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Parametric Integer Optimization

B. Bank / R. Mandel

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Parametric Integer Optimization
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by
B. Bank
and
R. Mandel

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by

Bernd Bank

and

Reinhard Mandel

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Autoren:

Doz. Dr. sc. nat. Bernd Bank
Dr. sc. nat. Reinhard Mandel
Humboldt-Universität zu Berlin
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Preface.

The subject matter of this volume is concerned with optimization problems in which some of the data involved are seen as parametric quantities and some or all of the variables are required to be integer.
The characterization of the parameter-dependent behaviour of the feasible region, the extreme value, and the set of optimal points elaborated here is of great importance both in theory and practice. This monograph is intended to be a first comprehensive contribution to the theory of parametric nonlinear and linear integer programming and, it is directed to mathematicians as well as to researchers, practical workers and students who are familiar with the basic knowledge of mathematical programming.
The book mainly contains research results obtained by the authors during the last five years and reflects the important contributions to the theory of parametric integer programming due to C.E. Blair and R.G. Jeroslow and that of R. Hansel to the quadratic integer case.
Our work was essentially influenced by fruitful discussions with several colleagues, including, above all, Professor F. Nožička, of the division Mathematische Optimierung at the Sektion Mathematik of the Humboldt-Universität zu Berlin, and stimulated by the collaboration with a research group at the Lomonossow University Moscow headed by E.G. Belousov. In particular, the authors would like to express their deep gratitude to D. Klatte and B. Kummer for their insightful discussions and helpful comments, and to J. Guddat, who steadily encouraged our interest in this subject and writing this monograph.
We have become greatly indebted to J. Kerger, Ch. Reimann and M. Willenberg for the assistance in preparing the final version of the manuscript, to S. Schmidt for her careful typing, and we would like to thank R. Höppner and G. Reiher of the Akademie-Verlag for their patience and support.
Finally, we welcome all comments, criticisms and suggestions, which should be directed to the authors:

Humboldt-Universität Berlin
Sektion Mathematik
PSF 1297
Berlin, DDR
1086

Bernd Bank
Reinhard Mandel

Berlin, August 1986

Contents

1. Introduction

Stability theory and the parametrization of optimization problems are of great importance both in theory and applications. Optimization models of real-life problems, like any mathematical model of such problems, only touch, in a certain approximative sense, the surface of the relations in the reality. For instance, if the functional dependences in the optimization problem are sufficiently well seized, then the applicability of the model essentially depends on the accuracy of the initial data and, a possible methodology to treat this problem is obtained by parametrization of the data in the optimization models. On the other hand, a parametrization of the data in an optimization problem reflects a typically mathematical framework, successfully applied in almost all branches of mathematics and their applications. During the last decade a remarkable success has been made on stability analysis and parametrization of the models in nonlinear programming. For a comprehensive survey we refer the reader to the monographs: BANK/GUDDAT/KLATTE/KUMMER/TAMMER (1982) (for shortness in the following: BANK et al. (1982)), BROSOWSKI (1982), DINKELBACH (1969), DONTCHEV (1983), FIACCO (1983), GAL (1973,1979), NOŽIČKA/GUDDAT/HOLLATZ/BANK (1974) and the Mathematical Programming Studies 10 (1979), 19(1982) and 21(1984).
But, if the parametric nonlinear program contains integer requirements on the variables (or on some of them), then, up to now, only a few results can be listed. One reason for this situation is that one has to handle the difficulties of nonlinear programming as well as those which arise by the fact that just little is known about (mixed-) integer solutions of nonlinear equations and inequalities and their structure.

In this book we focus our interest on parametric (nonlinear mixed-) integer optimization problems which, in general, can be written in the form (used symbols and notations are explained at the end of the Introduction)

$$P(\lambda) \quad \inf\{f(x,\lambda) \;/\; x \in M(\lambda),\ x_1,\dots,x_s \text{ integer}\}\ ,\ \lambda \in \Lambda$$

where the parameter set Λ is a subset of R^m, $f\colon R^n \times \Lambda \longrightarrow R$ is the objective function and the constraint set $M(\lambda)$, for every $\lambda \in \Lambda$, is a subset of R^n. (As usually $P(\lambda)$ is called a mixed-integer, a pure integer or "continuous" optimization problem if $0 < s < n$, $s = n$ or $s = 0$, respectively.)
The behaviour of the problem $P(\lambda)$ is essentially characterized by the properties of the constraint set-mapping

$$M\colon \Lambda \longrightarrow 2^{R^n} \text{ given by } M(\lambda) \subset R^n,$$

the value function

$$\varphi\colon \Lambda \longrightarrow R \cup \{+\infty,-\infty\} \text{ defined by } \varphi(\lambda) = \inf_{x \in M(\lambda)} f(x,\lambda)$$

and the optimal set-function

$$\psi\colon \Lambda \longrightarrow 2^{R^n} \text{ defined by } \psi(\lambda) = \{x \in M(\lambda) \;/\; f(x,\lambda) = \varphi(\lambda)\}.$$

