A. Liemant · K. Matthes · A. Wakolbinger

Equilibrium Distributions of Branching Processes

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Alfred Liemant
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Autoren:
Dr. sc. Alfred Liemant
Prof. Dr. rer.nat.habil. Klaus Matthes
Karl-Weierstraß-Institut für Mathematik
der Akademie der Wissenschaften der DDR, Berlin

Doz. Dr. Anton Wakolbinger
Institut für Mathematik der
Johannes-Kepler-Universität, Linz

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Preface

The present monograph deals mainly with one aspect of stochastic branching models, namely their equilibrium distributions. Any model of this kind has the trivial equilibrium distribution, which is concentrated on the "void population". Unless, however, the branching model corresponds to an "independent substochastic translation" or may be reduced in a certain sense to this case (cf. section 2.9.), *infinite* populations occur with positive probability in any non-trivial equilibrium situation.The results presented here are thus in the intersection of two theories: stochastic infinite particle systems and branching models.

From the point of view of stochastic infinite particle systems our model is elementary, because the independence hypotheses, which are characteristic of branching models, are rather restrictive. These restrictive hypotheses, however, admit a number of conclusions, which, at least partially, can serve as a "case study" for more complicated stochastic evolutions.

On the other hand, from the viewpoint of classical branching models, the introduction of infinite particle systems leads to significant complications. Nevertheless the subject treated here should be regarded as a natural one in the theory of branching models.

The theory of random point fields (and in particular that of infinitely divisible distributions of random point fields) is an important *tool* in the present investigations.

The whole monograph is restricted to models in discrete time, in order to avoid questions connected with the existence and the description of stochastic branching dynamics in continuous time (cf. Ikeda, Nagasawa, Watanabe (1968,1969), Nagasawa (1977), Asmussen, Hering (1983)).

Also, we do not touch upon the theory of measure valued branching processes. Parts of that theory are parallel to our setting (cf. Hermann (1981)), to which it is also connected through asymptotic considerations.

From the historic viewpoint, the present investigations have two sources. The first is the theory of spatially homogeneous branching models, which has been presented in detail in Matthes, Kerstan, Mecke (1978) (quoted in the sequel by [MKM]). The second is the general theory of spatially inhomogeneous substochastic translations, especially the papers of Kerstan et al. (starting with Kerstan, Debes (1969)), Shiga and Takahashi (1974) and Liggett (1978). The spatially homogeneous branching models with different "types" of individuals (cf. eg. Prehn, Röder (1977)) can be considered as an intermediate stage between the spatially homogeneous case and the general case. The present text contains no specific chapter on the spatially homogeneous theory. This is for the sake of conciseness and also because a number of important questions in the spatially homogeneous case still remain open. This case has yet to be studied from the point of view of the advanced general theory. A first impression of how the general theory developped here affects the spatially homogeneous case is provided by Fleischmann, Hermann, Matthes (1982).

The results presented in this text considerably improve upon the hitherto existing publications on the subject. As a systematic presentation, it clarifies the theory, eliminates superfluous assumptions of past treatments, and creates some useful concepts such as that of a "family equilibrium distribution." Terminology and notation are modified in order to facilitate reading. Examples play an important role in the whole presentation, some of which are unbroken threads running throughout the text. In the first chapter basic facts from the theory of random point fields are reviewed, for the most part without proof, but with reference to [MKM] and to Kallenberg (1983) (cited by [K]).

References and attributions of results are collected in an appendix "Comments and References", unless these results are "already standard" or are due to some of the present authors (whose relevant publications, however, are included in the list of references).

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Berlin and Linz, March 1987

The authors

Contents

1. AN INTRODUCTION TO RANDOM POINT FIELDS AND BRANCHING DYNAMICS

Let us consider a system of particles or rather a population of individuals which are indistinguishable up to their location, and assume that the population is subject to a stochastic branching dynamics in discrete time in the following way: each individual in the population produces, independently of the other individuals, a random offspring, whose distribution depends on the individual's location, and the population in the subsequent generation is the superposition of all these offsprings. This monograph deals with the time evolution of such systems, in particular investigating the structure of equilibrium distributions and establishing convergence-to-equilibrium results for certain initial distributions.

As a "phase space" (i.e. the space which the particles or individuals live in) we admit an arbitrary complete, separable metric space $[A, \rho_A]$. A population of individuals in a phase space $[A, \rho_A]$ might be described by the set S of the individuals' locations. Thus, as a mathematical model of populations all those subsets S of A could be considered which obey the finiteness condition $\#(S \cap X) < +\infty$ for all bounded subsets X of A. Such a modelling, however, does not allow to describe populations with more than one individual at a site. Even if such multiplicities do not play much role in the modelling of "real" populations, they do occur in certain mathematical idealizations, e. g. in stochastic branching models with countable phase space. It is therefore appropriate to associate, with an individual at the site $a \in A$, not the set $\{a\}$ but the corresponding Dirac measure δ_a. The whole population is then described by the sum of those δ_a's, i.e. by a measure Φ on the σ-algebra $\mathcal{A}$ of Borel subsets of the phase space. The multiplicity of individuals at some site $a \in A$ is now given by $\Phi(\{a\})$; this is how often the summand δ_a occurs in Φ.

Populations are thus modelled by measures Φ which take values in $\mathbf{Z}_+$ on bounded Borel subsets of the phase space; such measures are called counting measures. The superposition of populations is now described no more by the union of sets, but by the addition of the corresponding counting measures. The set of counting measures will be equipped with a natural topological and measurable structure.

The metric ρ_A comes into the above concepts only via the topology which it generates and the ρ_A-boundedness of subsets of A. If two complete metrics coincide in these respects, then they are equivalent for the whole theory.

According to the above statements, random counting measures - also called random point fields - may be understood as special random measures. Many concepts and results from the theory of random point fields may be extended in a natural way to random measures. This is more than a formal generalization: it turns out - also in the present monograph - that the study of random counting measures leads to random measures in its own right, e. g. via the Cox transformation.

A stochastic branching dynamics is constructed from a so called clustering field $(\kappa_{(a)})_{a\in A}$, where $\kappa_{(a)}$ is the distribution of the random offspring of one individual at site $a\in A$. The basic assumption for a stochastic branching dynamics is that the offsprings of the single individuals in the system are generated in a stochastically independent way according to the given clustering field. This independence assumption is the principal reason why favorite candidates for equilibrium distributions of stochastic branching dynamics are the so called infinitely divisible distributions. These are probability distributions of such random populations which may be represented as an independent superposition of an arbitrary number of identically distributed random populations. Via the canonical representation of infinitely divisible distributions, also infinite measures on the space of counting measures come into the theory.

Chapter 1 is an introduction into those parts of the theory of random point fields which are relevant for our purposes. Concerning the proofs, we usually refer to the comprehensive monographs [MKM] or [K]. There, one finds also detailed historical remarks.

In the subsequent chapters, basic concepts and standard results will be used sometimes without mentioning this introduction explicitely. It is also the purpose of the introduction to explain certain procedures in an exemplaric way, in particular the definition of new phase spaces (e. g. $[\mathbf{A}, \rho_{\mathbf{A}}]$ in 1.6., $[A^\wedge, \rho_{A^\wedge}]$ in 1.7.) in order to draw new consequences from already known facts.

The reader's attention should be drawn in particular to the presentation of Kallenberg's "backward technique" in 1.9., where special emphasis is put on the intuitive content of the formulae.

