I.4.2 for further details. The following example illustrates the use of these formulae to estimate model (II.1.2) for two US stocks, using the S&P 500 index as the risk factor.

EXAMPLE II.1.1: OLS ESTIMATES OF ALPHA AND BETA FOR TWO STOCKS

Use weekly data from 3 January 2000 until 27 August 2007 to estimate a single factor model for the Microsoft Corporation (MSFT) stock and the National Western Life Insurance Company (NWL) stock using the S&P 500 index as the risk factor.[4]

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[3] See Section I.4.2.2.
[4] Dividend adjusted data were downloaded from Yahoo! Finance.

(a) What do you conclude about the stocks' characteristics?
(b) Assuming the stocks' specific returns are uncorrelated, what are the characteristics of a portfolio with 70% of its funds invested in NWL and 30% invested in MSFT?

SOLUTION The spreadsheet for this example computes the weekly returns on the index and on each of the stocks and then uses the Excel regression data analysis tool as explained in Section I.4.2.7. The results are

$$\begin{aligned} R_{\mathrm{NWL}} &= \underset{(2.224)}{0.00358} + \underset{(7.129)}{0.50596}\, R_{\mathrm{SPX}}, \quad s_{\mathrm{NWL}} = 0.03212, \\ R_{\mathrm{MSFT}} &= \underset{(-0.3699)}{-0.00066} + \underset{(14.002)}{1.10421}\, R_{\mathrm{SPX}}, \quad s_{\mathrm{MSFT}} = 0.03569, \end{aligned} \tag{II.1.7}$$

where the figures in parentheses are the t ratios. We conclude the following:

- Since $\hat{\alpha}_{\mathrm{NWL}} = 0.00358$ and this is equivalent to an average outperformance of 18.6% per annum, NWL is a stock with a significant alpha. It also has a low systematic risk because $\hat{\beta}_{\mathrm{NWL}} = 0.50596$, which is much less than 1. Its specific risk, expressed as an annual volatility, is $0.03212 \times \sqrt{52} = 23.17\%$.
- Since the t ratio on $\hat{\alpha}_{\mathrm{MSFT}}$ is very small, MSFT has no significant outperformance or underperformance of the index. It also has a high systematic risk because the beta is slightly greater than 1 and a specific risk of $0.03569 \times \sqrt{52} = 25.74\%$, which is greater than the specific risk of NWL.

Now applying (II.1.4) gives a portfolio with the following characteristics:

$$\begin{aligned} \hat{\alpha} &= 0.7 \times 0.00358 - 0.3 \times 0.00066 = 0.00231, \\ \hat{\beta} &= 0.7 \times 0.50596 + 0.3 \times 1.10421 = 0.68543, \end{aligned}$$

and assuming the specific returns are uncorrelated implies that we can estimate the specific risk of the portfolio as

$$s = \sqrt{0.7^2 \times 23.17^2 + 0.3^2 \times 25.74^2} = 17.96\%.$$

The next example shows that it makes no difference to the portfolio alpha and beta estimates whether we estimate them:

- from the OLS regressions for the stocks, applying the portfolio weights to the stocks alphas and betas using (II.1.4) as we did above;
- by using an OLS regression of the form (II.1.3) on the constant weighted portfolio returns.

However, it does make a difference to our estimate of the specific risk on the portfolio!

EXAMPLE II.1.2: OLS ESTIMATES OF PORTFOLIO ALPHA AND BETA

A portfolio has 60% invested in American Express (AXP) stock and 40% invested in Cisco Systems (CSCO). Use daily data from 3 January 2000 to 31 December 2007 on the prices of these stocks and on the S&P 100 index (OEX) to estimate the portfolio's characteristics by:[5]

[5] Data were downloaded from Yahoo! Finance. The reason we use log returns in this example is explained in Section I.1.4.4.

(a) applying the same method as in Example II.1.1;
(b) regressing the constant weighted returns series {0.6 × Amex Return + 0.4 × Cisco Return} on the index returns.

SOLUTION The results are computed using an OLS regression of each stock return and of the constant weighted portfolio returns, and the alpha and beta estimates are summarized in Table II.1.1. Note that for the first two rows the last column is a weighted sum of the first two. That is, the portfolio's alpha could equally well have been calculated by just taking the weighted sum of the stocks' alphas, and similarly for the beta. However, if we compute the specific risk of the portfolio using the two methods we obtain, using method (a),

$$s_P = \sqrt{0.6^2 \times 0.01416^2 + 0.4^2 \times 0.02337^2} \times \sqrt{250} = 19.98\%.$$

But using method (b), we have

$$s_P = 0.01150 \times \sqrt{250} = 18.19\%.$$

The problem is that the specific risks are *not* uncorrelated, even though we made this assumption when we applied method (a).

