THEORETICAL AND APPLIED MATHEMATICS

THE IMPOSSIBLE QUINTIC

Made as Simple as Possible

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Preface

About 4000 years ago the ancient people of Iraq could solve quadratic equations. The next step in complexity - the solution of cubic equations - took more than 3000 years, but was solved in the early 1500's. The solution of the quartic (an equation with an x^4 term) came only a few years later. The race was on to find a solution to the quintic (an equation with an x^5 term), but with no result despite 300 years of effort. Everiste Galois was born in 1812 and in his late teenage years submitted a paper to a major mathematics journal in which he showed that the reason there had been no solution to the quintic was that a solution was absolutely impossible. He gave a proof of this impossibility. A mathematician a few years earlier had already shown this – but Galois put his proof of the insolubility of the quintic into a more general framework – a framework which is now valuable in quantum physics and coding theory. Unfortunately, Galois' proof was too abbreviated for other mathematicians to understand and the paper was sent back to Galois for revision. Meanwhile Galois had got into some sort of trouble and was challenged to a duel. He rewrote the paper the night before the duel. Fearing the worst, he asked his friends to persist in presenting it to mathematicians. He died from the duel. The paper was not recognised as the work of a genius, until 14 years later.

This book presents a simple version of Galois' proof. It is largely based on an excellent book "Galois Theory" by H M Edwards [1]. However, Edwards' book covers a wider area of mathematics than is required for a proof of the insolubility of the quintic and doesn't actually complete the insolubility proof. Unfortunately, most books on this area of mathematics seem to replicate to some degree the initial problem encountered by Galois. Galois, the cleverest of mathematicians, left gaps in reasoning. To him the steps to fill in those gaps were obvious, even trivial. He had trouble understanding that more ordinary mathematicians would have difficulty bridging the gaps and would need the details spelt out. This problem in Galois theory continues. Mathematicians who write on Galois theory may be so far ahead of their readership, that gaps are left that are hard for ordinary readers to bridge. Edwards' book follows an older, less abstract approach to Galois theory, but despite this, it too is not always easy to understand. A simpler proof of the insolubility of the quintic could be of interest to many people who have no more than university entrance level mathematics. Hopefully, this book will be of use to those people.

A proof that something cannot exist, seems beguiling. One can imagine the following conversation:

"There is no formula to give the roots of a quintic."

"No-one has found one yet?"

"No, it can't be done."

"Perhaps in future, better mathematicians with modern methods will come up with a formula?"

"No! If someone ever suggested a formula, we would know straight away that it is wrong – because it would be in contradiction of a logical proof that such a formula cannot exist."

"How can that be?"

It is the task of this book to answer that last question as simply as possible.

Chapter 1

Introduction to Galois' Proof

1. Radicals and Symmetry

What Galois showed was that in general it is not possible to express the solution to the quintic with square roots, cube roots, fourth roots and fifth roots of terms involving the coefficients of the quintic. In mathematical language, we say the general quintic is not solvable by radicals. In other words it is not soluble by using the coefficients and combining them somehow using the simple operations of arithmetic – add/subtract and multiply/divide, together with the taking of various roots/radicals. The roots of the quintic exist and can be approximated numerically, but are not expressible by a formula. By contrast the two roots of the quadratic are given by the radical expression $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ and roots of the cubic are given by a more complex radical expression containing several radical terms including

$$\sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)+\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}}$$

There is an even more complex radical expression for the solution of the quartic equation. There is no radical expression for the roots of the quintic. We will show that this is not because it is so complex that it has not yet been discovered. However, since the five roots of the quintic exist, we will call them r_1, r_2, r_3, r_4, r_5. We will exclude cases where roots are equal*. For the time being, we will also require the coefficients of the x^i terms of the quintic to be rational.