For the problem $P(\lambda)$ it is desirable that these mappings possess certain continuity or Lipschitzian properties (of global or local kind). The semi-continuity analysis of the above mappings, which will play a central role in our considerations, permits answers to the questions: 1. Does the accuracy of the solutions and extremal values increase with the degree of approximation of the problem data λ ? 2. Can a solution of $P(\lambda^o)$ obtained for a fixed λ^o be considered an approximative solution for the problems $P(\lambda)$ occurring under small perturbation of the data λ? We shall use the notation "stability" of $P(\lambda)$, which is not of a uniform meaning, in the literature to optimization in order to announce that certain semicontinuity properties of the multifunctions above are ensured.

First papers related to parametric (mixed-) integer linear optimization problems were published in the sixties by GOMORY (1965), FRANK (1967) and NOLTEMEIER (1970). The further development of this direction was mainly influenced by the contributions of MEYER (1974,1975,1976), BANK (1977), GEOFFRION/NAUSS (1977) and BLAIR/JEROSLOW (1977,1979,1982). While the first three papers were essentially oriented to solution methods for one- and more-parametric linear problems with parametrized linear objective functions and right-hand sides of the linear constraints, MEYER was the first who introduced the multifunctional considerations into the investigation of parametric integer problems in order to analyse the existence of solutions and the stability behaviour of such models. A comprehensive global and local analysis of the constraint set-mapping, the value function and the optimal set-function for (mixed-) integer linear problems with variable objective functions and right-hand sides of the constraints is due to BANK. BLAIR/JEROSLOW obtained an important quantitative characterization of (mixed-) integer linear problems with variable right-hand sides in terms of a kind of Lipschitzian properties for the value function and the optimal set-function. In 1982 BLAIR/JEROSLOW could extend their results and show that the value function of pure integer linear problems with variable right-hand sides can be identified with the so-called GOMORY-functions and, thus they obtained an algorithmic insight into the character of these value functions. All these results mentioned before can be considered to be a comprehensive theoretical background for (mixed-) integer linear problems (with a fixed constraint matrix; in the case that the constraint matrix is parametrized, no deep results are known up to now).

Numerical procedures for the analysis of parameter-dependent (mixed-) integer linear optimization problems are considered from different points of view in several papers, e.g. FRANK (1967), NOLTEMEIER (1970), MARSTEN/MORIN (1977), SHAPIRO (1977), HOLM/KLEIN (1978), SEELÄNDER (1980, 1980a).

Similar as in the case of "continuous" optimization problems the para-

metric framework related with (mixed-) integer problems shows a great importance for the development of a duality theory and solution methods. Remarkable progress in the (mixed-) integer linear case can be realized in the subadditive duality approach, cutting-plane theory, for penalty functions and partitioning procedures, see e.g. GOMORY (1965,1969), CHVÁTAL (1973), BURDET/JOHNSON (1974), JEROSLOW (1978,1979), SCHRIJVER (1979), BACHEM/SCHRADER (1978), BLAIR/JEROSLOW (1981,1982), BENDERS (1962), BANK/MANDEL/TAMMER (1979), TIND/WOLSEY (1981).
For postoptimality and sensitivity analyses in (mixed-) integer linear optimization we refer the reader to the articles due to GEOFFRION/NAUSS (1977) and WOLSEY (1981). Various papers deal with parametric combinatorial problems (0-1 problems), which will not be treated in this book; for references see the BIBLIOGRAPHIES I., II., III.