Table II.1.1 OLS alpha, beta and specific risk for two stocks and a 60:40 portfolio

	Amex	Cisco	Portfolio
Alpha	0.00018	−0.00022	0.00002
Beta	1.24001	1.76155	1.44863
Regression standard error	0.01416	0.02337	0.01150
Specific risk	22.39 %	36.96 %	18.19 %

We conclude that to estimate the specific risk of a portfolio we need to apply method (b). That is, we need to reconstruct a constant weighted portfolio series and calculate the specific risk from that regression. Alternatively and equivalently, we can save the residuals from the OLS regressions for each stock return and calculate the covariance matrix of these residuals. More details are given in Section II.1.3.3 below.

II.1.2.3 Estimating Portfolio Risk using EWMA

Whilst OLS may be adequate for asset managers, it is not appropriate to use a long price history of monthly or weekly data for the risk management of portfolios. Market risks require monitoring on a frequent basis – daily and even intra-daily – and the parameter estimates given by OLS will not reflect current market conditions. They merely represent an *average* value over the time period covered by the sample used in the regression model.

So, for the purpose of mapping a portfolio and assessing its risks, higher frequency data (e.g. daily) could be used to estimate a time varying portfolio beta for the model

$$Y_t = \alpha_t + \beta_t X_t + \varepsilon_t, \qquad \text{(II.1.8)}$$

where X_t and Y_t denote the returns on the market factor and on the stock (or portfolio), respectively, at time t. In this model the systematic and specific risks are no longer assumed constant over time. The time varying beta estimates in (II.1.8) better reflect the current risk factor sensitivity for daily risk management purposes. To estimate time varying betas we cannot apply OLS so that it covers only the recent past. This approach will lead to very significant problems, as demonstrated in Section II.3.6. Instead, a simple time varying model for the covariance and variance may be applied to estimate the parameters of (II.1.8). The simplest possible time varying parameter estimates are based on an *exponentially weighted moving average* (EWMA) model. However the EWMA model is based on a very simple assumption, that returns are i.i.d. The EWMA beta estimates vary over time, even though the model specifies only a constant, unconditional covariance and variance. More advanced techniques include the class of *generalized autoregressive conditional heteroscedasticity* (GARCH) models, where we model the *conditional* covariance and variance and so the true parameters as well as the parameter estimates change over time.[6]

A time varying beta is estimated as the covariance of the asset and factor returns divided by the variance of the factor returns. Denoting the EWMA smoothing constant by λ, the EWMA estimate of beta that is made at time t is

$$\hat{\beta}_t^{\lambda} = \frac{\mathrm{Cov}_{\lambda}(X_t, Y_t)}{V_{\lambda}(X_t)}. \tag{II.1.9}$$

That is, the EWMA beta estimate is the ratio of the EWMA covariance estimate to the EWMA variance estimate with the *same* smoothing constant. The modeller must choose a value for λ between 0 and 1, and values are normally in the region of 0.9–0.975. The decision about the value of λ is discussed in Section II.3.7.2.

We now provide an example of calculating the time varying EWMA betas for the portfolio in Example II.1.2. Later on, in Section II.4.8.3 we shall compare this beta with the beta that is obtained using a simple bivariate GARCH model. We assume $\lambda = 0.95$, which corresponds to a half-life of approximately 25 days (or 1 month, in trading days) and compare the EWMA betas with the OLS beta of the portfolio that was derived in Example II.1.2. These are shown in Figure II.1.1, with the OLS beta of 1.448 indicated by a horizontal grey line. The EWMA beta, measured on the left-hand scale, is the time varying black line. The OLS beta is the average of the EWMA betas over the sample. Also shown in the figure is the EWMA estimate of the systematic risk of the portfolio, given by

$$\text{Systematic Risk} = \hat{\beta}_t^{\lambda}\sqrt{V_{\lambda}(X_t)} \times \sqrt{h}, \tag{II.1.10}$$

where h denotes the number of returns per year, assumed to be 250 in this example.

