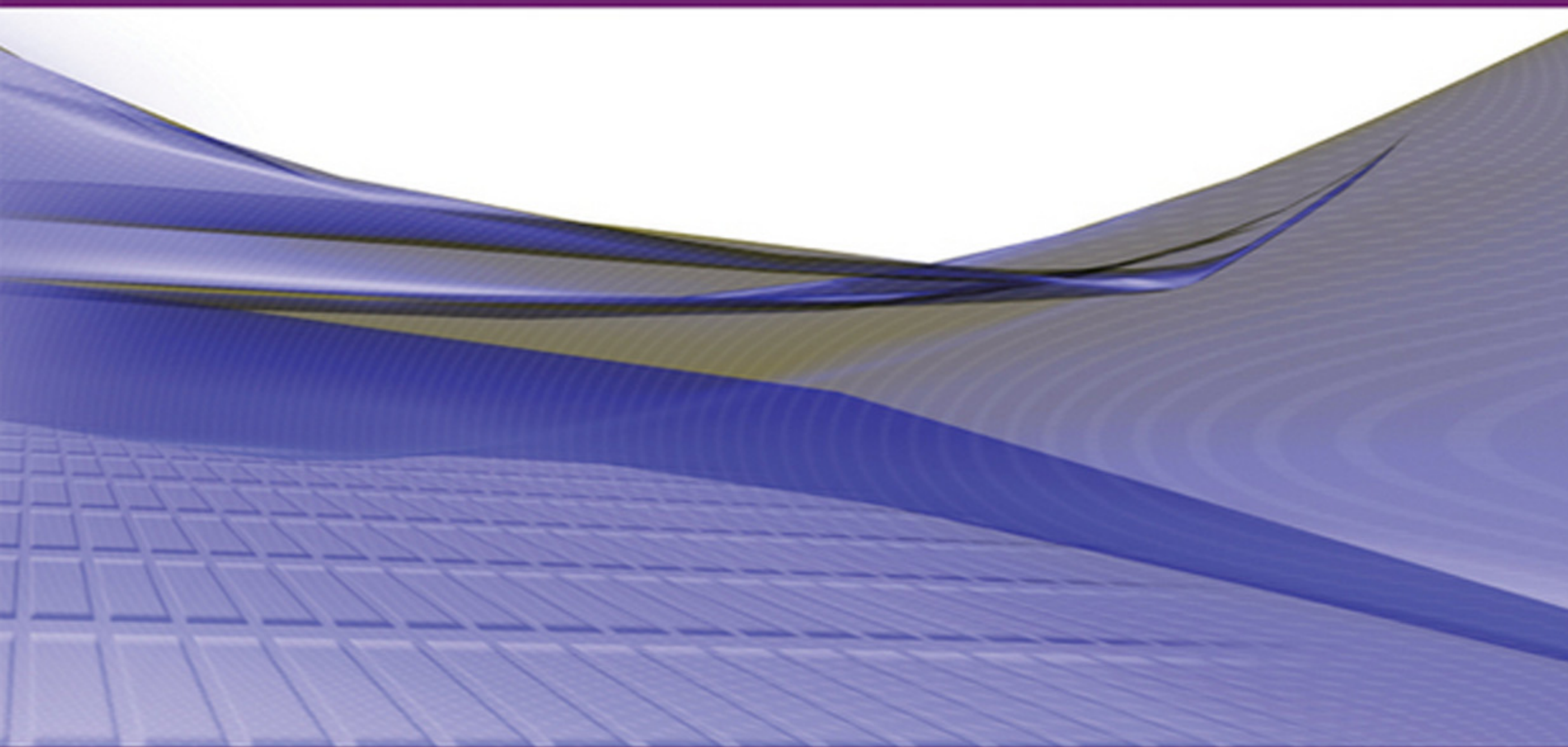
Mechanical Characterization of Materials and Wave Dispersion

Edited by

Yvon Chevalier and Jean Vinh Tuong



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Preface

The mechanical testing of material is an important activity in research and industry. Scientists, engineers and technicians in a large range of domains (such as chemistry, metallurgy, mechanics, physics, polymer science, the rubber industry, aerospace and aeronautical industries, etc.) are interested in the technology used to investigate the mechanical properties of materials.

Static and dynamic tests are complementary and used concurrently. Static tests are often used in industry. Dynamic tests, however, are becoming more popular and, surprisingly, in many cases are easier to use than static ones, at least at lower frequencies. Let us take an example concerning the measurement of elastic Young's modulus or the shear modulus of a steel rod. In (nearly) static tests, glued strain gauges or special micro-displacement transducers are used to measure, the displacement of the sample in two or three directions at once, which enable us to evaluate the strains. With the measurement of applied force or torque, these two moduli are deduced from the basic definitions relating to stress and strain. There are a succession of measurements and calculations from the stress *versus* strain curves.

To obtain such elastic moduli using dynamic tests, evaluation of resonance frequencies only is required; dimensions and geometry of the sample and its weight, as well as boundary conditions, being known.

The main interest in dynamic testing, however, resides in characterization of the viscoelastic properties of materials, i.e. the dependency of technical moduli (or relaxation, creep functions) *versus* the frequency (or time).

Growing interest in dynamic tests

In industrially advanced countries, societies for material testing regularly publish recommendations concerning mechanical tests with indications on methods and test procedures. Over the last five decades, the methods of investigating dynamic mechanical properties have made significant progress. In the scientific and technical literature devoted to this problem, various viewpoints have been adopted. For example, the dynamic tests that interpret materials at a molecular level, i.e. structural factors, molecular weight, cross-linking, crystallinity, etc., constitute tools in the chemistry and physics of polymers. The science of rheology is being more frequently adopted in order to obtain technical moduli (or time functions) which serve in viscoelastic constitutive equations relating stress components to strain components.

Composite materials cover a large domain including laminated plastics and panels for the building industry. Special composite materials were initially designed and fabricated for advanced applications in the aeronautical and aerospace industries in the 1970s. The anisotropic properties of such materials are obtained by the appropriate arrangement of high-strength fibers in metallic or non-metallic matrices of the layers or by the orientation of the layers in the structural composite. The mechanical characterization of such anisotropic materials consequently requires special testing procedures which are more elaborate than the ones devoted to isotropic materials.

Characterization of metallic and non-metallic material damping capacities is of interest to specialists in chemistry and physics as well as in mechanics.

Measuring damping

The measurement of damping coefficients of mechanical structures gives rise to a large variety of methods in structural dynamics that deserve the attention of specialists in material testing. The transposition of those methods into rheology, however, requires some caution and adaptations. The damping of a mechanical structure depends on damping of the materials used in the structure and the geometry of the structure itself. Consequently it is necessary to have this distinction in mind. Material damping can be deduced from structural damping on the condition that the relationship between these two kinds of damping is known.

Damping capacities of materials cover a much larger range than structural damping. Globally, damping capacities of materials (defined as the quotient of the imaginary to real part of a complex modulus) can be divided into three classes:

a) low damping – $\tan\delta < 10^{-3}$;

b) medium damping – $10^{-3} < \tan\delta < 5.10^{-2}$;

c) high damping – $5.10^{-2} < \tan\delta < 10$.

Mechanical structural damping, in the majority of cases, concerns class (b) and no caution concerning measurement technique is needed. Class (a) concerns steel or special metals and requires special caution when taking measurements in order to eliminate the predominant influence of air damping on the sample. Class (c) concerns some rubbers or special blends of polymers and metal powders. The usual methods adopted in structural dynamics require special adaptations.

Size and shape of the sample

In many circumstances, analysts have to deal with samples with special or unusual shapes and sizes. The sample cut off from a hollow cylinder is curved and necessitates a special sample holder. The sample can be very small and therefore commercially available apparatus cannot be used. This is the case in biomechanics, for example, where the sample is a cut-off from a small bone. Analysts then have to come up with and devise a special set up.

Appropriate knowledge on the elastodynamics of bounded media

Let us begin with some remarks about currently available instruments. In some apparatuses, the mechanical part and adopted loading system are designed in such a way that vibrations imposed on the sample are far from simple. It eventually gives rise to different kinds of vibrations (extensional, bending, torsion) which are coupled in the sample itself. Coupling of such vibrations is often neglected in proposed formulae giving the moduli. The last formulae are deduced from the elementary theory of vibration using localized mechanical parameters that are not necessarily valid for short and thick samples. Mechanical effects (such as shear and inertia effects) are not taken into account. When experiments are conducted in higher frequency ranges, wave dispersion phenomenon (which describes the variation of the wave velocity in the sample *versus* the frequency) is rarely taken into account.

Attachment of the sample by clamping, gluing or screw tightening creates zones where there is a three-dimensional state of stress that can be localized in the sample submitted to compression forces and also beyond the contact zone between the sample and holder system. This effect is particularly pronounced for a short sample.

These remarks, among others, show that confidence granted to an apparatus must not exclude critical thought and a mechanical background.

The book

I will now present and comment on the chapters in this book.

The authors have intentionally situated dynamic testing of materials in the context of bounded medium elastodynamics. The measurements of dynamic responses of the sample in a large range of frequencies are interesting for analysts who want to obtain viscoelastic complex moduli. Rheologists[1] are interested in the relationship between various resonance peaks of dynamic responses *versus* frequency and micromechanisms of the polymer being tested. Mechanical engineers wish to obtain the curves of complex moduli at low and high frequencies so as to include them in calculation of the dynamic responses of the mechanical structure.

The motion equations of the sample must be carefully chosen, taking into account the frequency range. The upper frequency guides the choice of degree of approximation, which is related to the set of motion equations in view of *wave dispersion* characterization at the chosen upper frequency range. Wave dispersion is not the only effect we need to account for. There is another dispersion phenomenon: the *viscoelastic dispersion*, which is also frequency dependent.