1.1. Random Measures

Let $[A, \rho_A]$ be a complete separable metric space, called **phase space**, $\mathcal{A}$ the corresponding σ-field of all Borel subsets of A, and $\mathcal{B}$ the ring of all bounded sets X in $\mathcal{A}$. Further, let N denote the set of all measures ν on $\mathcal{A}$ taking only finite values on $\mathcal{B}$. Any $\nu\in N$ may be viewed as a "mass allocation" on A. In the sequel, the set N will be furnished with order, topological and measurable structure.

With the natural semi-ordering $\leq$ of measures, N forms a conditionally complete distributive lattice with zero measure o on $\mathcal{A}$ as smallest element. The greatest lower bound resp. least upper bound of $\{\sigma, \rho\}$ in $[N, \leq]$ is denoted by $\sigma \wedge \rho$ resp. $\sigma \vee \rho$. Any decreasing sequence (ν_n) of measures in N has a greatest lower bound ν. In this case there holds $\nu_n(X) \downarrow \nu(X)$, $X \in \mathcal{B}$, and we write $\nu_n \downarrow \nu$. The least upper bound of any increasing sequence (γ_n) exists in $[N, \leq]$ iff $\sup\limits_{n=1,2..} \gamma_n(X) < +\infty$, $X \in \mathcal{B}$. In this case there holds $\gamma_n(X) \uparrow \gamma(X)$, $X \in \mathcal{A}$, and we write $\gamma_n \uparrow \gamma$.

Let F denote the set of all $\mathcal{A}$-measurable mappings from A into $[0,+\infty]$, and F_c be the subset of all bounded, ρ_A-continuous mappings $f \in F$ with bounded support supp f. For all $f \in F$ and all measures ν on $\mathcal{A}$ we put

$$\langle \nu, f \rangle := \int f(a)\, \nu(da) .$$

Let $\mathcal{N}$ be the σ-field on N generated by the mappings $\nu \rightarrow \nu(X)$, $X \in \mathcal{B}$, and let τ_N denote the **vague topology** on N, i. e. the topology generated by all mappings $\nu \rightarrow \langle \nu, f \rangle$, $f \in F_c$; $\xrightarrow{N}$ will stand for the corresponding convergence.

1.1.1. [MKM, Prop. 3.2.1., 3.2.3.] $[N, \tau_N]$ is a Polish space (i. e. τ_N is generated by a complete, separable metric). The corresponding σ-algebra of Borel subsets coincides with $\mathcal{N}$.

For all measures ν on $\mathcal{A}$, the ring of all $X \in \mathcal{B}$ with the property $\nu(\partial X) = 0$ is denoted by $\mathcal{B}_\nu$.

1.1.2. [MKM, Prop. 3.2.2.] $\nu_n \xrightarrow{N} \nu$ iff $\nu_n(X) \rightarrow \nu(X)$, $X \in \mathcal{B}_\nu$.

1.1.3. [MKM, Prop. 3.2.5.] A subset Y of N is τ_N-relatively compact iff it obeys, for all ρ_A- bounded and closed subsets X of A, the following conditions:

a) $\sup\limits_{\nu \in Y} \nu(X) < +\infty$.

b) For any $\varepsilon > 0$ there exists a ρ_A-compact subset B of X such that $\sup\limits_{\nu \in Y} \nu(X \setminus B) < \varepsilon$.

$\nu_n \uparrow \nu$ as well as $\nu_n \downarrow \nu$ imply the convergence $\nu_n \xrightarrow{N} \nu$. Moreover, $\nu_n \xrightarrow{N} \nu$, $\gamma_n \xrightarrow{N} \gamma$, $\nu_n \leq \gamma_n$, n=1,2,..., leads to $\nu \leq \gamma$.

In virtue of 1.1.3. this yields

1.1.4. For all $\sigma \in N$ the set $\{\nu: \nu \in N, \nu \leq \sigma\}$ is τ_N - compact.

By a **random measure with phase space** $[A, \rho_A]$ we mean a random element of the measurable space $[N, \mathcal{N}]$; this can be interpreted as a random mass allocation on the phase space. Any random measure induces a probability distribution on $\mathcal{N}$; the probability distributions and - more generally - the measures on $\mathcal{N}$ will be our primal object of interest for the rest of this section.

For all natural numbers k, $\nu \to \nu^{\otimes k}$ is a mapping from N into the set of measures on $\mathcal{A}^{\otimes k}$. As the mapping $\nu \to \nu^{\otimes k}(S)$ is $\mathcal{N}$-measurable for all $S \in \mathcal{A}^{\otimes k}$, we may form, for each measure H on $\mathcal{N}$, the measure

$$\Lambda^{(k)}_H(.) := \int \nu^{\otimes k}(.)\, H(d\nu)$$

on $\mathcal{A}^{\otimes k}$, which is called **k^{th} moment measure** of H. We say that H is **of k^{th} order** if $\Lambda^{(k)}_H(X^k)$ is finite for all $X \in \mathcal{B}$. For k=1, we write Λ_H instead of $\Lambda^{(1)}_H$ and speak of the **intensity measure** of H. For all $f \in F$ there holds

$$\langle \Lambda_H , f \rangle = \iint f(a)\, \nu(da)\, H(d\nu) \ .$$

If H is interpreted as the law of a random mass allocation, then $\Lambda_H(X)$ is the expectation of the random mass in X. For simplicity, we further put $\mathcal{B}_H := \mathcal{B}_{\Lambda_H}$.

Let **G** denote the linear space of all finite signed measures on $\mathcal{N}$. Together with the **convolution operation**

$$G_1 * G_2 := (G_1 \otimes G_2)((\nu_1+\nu_2) \in (.)) \qquad (G_1, G_2 \in \mathbf{G}) \ ,$$

G becomes a commutative real algebra with unit element δ_o. For all $G \in \mathbf{G}$ and $k \in \mathbf{Z}_+$, we denote the k-th convolution power of G by G^k. In particular, there holds $G^0 = \delta_o$.

For abbreviation, we put, for all $X \in \mathcal{A}$ and all measures ν on $\mathcal{A}$,

$${}_X\nu := \nu((.) \cap X) \ .$$

For all $X \in \mathcal{B}$, the mapping

$$G \to {}_XG := G({}_X\nu \in (.))$$

is a homomorphism of the algebra **G** into itself.

We will always write $\|q\|$ for the **total variation** of a finite signed measure q on an arbitrary measurable space.

With the norm $\| \ \|$, **G** is a complete normed algebra. Instead of $\|{}_XG\|$ we also write ${}_X\|G\|$ $(G \in \mathbf{G}, X \in \mathcal{A})$. $\mathbf{G}_+$ denotes the set of all finite measures on $\mathcal{N}$, $\tau_{\mathbf{G}_+}$ stands for the weak topology on $\mathbf{G}_+$, and $\Rightarrow$ for the corresponding convergence. $[\mathbf{G}_+, \tau_{\mathbf{G}_+}]$ is again a Polish space.

For all finite sequences $X_1,\dots,X_m$ of sets in $\mathcal{B}$ and all $G \in \mathbf{G}$ we put

$$G_{X_1,\dots,X_m} := G([\nu(X_1),\dots,\nu(X_m)] \in (.)) \ .$$

Obviously there holds

$$(G_1 * G_2)_{X_1,\dots,X_m} = (G_1)_{X_1,\dots,X_m} * (G_2)_{X_1,\dots,X_m} \quad (G_1, G_2 \in \mathbf{G}_+) \ ,$$

where $*$ on the right hand side means the usual convolution of finite measures on the σ-algebra $(\mathcal{B}_+)^{\otimes m}$ of Borel sets on $\mathbf{R}_+{}^m$.