During 2001 the portfolio had a beta much greater than 1.448, and sometimes greater than 2. The opposite is the case during the latter part of the sample. But note that this remark does depend on the choice of λ: the greater the value of λ the smoother the resulting series, and when $\lambda = 1$ the EWMA estimate coincides with the OLS estimate. However, when $\lambda < 1$ the single value of beta, equal to 1.448, that is obtained using OLS does not reflect the day-to-day variation in the portfolio's beta as measured by the EWMA estimate.

A time varying estimate of the systematic risk is also shown in Figure II.1.1. The portfolio's systematic risk is depicted in the figure as an annualized percentage, measured on the right-hand scale. There are two components of the systematic risk, the beta and the volatility of the market factor, and the systematic risk is the *product* of these. Hence the systematic risk was relatively low, at around 10% for most of the latter part of the sample even though the

[6] EWMA and GARCH models are explained in detail in Chapters II.3 and II.4.

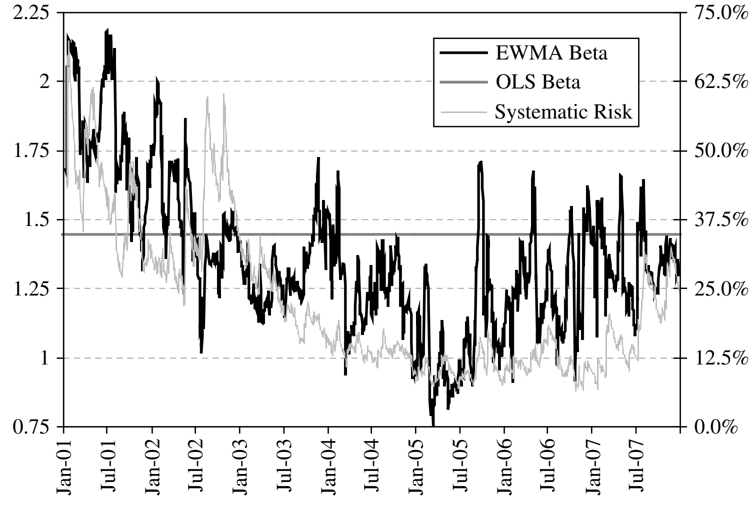

Figure II.1.1 EWMA beta and systematic risk of the two-stock portfolio

portfolio's beta was greater than 1, because the S&P 100 index had a very low volatility during this period. On the other hand, in August and October 2002 the portfolio had a high systematic risk, not because it had a high beta but because the market was particularly volatile then. By contrast, the OLS estimate of systematic risk is unable to reflect such time variation. The average volatility of the S&P 100 over the entire sample was 18.3% and so OLS produces the single estimate of $18.3\% \times 1.448 = 26.6\%$ for systematic risk. This figure represents only an average of the systematic risk over the sample period.

II.1.2.4 Relationship between Beta, Correlation and Relative Volatility

In the single index model the beta, market correlation and relative volatility of an asset or a portfolio with return Y when the market return is X are defined as

$$\beta = \frac{\mathrm{Cov}(X,Y)}{V(X)}, \quad \varrho = \frac{\mathrm{Cov}(X,Y)}{\sqrt{V(X)V(Y)}}, \quad \nu = \sqrt{\frac{V(Y)}{V(X)}} \tag{II.1.11}$$

Hence,

$$\beta = \varrho\nu, \tag{II.1.12}$$

i.e. the *equity beta* is the product of the *market correlation* ϱ and the *relative volatility* ν of the portfolio with respect to the index or benchmark.

The correlation is bounded above and below by $+1$ and -1 and the relative volatility is always positive. So the portfolio beta can be very large and negative if the portfolio is negatively correlated with the market, which happens especially when short positions are held. On the other hand, very high values of beta can be experienced for portfolios containing many risky stocks that are also highly correlated with the market.