These two effects sometimes act in the same sense with respect to frequency, and *vice versa* regarding sample responses, depending on the type of stationary wave in the sample and the working frequency. This is the reason main wave dispersion should be taken into account and raises the delicate problem of reasoned choice of appropriate equations of motion, compatible with tractable numerical exploitation of experimental results.

Recently, specialists in structural computing science have focused on the *continuous element method* which permits structure calculation even in the ultrasonic frequency range. This method presents advantages and constitutes a serious competitor for *classical finite element method*. The elastodynamics of bounded media precisely furnishes theoretical foundations, particularly in the domain of wave dispersion. Consequently, this last topic is treated in detail for various wave types adopted in samples.

1 Rheology designates the science which studies the flow (Greek radical *Rheos*) of solid or liquid materials.

One of the new methods of treating viscoelastic material characterization is to use continuous elements as a tool to numerically solve an inverse problem without recourse to closed-form eigenvalue solutions of boundary equations.

We try to bridge the gap between theoretical academic works on wave dispersion and practical applications that do not yet sufficiently exploit the literature. Many significant theoretical contributions concerning wave dispersion in bounded media during the last three decades merit being brought together, classified and examined in view of applications.

Part A is devoted to continuum mechanics (constitutive equations of materials including *anisotropic materials*). Chapter 1 covers linear and applied viscoelasticity. Chapter 2 looks at the principle of correspondence that permits the conversion of elastic equations of motion into viscoelastic ones, with the condition that boundary conditions and sample geometry remain the same.

Chapter 3 is devoted to Williams-Landel-Ferry's (WLF) method, which is very popular in the field of polymer chemistry and deserves the attention of mechanical engineers. It permits artificial enlargement of the modulus (or compliance) curve in an unusually large frequency range (often more than eight decades) on the condition that *the superposition principle temperature-frequency is applicable.*

Serious limitations of WLF's method must be taken into account when dealing with anisotropic artificial composite materials. The superposition principle may not be valid for such materials. The remaining possibility is to directly evaluate complex modulus (or compliance) over a large frequency range. This is the main reason to resort to appropriate wave dispersion theories for these materials.

The closed-form expression of viscoelastic modulus (or compliance) is often necessary in computer codes to evaluate the damping responses of structures. Examining this problem from a practical point of view, we notice that analog models, usually proposed in textbooks and publications, with a reduced number of springs and dashpots cleverly arranged in series and/or in parallel, indeed help the reader "visualize" the material.

For a given experimental dynamic curve, however, we do not know in advance how many associated mechanical elements (springs and dashpots) will be adopted, particularly when the frequency range is large. The appropriate model is often more complicated than the simple academic models indicated above. This unknown model belongs to the "*black box*" constituted by the material in the usual mechanical inverse problem to be solved![2]

Some methods are then proposed to obtain a closed-form expression of modulus (or compliance) *versus* frequency by quotient of polynomials of the same degree (without *a priori* assumption of its degree) or by fractional derivatives whose interest resides in the condensed mathematical expression.

In Chapter 4, various formulations of equations of motion are presented. As we have to deal with bounded medium and finite sample length, no exact equations are available: approximate equations of motion are to be found. The main question is: *what is the degree of approximation we must adopt*? This question raises a subsidiary question: how many generalized displacement components and generalized force components are to be adopted to fully cover the mechanical behavior of the sample?

All the methods presented in dynamics textbooks can be utilized. D'Alembert's principle and Lagrange's equations constitute the first group of methods. The second group includes Hamilton's variational principle using simple displacement field. Love's variational principle can be considered as derived from Hamilton's one.[3] Mixed field Reissner's principle is, in some cases, useful for correctly portraying the dispersion curve of the sample. This variational method is referred to in an accurate analysis of vibrations of an anisotropic rod.

Part B concerns various types of rod vibration: extension, bending and torsion. Vibration modes are a source of vocabulary confusion for analysts. Let us clarify some different definitions.

Vibration modes might concern the *nature of the vibration* as mentioned above. The nature of the vibration is related to the predominant strain in the sample, i.e. the extensional strain along the rod axis in longitudinal motion, shear strain in rod torsion, and axial strain in bending test.

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2 Recently in electrical engineering, as well as in mechanical engineering, attention has focused on distributed models in a ladder using linear elementary models (springs and dashpots) or fractional derivatives which constitute an elegant method to characterize materials in a large frequency range with minimum parameters.

3 Using variational principles and integrating by parts, we directly obtain equations of motion and natural boundary conditions as well.

In structural dynamics, vibration mode is related to *eigenfrequencies* and *eigenvectors,* which are portrayed by nodal lines on the sample surface whose density increases with frequency.

Elastodynamic vocabulary: attention is focused not only on the representation of nodal lines on the lateral surface of the sample, but also on the sample thickness itself. Let us take an example: the bending test on a rod with a rectangular cross-section. There is a neutral line in the thickness whose motion is representative of bending motion in the first elastodynamic mode. Higher elastodynamic modes correspond to discontinuous or undulating neutral lines in the cross-section. To create such modes, a special array of small piezoelectric exciters can be used. For the usual characterization of material, higher elastodynamic modes are rarely used, although they might constitute a good tool in fracture mechanics. To avoid confusion on the signification of vibration mode, additional indications between brackets will be used: (nature), (eigenfrequency or eigenvalue), (elastodynamics).

In some chapters, theoretical works are presented with proofs so as to facilitate the reader's consultation. Intentionally, Part B is presented with details in the theoretical formulation so as to facilitate the reader's work and reduce his/her burden in the search of scientific papers sometime published some centuries ago! Appendixes presented at the end of each chapter might help researchers to find the demonstration of formulae. For each kind of wave a collection of theories from elementary to sophisticated may present difficulties and a profusion of theories to a reader who approaches the problem for the first time. We have presented a set of theories as a toolbox: practitioners and researchers have to choose the appropriate tool for special applications themselves.

Some readers might be surprised by the unusual length of the chapters in Part B compared to a classical book devoted to the same topics. The authors' intention is to gather together all the possible groups of theories with various degrees of approximations, so the reader does not need to search elsewhere. The contributions of our research team are naturally presented with the intention of completing existing literature on the vibration of rods with finite and infinite lengths.

Coupled vibrations highlight the effect of non-diagonal elastic coefficients in the equations of motion. Coupled vibrations are intentionally used with an *off-axis anisotropic* rod. Matricial diagonal coefficients being known, such coupled vibrations permit us to evaluate *non-diagonal terms*.

Coupled vibrations exist even in an isotropic rod submitted to various vibration types, even for a closed section. In elementary theories these vibrations are neglected at lower frequencies. Shear effect in longitudinal and bending vibrations, however, occurs in equations of motion with higher degrees of approximation. Torsional rod vibration gives rise to axial strain and extensional vibration occurs. Consequently, two or more elastic (or viscoelastic) moduli are present in equations of motion.

The extensive utilization of rods in this book, instead of plates, necessitates explanation. In some technical and scientific publications, plate is indeed used to evaluate elastic (and/or viscoelastic) moduli. The objective of such works is to determine the whole set of elastic moduli. Elastic vibrations of plates necessitate measurements of vibration amplitude at many points and eventually for a certain number of (eigenfrequency) modes. On grounds of numerical calculation, optimization algorithms are referred to. The degree of complexity is considerably increased with respect to that concerning a rod. The challenge of adopting plate equations of motion is prohibitive compared to the one-dimensional equation for a rod with one, two or three displacement variables. The results obtained from plates in the scientific literature are unfortunately far from convincing, with the objective of solving an inverse problem to find moduli or stiffness coefficients of material[4].

Chapters 5, 6 and 7 present torsional, bending and extensional vibrations. In Chapter 5, a rod with rectangular cross-section is adopted, taking into account the ease of obtaining such a section. Warping of the cross-section is examined for isotropic and anisotropic materials. Saint Venant's dynamic equations of motion are presented as are the higher approximation equations of motion corresponding to more complex section warping.

Bending vibration in Chapter 6 concerns the elementary Bernoulli-Euler's equation of motion. Timoshenko's equation with a higher degree of approximation is preferred when working at a higher frequency. The bending vibration of an off-axis rod is also presented in order to evaluate the compliance a non-diagonal coefficient of anisotropic materials.

Extensional vibrations in a rod are presented in detail in Chapter 7. The longitudinal wave dispersion is surprisingly more difficult to apprehend than the one concerning the two aforementioned vibrations and requires a more elaborate displacement field. For application at higher frequency, the fourth-degree Bishop's equation of motion is not capable of correctly portraying the wave dispersion curve at higher frequencies. Touratier's formulation using internal constraints extends Volterra's work to anisotropic rods.

4 In Chapter 10, however, progressive waves are used in plates to obtain material stiffness coefficients at ultrasonic frequency range

Chapter 8 is devoted to Le Rolland-Sorin’s double pendulum working at very low frequency. This inventive, artful and simple method is practically unknown in English-speaking countries and deserves practicians’ attention in the sense it requires so few measuring instruments compared to other test methods. The functioning principle is unusual compared to existing methods used in dynamic tests.

Chapter 9 examines vibrations in rings and hollow cylinders. In many situations we have to deal with a curved rod or straight rod with curved cross-section.

Chapter 10 is devoted to the propagation of ultrasonic waves in thick plates. Ultrasonic progressive dilatational (and/or shear) wave can be chosen in advance as can the wave direction. The second-order equation of motion is simple to handle and, surprisingly, the interpretation of experimental results is much easier to obtain than rod vibrations at lower frequencies. Plate samples with a large thickness compared to wavelength are used to equate the plate with a semi-infinite medium. Progressive waves are used for this purpose.

Chapter 11 concerns evaluation of the viscoelastic complex modulus using characteristic (trigonometric and hyperbolic) functions to express displacement components. Transmissibility function (which relates output displacement to input displacement) is used in the framework of an inverse problem to evaluate complex moduli (or compliance). Methods using some special mathematical algorithms are presented in the framework of research of solutions to the important mathematical inverse problems.