In contrast to the situation just analysed rather little is known about nonlinear problems in parametric (mixed-) integer programming. The stability analyses explored in the articles by RADKE (1975) and ALLENDE (1980) are obtained under compactness requirements imposed on the constraint sets of the nonlinear problems. The basis of our considerations of parametric (mixed-) integer nonlinear optimization problems is given by the contributions of HANSEL (1980), BANK/HANSEL (1984), MANDEL (1985) and BANK/BELOUSOV/MANDEL/ČEREMNYCH/SHIRONIN (1986) (cited in the following as BANK/BELOUSOV et al. (1986)), where no compactness for the constraint sets is assumed.
The main part of our considerations of parametric (mixed-) integer nonlinear programming is related with the stability analysis of the general class of problems

$$P(p,b) \quad \inf\{f(x) + p^T x \;/\; x \in M(b), x_1,\dots,x_s \text{ integer}\},\; p \in R^n,\; b \in R^m,$$

where the fixed function $f : R^n \longrightarrow R$ is quadratic or convex polynomial (without a linear part in both cases) and the constraint sets $M(b)$ are of the form

$$M(b) = C(b) + V \subset R^n,\; b \in R^m, \tag{1}$$

and given by a compact-valued upper semicontinuous multifunction C and a fixed convex polyhedral cone V. Further explorations are carried out for problems P(p,b) with constraint sets of the implicit form

$$M(b) = \{x \in R^n \;/\; p_j(x) \leq b_j,\; j=1,\dots,m\},\; b \in R^m, \tag{2}$$

where $p_j(x)$, $j=1,\dots,m$, are quasiconvex polynomial functions on R^n. These latter constraint sets belong, under a certain assumption ((CPC), see below), to the class defined by (1).
The following three aspects of a stability theory for P(p,b) are treated in the book:
- Existence and stability of feasible points,
- Semicontinuity properties of the value functions of P(p,b), their

level-sets and optimal sets,
- Existence of optimal points.

Parametric (mixed-) integer linear programming is a special case of the general problems just explained, for which additional, more sophisticated results (quasi-Lipschitzian properties) can be derived. Further, for this class of problems several applications of stability results to duality and solution concepts are demonstrated.

Now we elucidate the arrangement and the contents of the single chapters. Chapter 2. After defining the different semicontinuities used for the multifunctions in our context we present some known basic results for parameter-dependent constraint sets in R^n without integer requirements which are fairly general and, in particular, closely related to (mixed-) integer nonlinear programming cases. Some of the theorems are valid in more general spaces (see BANK et al. (1982)), our limitation to R^n is natural and allows certain simplifications. The considerations of this chapter will be completed by the investigation of constraint sets M(b) described by quasiconvex polynomial inequalities (Chapter 4.) and linear constraints (Chapter 4. and 8.) where also Lipschitzian properties of the corresponding multifunctions are treated.

Chapter 3. Here, the aim is to give a survey on the known qualitative stability results for quite general nonlinear optimization problems in R^n in a form associated with (mixed-) integer problems (see BANK et al. (1982)).
This presentation is carried on by the results in Chapter 4. for (quasi) convex polynomial optimization problems as well as for the case $s = 0$ (i.e. no integer variables) in the (mixed-) integer problems P(p,b) from Chapter 6. and P(b) from Chapter 8.

Chapter 4. Firstly, we derive basic properties of quasiconvex polynomials and constraint sets M(b) of the form (2) (where we partially follow a similar way proposed by BELOUSOV (1977) for convex polynomial functions). Taking into account results and techniques from Chapter 2. these properties enable us to examine the constraint sets M(b) in view of the continuity of the corresponding multifunction M, the representability of M(b) as a sum of a compact set C(b) and the recession cone V and the identification of such inequalities $p_j(x) \leq b_j$ which only are essential for the existence of (mixed-) integer points in M(b).

An important result with respect to the stability study for (mixed-) integer problems P(p,b) in Chapter 6. consists in the fact that the constraint sets M(b) from (2) belong to the class defined by (1) if at least one set $M(b^0)$ is representable in the form (1) (condition (CPC)). Moreover, in (1) a compact-valued multifunction C which is continuous on the effective domain of M if the condition (CPC) is satisfied can be chosen. The proof of this result, which makes use of an idea of KUMMER

(1977) for the included special case of linear constraints $p_j(x) \leq b_j$, permits a linear programming proof of a version of HOFFMAN's (1952) well-known Lipschitzian result for the mapping M. Further, in this linear case we obtain that there is even a Lipschitzian multifunction C allowing the representation (1), which is a result of importance for our considerations in Chapter 8.
In order to be able to characterize (mixed-) integer points (Chapter 5.) in sets M(b) of the form (2) we investigate the so-called stable mapping M_{st} associated to M which respects only the inequalities $p_j(x) \leq b_j$ essential for the existence of such points.
Summarizing stability results for quasi-convex polynomial optimization problems we complete this chapter.