In Figures II.1.2 and II.1.3 we show the daily EWMA estimates of beta, relative volatility and correlation (on the right-hand scale) of the Amex and Cisco stocks between

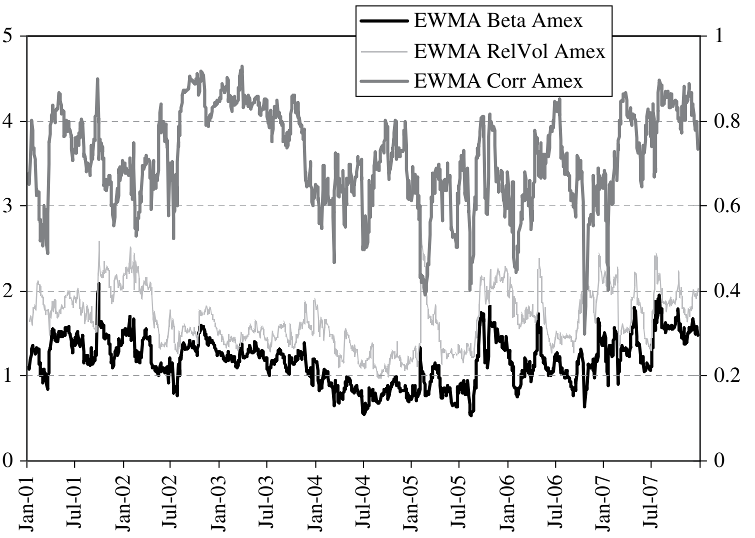

Figure II.1.2 EWMA beta, relative volatility and correlation of Amex ($\lambda = 0.95$)

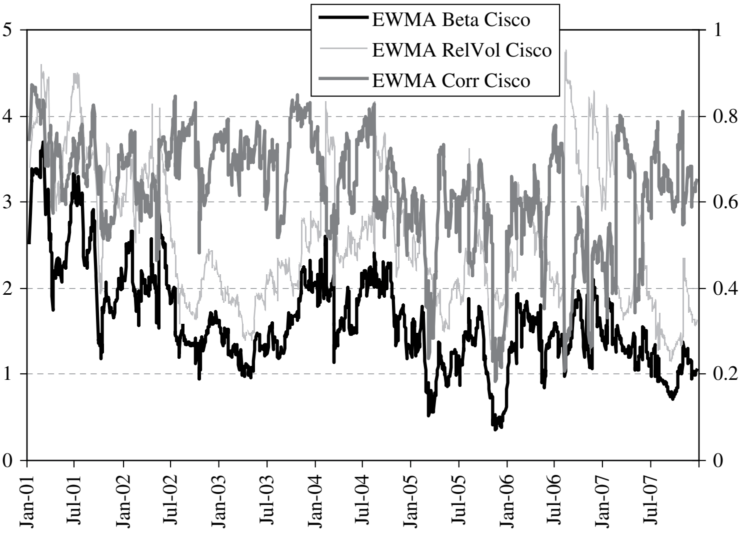

Figure II.1.3 EWMA beta, relative volatility and correlation of Cisco ($\lambda = 0.95$)

January 2001 and December 2007.[7] The same scales are used in both graphs, and it is clear that Cisco has a greater systematic risk than Amex. The average market correlation of both stocks is higher for Amex (0.713 for Amex and 0.658 for Cisco) but Cisco is much more volatile than Amex, relative to the market. Hence, EWMA correlation is more unstable and its EWMA beta is usually considerably higher than the beta on Amex.

[7] As before, $\lambda = 0.95$.

II.1.2.5 Risk Decomposition in a Single Factor Model

The principle of portfolio diversification implies that asset managers can reduce the specific risk of their portfolio by diversifying their investments into a large number of assets that have low correlation – and/or by holding long and short positions on highly correlated assets. This way the portfolio's specific risk can become insignificant. *Passive managers*, traditionally seeking only to track the market index, should aim for a net $\alpha = 0$ and a net portfolio $\beta = 1$ whilst simultaneously reducing the portfolio's specific risk as much as possible. *Active managers*, on the other hand, may have betas that are somewhat greater than 1 if they are willing to accept an increased systematic risk for an incremental return above the index.

Taking the expectation and variance of (II.1.3) gives

$$E(Y) = \alpha + \beta E(X). \tag{II.1.13}$$

If we assume $\mathrm{Cov}(X, \varepsilon) = 0$,

$$V(Y) = \beta^2 V(X) + V(\varepsilon). \tag{II.1.14}$$

It is very important to recognize that the total portfolio variance (II.1.14) represents the variance of portfolio returns around the expected return (II.1.13). It does not represent the variance about any other value! This is a common mistake and so I stress it here: it is statistical nonsense to measure the portfolio variance using a factor model and then to assume this figure represents the dispersion of portfolio returns around a mean that is anything other than (II.1.13). For example, the variance of a portfolio that is estimated from a factor model does *not* represent the variance about the target returns, except in the unlikely case that the expected return that is estimated by the model is equal to this target return.