Finally, Chapter 12 complements the preceding chapter, using so-called continuous elements. This method is interesting because it offers us the chance to obtain a response curve in a very large frequency range by numerical computation which takes much less time than the finite element method. In our opinion, the matricial formulation of the problem and integration of elastodynamic equations of motion constitute one of the best ways of tackling the inverse viscoelastic problem.

Yvon Chevalier and Jean Tuong Vinh

February 2010

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PART A

Constitutive Equations of Materials

A lot of theories, yes, but in view of applications

Henri Bouasse
French physicist, 1920

Chapter 1

Elements of Anisotropic Elasticity and Complements on Previsional Calculations

The objective of this chapter is to present in a concise form the constitutive equations that relate stresses to strains. Previsional calculations especially adapted for composite materials are succinctly approached in the second part of this chapter. Through experience, we are convinced that this constitutes a useful and practical tool with which to tackle the problem of artificial composite materials that are used in industrial applications.

When we have to deal with isotropic materials, the mathematical formulation of those relationships is reduced to simple expressions. The number of independent elastic constants is reduced to two, from a choice of five. The three remaining constants can be expressed against the two retained. Adoption of a given couple of elastic constants is dependent on various practical considerations, e.g. the shape and size of the sample, available method of testing (static or dynamic), nature of waves (stationary or progressive), etc.

Composite materials, whose utilization is becoming more widespread, are anisotropic in most cases. The formulation of constitutive equations requires more than two elastic constants. Symmetry considerations permit us to adopt the number of constants. Experimenters who want to mechanically characterize these materials cannot avoid these preliminary considerations.

Chapter written by Yvon CHEVALIER.

1.1. Constitutive equations in a linear elastic regime

Small (infinitesimal) strains are defined from displacements, with $\vec{u} = (u_i)$ [1] and $\vec{x} = (x_i)$ being coordinates:

$$\varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \quad i, j = 1, 2, 3 \qquad [1.1]$$

General Hooke's law expresses the proportionality between stress tensor σ_{ij} and strain tensor ε_{ij}:

$$\sigma_{ij} = c_{ijkl}\,\varepsilon_{kl} \qquad [1.2]$$

Equation [1.2] can be inverted and rewritten as:

$$\varepsilon_{ij} = s_{ijkl}\,\sigma_{kl} \qquad [1.3]$$

Equation [1.3] is used in static tests in which forces are input signals and consequently stresses, and output signals (responses) are displacements or strains. In wave propagation in an elastic medium, the strains are input signals whereas the stresses are responses. Equation [1.2] is then adopted.

Stress and strain tensors each have nine components. Symmetry consideration reduces the number of components to six.

1.1.1. *Symmetry applied to tensors s_{ijkl} and c_{ijkl}*

If the deformation energy is evaluated with the assumption that an elastic potential w (deformation energy density) exists so as:

$$\sigma_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}} \qquad [1.4]$$

or:

$$w(B) - w(A) = \int_A^B \sigma_{ij}\, d\,\varepsilon_{ij} \qquad [1.5]$$

taking into account equation [1.2], [1.4] shows that:

$$w = \frac{1}{2} c_{ijkl} \varepsilon_{ij} \varepsilon_{kl} = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \qquad [1.6]$$

The deformation energy density w can be evaluated either from strains or stresses:

$$\left.\begin{aligned} w_{\varepsilon}(\varepsilon) &= \frac{1}{2} c_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \\ w_{\sigma}(\varepsilon) &= \frac{1}{2} s_{ijkl} \sigma_{ij} \sigma_{kl} \end{aligned}\right\} \qquad [1.7]$$

Energy density being a scalar, in relation [1.7] symmetry of stiffness and compliance tensors are obtained:

$$\left.\begin{aligned} C_{iikl} &= C_{klii} \\ S_{ijkl} &= S_{klij} \end{aligned}\right\} \qquad [1.8]$$

1.1.2. *Constitutive equations under matrix form*

In practical calculations, it is convenient to adopt a matrix representation. Stress and strain tensors are represented by six components instead of nine. Only three components of shear strains and shear stresses are retained.

$$\{\sigma\} = \{C\} \ \{\varepsilon\} \qquad [1.9a]$$

$$\{\varepsilon\} = \{S\} \ \{\sigma\} \qquad [1.9b]$$

The following notations are adopted here[1]:

$$\{\sigma\} = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}\}^T \qquad [1.10]$$

$$\{\varepsilon\}= \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33,} 2\varepsilon_{33}, 2\varepsilon_{31}, 2\varepsilon_{12}\}^T \qquad [1.11]$$

---

1 $\{\sigma\} = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}\}^T$ column matrix notation.

Factor 2 is introduced in [1.11] for shear strains.

The indexes adopted in [1.10] and [1.11] for the three last components are (23), (31) and (12). That is a convention. In other publications, this convention can be replaced by another one. However, the aforementioned convention is the prevalent one.

REMARK ON TENSORIAL WRITING: going from tensorial writing to matrix writing, there is a kind of contraction of indexes. It is difficult to find the meaning of indexes such as 4, 5 and 6. In the study of wave propagation, the adoption of matrix writing gives rise to difficulties in the interpretation of wave characteristics, e.g. polarization of wave plane and direction of wave propagation. In this respect tensorial notations are preferred.

$C_{11} = c_{1111}$	$C_{12} = c_{1122}$	$C_{13} = c_{1133}$	$C_{14} = c_{1123}$	$C_{15} = c_{1131}$	$C_{16} = c_{1112}$
$C_{21} = c_{2211}$	$C_{22} = c_{2222}$	$C_{23} = c_{2233}$	$C_{24} = c_{2223}$	$C_{25} = c_{2231}$	$C_{26} = c_{2212}$
$C_{31} = c_{3311}$	$C_{32} = c_{3322}$	$C_{33} = c_{3333}$	$C_{34} = c_{3323}$	$C_{35} = c_{3331}$	$C_{36} = c_{3312}$
$C_{41} = c_{2311}$	$C_{42} = c_{2322}$	$C_{43} = c_{2333}$	$C_{44} = c_{2323}$	$C_{45} = c_{2331}$	$C_{46} = c_{2312}$
$C_{51} = c_{3111}$	$C_{52} = c_{3122}$	$C_{53} = c_{3133}$	$C_{54} = c_{3123}$	$C_{55} = c_{3131}$	$C_{56} = c_{3112}$
$C_{61} = c_{1211}$	$C_{62} = c_{1222}$	$C_{63} = c_{1233}$	$C_{64} = c_{1223}$	$C_{65} = c_{1231}$	$C_{66} = c_{1212}$

Table 1.1. *Matrix and tensor components of stiffness*

$S_{11} = s_{1111}$	$S_{12} = s_{1122}$	$S_{13} = s_{1133}$	$S_{14} = 2s_{1123}$	$S_{15} = 2s_{1131}$	$S_{16} = 2s_{1112}$
$S_{21} = s_{2211}$	$S_{22} = s_{2222}$	$S_{23} = s_{2233}$	$S_{24} = 2s_{2233}$	$S_{25} = 2s_{2231}$	$S_{26} = 2s_{2212}$
$S_{31} = s_{3311}$	$S_{32} = s_{3322}$	$S_{33} = s_{3333}$	$S_{34} = 4s_{3323}$	$S_{35} = 2s_{3331}$	$S_{36} = 2s_{3312}$
$S_{41} = 2s_{2311}$	$S_{42} = 2s_{2322}$	$S_{43} = 2s_{2333}$	$S_{44} = 4s_{2323}$	$S_{45} = 4s_{2331}$	$S_{46} = 4s_{2312}$
$S_{51} = 2s_{3111}$	$S_{52} = 2s_{3122}$	$S_{53} = 2s_{3133}$	$S_{54} = 4s_{3123}$	$S_{55} = 4s_{3131}$	$S_{56} = 4s_{3112}$
$S_{61} = 2s_{1211}$	$S_{62} = 2s_{1222}$	$S_{63} = 2s_{1233}$	$S_{64} = 4s_{1223}$	$S_{65} = 4s_{3112}$	$S_{66} = 4s_{1212}$

Table 1.2. *Matrix and tensor components of compliance – appearance of coefficient 2 or 4 is due to the adoption of definition of shear strains in equation [1.11]*

1.2. Technical elastic moduli

These are obtained by industrial tests which are often nearly static ones.

1.2.1. *Tension tests with one normal stress component σ*

$$\{\sigma\} = \{\sigma, 0, 0, 0, 0, 0\}^T \qquad [1.12]$$

is applied in direction 1. The stress state is supposed to be uniaxial and uniform in any sample section.

Bringing [1.12] into [1.9b], the following equations are obtained:

$$\varepsilon_{11} = S_{11}\, \sigma = \frac{\sigma}{E_1} \qquad [1.13a]$$

$$\varepsilon_{22} = S_{21}\, \sigma = -\frac{\nu_{21}}{E_2}\, \sigma \qquad [1.13b]$$

$$\varepsilon_{33} = S_{31}\, \sigma = -\frac{\nu_{31}}{E_3}\, \sigma \qquad [1.13c]$$

$$2\, \varepsilon_{23} = S_{41}\, \sigma = \frac{\eta_{1,23}}{G_{23}}\, \sigma \qquad [1.13d]$$

$$2\, \varepsilon_{13} = S_{51}\, \sigma = \frac{\eta_{1,13}}{G_{13}}\, \sigma \qquad [1.13e]$$

$$2\, \varepsilon_{12} = S_{61}\, \sigma = \frac{\eta_{1,12}}{G_{12}}\, \sigma \qquad [1.13f]$$

1.2.1.1. *Young's modulus E_1*

In such tension tests, a straight sample presented as a rod with uniform section is used for Young's modulus calculation. The lateral boundaries must be free surfaces without applied normal stresses [1.12]:[2]

---

2 Attempt to transform a tension test into a compression test with normal stress applied to lateral boundaries (so as to prevent buckling) may give rise to false measurement of E_1.