1.1.5. [MKM, Prop. 7.2.4.] In $\mathbf{G}_+$ there holds $G_n \Rightarrow G$ iff for all finite sequences $X_1,\dots,X_m$ of sets in $\mathcal{B}_G$ the weak convergence

$$(G_n)_{X_1,\dots,X_m} \Rightarrow G_{X_1,\dots,X_m}$$

in the space of all finite measures on $(B_+)^{\otimes m}$ takes place.

Due to $\mathcal{B}_{G_1 * G_2} = \mathcal{B}_{G_1} \cap \mathcal{B}_{G_2}$ there follows

1.1.6. In $\mathbf{G}_+$, $G_n \Rightarrow G$ and $H_n \Rightarrow H$ imply that $G_n * H_n \Rightarrow G * H$.

Moreover, from 1.1.5. it is easy to conclude

1.1.7. $G_n \Rightarrow G$ and $X \in \mathcal{B}_G$ imply that $\Lambda_G(X) \leq \liminf_{n\to\infty} \Lambda_{G_n}(X)$.

Another consequence of 1.1.5. is

1.1.8. For all measures G, G_1, G_2,... of first order in $\mathbf{G}_+$ such that $G_n \Rightarrow G$, the vague convergence $\Lambda_{G_n} \xrightarrow{N} \Lambda_G$ takes place iff for all $X \in \mathcal{B}$ the condition

$$\sup_{n=1,2,\dots} \int_{\{v: v(X)>c\}} v(X)\, G_n(dv) \xrightarrow[c\to\infty]{} 0$$

is fulfilled.

Let V denote the set of all probability distributions on $\mathcal{N}$ and V be the smallest σ-algebra of subsets of V with respect to which all mappings $Q \rightarrow Q(Y)$, $Y \in \mathcal{N}$, are measurable. V is a weakly closed subset of $\mathbf{G}_+$, and the σ-algebra V coincides with the σ-algebra of Borel subsets of V with respect to the weak topology τ_V, i.e. t he restriction of τ_{G_+} to V. Obviously, $v \rightarrow \delta_v$ is a homeomorphic imbedding of $[N, \tau_N]$ into $[V, \tau_V]$. The following simple estimate will sometimes be useful:

1.1.9. For all Q_1, $Q_2 \in$ V and all $X \in \mathcal{A}$ there holds

$$\begin{aligned} {}_X\|Q_1 - Q_2\| &\leq 2\,(Q_1(v(X)>0) + Q_2(v(X)>0)) \\ &\leq 2\,(\Lambda_{Q_1}(X) + \Lambda_{Q_2}(X)) . \end{aligned}$$

Together with Q, obviously all of the $Q_{X_1,\dots,X_m}$ are probability distributions. In this case, these are called **finite dimensional distributions** of Q. On the other hand, with any probability distribution p on $(B_+)^{\otimes m}$ we may associate via

$$Q := p\left(\sum_{1\leq i\leq m} x_i \delta_i \in (.)\right)$$

the distribution of a random measure with phase space $\{1,\dots,m\}$, for which then holds

$$p = Q_{\{1\},\dots,\{m\}} .$$

Hence one may identify random elements of $\mathbf{R}_+{}^m$ in a natural way with random measures on the phase space $\{1,\dots,m\}$.

In **V**, a semi-order $\lesssim$ will be introduced as follows: we say that $Q_1 \lesssim Q_2$ if there exists a distribution H on $\mathcal{N}^{\otimes 2}$ with the properties

$$H(\nu_i \in (.)) = Q_i \text{ for } i = 1,2 \quad ; \; H(\nu_1 \leq \nu_2) = 1 \quad .$$

This is equivalent to the existence of an $\mathcal{N}, \boldsymbol{V}$ - measurable mapping $\nu \to L(\nu)$ from N into **V** with the property $Q_1 = \int L(\nu)\,(.)\, Q_2\,(d\nu)$; $(L(\nu))\,(\gamma \leq \nu) = 1$ for all $\nu \in N$.

1.1.10. $V_n \Rightarrow V$, $Q_n \Rightarrow Q$ and $V_n \lesssim Q_n$ for n = 1,2,... imply that $V \lesssim Q$.

If A coincides with {1,...,m} and **V** is identified in the above mentioned sense with the set of all distributions on $(\boldsymbol{B}_+)^{\otimes m}$, then $\lesssim$ is just the well-known relation "stochastically smaller" (cf. [Stoyan (1983)]), which we continue to denote by $\lesssim$:

$p_1 \lesssim p_2$ iff there exists a distribution q on $(\boldsymbol{B}_+)^{\otimes 2m}$ such that

$$q([x_1,...,x_m] \in (.)) = p_1 \;;\; q([x_{m+1},...,x_{2m}] \in (.)) = p_2 \;;\; q(x_i \leq x_{i+m} \text{ for } 1 \leq i \leq m) = 1.$$

By means of 1.1.10. there results

1.1.11. In **V** there holds $V \lesssim Q$ iff for all finite sequences $X_1,...,X_m$ of sets in $\mathcal{B}$ the inequality

$$V_{X_1,...,X_m} \lesssim Q_{X_1,...,X_m}$$

holds true.

Now let $(Q_i)_{i\in I}$ be some at most countable family of distributions on $\mathcal{N}$. For $I=\emptyset$ we define $\underset{i\in I}{\ast} Q_i$ as the unit element δ_o of **G**. Otherwise we form $H := \bigotimes_{i\in I} Q_i$ and put

$$\underset{i\in I}{\ast} Q_i \; := H\,((\sum_{i\in I} \nu_i) \in (.)) \;,$$

provided that $H\,((\sum_{i\in I}\nu_i) \in N) = 1$. (With this definition, $\underset{i\in I}{\ast} Q_i$ is the distribution of the sum of independent random measures with distribution Q_i, $i\in I$.) The intensity measure of $Q = \underset{i\in I}{\ast} Q_i$ is $\Lambda_Q = \sum_{i\in I} \Lambda_{Q_i}$. On the other hand, the convolution $\underset{i\in I}{\ast} Q_i$ certainly exists if $\sum_{i\in I} \Lambda_{Q_i}$ belongs to N.

If V is a distribution on $\mathcal{N}$ and k is a natural number, then there exists at most one convolution root of V in **V**. If there exists a k-th convolution root for any $k\in \mathbf{Z}_+$, then V is called **infinitely divisible.** Let **I** denote the set of all infinitely divisible V in **V**.

With any measure G in $\mathbf{G}_+$, we associate in form of the "Poissonian mixture"

$$\mathcal{E}_G := e^{-G(N)} \sum_{k>0} (k!)^{-1}\, G^k$$

a distribution in **I**, satisfying the relations

$$\mathcal{E}_{G_1} * \mathcal{E}_{G_2} = \mathcal{E}_{G_1+G_2} \quad \text{for all } G_1, G_2 \in \mathbf{G}_+ \;;\; \Lambda_{\mathcal{E}_G} = \Lambda_G \quad \text{for all } G \in \mathbf{G}_+ \;.$$

In view of $\mathcal{E}_G = \mathcal{E}_{G((.)\setminus\{o\})}$ we may restrict ourselves here to measures G in $\mathbf{G}_+$ which have no mass in o.