In a sense, it is somewhat arbitrary to say a quintic is not soluble in terms of the coefficients. We are happy to say a quantity like $\sqrt{2}$ is a solution of $x^2 = 2$, when, although it can be calculated to as many decimal places as we wish, we cannot define it more precisely than by saying it is the positive solution to $x^2 = 2$. Likewise, just as we can calculate $\sqrt{2}$, we could calculate any root of any quintic as accurately as we like using numerical methods, but in general the answer we obtain would not be absolutely precise. The difference here from the situation of $x^2 = 2$ having a solution $\sqrt{2}$, is that we don't have a symbol for a root of a general quintic, whilst we have a symbol $\sqrt{n}$ for the solution of $x^2 = n$. In that sense, it could be regarded as somewhat arbitrary to say that we cannot generally solve a quintic whereas we can always solve a quadratic. However, it is of interest that our intuition that the quintic could be solved in terms of the coefficients just by adding in operations such as $\sqrt[5]{\ }$, $\sqrt[4]{\ }$ and so on, turns out to be wrong in general.

It turns out that the reason the solution to a general quintic cannot be expressed in terms of square, third, fourth and fifth roots, has something to do with symmetry – the symmetry of arithmetic expressions that involve the roots of the quintic. A tenuous hint in this direction is obtained by considering the two roots ($+i$ and $-i$) of the equation $x^2 + 1 = 0$. If we perform the elementary operations of arithmetic (plus/minus, multiply/divide) on an arithmetic expression involving these two roots, it does not matter whether we swop the roots around (taking the complex conjugate) before or after performing the arithmetic operations – we get the same answer. There is something in the idea here which can be used as a strategy to obtain the quadratic formula. Rather than explore this in the context of the quadratic, we will proceed directly to study the symmetries involved in proving that there is no formula possible to solve the general quintic.

The overall proof of the insolubility of the quintic given here is intended to be accessible to anyone with good university entrance level mathematics. As in the paragraph above, it does require the reader to be aware of the existence of complex numbers. Awareness of the mathematical idea of a group may also be helpful. Matrices and determinants are used, including Cramer's rule, something that might not be encountered until second year university mathematics. However, matrices,

*Our aim is to show only that there are at least some quintics whose solutions are not given by a radical expression determinants and Cramer's rule are covered in the appendix.

Since Galois, an approach to the subject using abstract algebra beyond group theory has become mainstream and the subject area has developed further. Many texts giving the modern approach to Galois theory are available of which the book by the Maxfields is perhaps the easiest [2]. However, Galois' original approach requires us to gain familiarity with fewer abstract constructs and so may be more memorable. Galois' original approach also impresses most dramatically with the astounding brilliance of his ideas.

2. Book Outline

The mathematics starts in the next chapter with a proof about the expansion of a construct called a Vandermonde determinant. This proof is used just in the following two initial theorems, but it illustrates the power of symmetry. The first of the theorems following the proof about the expansion of Vandermonde determinants, is a generalisation of the idea that all symmetric expressions in a and b, such as $a^n + (ab)^m + b^n$ can be expressed in terms of what are called elementary symmetric multinomials. (For two variables, these elementary symmetric multinomials are $a+b$ and ab). This result follows from a long and messy proof that shows that there exists a theoretical, but unwieldy, process for building expressions like $a^n + (ab)^m + b^n$ from the elementary symmetric expressions. The proof could seem intimidating but should be less so when one realises that it is intended only as a proof of principle and the lengthy manipulations are not required in practice. If one can accept the plausible result that all symmetric expressions can be expressed in terms of elementary symmetric multinomials, then this preliminary proof could be omitted on first reading. Much the same applies to the second initial proof – the theorem by Lagrange about resolvents. The resolvent is an amalgam of irrational roots formed in such a way that the process is able to be reversed so that each of the roots can be obtained from the resolvent. The definition of a resolvent and the result that they can always be found, needs to be understood. Unlike the previous proof on symmetric polynomials, the result is not immediately plausible. However like the previous proof, it is a long and messy proof and could also be omitted at first reading.

The following chapter goes back nearly 2400 years to Euclid's algorithm for finding greatest common divisors. This theorem is then generalised to finding