$\sigma_{22} = \sigma_{33} = 0$

No shear stresses:

$\sigma_{23} = \sigma_{31} = \sigma_{12} = 0.$

1.2.1.2. *Poisson's coefficients*

In equations [1.13b] and [1.13c], ν_{21} and ν_{31} are Poisson's numbers. Symmetry of the compliance matrix implies:

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \qquad [1.14]$$

In general ν_{ij} with i ≠ j represents the contraction of the thickness in direction j with a normal stress applied in direction i:

$$\varepsilon_{22}/\varepsilon_{11} = -\nu_{21}\frac{E_1}{E_2} = -\nu_{12}\,.$$

1.2.1.3. *Shear moduli*

G_{ij} with i ≠ j are used in the last three equations of [1.13]. They are directly evaluated by other tests.

1.2.1.4. *Lekhnitskii's coefficients $\eta_{i,ij}$ with $i \neq j$*

In [1.13d] rewritten here:

$$2\varepsilon_{23} = \frac{\eta_{1,23}}{G_{23}}\,,$$

coefficient $\eta_{1,23}$ describes a shear strain in the plane (2, 3): $2\varepsilon_{23}$ when a normal stress $\sigma_{11} = \sigma$ is applied in the direction 1.

It describes a coupling (tension shear) which happens in a special direction of anisotropic material.

This direction does not coincide with a direction of symmetry.

1.2.2. *Shear test*

Appropriate loading is applied so the following simple state of stress is obtained:

$$\{\sigma\} = \{0, 0, 0, 0, 0, \tau\}^T \tag{1.15}$$

Bringing [1.15] into [1.9b]:

$$\varepsilon_{11} = S_{16}\tau = \frac{\eta_{12,1}\tau}{E_1} \tag{1.16a}$$

$$\varepsilon_{22} = S_{26}\tau = \frac{\eta_{12,2}\tau}{E_2} \tag{1.16b}$$

$$\varepsilon_{33} = S_{36}\tau = \frac{\eta_{12,3}\tau}{E_3} \tag{1.16c}$$

$$2\varepsilon_{23} = S_{46}\tau = \frac{\mu_{12,23}\tau}{G_{23}} \tag{1.16d}$$

$$2\varepsilon_{13} = S_{56}\tau = \frac{\mu_{12,13}\tau}{G_{13}} \tag{1.16e}$$

$$2\varepsilon_{12} = S_{66}\tau = \frac{\tau}{G_{12}} \tag{1.16f}$$

1.2.2.1. *Shear or Coulomb's moduli*

Equation [1.16f] permits the evaluation of shear modulus. Shear stress $\sigma_{12} = \tau$ is applied in the plane (1, 2) and strains $2\varepsilon_{12}$ are evaluated in the same plane.

The three shear moduli G_{23}, G_{31}, G_{12} are the inverse of the last three diagonal components of a compliance matrix in the case of orthotropic materials.

$$G_{ij} = \frac{1}{C_{pp}}, i \neq j, \ p = 9 - (i + j) \qquad [1.17]$$

Let us mention that for anisotropic material, two indexes ij ($i \neq j$) are used to describe the plane in which shear stress is applied[3].

1.2.2.2. *The Chentsov coefficient*

$\mu_{ij,\,kl}$ is the mutual influence coefficient describing shear strain appearing in the plane (k, l) when a shear stress is applied in the plane (i, j).

1.2.2.3. *Mutual influence coefficient of the first kind*

$\eta_{\,ij,\,k}$ describes the appearance of normal strain ε_{kk} when a shear stress $\tau = \tau_{ij}$ is applied in the plane (i, j) (relationship [1.16a, b and c]).

1.2.2.4. *Mutual influence coefficient of the second kind*

$\eta_{i,\,jk}$ describes the appearance of a shear strain ε_{jk} when a normal stress σ_{ii} is applied in the direction i (relationship [1.13d, e and f]).

Chentsov's and Lekhnitskii's coefficients are not zero where reference axes are not coincident with the symmetry axes of the material. Tables 1.1 and 1.2 provide the tensorial and matricial definitions of elastic coefficients.

1.3. Real materials with special symmetries

Real natural or artificial materials have some specific symmetries that reduce the number of independent elastic coefficients. Wood is a natural anisotropic material which is orthotropic. It possesses two orthogonal planes of symmetry. Changing reference axes and taking into account those symmetry planes, we must obtain the same elastic coefficients in the representation of compliance matrix {S} or stiffness matrix {C}, see Figure 1.1.

Artificial composite fabricated from a thick layer of unidirectional fibers, which are regularly distributed in the thickness, may be considered as a transverse isotropic material.

The number of elastic constants defined along a symmetry axes is five.

A quasi-transverse isotropic material exists with 90° of rotation around an axis. It possesses six elastic moduli, see Figure 1.2.

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3 For isotropic materials, the two indexes are suppressed.

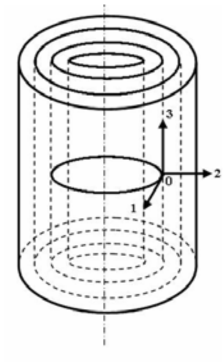

Figure 1.1. *Wood is a natural orthotropic material. It has three orthogonal symmetry planes (1, 3) and (2, 3)*

To define the elastic constants of such materials, we shall take general matrices {S} or {C} and by using an appropriate conversion matrix with specific degree of degree, we will obtain new matrices in which the components must be invariant. In what follows, thanks to the change of reference axes, we shall sweep all the possible symmetries that can exist in the materials. We shall successively define various materials in the mechanical classification framework[4].

1.3.1. *Change of reference axes*

Let the material initially be characterized in reference axes (x, y, z). Let us adopt new reference axes (x′, y′, z′) which are more appropriate for calculations. In Figure 1.3, a flat unidirectional fiber composite and cylindrical shell obtained by filament helicoidal winding is represented with old and new reference axes.

1.3.1.1. *Transformation tensors*

S and C are of fourth order. Change of reference axes applied to C tensor requires the use of direction cosines P_{ij}. The new component c'_{ijkl} is expressed *versus* the old components c_{pqrs} as:

$$\left.\begin{aligned} c'_{ijkl} &= c_{pqrs}\, P_{pi}\, P_{qj}\, P_{rk}\, P_{sl} \\ c_{ijkl} &= c'_{pqrs}\, P_{ip}\, P_{jq}\, P_{kr}\, P_{ls} \end{aligned}\right] \qquad [1.18]$$

4 Another classification exists in crystallography using another vocabulary.

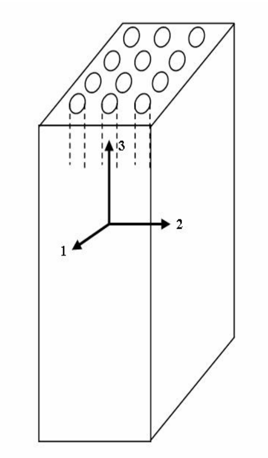

Figure 1.2. *Artificial composite made with unidirectional fibers regularly distributed in the plane (1, 2). It has a symmetry axis (0, 3). In the plane (1, 2) orthogonal to the symmetry axis, the material is isotropic. The material is a transverse isotropic one*

In spite of its apparent simplicity, equation [1.18] requires attention and in the second member there is a sum of terms. Often in the calculation some of the terms are omitted. In practice, using matrices is easier when carrying out manual calculation as well as computer code calculation.

1.3.1.2. *Passage matrices for stress and strain*

Stress and strain are second-order tensors. Changing reference axes, new stresses σ'_{ij} and new strains ε'_{ij} are:

$$\sigma'_{ij} = P_{ki}\ P_{lj}\ \sigma_{kl}\ ,\ \sigma_{ij} = P_{ik}\ P_{jl}\ \sigma'_{kl} \qquad [1.19a]$$

$$\varepsilon'_{ij} = P_{ki}\ P_{lj}\ \varepsilon_{kl}\ ,\ \varepsilon_{ij} = P_{ik}\ P_{jl}\ \sigma'_{kl} \qquad [1.19b]$$

The following transformation matrix is used:

$$[\,P\,] = \begin{array}{c} \\ x' \\ y' \\ z' \end{array} \left[\begin{array}{ccc} x & y & z \\ P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{array} \right] \qquad [1.20]$$

to go from references (x, y, z) to new references (x′, y′, z′). Rotation is with angle α around axis z.

The transformation matrix is:

$$[P] = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [1.21]$$

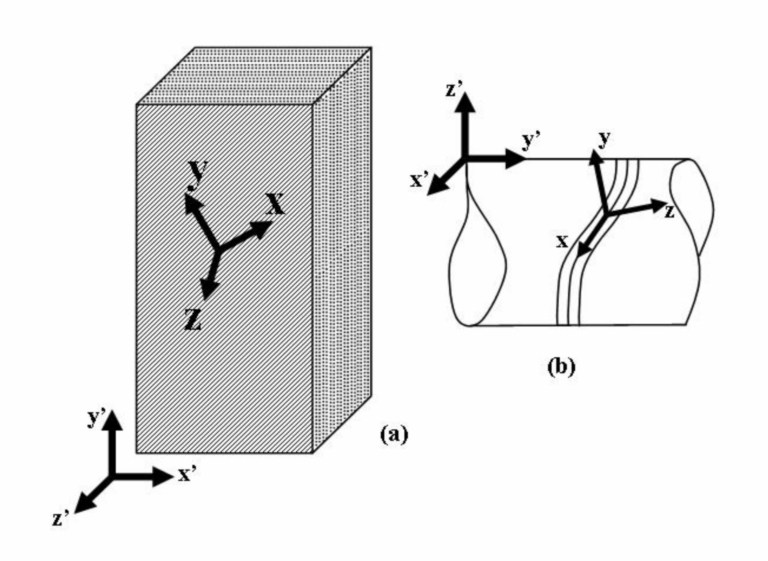

Figure 1.3. *a) Off-axis rod with unidirectional fibers; b) Cylindrical shell with helicoidal winding*