Let $\mathbf{U}$ denote the set of all possibly infinite measures on $\mathcal{N}$ satisfying the conditions

$$U(\{o\}) = 0 \quad ; \quad \int (1-e^{-\nu(X)})\, U(d\nu) < \infty \ \text{ for all } X \in \mathcal{B} \quad .$$

It can be shown that for an arbitrary at most countable family $(G_i)_{i\in I}$ of measures in $\mathbf{G}_+$ with the property $G_i(\{o\}) = 0$, $i\in I$, the convolution $V := \mathop{*}_{i\in I} \mathcal{E}_{G_i}$ exists iff $U := \sum_{i\in I} G_i$ belongs to $\mathbf{U}$. In this case, V depends only on U and not on the special representation of U. Taking into consideration that any $U\in\mathbf{U}$ has such a representation $U = \sum_{i\in I} G_i$, we may define, for all $U\in\mathbf{U}$, the distribution $\mathcal{E}_U$ by $\mathcal{E}_U := V$. Again there holds

1.1.12. $\mathcal{E}_{U_1} * \mathcal{E}_{U_2} = \mathcal{E}_{U_1+U_2} \quad (U_1, U_2 \in \mathbf{U}) \quad ; \quad \Lambda_{\mathcal{E}_U} = \Lambda_U \quad (U\in\mathbf{U})$.

Hence all distributions $\mathcal{E}_U$, $U\in\mathbf{U}$, are infinitely divisible.

In view of $\delta_\nu = (\delta_{k^{-1}\nu})^k$ for $\nu\in N$, $k = 1,2,\ldots$, also δ_ν, $\nu\in N$, is an element of **I**.

From the two above mentioned special types of infinitely divisible distributions on $\mathcal{N}$, all distributions in **I** may be obtained in the following way:

1.1.13. [K, Prop.6.1.] The mapping $[\nu,U] \to \delta_\nu * \mathcal{E}_U$ establishes a 1-1 correspondence between $N\times\mathbf{U}$ and **I**, the (unique) factorization

$$V = \delta_\nu * \mathcal{E}_U \qquad (\nu\in N,\ U\in\mathbf{U})$$

is called **canonical representation** of V, and the measure U figuring there is called **canonical measure** of V, written as $U=\widetilde{V}$. Obviously there holds $\Lambda_V = \nu + \Lambda_{\widetilde{V}}$.

1.2. Random Point Fields

A measure Φ in N is called **counting measure** if it takes only nonnegative integer values on $\mathcal{B}$. The set of all counting measures in N will be denoted by M.

1.2.1. [MKM, Prop.1.1.2.] Any $\Phi\in M$ is of the form $\Phi = \sum_{a\in \operatorname{supp}\Phi} \Phi(\{a\})\delta_a$.

Hence any counting measure Φ may be interpreted as an (at most countable) system of identical particles or individuals in the phase space $[A, \rho_A]$. For all $X\in\mathcal{A}$, $\Phi(X)$ "counts" the number of particles in the system which are situated in X. Specifically, $\Phi(\{a\})$ is the number of particles at the site $a\in A$. If, for all $a\in A$, $\Phi(\{a\})$ is either zero or one, then Φ is called **simple**. Obviously, the mapping $\Phi \to \operatorname{supp}\Phi$ establishes a 1-1 correspondence between the set of all simple counting measures and the system of all subsets X of A with the property $\#(X\cap S) < +\infty$, $S\in\mathcal{B}$.

Being a subset of N, the set M inherits the order, topological and measurable structure of N defined in 1.1. Let $\mathcal{M}$ be the σ-algebra on M generated by the mappings $\Phi \rightarrow \Phi(X)$, $X \in \mathcal{B}$, and let τ_M denote the **vague topology** on M, i. e. the topology generated by all the mappings $\Phi \rightarrow \langle \Phi, f \rangle$, $f \in F_c$; $\xrightarrow{M}$ will stand for the corresponding convergence. Obviously, τ_M is the restriction of τ_N to M, and $\mathcal{M}$ is the restriction of $\mathcal{N}$ to M.

1.2.2. [MKM, Prop. 3.2.4.] The set M is τ_N-closed.

This together with 1.1.1. shows:

1.2.3. $[M,\tau_M]$ is a Polish space. The corresponding σ-algebra of Borel subsets coincides with $\mathcal{M}$.

The mapping $a \rightarrow \delta_a$ provides a homeomorphic imbedding of the phase space into $[M,\tau_M]$. One may conclude easily from 1.1.2. that vague convergence $\nu_n \xrightarrow{N} \nu$ in $[N,\tau_N]$ can be characterized as follows : Let (X_m) be some ascending sequence in $\mathcal{B}_V$ such that any $X \in \mathcal{B}$ is covered by some X_m. Then $\nu_n \xrightarrow{N} \nu$ is equivalent to the weak convergence of $(\chi_{X_m} \nu_n)$ towards $(\chi_{X_m} \nu)$ in the space of finite measures on $\mathcal{A}$ for all $m \in \mathbf{Z}_+$.

On the other hand , weak convergence of finite counting measures can be characterized in an intuitive way:

1.2.4. [Kerstan, Matthes, Mecke(1982), Prop. 1.9.10.] In the set of finite counting measures on $\mathcal{A}$, weak convergence $\Phi_n \Rightarrow \Phi$ takes place iff, for all suitably large natural numbers n, $\Phi_n(A) = \Phi(A)$, and there exist representations $\Phi_n = \sum_{1 \le i \le k} \delta_{a_{n,i}}$ such that

$a_{n,i} \xrightarrow[n \to \infty]{} a_i$, $1 \le i \le k$, and $\Phi = \sum_{1 \le i \le k} \delta_{a_i}$.

By a **random point field** (or random counting measure) **with phase space** $[A,\rho_A]$ we mean a random element of the measurable space $[M, \mathcal{M}]$. Because of $M \in \mathcal{N}$ and $\mathcal{M} = M \cap \mathcal{N}$ we may interpret the random point fields as random measures whose realizations are almost surely counting measures. Any random point field induces a distribution on $\mathcal{M}$; for the rest of this section we will deal with distributions and, more generally, with measures on $\mathcal{M}$.

Let **E** denote the linear space of all finite signed measures on $\mathcal{M}$. Together with the **convolution operation**

$$E_1 * E_2 := (E_1 \otimes E_2)((\Phi_1 + \Phi_2) \in (.)) \qquad (E_1, E_2 \in \mathbf{E})$$

and the variation norm $\| \; \|$, **E** becomes a commutative, complete normed real algebra, which may be identified with the subalgebra of **G** consisting of all G which are concentrated on M. In the same way, the set $\mathbf{E}_+$ of all finite measures on $\mathcal{M}$ will be viewed as a subset of $\mathbf{G}_+$.

Let τ_{E_+} denote the weak topology on $\mathbf{E}_+$ and $\Rightarrow$ the corresponding convergence. The set $\mathbf{E}_+$ is τ_{G_+}-closed, and τ_{E_+} coincides with the restriction of τ_{G_+} to $\mathbf{E}_+$.

1.2.5. [MKM, Prop. 3.2.7.] A subset **L** of $\mathbf{E}_+$ is τ_{E_+}-relatively compact iff it has the following two properties:

a) $\sup_{L\in \mathbf{L}} L(M)<+\infty$,

b) for each closed set X in $\mathcal{B}$ and each $\eta>0$ there exists a natural number $n_{X,\eta}$ as well as a compact subset $B_{X,\eta}$ of X such that

$$\sup_{L\in \mathbf{L}} L(\Phi(X) \geq n_{X,\eta}) < \eta \ ; \ \sup_{L\in \mathbf{L}} L(\Phi(X\setminus B_{X,\eta})>0) < \eta \ .$$

Let **P** denote the set of all distributions on $\mathcal{M}$ and $\boldsymbol{P}$ denote the σ-algebra generated by the mappings $P \to P(Y)$, $Y\in \mathcal{M}$. In the above mentioned sense **P** is viewed as a subset of **V**, which leads to the equality $\boldsymbol{P} = \mathbf{P}\cap \boldsymbol{V}$.