The first term in (II.1.14) represents the systematic risk of the portfolio and the second represents the specific risk. When risk is measured as standard deviation the systematic risk component is $\beta\sqrt{V(X)}$ and the specific risk component is $\sqrt{V(\varepsilon)}$. These are normally quoted as an annualized percentage, as in the estimates given in the examples above.

From (II.1.14) we see that the volatility of the portfolio return – about the expected return given by the factor model – can be decomposed into three sources:

- the sensitivity to the market factor beta,
- the volatility of the market factor, and
- the specific risk.

One of the limitations of the equity beta as a risk measure is that it ignores the other two sources of risk: it says nothing about the risk of the market factor itself or about the specific risk of the portfolio.

We may express (II.1.14) in words as

$$\text{Total Variance} = \text{Systematic Variance} + \text{Specific Variance} \tag{II.1.15}$$

or, since risk is normally identified with standard deviation (or annualized standard deviation, i.e. volatility),

$$\text{Total Risk} = (\text{Systematic Risk}^2 + \text{Specific Risk}^2)^{1/2}. \tag{II.1.16}$$

Thus the components of risk are *not* additive. Only variance is additive, and then only under the assumption that the covariance between each risk factor's return and the specific return is 0.

II.1.3 MULTI-FACTOR MODELS

The risk decomposition (II.1.14) rests on an assumption that the benchmark or index is uncorrelated with the specific returns on a portfolio. That is, we assumed in the above that $\mathrm{Cov}(X, \varepsilon) = 0$. But this is a very strong assumption that would not hold if there were important risk factors for the portfolio, other than the benchmark or index, that have some correlation with the benchmark or index. For this reason single factor models are usually generalized to include more than one risk factor, as assumed in the *arbitrage pricing theory* developed by Ross (1976). By generalizing the single factor model to include many risk factors, it becomes more reasonable to assume that the specific return is not correlated with the risk factors and hence the risk decomposition (II.1.16) is more likely to hold.

The success of multi-factor models in predicting returns in financial asset markets and analysing risk depends on both the choice of risk factors and the method for estimating factor sensitivities. Factors may be chosen according to fundamentals (price–earning ratios, dividend yields, style factors, etc.), economics (interest rates, inflation, gross domestic product, etc.), finance (such as market indices, yield curves and exchange rates) or statistics (e.g. principal component analysis or factor analysis). The factor sensitivity estimates for fundamental factor models are sometimes based on cross-sectional regression; economic or financial factor model betas are usually estimated via time series regression; and statistical factor betas are estimated using statistical techniques based on the analysis of the eigenvectors and eigenvalues of the asset returns covariance or correlation matrix. These specific types of multi-factor models are discussed in Sections II.1.4–II.1.6 below. In this section we present the general theory of multi-factor models and provide several empirical examples.

II.1.3.1 Multi-factor Models of Asset or Portfolio Returns

Consider a set of k risk factors with returns $X_1, \ldots, X_k$ and let us express the systematic return of the asset or the portfolio as a weighted sum of these. In a multi-factor model for an asset return or a portfolio return, the return Y is expressed as a sum of the systematic component and an idiosyncratic or specific component ε that is not captured by the risk factors. In other words, a multi-factor model is a multiple regression model of the form[8]

$$Y_t = \alpha + \beta_1 X_{1t} + \ldots + \beta_k X_{kt} + \varepsilon_t. \tag{II.1.17}$$

In the above we have used a subscript t to denote the time at which an observation is made. However, some multi-factor models are estimated using cross-sectional data, in which case the subscript i would be used instead.

Matrix Form

It is convenient to express (II.1.17) using matrix notation, but here we use a slightly different notation from that which we introduced for multivariate regression in Section I.4.4.2. For reasons that will become clear later, and in particular when we analyse the Barra model, it helps to isolate the constant term alpha in the matrix notation. Thus we write

$$\mathbf{y} = \boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \varepsilon_t \sim \text{i.i.d.}(0, \sigma^2), \tag{II.1.18}$$

[8] In this chapter, since we are dealing with *alpha models*, it is convenient to separate the constant term alpha from the other coefficients. Hence we depart from the notation used for multiple regression models in Chapter I.4. There the total number of coefficients including the constant is denoted k, but here we have $k+1$ coefficients in the model.