Matrix [P] is orthogonal, which means that the column vectors (or line vectors) are orthonormal: $[P]^T$ = transpose of [P] with an interchange of lines and columns.

$$[P]^{-1} = [P]^T \Rightarrow \left[P^{-1}\right]_{ij} = P_{ji} \quad [1.22]$$

1.3.1.3. *Change of axes for second-order tensors*

$$[\sigma]' = [P]^{-1}\,[\sigma]\,[P] \quad [1.23]$$

$$[\varepsilon]' = [P]^{-1}\ [\varepsilon]\ [P] \qquad [1.24]$$

and then:

$$\{\sigma'\} = \{M_\sigma\}\ \{\sigma\} \qquad [1.25]$$

$$\{\varepsilon'\} = \{M_\varepsilon\}\ \{\varepsilon\} \qquad [1.26]$$

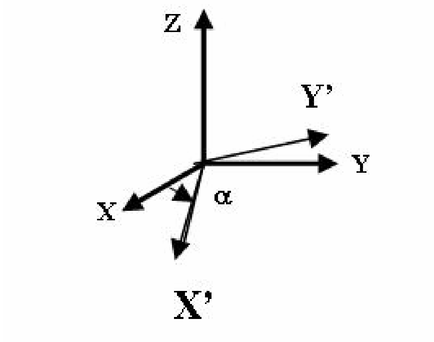

Figure 1.4. *Rotation around z axis*

The 6 x 6 matrices $\{M_\sigma\}$ and $\{M_\varepsilon\}$ are defined as follows:

$$\{M_\sigma\} = \begin{bmatrix} [D_1] & 2[A] \\ [B] & [D_2] \end{bmatrix},\ \{M_\varepsilon\} = \begin{bmatrix} [D_1] & [A] \\ 2[B] & [D_2] \end{bmatrix} \qquad [1.27]$$

In equation [1.27], square sub-matrices are defined as:

$$[A] = \begin{bmatrix} P_{21}\ P_{31} & P_{11}P_{31} & P_{11}P_{21} \\ P_{22}\ P_{32} & P_{12}\ P_{32} & P_{12}P_{22} \\ P_{23}\ P_{33} & P_{13}\ P_{33} & P_{13}\ P_{23} \end{bmatrix},$$

$$[B] = \begin{bmatrix} P_{12}\ P_{13} & P_{22}\ P_{23} & P_{32}\ P_{33} \\ P_{11}\ P_{13} & P_{21}\ P_{23} & P_{31}\ P_{33} \\ P_{11}\ P_{12} & P_{21}\ P_{22} & P_{31}\ P_{32} \end{bmatrix}, \qquad [1.28]$$

$$[D_1] = \begin{bmatrix} P_{11}^2 & P_{21}^2 & P_{31}^2 \\ P_{12}^2 & P_{22}^2 & P_{32}^2 \\ P_{13}^2 & P_{23}^2 & P_{33}^2 \end{bmatrix},$$

$$[D_2] = \begin{bmatrix} P_{22}P_{33} + P_{32}P_{23} & P_{12}P_{33} + P_{32}P_{13} & P_{12}P_{23} + P_{22}P_{13} \\ P_{21}P_{33} + P_{31}P_{23} & P_{11}P_{33} + P_{31}P_{13} & P_{11}P_{23} + P_{21}P_{13} \\ P_{21}P_{32} + P_{31}P_{22} & P_{11}P_{32} + P_{31}P_{12} & P_{11}P_{22} + P_{21}P_{12} \end{bmatrix},$$

Equations [1.19a] and [1.19b] show that:

$$\{M_\sigma\}^{-1} = \{M_\varepsilon\}^T = \begin{bmatrix} [D_1]^T & 2[B]^T \\ [A]^T & [D_2]^T \end{bmatrix}$$

[1.29]

$$\{M_\varepsilon\}^{-1} = \{M_\sigma\}^T = \begin{bmatrix} [D_1]^T & [B]^T \\ 2[A]^T & [D_2]^T \end{bmatrix}$$

Matrices $\{M_\sigma\}$ and $\{M_\varepsilon\}$ are not identical because of the adopted definition of the strain vector (equation [1.11]). In the last three components, factor 2 is introduced to describe shear strains.

Rewriting [1.25], [1.26] and [1.27] is accounted for:

$$\{\sigma'\} = \{M_\sigma\}\{\sigma\}, \qquad \{\sigma\} = \{M_\varepsilon\}^T\{\sigma'\}$$

[1.30]

$$\{\varepsilon'\} = \{M_\varepsilon\}\{\varepsilon\}, \qquad \{\varepsilon\} = \{M_\sigma\}^T\{\varepsilon'\}$$

1.3.1.4. *Change of reference axes for {C} and {S} matrices*

$$\{\sigma\} = \{C\}\{\varepsilon\} \qquad \{\varepsilon\} = \{S\}\{\sigma\}$$

$$\{\sigma'\} = \{C'\}\{\varepsilon'\} \qquad \{\varepsilon'\} = \{S'\}\{\sigma'\} \qquad [1.31]$$

Bringing [1.30] into [1.31], we get:

$$\{C'\} = \{M_\sigma\} \{C\} \{M_\sigma\}^T$$

$$\{S'\} = \{ M_\varepsilon\} \{C\} \{M_\varepsilon\}^T \qquad [1.32]$$

Matricial equalities in [1.32] portray tensorial laws defined in equations [1.19a] and [1.19b].

1.3.1.5. *Rotation around axis z*

Equation [1.21] gives the [P] matrix, from which submatrices [A] [B] [D] and [D_2] are evaluated, see Figure 1.4.

$$[A] = \begin{bmatrix} 0 & 0 & \sin\alpha\cos\alpha \\ 0 & 0 & -\sin\alpha\cos\alpha \\ 0 & 0 & 0 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\sin\alpha\cos\alpha & \sin\alpha\cos\alpha & 0 \end{bmatrix} \qquad [1.33]$$

$$[D_1] = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & 0 \\ \sin^2\alpha & \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[D_2] = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & \cos 2\alpha \end{bmatrix}$$

1.3.2. *Orthotropic materials possess two orthogonal planes of symmetry*

Passage matrix has diagonal form (see Figure 1.1):

$$[P] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad [1.34]$$

$\{M_\sigma\}$ and $\{M_\varepsilon\}$ defined in equation [1.27] are diagonal and identical, with the exception of the fourth and fifth terms, where it is equal to -1. Post multiplication of the stiffness matrix by $\{M_\sigma\}^T$ changes the signs of the fourth and fifth columns of this last matrix. Pre-multiplication $[M_\sigma]$, however, changes the sign of the fourth and fifth lines:

$$\{C'\} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & -C_{14} & -C_{15} & C_{16} \\ & C_{22} & C_{23} & -C_{24} & -C_{25} & C_{26} \\ & & C_{33} & -C_{34} & -C_{35} & C_{36} \\ \text{sym} & & & C_{44} & C_{45} & -C_{46} \\ & & & & C_{55} & -C_{56} \\ & & & & & C_{66} \end{bmatrix}, \qquad [1.35]$$

the plane (1, 2) being symmetric $\{C'\} = \{C\}$.

Comparing original matrix $\{C\}$ with equation [1.34], we must set components that change sign to zero.

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & 0 & 0 & C_{36} \\ \text{sym} & & & C_{44} & C_{45} & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \qquad [1.36]$$

A second symmetry with respect to plane (2, 3) or (y, z) with similar reasoning gives rise to the following matrix:

$$\{C\} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ \text{sym} & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \qquad [1.37]$$

Orthotropic material has three orthogonal planes of symmetry and is characterized by nine independent moduli, nine stiffness [1.37], or nine technical moduli [1.38]:

$$\{S\}=\{C\}^{-1} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ & & \frac{1}{E_3} & 0 & 0 & 0 \\ & & & \frac{1}{G_{23}} & 0 & 0 \\ & \text{sym} & & & \frac{1}{G_{13}} & 0 \\ & & & & & \frac{1}{G_{12}} \end{bmatrix} \qquad [1.38]$$

An example of a natural orthotropic material is Douglas pinewood.

Samples are taken far from the axes of the trunk body. The stiffness matrix was evaluated and components of {C} are expressed in MPa.

$$\{C\}_{MPa} = \begin{bmatrix} 17{,}000 & 2{,}950 & 1{,}500 & 0 & 0 & 0 \\ 2{,}950 & 2{,}350 & 1{,}100 & 0 & 0 & 0 \\ 1{,}500 & 1{,}100 & 2{,}800 & 0 & 0 & 0 \\ 0 & 0 & 0 & 250 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1{,}100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1{,}550 \end{bmatrix} \qquad [1.39]$$

Technical elastic moduli were experimentally evaluated, from which rigidity matrix [1.39] is deduced.