Chapter 5. is concerned with (mixed-) integer constraint sets

$$G(b) = \{x \in R^n \;/\; x \in M(b),\ x_1, \ldots, x_s \text{ integer}\},\ b \in R^m, \tag{3}$$

First we discuss the 'uniform distribution' of (mixed-) integer points in subsets of the space R^n which have a representation as the sum of a bounded set and a convex cone. We extend some results of BELOUSOV (1977) from the pure integer case to the mixed-integer one, where we restrict ourselves to such questions necessary for our further considerations; a comprehensive study of the distribution of integer points in convex sets in R^n can be found in SHIRONIN (1980). For the case that M(b) in (3) has the form (1) we can characterize the structure of the sets G(b) and their convex hulls and show that the corresponding multifunctions are upper semicontinuous on their common domain if the included polyhedral cone V from (1) has a system of mixed-integer generators (condition (MIG)).
The second part of this chapter deals with sets G(b) where M(b) is given by quasi-convex polynomial inequalities (i.e. M(b) from (2)). In particular, we present conditions for the existence of (mixed-) integer points in sets M(b). Under the assumptions (CPC) and (MIG) we can expose subsets of the effective domain of the multifunction G on which this mapping shows continuity. A result due to TARASOV/KHACHIYAN (1980) giving an estimate of the norm of at least one point belonging to G(0) (where M(b) from (2) is described by convex polynomial functions with integer coefficients) completes this part of Chapter 5., which can be considered a first more comprehensive study of (mixed-) integer quasiconvex polynomial inequality systems.

The third section is related with the particular linear diophantine case intensively studied during the seventies, where we give a quasi-Lipschitzian analysis of the multifunction G and related compact-valued mappings. The chapter terminates with a characterization of the vertices of the convex hull of G(b) (where M(b) has the standard form of linear programming with integer data), which is similar to that for a feasible basic point in linear programming.

Chapter 6. contains the stability analysis of the parametric (mixed-) integer problems P(p,b) (as introduced above). We characterize the continuity behaviour of the value functions φ, the optimal set-functions ψ and the ε-optimal set-functions $\tilde{\psi}$. Most continuity results for both kinds of the fixed part f(x) in the objective function are derived for the general class of constraint sets given by (1). The more special case of constraint sets M(b) defined in (2) where the condition (CPC) is fulfilled has only to be considered in order to guarantee the upper semicontinuity of the ε-optimal set-function $\tilde{\psi}$ on certain subsets of its domain. Already the following simple example of a parametric linear integer problem shows that neither the value function is upper semi-continuous nor the optimal set-function is upper semi-continuous if the constraint set-mapping is not continuous.

$$P(b) \ \inf\{-x \ / \ 0 \leq x \leq b, \ x \text{ integer}\}, \ b \in R.$$

For $\{b_t\}$ with $b_t \underset{Df}{=} 1 - \frac{1}{t} \longrightarrow b_o = 1$ one has $\varphi(b_t) = 0$ and $\varphi(b_o) = -1$ and, further, $\psi(b_t) = \{0\}$ and $\psi(b_o) = \{1\}$.

The level set-mapping N corresponding to the problem P(p,b) defined by

$$N(p,b,a) = \{x \in R^n \ / \ x \in G(b), f(x)+p^Tx \leq a\}$$

plays an important role for the derivation of results as well as by its own rights in a characterization of the stability behaviour of optimization problems, which is in particular true for (mixed-) integer problems. HANSEL (1980) was the first who used the level sets in the analysis of parametric quadratic (mixed-) integer problems and our proof is a modified version of HANSEL's one corresponding to our case.

For the problems P(p,b) in Chapter 6. the question for the existence of optimal points is also answered. This subject of general interest is more intensively investigated in Chapter 7. Obviously, a (mixed-) integer problem

$$\inf\{f(x) \ / \ x \in M, \ x_1,\dots,x_s \text{ integer}\}, \tag{4}$$

where $M \subset R^n$ and $f : M \longrightarrow R$, has an optimal point if the trivial necessary conditions:

$$\text{(NC)} \quad \begin{aligned} & M_s = \{x \in M \ / \ x_1,\dots,x_s \text{ integer}\} \neq \emptyset \\ & \inf\{f(x) \ / \ x \in M_s\} > -\infty \end{aligned}$$

are satisfied. We point out for which classes of problems (4) the conditions (NC) are also sufficient for the existence of an optimal point (if only fairly weak additional assumptions are allowed). The results to this question for (mixed-) integer problems included here reflect that a level of knowledge which is comparable to that known in nonlinear optimization has been reached.

Chapter 8. Here we treat (mixed-) integer linear optimization problems