$\tau_{\mathbf{P}}$ denotes the weak topology on **P**, i. e. the restriction of τ_{E_+} to the τ_{E_+}-closed set **P**. The σ-algebra of Borel subsets of the Polish space $[\mathbf{P}, \tau_{\mathbf{P}}]$ coincides with $\boldsymbol{P}$.

1.2.6. [MKM, Prop. 3.2.8.] If **L** is a subset of **P** such that $\{\Lambda_L\}_{L\in \mathbf{L}}$ is a τ_N-relatively compact subset of N, then **L** itself is $\tau_{\mathbf{P}}$-relatively compact.

In view of 1.1.4. this leads to

1.2.7. For all $\sigma\in N$, $\{P:P\in \mathbf{P}, \Lambda_P\leq\sigma\}$ is $\tau_{\mathbf{P}}$-compact.

Note that, for $P\in \mathbf{P}$, the **finite dimensional distributions** $P_{X_1,\dots,X_m}$ may be viewed as distributions on $\mathcal{P}(\mathbf{Z}_+{}^m)$; on the other hand the distributions on $\mathcal{P}(\mathbf{Z}_+{}^m)$ can be identified with the distributions of random point fields with phase space $\{1,\dots,m\}$.

If a distribution P on $\mathcal{M}$ has, for a natural number k, a k-th convolution root L in **P**, then we write $L=:\sqrt[k]{P}$. It may happen that a k-th convolution root of P exists in **V** but not in **P**. For any counting measure Φ, δ_Φ is an element of **I**, but $\sqrt[k]{\delta_\Phi}$ exists iff for all points $a \in \mathrm{supp}\Phi$ the number $\Phi(\{a\})$ is an integer multiple of k. **T** will denote the set of all **distributions** on $\mathcal{M}$ which are **infinitely divisible** in **P**, i. e. the set of all $P\in \mathbf{P}$ for which all convolution roots $\sqrt[k]{P}$, k=1,2,..., exist. The set **T** is contained in **I**, but, as mentioned above, is different from $\mathbf{I}\cap\mathbf{P}$.

1.2.8. [MKM, Prop. 3.2.11.] The set **T** is $\tau_{\mathbf{P}}$-closed.

Let **W** denote the set of all possibly infinite measures W on $\mathcal{M}$ obeying the two conditions

$$W(\{o\}) = 0 \quad ; \quad W(\psi(X)>0) < +\infty \text{ for all } X\in\mathcal{B} \quad .$$

As a counterpart to 1.1.13. there holds:

1.2.9. [MKM, Prop. 2.1.10.] The mapping $W \to \mathcal{E}_W$ provides a 1-1 correspondence between the sets **W** and **T**.

Obviously **W** may be identified with the set of those $U\in\mathbf{U}$ which are concentrated on M. Instead of $P=\mathcal{E}_W$ we also write $W=:\widetilde{P}$, calling $\widetilde{P}$ the **canonical measure** of P.

For all $P\in\mathbf{T}$ and $r\in\mathbf{R}_+$, one puts $P^r := \mathcal{E}_{r\widetilde{P}}$, thus obtaining a one-parameter convolution semigroup $(P^r)_{r\in\mathbf{R}_+}$ (cf. section 2.4. in [MKM]).

Because of $\Lambda_{\widetilde{P}} = \Lambda_P$, a distribution P in **T** is of first order iff its canonical measure is of first order. Further note that a measure H on $\mathcal{M}$ of first order is contained in **W** iff it has no mass in o.

1.2.10. [MKM, Prop. 2.2.4.] For all $P\in\mathbf{T}$ and all $X\in\mathcal{A}$, also ${}_XP$ is an element of **T** and there holds

$$\widetilde{({}_XP)} = \widetilde{P}({}_X\psi\in(.), \psi(X)>0) \quad .$$

In virtue of 1.2.10. and the definition of $W \to \mathcal{E}_W$, for any given $P\in\mathbf{T}$ and $X\in\mathcal{A}$ such that $P(\Phi(X)=0) = \exp(-\widetilde{P}(\psi(X)>0)) > 0$, the distribution ${}_XP$ may be represented as Poissonian mixture

$${}_XP = e^{-Q(M)} \sum_{k\geq 0} (k!)^{-1}Q^k \ , \quad Q := \widetilde{P}({}_X\Psi\in(.), \Psi(X)>0)$$

1.2.11. [MKM, Prop. 2.2.6.] For all distributions $P\in\mathbf{T}$ of first order there holds

$$\Lambda^{(2)}{}_P = \Lambda^{(2)}{}_{\widetilde{P}} + \Lambda_P\otimes\Lambda_P \ .$$

With any measure $\nu\in N$ we associate via

$$\mathcal{Q}_\nu := \nu(\delta_a\in(.))$$

a measure $\mathcal{Q}_\nu \in \mathbf{W}$ with intensity measure ν, and call

$$\Pi_\nu := \mathcal{E}_{\mathcal{Q}_\nu}$$

the **Poisson distribution with intensity measure** ν.

1.2.12. [MKM, Prop. 2.2.14.] For all $\nu\in N$, a distribution P on $\mathcal{M}$ coincides with Π_ν iff for any finite sequence $X_1,\dots,X_m$ of pairwise disjoint sets in $\mathcal{B}$ there holds

$$P_{X_1,\dots,X_2} = \pi_{\nu(X_1)} \otimes\dots\otimes \pi_{\nu(X_m)} \ .$$

1.2.13. [MKM, Prop. 3.3.7.] The mapping $\nu \to \Pi_\nu$ provides a homeomorphic imbedding from $[N, \tau_N]$ into $[\mathbf{P}, \tau_\mathbf{P}]$.

With any $G\in\mathbf{G}$, we associate via

$$\mathfrak{I}(G) := \int \Pi_v(.)\,G(dv)$$

a signed measure $\mathfrak{I}(G)$ on $\mathcal{M}$, the so called **Cox transform** of G.

1.2.14. [MKM, Prop. 7.1.2.] The Cox transform $\mathfrak{I}$ is an isomorphism from the algebra **G** onto a subalgebra of **E**.

Obviously, $\mathfrak{I}(G)$ is in $\mathbf{E}_+$ resp. in **P**, if G is in $\mathbf{G}_+$ resp. in **V**. In this case there holds $\Lambda_{\mathfrak{I}(G)} = \Lambda_G$. Distributions of the form $\mathfrak{I}(G)$, $G\in\mathbf{V}$, are called **Cox distributions.**

1.2.15. [MKM, Prop. 7.2.1., 7.2.2.] The Cox transform $\mathfrak{I}$ provides a homeomorphism from $[\mathbf{G}_+, \tau_{\mathbf{G}_+}]$ into a closed subset of $[\mathbf{E}_+, \tau_{\mathbf{E}_+}]$.

Any counting measure Φ may be written in the form $\Phi = \sum_{i\in I} \delta_{a_i}$.
For all natural numbers k we put

$$\Phi^{(k)} := \sum_{i_1,\ldots,i_k\in I \text{ pairwise distinct}} \delta_{[a_{i_1},\ldots,a_{i_k}]} \;.$$

Obviously there holds $\Phi^{(k)} \le \Phi^{\otimes k}$.