$E_1 = 13{,}300$ MPa $\quad E_2 = 1{,}550$ MPa $\quad E_3 = 2{,}280$ MPa

$\nu_{12} = 1.23 \quad \nu_{13} = 0.052 \quad \nu_{23} = 0.314$

$\nu_{21} = 0.145$ $\nu_{31} = 0.009$ $\nu_{32} = 0.456$

$G_{23} = 250$ MPa $G_{13} = 1{,}100$ MPa $G_{12} = 1{,}350$ MPa

1.3.3. *Quasi-isotropic transverse (tetragonal) material*

In Figure 1.3a (the off-axis rod with unidirectional fibers) the plane of the unidirectional layers are superposed respectively at 0° and 90°. This geometry concerns orthotropic material that remains invariant with 90° rotation around an axis perpendicular to the layers (axis 3). We start with matrix {C} in [1.36] for an orthotropic material. Axis 3 being used for $\pi/2$ rotation, we must obtain six independent elastic moduli.

$$\{C\} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ \text{sym} & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & C_{66} \end{bmatrix} \quad [1.40]$$

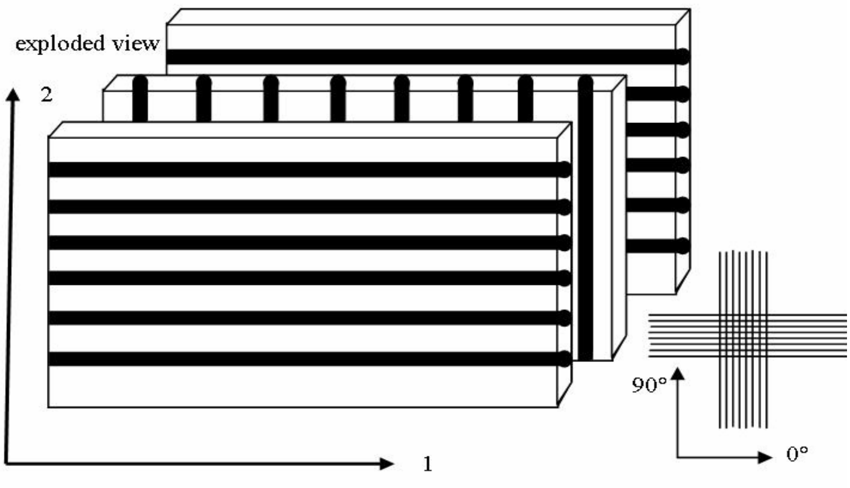

Figure 1.5. *Quasi-isotropic transverse artificial material made with successive layers at 0° and 90°*

An example of this is a multilayered composite made with taffeta tissues – carbon-epoxy with high-strength fibers being 56 % in volume.

$E_1 = E_2 = 57.8$ GPa, $E_3 = 6.9$ GPa

$\nu_{12} = \nu_{21} = 0.025$ $\nu_{13} = \nu_{23} = 0.585$

$\nu_{31} = \nu_{32} = 0.070$

$G_{23} = G_{13} = 3.3$ GPa $G_{13} = 18.0$ GPa

1.3.4. *Transverse isotropic materials (hexagonal system)*

Figure 1.6 represents such an artificial material.

The plane representing a section is a plane of symmetry.

We can consider this to be a special orthotropic material, such that a rotation around z axis with any angle does not modify the elastic constants.

That is:

$$C_{11} = C_{22} \qquad [1.41]$$

and:

$$C_{66} = 2\,(C_{11} - C_{12} - 2\,C_{66}) \sin^2 \alpha \cos^2 \alpha + C_{66} \qquad [1.42]$$

If we make the bracket equal to zero:

$$C_{66} = \frac{C_{11} - C_{12}}{2} \qquad [1.43]$$

we get the transverse isotropic relationship.

An example of this is a glass-epoxy transverse isotropic composite, 65% of fiber in volume.

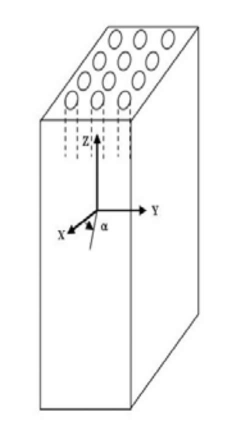

Figure 1.6. *Transverse isotropic material. z is the fiber axis*

$$\{C\} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ \text{sym} & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & C_{66} = \dfrac{C_{11} - C_{12}}{2} \end{bmatrix} \qquad [1.44]$$

Elastic moduli:

$E_1 = E_2 = 18.5$ GPa $\quad E_3 = 58.9$ GPa

$\nu_{12} = \nu_{12} = 0.425$, $\quad \nu_{13} = \nu_{23} = 0.038$, $\quad \nu_{31} = \nu_{32} = 0.122$

$G_{23} = G_{13} = 7.25$ GPa $\quad G_{12} = 6.5$ GPa

1.3.5. *Quasi-isotropic material (cubic system)*

Such a material has three principal orthogonal axes of symmetry. A rotation with any angle around one of those axes must give rise to the same material. We take an orthotropic material and operate rotation:

$$\{C\} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ \text{sym} & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & C_{44} \end{bmatrix} \qquad [1.45]$$

Such a material has three independent elastic moduli. A three-dimensional composite with reinforcement in three orthogonal directions has special application in aeronautics.

1.3.6. *Isotropic materials*

We start with [1.45] concerning a quasi-isotropic material and we apply the transverse isotropic relationship [1.41] in order to get sheer stiffness coefficients.

$$\{C\} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & \frac{C_{11} - C_{12}}{2} = C_{44} & 0 & 0 \\ \text{sym} & & & & \frac{C_{11} - C_{12}}{2} = C_{55} & 0 \\ & & & & & \frac{C_{11} - C_{12}}{2} = C_{66} \end{bmatrix}$$

Two independent elastic constants C_{11} and C_{12} are used – stiffness matrix components *versus* Lamé's coefficients. Lamé proposed the two independent elastic constants λ and μ:

$$C_{11} = \lambda + 2\,\mu$$

$$C_{12} = \lambda$$

Then when $C_{ii} = \mu = G$ with $i \geq 4$, $\mu = G$, the shear modulus is called Coulomb's modulus.

Compliance matrix [S] is:

$$[S] = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ & & \frac{1}{E} & 0 & 0 & 0 \\ & & & \left(\frac{2(1+\nu)}{E} = \frac{1}{G}\right) & 0 & 0 \\ & & & & \left(\frac{2(1+\nu)}{E} = \frac{1}{G}\right) & 0 \\ & & & & & \left(\frac{2(1+\nu)}{E} = \frac{1}{G}\right) \end{bmatrix}$$

Usually the components of a compliance matrix are written *versus* the two independent elastic moduli, E and ν being the Poisson's number.

1.4. Relationship between compliance S_{ij} and stiffness C_{ij} for orthotropic materials

Matrix inversion permits the calculation of compliances:

$$S_{11} = \frac{C_{22}C_{33} - C_{23}^2}{\Delta C} = \frac{1}{E_1}, \; S_{12} = \frac{C_{23}C_{13} - C_{33}C_{12}}{\Delta C} = -\frac{\nu_{12}}{E_1}$$

$$S_{22} = \frac{C_{33}C_{11} - C_{13}^2}{\Delta C} = \frac{1}{E_2}, \; S_{13} = \frac{C_{12}C_{23} - C_{22}C_{13}}{\Delta C} = -\frac{\nu_{13}}{E_1}$$

$$S_{33} = \frac{C_{11}C_{22} - C_{12}^2}{\Delta C} = \frac{1}{E_3}, \; S_{23} = \frac{C_{13}C_{12} - C_{11}C_{23}}{\Delta C} = -\frac{\nu_{23}}{E_1}$$

$$S_{44} = \frac{1}{C_{44}} = \frac{1}{G_{23}}, \quad S_{55} = \frac{1}{C_{55}} = \frac{1}{G_{31}}, \quad S_{66} = \frac{1}{C_{66}} = \frac{1}{G_{12}}$$

with

$$\Delta C = C_{11}C_{22}C_{33} + 2\,C_{12}C_{23}C_{31} - C_{12}^2C_{33} - C^2_{23}C_{11} - C_{13}^2C_{22} \qquad [1.46]$$

Permutation of symbols S and C in [1.42] enables us to obtain the rigidity (or stiffness) matrix *versus* compliance matrix:

$$C_{11} = \frac{1 - \nu_{23}\,\upsilon_{32}}{E_2\,E_3\,\Delta S}, \qquad C_{12} = \frac{\nu_{21} + \nu_{23}\,\nu_{31}}{E_2\,E_3\,\Delta S}$$

$$C_{22} = \frac{1 - \nu_{31}\,\upsilon_{13}}{E_3\,E_1\,\Delta S}, \qquad C_{13} = \frac{\nu_{13} + \nu_{12}\,\nu_{23}}{E_1\,E_2\,\Delta S}$$

$$C_{23} = \frac{1 - \nu_{12}\,\nu_{21}}{E_1\,E_2\,\Delta S}, \qquad C_{23} = \frac{\nu_{32} + \nu_{31}\,\nu_{12}}{E_3\,E_1\,\Delta S}$$

$$C_{44} = \frac{1}{S_{44}} = G_{23}\,,\ C_{55} = \frac{1}{S_{55}} = G_{13}\,,\ C_{66} = \frac{1}{S_{66}} = G_{12}$$

$$\Delta S = \frac{1 - \nu_{12}\,\nu_{21} - \nu_{23}\,\nu_{32} - \nu_{31}\,\nu_{13} - 2\nu_{21}\,\nu_{13}\,\nu_{32}}{E_1\,E_2\,E_3} \qquad [1.47]$$

with:

$$\nu_{12}\,\nu_{23}\,\nu_{31} = \nu_{21}\,\nu_{13}\,\nu_{32}.$$

1.5. Useful inequalities between elastic moduli

Elastic systems are stable. This means that the deformation energy of such systems in order to change from natural state to deformed state must be positive. Consequently, the stiffness matrix as well as the compliance matrix must be positive.

These considerations, presented below, are useful for practitians when checking the calculations of components of stiffness and compliance matrices.

1.5.1. *Orthotropic materials*

In [1.37] and [1.38], the following inequalities are obtained:

$$C_{ij}^2 < C_{ii}\ C_{jj}$$

$$S_{ij}^2 < S_{ii}\ S_{jj} \quad i, j = 1, 2, \ldots, 5, 6 \qquad [1.48]$$

From [1.48] and [1.14] we must obtain:

$$1 - \nu_{ij}\, \nu_{ji} > 0 \quad i, j = 1, 2, 3 \qquad [1.49]$$

and also:

$$\frac{1}{E_i\ E_j} > \frac{\nu_{ij}\ \nu_{ji}}{E_i\ E_j} = \frac{\nu_{ij}^2}{E_i^2}$$

or:

$$\nu_{ij}^2 < \frac{E_i}{E_j} \qquad i, j = 1, 2, 3 \qquad [1.50]$$

Matrix S is positive if and only if its eigenvalues are positive.

Determinants Δ C and Δ S in [1.46] and [1.47], being respectively the products of eigenvalues of stiffness and compliances, are positive:

$$1 - \nu_{12} - \nu_{21} - \nu_{23} - \nu_{32} - \nu_{31} - \nu_{13} - 2\ \nu_{12}\ \nu_{32}\ \nu_{13} > 0$$

This equation can be rearranged as p – p:

$$\nu_{12}\ \nu_{13}\ \nu_{32} < \frac{1 - \nu_{12}\ \nu_{21} - \nu_{23}\ \nu_{32} - \nu_{31}\nu_{13}}{2} < \frac{1}{2}$$

Poisson's number being positive, and finally:

$$\nu_{12}\ \nu_{23}\ \nu_{31}\ =\ \nu_{21}\ \nu_{13}\ \nu_{32}\ <\ \frac{1}{2} \qquad [1.51]$$

1.5.2. *Quasi-transverse isotropic materials*

If the plane (1, 2) is the quasi-isotropic one, the first three eigenvalues of the stiffness matrix are the solution of the third-degree equation:

$$\left(C_{11}-C_{12}-\lambda\right)\left\{\lambda^2-\left(C_{11}+C_{12}+C_{33}\right)\lambda+C_{33}\left(C_{11}+C_{12}\right)-2C_{13}^2\right\}$$

The eigenvalues are positive if and only if:

$$\begin{aligned} &C_{11} > C_{12} \\ &C_{44} = C_{55} > 0 \\ &C_{11} + C_{12} + C_{33} > 0 \\ &C_{11} + C_{12} > \frac{2C_{13}^2}{C_{33}} \end{aligned} \qquad [1.52]$$

A similar relationship can be obtained for the compliance matrix by substituting symbol S for C. The last two inequalities of [1.52] give rise to the following inequalities:

$$\begin{aligned} &\nu_{12} < 1 + \frac{E_1}{E_3} \\ &\nu_{13}\ \nu_{31} < \frac{1-\nu_{12}}{2} \end{aligned} \qquad [1.53]$$

1.5.3. *Transverse isotropic, quasi-isotropic, and isotropic materials*

Equations [1.47] and [1.48] are satisfied for the three classes of materials, taking into account components of matrices C and S for each type of material.

For isotropic and quasi-isotropic materials, the second part of equation [1.48] shows that $\nu < 0.5$.

1.6. Transformation of reference axes is necessary in many circumstances

1.6.1. *Practical examples*

Samples of composite materials are intentionally tailored in such a way that the axes of the samples do not coincide with the natural axes of the material. Stiffness or compliance matrices are consequently evaluated with sample reference axes.

Multilayered artificial composites for aerospace applications are made with superposition of a certain number of layers glued together. Each layer has its own orientation of reinforced fibers. In the calculation of global stiffness or compliance of the composites, transformation of the reference axis in each layer is necessary for the computation, using finite elements of the structure.

In a mechanical structure using multilayered composite materials, some components of the elastic compliance matrix of the materials are required. The problem is finding an optimized multilayered composite with the relevant number of layers and orientation of fibers in each layer.

In the framework of matrix calculations, this problem is presented in section 1.3.1, equations [1.17] and [1.24]. It is detailed in equations [1.26] to [1.30].

1.6.2. *Components of stiffness and compliance after transformation*

We often have to deal with the problem of rotation around an axis. Below are three tables that will be useful in Chapters 6 to 12 of this book.

If we compare Table 1.3 with Table 1.4, we see that components of new matrix {C} after transformation do not necessarily have the same coefficients. The reason is, if we refer to transformation matrices, that equations [1.24] and [1.25], $\{M_\sigma\}$ and $\{M_\varepsilon\}$ are not the same. We recall that this is due to the definition or technical shear strains in equation [1.11].

However, for the purposes of computation, in the fabrication of codes the following remarks permit the utilization of a unique computer code. Equation [1.53] points out a connection between stiffness and compliance.

If three rotations are effected around three principal axes of an orthotropic material, the problem is reduced to index permutation, a rotation around z axis, the inverse permutation being effected in the next operation.

Table 1.5. concerns elastic technical constants.

$$\left\{\begin{array}{ll} S_{ij} = C_{ij} & i, j = 1, 2, 3 \\ S_{pp} = 4\, C_{pp} & p = 4, 5, 6 \\ S'_{ij} = C'_{ij} & i, j = 1, 2, 3 \\ S'_{pp} = 4\, C'_{pp} & p = 4, 5, 6 \\ S'_{45} = 4\, C'_{45} & \\ S'_{i6} = 2\, C'_{i6} & i = 1, 2, 3 \end{array}\right\} \qquad [1.54]$$

1.6.3. *Remarks on shear elastic moduli G_{ii} (ij = 23, 31, 12) and stiffness constants C_{ii} (with i = 4, 5, 6)*

Comparing Table 1.5 to Table 1.3, we notice that:

$$G'_{23} \neq C'_{44},\quad G'_{31} \neq C'_{55},\quad G'_{12} \neq C'_{12}$$

1.6.4. *The practical consequence of a transformation of reference axes*

This problem concerns only anisotropic materials. Compliance and stiffness tensors are of fourth order:

$$c'_{ijkl} = P_{pi} \,.\, P_{qj} \,.\, P_{rk} \,.\, P_{sl} \,.\, c_{pqrs} \qquad [1.55]$$

$P_{\alpha\beta}$ being direction cosine.

In Tables 1.3, 1.4 and 1.5 the consequence of the rank of those tensors is that the power of coefficients in each tensor component is 4. This raises the problem of accuracy and errors in measurements of the new components. Any error in angle α has a strong influence on the evaluated components of the two matrices. In some circumstances, before fabrication of the sample, the angle α has to be optimized.

1.7. Invariants and their applications in the evaluation of elastic constants

In textbooks devoted to continuum mechanics, invariants are extensively used in the study of stress and strain tensors. By definition, invariants are scalars obtained by a combination of matrix (or tensor) elements that remain constant by transformation of the reference axes. For second-order tensors concerning stress and strain, the three invariants are the coefficients of characteristic equation:

$$\det([\sigma] - \lambda\, [I]) = 0 \qquad [1.56]$$

Matrix of rigidity

$$\{C'\} = \{M_\sigma\}\{C\}\{M_\sigma\}^t$$

$$C'_{11} = C_{11}\cos^4\alpha + 2\left(C_{12} + 2C_{66}\right)\sin^2\alpha\cos^2\alpha + C_{22}\sin^4\alpha$$

$$C'_{12} = \left(C_{11} + C_{12} - 4C_{66} - 2C_{12}\right)\sin^2\alpha\cos^2\alpha + C_{12}$$

$$C'_{13} = C_{13}\cos^2\alpha + C_{23}\sin^2\alpha$$

$$C'_{14} = C'_{15} = 0$$

$$C'_{16} = \left[C_{22}\sin^2\alpha - C_{11}\cos^2\alpha + \left(C_{12} + 2C_{66}\right)\left(\cos^2\alpha - \sin^2\alpha\right)\right]\sin\alpha\cos\alpha$$

- -

$$C'_{22} = C_{11}\sin^4\alpha + 2\left(C_{12} + 2C_{66}\right)\sin^2\alpha\cos^2\alpha + C_{22}\cos^4\alpha$$

$$C'_{23} = C_{13}\sin^2\alpha + C_{23}\cos^2\alpha$$

$$C'_{24} = C'_{25} = 0$$

$$C'_{26} = \left[C_{22}\cos^2\alpha - C_{11}\sin^2\alpha - \left(C_{12} + 2C_{66}\right)\left(\cos^2\alpha - \sin^2\alpha\right)\right]\sin\alpha\cos\alpha$$

- -

$$C'_{33} = C_{33}$$

$$C'_{34} = C'_{35} = 0$$

$$C'_{36} = \left(C_{23} - C_{13}\right)\sin\alpha\cos\alpha$$

- -

$$C'_{44} = C_{44}\cos^2\alpha + C_{55}\sin^2\alpha$$

$$C'_{45} = \left(C_{44} - C_{55}\right)\sin\alpha\cos\alpha$$

$$C'_{46} = 0$$

- -

$$C'_{55} = C_{44}\sin^2\alpha + C_{55}\cos^2\alpha$$

$$C'_{56} = 0$$

- -

$$C'_{66} = \left(C_{11} + C_{12} - 2C_{12} - 4C_{66}\right)\sin^2\alpha\cos^2\alpha + C_{66}$$

Table 1.3. *Rotation with an angle α around z axis and its influence on stiffness matrix {C} of an orthotropic material*

Compliance matrix

$$\{S'\} = \{M_\varepsilon\}\{S\}\{M_\varepsilon\}^t$$

$$S'_{11} = S_{11}\cos^4\alpha + (S_{66} + 2S_{12})\sin^2\alpha\cos^2\alpha + S_{22}\sin^4\alpha$$

$$S'_{12} = (S_{11} + S_{22} - S_{66} - 2S_{12})\sin^2\alpha\cos^2\alpha + S_{12}$$

$$S'_{13} = S_{13}\cos^2\alpha + S_{23}\sin^2\alpha$$

$$S'_{14} = S'_{15} = 0$$

$$S'_{16} = 2\left[S_{22}\sin^2\alpha - S_{11}\cos^2\alpha + \left[S_{12} + \frac{S_{66}}{2}\right]\left(\cos^2\alpha - \sin^2\alpha\right)\right]\sin\alpha\cos\alpha$$

- -

$$S'_{22} = S_{11}\sin^4\alpha + (S_{66} + 2S_{12})\sin^2\alpha\cos^2\alpha + S_{22}\cos^4\alpha$$

$$S'_{23} = S_{13}\sin^2\alpha + S_{23}\cos^2\alpha$$

$$S'_{12} = S'_{25} = 0$$

$$S'_{26} = 2\left[S_{22}\cos^2\alpha - S_{11}\sin^2\alpha - \left[S_{12} + \frac{S_{66}}{2}\right]\left(\cos^2\alpha - \sin^2\alpha\right)\right]\sin\alpha\cos\alpha$$