1.2.16. For all simple $\Phi\in M$ we have

$$\Phi^{(k)} = \Phi^{\otimes k}\,((.)\cap\{[x_1,\ldots,x_k]\in A^k : x_i \neq x_j \text{ for } 1\le i<j\le k\}) \;.$$

Let $\rho_{\mathbf{Z}_+}$ be the metric of $\mathbf{Z}_+$ defined by $\rho_{\mathbf{Z}_+}(i,j) = 0$ if $i=j$ and 1 if $i\neq j$, and denote, for the moment, the direct product of $[A, \rho_A]$ and $[\mathbf{Z}_+, \rho_{\mathbf{Z}_+}]$ by $[A', \rho_{A'}]$. (All objects belonging to the new phase space A' will be distinguished by a prime.)

1.2.17. The mapping

$$\Phi \to \Phi' := \sum_{a\in\operatorname{supp}\Phi}\; \sum_{0\le j<\Phi(\{a\})} \delta_{[a,j]}$$

is a bijective and in both directions continuous mapping from M onto the set of simple counting measures in M'.

Noting that $\Phi^{(k)}$ may be reconstructed from $(\Phi')^{(k)}$ in an easy way, we conclude from 1.2.16. and 1.2.17. that the mapping $\Phi \to \Phi^{(k)}(S)$ is $\mathcal{M}$-measurable for all $S\in\mathcal{A}^{\otimes k}$. Hence we may form, for any measure H on $\mathcal{M}$ and all natural numbers k, by

$$\Gamma^{(k)}{}_H := \int \Phi^{(k)}(.)\,H(d\Phi)$$

the k-th **factorial moment measure** $\Gamma^{(k)}{}_H$ of H. Obviously there holds

1.2.18. $\quad \Lambda_H = \Gamma^{(1)}{}_H \quad , \quad \Lambda^{(2)}{}_H = \Gamma^{(2)}{}_H + \int \delta_{[a,a]}(.)\,\Lambda_H(da) \;.$

Together with 1.2.11. this implies

1.2.19. For all distributions P of fist order in **T** there holds

$$\Gamma^{(2)}{}_P = \Gamma^{(2)}{}_{\tilde{P}} + \Lambda_P \otimes \Lambda_P \;.$$

1.3. The Laplace Transformation

For all $G \in \mathbf{G}_+$ we define via

$$\mathcal{L}_G(f) := \int \exp(-\langle \nu, f\rangle) G(d\nu) \qquad (f \in F)$$

its **Laplace transform** $\mathcal{L}_G$ (where $\exp(-\infty) := 0$). It is immediate from the definition that

1.3.1. $\mathcal{L}_{c_1 G_1 + c_2 G_2}(f) = c_1 \mathcal{L}_{G_1}(f) + c_2 \mathcal{L}_{G_2}(f) \qquad (f \in F\,;\, c_1, c_2 \in \mathbf{R}_+\,;\, G_1, G_2 \in \mathbf{G}_+)$.

1.3.2. $\mathcal{L}_{G_1 * G_2}(f) = \mathcal{L}_{G_1}(f) \cdot \mathcal{L}_{G_2}(f) \qquad (f \in F\,;\, G_1, G_2 \in \mathbf{G}_+)$.

The mapping $G \to \mathcal{L}_G$ is 1-1; there even holds

1.3.3. For all $G_1, G_2 \in \mathbf{G}_+$,

$$\mathcal{L}_{G_1}(f) = \mathcal{L}_{G_2}(f) \qquad (f \in F_c)$$

implies that $G_1 = G_2$.

By the Laplace transformation, weak convergence in $\mathbf{G}_+$ is carried into pointwise convergence for all $f \in F_c$. Sharpening 1.3.3., there holds

1.3.4. [K, Prop. 4.2.] For all $G, G_1, G_2, \ldots$ in $\mathbf{G}_+$, weak convergence $G_n \Rightarrow G$ is equivalent to the validity of

$$\mathcal{L}_{G_n}(f) \to \mathcal{L}_G(f) \qquad (f \in F_c) \ .$$

For all measures ν on $\mathcal{A}$, let F_ν denote the set of all $f \in F$ with the property $\langle \nu, f\rangle < +\infty$. For all $G \in \mathbf{G}_+$ and all $f, g \in F_{\Lambda_G}$ there holds

$$\left| \int \exp(-\langle \nu, f\rangle)\, G(d\nu) - \int \exp(-\langle \nu, g\rangle) G(d\nu) \right|$$
$$\leq \int \left| \langle \nu, f\rangle - \langle \nu, g\rangle \right| G(d\nu) \quad \leq \quad \int \langle \nu, |f-g|\rangle \cdot G(d\nu) \ ,$$

which leads to the elementary inequality

1.3.5. For all $G \in \mathbf{G}_+$ and all $f, g \in F_{\Lambda_G}$ there holds

$$\left| \mathcal{L}_G(f) - \mathcal{L}_G(g) \right| \leq \langle \Lambda_G, |f-g| \rangle \ .$$

For all $G \in \mathbf{G}_+$ and all $X_1, \ldots, X_m \in \mathcal{B}$ we obtain for all $s_1, \ldots, s_m \in \mathbf{R}_+$:

$$\mathcal{L}_{G_{X_1,\ldots,X_m}}(s_1, \ldots, s_m)$$
$$= \int (\exp(-\sum_{1 \leq i \leq m} s_i x_i) G_{X_1,\ldots,X_m}(d[x_1, \ldots, x_m]) = \mathcal{L}_G(\sum_{1 \leq i \leq m} s_i 1_{X_i}) \ .$$

Hence follows

1.3.6. For all $G, G_1, G_2, \ldots \in \mathbf{G}_+$ and all finite sequences $X_1, \ldots, X_m$ of sets in $\mathcal{B}$,

$$(G_n)_{X_1,\ldots,X_m} \Rightarrow G_{X_1,\ldots,X_m}$$

is equivalent to

$$\mathcal{L}_{G_n}(\sum_{1 \leq i \leq m} s_i 1_{X_i}) \to \mathcal{L}_{G_n}(\sum_{1 \leq i \leq m} s_i 1_{X_i}) \qquad (s_1, \ldots, s_m \in \mathbf{R}_+) \ .$$

Assume now that in G_+ the convergence relation $G_n \Rightarrow G$ holds true, and that, for some $\nu \in N$, the inequalities $\Lambda_{G_n} \le \nu$, $n = 1, 2, \ldots$, are fulfilled. Subject to these assumptions, also $\Lambda_G \le \nu$ holds true. Furthermore, for any $f \in F_\nu$ and any $\varepsilon > 0$ there exists some $g \in F_c$ such that $\langle \nu, |f-g| \rangle < \varepsilon$, and we obtain by 1.3.4. and 1.3.5.

$$\limsup_{n\to\infty} | \mathcal{L}_{G_n}(f) - \mathcal{L}_G(f) |$$

$$\le \limsup_{n\to\infty} | \mathcal{L}_{G_n}(f) - \mathcal{L}_{G_n}(g) | + \lim_{n\to\infty} | \mathcal{L}_{G_n}(g) - \mathcal{L}_G(g) | + | \mathcal{L}_G(f) - \mathcal{L}_G(g) |$$

$$\le \limsup_{n\to\infty} \langle \Lambda_{G_n}, |f-g| \rangle + 0 + \langle \Lambda_G, |f-g| \rangle \le 2\langle \nu, |f-g| \rangle | \le 2\varepsilon .$$

Hence we recognise the validity of

1.3.7. If in G_+ the weak convergence $G_n \Rightarrow G$ takes place, and if for some $\nu \in N$ the inequalities $\Lambda_{G_n} \le \nu$, $n=1,2,\ldots$, are fulfilled, then there holds

$$\mathcal{L}_{G_n}(f) \to \mathcal{L}_G(f) \quad (f \in F_\nu) .$$

In view of 1.3.6. one further gets:

1.3.8. Subject to the assumptions of 1.3.7. there holds $(G_n)_{X_1,\ldots,X_m} \Rightarrow G_{X_1,\ldots,X_m}$ for any finite sequence $X_1, \ldots, X_m$ of sets in $\mathcal{B}$.