- -

$$S'_{33} = S_{33}$$

$$S'_{34} = S'_{35} = 0$$

$$S'_{36} = 2(S_{23} - S_{13})\sin\alpha\cos\alpha$$

- -

$$S'_{44} = S_{44}\cos^2\alpha + S_{55}\sin^2\alpha$$

$$S'_{45} = (S_{44} - S_{55})\sin\alpha\cos\alpha$$

$$S'_{46} = 0$$

- -

$$S'_{55} = S_{44}\sin^2\alpha + S_{55}\cos^2\alpha$$

$$S'_{56} = 0$$

- -

$$S'_{66} = 4(S_{11} + S_{22} - 2S_{12} - S_{66})\sin^2\alpha\cos^2\alpha + S_{66}$$

Table 1.4. *Rotation with an angle α around z axis and its influence on compliance matrix {S} of an orthotropic material*

Technical elastic moduli

$$\frac{1}{E_1'} = S_{11}' = \left(\frac{\cos^4 \alpha}{E_1}\right) + \frac{\sin^4 \alpha}{E_2} + \frac{\sin^2 \alpha \cos^2 \alpha}{G_{12}} - \frac{2 \nu_{12}}{E_1} \sin^2 \alpha \cos^2 \alpha$$

$$\frac{1}{E_2'} = S_{22}' = \frac{\sin^4 \alpha}{E_1} + \frac{\cos^4 \alpha}{E_2} + \frac{\sin^2 \alpha \cos^2 \alpha}{G_{12}} - \frac{2 \nu_{12}}{E_1} \sin^2 \alpha \cos^2 \alpha$$

$$E_3' = E_3$$

- -

$$-\nu_{12}' = E_1' \left[\left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}} + \frac{2 \nu_{12}}{E_1}\right) \sin^2 \alpha \cos^2 \alpha - \frac{\nu_{12}}{E_1}\right]$$

$$\nu_{13}' = \frac{E_1'}{E_3} \left(\nu_{31} \cos^2 \alpha + \nu_{32} \sin^2 \alpha\right)$$

$$\nu_{23}' = \frac{E_2'}{E_3} \left(\nu_{31} \sin^2 \alpha + \nu_{32} \cos^2 \alpha\right)$$

- -

$$G_{23}' = \frac{1}{S_{44}'} = \left[\frac{\cos^2 \alpha}{G_{23}} + \frac{\sin^2 \alpha}{G_{13}}\right]^{-1}$$

$$G_{13}' = \frac{1}{S_{55}'} = \left[\frac{\sin^2 \alpha}{G_{23}} + \frac{\cos^2 \alpha}{G_{13}}\right]^{-1}$$

$$G_{12}' = \frac{1}{S_{66}'} = \frac{1}{4} \left[\begin{array}{l} \left(\frac{1}{E_1} + \frac{1}{E_2} + \frac{2\upsilon_{12}}{E_1}\right) \sin^2\alpha \cos\alpha + \\ \frac{1}{G_{12}} \left(\frac{1}{4} - \sin^2\alpha \cos^2\alpha\right) \end{array}\right]^{-1}$$

Table 1.5. *Rotation with an angle α around z axis and its incidence on technical elastic constants*

[I] being the identity matrix:

$$[\,I\,] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For the stress matrix we have[5]:

$$I\,(\sigma) = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} \qquad \text{linear invariant}$$

$$II\,(\sigma) = \frac{1}{2}\left(\sigma_{ij}\ \sigma_{ji} - \sigma_{ij}\ \sigma_{ij}\right) \qquad \text{quadratic invariant}$$

$$III(\sigma) = \det\,[\sigma] \qquad \text{cubic invariant (determinant)}$$

In a plane stress state, the two first invariants are graphically represented by the Mohr circle [TSA 80]. This representation is convenient for the experimental evaluation of strains by strain gauges.

For fourth rank tensors, calculation of invariants is similar to that which is presented above. The number of invariants is higher. There are five invariants of the first order, two invariants of the second order, and one of the third order.

Details of the calculation can be found in [CHE 96, JON 75, LEK 60, TSA 80].

1.7.1. *Elastic constants versus invariants*

Table 1.6 collects the invariants of the three-species material, which when examined are orthotropic. It is easy to convert the expressions for another type of material (quasi-isotropic transverse, isotropic transverse or cubic ones).

---

5 Developing $II\,(\sigma)$, $III\,(\sigma)$ we obtain:

$$II\,(\sigma) = \sigma_{11}\ \sigma_{22} + \sigma_{22}\ \sigma_{33} + \sigma_{33}\ \sigma_{11} - \sigma_{13}^2 - \sigma_{21}^2 - \sigma_{32}^2$$

$$III\,(\sigma) = \sigma_{11}\ \sigma_{22}\ \sigma_{33} + 2\,\sigma_{12}\ \sigma_{13}\ \sigma_{23} - \sigma_{22}\ \sigma_{13}^2 - \sigma_{33}\ \sigma_{12}^2 - \sigma_{11}\ \sigma_{23}^2$$

Components of stiffness matrix {C} are expressed against the invariants as follows:

$$
\begin{aligned}
C'_{11} &= U_1 + U_2 \cos 2\alpha + U_3 \cos 4\alpha \\
C'_{12} &= U_4 - U_3 \cos 4\alpha \\
C'_{13} &= V_1 + V_2 \cos 2\alpha \\
C'_{16} &= -\frac{U_2}{2} \sin 2\alpha - U_3 \sin 4\alpha \\
C'_{22} &= U_1 - U_2 \cos 2\alpha + U_3 \cos 4\alpha \\
C'_{23} &= V_1 - V_2 \cos 2\alpha \\
C'_{26} &= -\frac{U_2}{2} \sin 2\alpha + U_3 \sin 4\alpha \\
C'_{36} &= -\frac{V_2}{2} \sin 2\alpha \\
C'_{44} &= W_1 + W_2 \cos 2\alpha \\
C'_{45} &= W_2 \sin 2\alpha \\
C'_{55} &= W_1 - W_2 \cos 2\alpha \\
C'_{66} &= U_5 - U_3 \cos 4\alpha
\end{aligned}
\qquad [1.57]
$$

with:

$$U_1 = \frac{3\,C_{11} + 3\,C_{22} + 2\,C_{12} + 4\,C_{66}}{8}$$

$$U_2 = \frac{C_{11} - C_{22}}{2}$$

$$U_3 = \frac{C_{11} + C_{22} - 2\,C_{12} - 4\,C_{66}}{8}$$

$$U_4 = \frac{C_{11} + C_{22} + 6\,C_{12} - 4\,C_{66}}{8}$$

$$U_5 = \frac{C_{11} + C_{22} - 6\,C_{12} - 4\,C_{66}}{8} \qquad [1.58]$$

$$V_1 = \frac{C_{13} + C_{23}}{2}$$

$$V_2 = \frac{C_{13} - C_{23}}{2}$$

$$W_1 = \frac{C_{44} + C_{55}}{2}$$

$$W_2 = \frac{C_{44} - C_{55}}{2}$$

The invariants presented in Table 1.6 are independent. By combination, other families of invariants can be obtained.

U_1, U_2, U_3 and U_4 in [1.58] can be considered to be a linear combination of invariants or power of invariants.

$$\frac{3\,C'_{11} + 3\,C'_{22} + 2\,C'_{12} + 4\,C'_{66}}{8} = \frac{I_1(C) + 2\,I_2(C)}{8} = U_1$$

$$\frac{C'_{11} + C'_{22} + 6\,C'_{12} - 4\,C'_{66}}{8} = \frac{3\,I_1(C) - 2\,I_2(C)}{8} = U_4 \qquad [1.59]$$

$$\frac{C'_{11} + C'_{22} - 2\,C'_{12} + 4\,C'_{66}}{8} = \frac{2\,I_1(C) - 2\,I_2(C)}{8} = U_5 = \frac{U_1 - U_4}{2}$$

$$C'_{11}C'_{22} + 2\,C'_{66}\left(C'_{11} + C'_{22}\right) - C'^2_{12} - 2C'^2_{16} - 2C'^2_{26} = U_1^2 - U_2^2 - U_4^2 - 4U_3^2$$

1.7.2. *Practical utilization of invariants in the evaluation of elastic constants*

Measurements of elastic constants are subject to errors of different kinds:

– error of angle α due to the fabrication of a sample's dimension error;

– dimension errors of the samples;

– errors in measurement during static or dynamics tests.

Evaluation of elastic constants cannot be reduced to a restricted number of measurements. We have to deal with an optimization problem with α as a parameter.

The first stage concerns evaluation of elastic constants (C_{ij} or S_{ij}). Each set of experimental results corresponds to an angle α in equations [1.57] to [1.58].

In the second stage, the calculation of invariants in Table 1.6 is carried out. Optimization of invariants constitutes the third step. From optimized invariants, elastic constants are calculated from [1.57].

1.8. Plane elasticity

In many mechanical structures, plate and/or shell elements are used. In such elements, the thickness is small compared to other dimensions. For plates, the number of stress and strain components is reduced to three.

In Figure 1.7, axis 3 is directed through the thickness.

1.8.1. *Expression of plane stress stiffness versus compliance matrix*

Components S_{ij}:

$$\{\sigma\}^T = \left(\sigma_{11} \ \ \sigma_{22} \ \ 0 \ \ 0 \ \ 0 \ \ \sigma_{12}\right) \qquad [1.60]$$

is the plane stress vector applied in plane (1, 2), then for orthotropic material:

$$\{\varepsilon\}^T = \left(\varepsilon_{11} \ \ \varepsilon_{22} \ \ \varepsilon_{33} \ \ 0 \ \ 0 \ \ 2\varepsilon_{12}\right)$$