The canonical representation of infinitely divisible distributions on $\mathcal{N}$ now takes the following form:

1.3.9. [K, Prop. 6.1.] For all $\nu \in N$ and all $U \in \mathbf{U}$ there holds

$$\mathcal{L}_{\delta_\nu * \mathcal{E}_U}(f) = \exp(-\langle \nu, f \rangle - \int (1 - \exp(-\langle \gamma, f \rangle)) U(d\gamma)) .$$

Hence results

1.3.10. For all $P \in \mathbf{T}$ one has

$$\mathcal{L}_P(f) = \exp(-\int (1 - \exp(-\langle \psi, f \rangle)) \tilde{P}(d\psi)) \qquad (f \in F) .$$

Particulary there hold

1.3.11. $\mathcal{L}_{\Pi_\nu}(f) = \exp(-\int (1 - \exp(-f(a))) \nu(da)) \qquad (\nu \in N;\ f \in F)$.

1.3.12. $\mathcal{L}_{\mathfrak{Z}(Q)}(f) = \mathcal{L}_Q(1 - \exp(-f)) \qquad (Q \in \mathbf{V};\ f \in F)$.

In $\mathbf{V}$, 1.3.2. generalizes to infinite convolution products:

1.3.13. In $\mathbf{V}$, the relation $Q = \mathop{\ast}_{i \in I} Q_i$ implies

$$\mathcal{L}_Q(f) = \prod_{i \in I} \mathcal{L}_{Q_i}(f) \quad (f \in F) .$$

For some computations, the set F will be too large and will be replaced by the set F_b of all $\mathcal{A}$-measurable bounded nonnegative real-valued functions on A with bounded support. Obviously there holds

$$0 < \mathcal{L}_Q(f) \le 1 \qquad (Q \in \mathbf{V};\ f \in F_b) .$$

1.4. Clustering Fields and Branching Dynamics

Let $[A', \rho_{A'}]$ be some other complete, separable metric space. All objects which refer to the phase space $[A', \rho_{A'}]$ will be indicated by a prime (e.g. $\mathcal{A}'$, $\mathcal{B}'$,...)

A **clustering field on** $[A, \rho_A]$ **with phase space** $[A', \rho_{A'}]$ is an $\mathcal{A}, \boldsymbol{P}'$-measurable mapping $\kappa_{(.)}$ from A into $\mathbf{P}'$: with each site in A one associates a distribution of a random element (a "cluster") χ_a in $[M', \mathcal{M}']$. χ_a may be interpreted as random offspring of an individual at site a. A clustering field κ will be called **stochastic** resp. **substochastic** if $\kappa_{(a)}(\chi(A)=1) = 1$ resp. $\kappa_{(a)}(\chi(A)\leq 1) = 1$ for all $a\in A$.

Having in mind the interpretation of $\kappa_{(a)}$ as an "individual's random offspring distribution", we put for $\Phi = \sum_{i\in I} \delta_{a_i}$

$$\kappa_{(\Phi)} := \mathop{*}_{i\in I} \kappa_{(a_i)} ,$$

if the convolution on the right hand side exists. Of course one may avoid a particular indexing of points in Φ, putting

$$\kappa_{(\Phi)} := \mathop{*}_{a\in \operatorname{supp}\Phi} (\kappa_{(a)})^{\Phi(\{a\})} .$$

Our definition of the "population's random offspring distribution" $\kappa_{(\Phi)}$ is based on an **assumption** which is typical for stochastic branching models , namely , that the random offspring χ of the population Φ is an independent superposition $\sum_{i\in I}\chi_{a_i}$ of the random offsprings χ_{a_i} of the individuals in Φ, the family $(\chi_{a_i})_{i\in I}$ being distributed according to $\bigotimes_{i\in I}\kappa_{(a_i)}$, i. e. the offsprings of the different individuals in Φ are generated in a stochastically independent way according to the given clustering field κ. If κ is stochastic rep. substochastic, then the random offspring χ arises from Φ via a so called **stochastic** resp. **substochastic translation**.

If, for some $\Phi\in M$, $\kappa_{(\Phi)}$ does not exist, then this means that the superposition $\sum_{i\in I} \chi_{a_i}$ violates, with positive probability, the finiteness condition which is characteristic for measures in N, i. e. in this case there exists an $X\in \mathcal{B}$ such that

$$(\bigotimes_{i\in I}\kappa_{(a_i)})(\sum_{i\in I}\chi_{a_i}(X) =+\infty) > 0 .$$

Let ${}_{\kappa}M$ denote the set of all $\Phi\in M$ for which $\kappa_{(\Phi)}$ is defined (which are, so speak, "sufficiently thin").

1.4.1. [MKM, Prop. 4.1.2.] Let κ be a clustering field on $[A, \rho_A]$ with phase space $[A',\rho_{A'}]$.Then $\Phi\in M$ belongs to ${}_{\kappa}M$ iff $\langle\Phi, \kappa_{(.)}(\chi(X)>0)\rangle < +\infty$ for all $X\in \mathcal{B}$.

Obviously, ${}_{\kappa}M$ is a descending set relative to the natural semi-order $\leq$. 1.4.1. shows that ${}_{\kappa}M$ belongs to $\mathcal{M}$; there even holds

1.4.2. [MKM, Prop. 4.1.3.] Subject to the assumptions of 1.4.1., $\Phi \to \kappa_{(\Phi)}$ is an $\mathcal{M}, \boldsymbol{P}'$-measurable mapping from ${}_{\kappa}M$ into $\mathbf{P}'$.

The mapping $\Phi \to \kappa_{(\Phi)}$ is called **stochastic branching dynamics** on $[M, \mathcal{M}]$ induced by the clustering field κ. If H is some measure on $\mathcal{M}$ with the property $H(\Phi \notin {}_{\kappa}M) = 0$, then we form via

$$H_{\kappa} := \int \kappa_{(\Phi)}(.)\, H(d\Phi)$$

the "clustered" measure H_{κ} on $\mathcal{M}'$. If a distribution $P \in \mathbf{P}$ is interpreted as a distribution of a random population Φ_0 of individuals in the phase space $[A, \rho_A]$, then P_{κ}, if it exists, may be interpreted as distribution of the succeeding generation Φ_1.

The set $\{P : P \in \mathbf{P}, P({}_{\kappa}M) = 1\}$ of those distributions P on $\mathcal{M}$, for which P_{κ} exists, belongs to $\boldsymbol{P}$. There even holds

1.4.3. [MKM, Prop. 4.2.4.] Subject to the assumptions of 1.4.1., $P \to P_{\kappa}$ is a $\boldsymbol{P}, \boldsymbol{P}'$-measurable mapping from $\{P : P \in \mathbf{P},\ P({}_{\kappa}M = 1)\}$ into $\mathbf{P}'$.

The clustering operation is monotone in the following sense:

1.4.4. Let P_1, P_2 be distributions in $\mathbf{P}$ such that $P_1 \lesssim P_2$ and $(P_2)_{\kappa}$ exists. Then also $(P_1)_{\kappa}$ exists, and $(P_1)_{\kappa} \lesssim (P_2)_{\kappa}$.

Clustering commutes with convolution:

1.4.5. [MKM, Prop. 4.3.1.] If κ is a clustering field on $[A, \rho_A]$ with phase space $[A', \rho_{A'}]$, and $(P_i)_{i \in I}$ is an at most countable family of distribution on $\mathcal{M}$ whose convolution P exists, then there holds

$$P_{\kappa} = \mathop{\ast}_{i \in I} (P_i)_{\kappa} \quad ,$$

where the existence of one side implies the existence of the other.

A simple but frequently used tool in dealing with clustered infinitely divisible distributions is the so called "clustering theorem":

1.4.6. [MKM, Prop. 4.3.3.] Let P be a distribution in $\mathbf{T}$ and κ be a clustering field on $[A, \rho_A]$ with phase space $[A', \rho_{A'}]$. Then P_{κ} exists iff $\tilde{P}_{\kappa}$ exists and $\tilde{P}_{\kappa}((.)\setminus\{o\})$ belongs to $\mathbf{W}'$. In this case P_{κ} belongs to $\mathbf{T}'$, and there holds

$$\widetilde{P_{\kappa}} = \tilde{P}_{\kappa}((.)\setminus\{o\}) \ .$$

By 1.3.13. there follows

1.4.7. If P is a distribution on $\mathcal{M}$ and κ is a clustering field on $[A, \rho_A]$ with phase space $[A', \rho_{A'}]$ such that P_{κ} exists, then there holds

$$\mathcal{L}_{P_{\kappa}}(f) = \mathcal{L}_{P}(-\ln \mathcal{L}_{\kappa_{(.)}}(f)) \quad , \quad f \in F , \quad \text{where } \ln 0 := -\infty \ .$$

In view of 1.3.11. one gets

1.4.8. If, under the assumptions of 1.4.7., P is a Poisson distribution Π_ν, then one has

$$\mathcal{L}_{(\Pi_\nu)_\kappa}(f) = \exp(-\langle \nu, 1-\mathcal{L}_{\kappa_{(.)}}(f)\rangle) \qquad (f \in F) \ .$$

Let now, in addition to $[A, \rho_A]$ and $[A', \rho_{A'}]$, a third complete separable metric space $[A'', \rho_{A''}]$ be given. If κ is a clustering field on $[A, \rho_A]$ with phase space $[A', \rho_{A'}]$ and ω is a clustering field on $[A', \rho_{A'}]$ with phase space $[A'', \rho_{A''}]$ such that $(\kappa_{(a)})_\omega$ exists for all $a \in A$, then we put

$$(\omega \circ \kappa)_{(a)} := (\kappa_{(a)})_\omega \qquad (a \in A) \ ,$$

thus obtaining, in virtue of 1.4.2. and 1.4.3., a clustering field $\omega \circ \kappa$ on $[A, \rho_A]$ with phase space $[A'', \rho_{A''}]$. We may interprete $(\omega \circ \kappa)_{(\Phi)}$ as distribution of the random "grandchildren's generation" of a population Φ, i. e. $\omega \circ \kappa$ may be conceived as a composition of the stochastic branching dynamics κ and ω.

1.4.9. [MKM, Prop. 4.3.4.] Let H be a measure on $\mathcal{M}$, κ be a clustering field on $[A, \rho_A]$ with phase space $[A', \rho_{A'}]$ and ω be a clustering field on $[A', \rho_{A'}]$ with phase space $[A'', \rho_{A''}]$. If both the clustering field $\omega \circ \kappa$ and the measure H_κ exist, then there holds

$$(H_\kappa)_\omega = H_{\omega \circ \kappa} \ ,$$

where the existence of one side entails the existence of the other.

In the special case $[A', \rho_{A'}] = [A, \rho_A]$ we simply speak of **clustering fields κ on $[A, \rho_A]$**.

For a clustering field κ on $[A, \rho_A]$, we introduce step by step its **clustering powers** $\kappa^{[n]}$, $n \in \mathbf{Z}_+$:

$$\kappa^{[0]}{}_{(a)} := \delta_{\delta_a} \quad (a \in A) \ ; \quad \kappa^{[n+1]} := \kappa \circ \kappa^{[n]} \quad (n \in \mathbf{Z}_+) \ .$$

It can easily be shown by 1.4.9. that

$$\kappa^{[m]} \circ \kappa^{[j]} = \kappa^{[m+j]} \qquad (j, m \in \mathbf{Z}_+) \ ,$$

provided that all clustering powers of κ exist.

1.5. Intensity Kernels

A **kernel** from a measurable space $[E_1, \mathcal{E}_1]$ to a measurable space $[E_2, \mathcal{E}_2]$ is a mapping K from $E_1 \times \mathcal{E}_2$ into $[0, +\infty]$ with the two properties

a) For all $e \in E_1$, $K(e,(.))$ is a measure on $\mathcal{E}_2$.

b) For all $X \in \mathcal{E}_2$, $K((.),X)$ is an $\mathcal{E}_1$-measurable mapping from E_1 into $[0, +\infty]$.

For any measure γ on $\mathcal{E}_1$ and any $\mathcal{E}_2$-measurable mapping f from E_2 into $[0, +\infty]$, one puts

$$(\gamma * K)(.) := \int K(e,(.))\gamma(de) \quad ; \quad (K*f)(.) := \int f(z) K((.),dz) \quad ,$$

thus obtaining a measure $\gamma * K$ on $\mathcal{E}_2$ and an $\mathcal{E}_1$-measurable mapping $K*f$ from E_1 into $[0,+\infty]$. There always holds

1.5.1. $\int f(z)\,(\gamma * K)(dz) \;=\; \int (K*f)(e)\,\gamma(de)$.

If J is a kernel from $[E_2, \mathcal{E}_2]$ to a measurable space $[E_3, \mathcal{E}_3]$, then we obtain via

$$(K*J)(e,(.)) := K(e,(.))*J \qquad (e \in E_1)$$

a kernel $K*J$ from $[E_1, \mathcal{E}_1]$ to $[E_3, \mathcal{E}_3]$.

A kernel K is called **stochastic** resp. **substochastic** if for all $e \in E_1$ there hold $K(e,E_2) = 1$ resp. $K(e,E_2) \leq 1$.

If K is a kernel from $[E, \mathcal{E}]$ to $[E, \mathcal{E}]$, then the kernels $K^{[n]}$, $n \in \mathbf{Z}_+$, are defined by

$$K^{[0]}(a,(.)) := \delta_a \qquad (a \in A) \quad ; \quad K^{[n+1]} := K*K^{[n]} \qquad (n \in \mathbf{Z}_+) \ .$$

Obviously there holds

$$K^{[m]}*K^{[k]} = K^{[m+k]} \qquad (m,k \in \mathbf{Z}_+) \ .$$

For any clustering field κ on $[A, \rho_A]$ with phase space $[A', \rho_{A'}]$, we define a kernel J_κ from $[A, \mathcal{A}]$ to $[A', \mathcal{A}']$ by

$$J_\kappa(a,(.)) := \Lambda_{\kappa_{(a)}} \quad ,$$

calling J_κ the **intensity kernel** of κ.

If a measure L on $\mathcal{M}$ is "clustered" by κ, then this corresponds, on the level of intensity measures, to a transformation of Λ_L by J_κ:

1.5.2. [MKM, Prop. 4.2.2.] If κ is a clustering field on $[A, \rho_A]$ with phase space $[A', \rho_{A'}]$ and L is a measure on $\mathcal{M}$ such that L_κ exists, then

$$\Lambda_{L_\kappa} = \Lambda_L * J_\kappa \ .$$

Conversely, we can conclude the existence of L_κ by inspection of J_κ and Λ_L: