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Preface

The monograph is devoted to spherical and plane integral operators for high-dimensional boundary value problems of mathematical physics. The derived integral operators are used to provide equivalent integral formulations of the boundary value problems. Direct and converse mean value theorems are proved for scalar elliptic equations such as the Laplace, Helmholtz, and diffusion equations, parabolic equations, high-order elliptic equations, e.g. the biharmonic and metaharmonic equations, and systems of elliptic equations like the Lamé equation and other systems of elasticity equations. These results are presented in the first part of the book, which includes Chapters 1–8 and follow mainly our book *Spherical Means for PDEs* published in 1997 by VSP [167]. The second part, Chapters 9–13, deals with the applications of the developed integral operator relations to numerics for PDEs. The integral operators defined on disks, spheres, 2D half-planes, 3D half-spaces, and some other domains are used to construct new numerical methods for solving relevant boundary value problems for a wide class of domains. We consider mainly two basic approaches: the first is developed for domains that can be composed as a union of overlapped disks, spheres, half-planes and half-spaces. The second is similar to the method of fundamental solutions, but is based on a numerical inversion of the integral operators by a randomized spectral projection method. We also show how the integral operators can be used to solve PDEs with random loads and stochastic boundary conditions.

The book is written for mathematicians who work in the field of partial differential and integral equations, physicists, and engineers dealing with computational methods and applied probability, for students and postgraduates studying mathematical physics and numerical mathematics.

Contents

1 Introduction

It is well known that many boundary value problems for partial differential equations can be reformulated in the integral form. We mention the classical example from potential theory where the original boundary value problems are reformulated in terms of equivalent integral equations (on the boundary, or in the volume, as in the Green formula, see, e.g. [33; 75; 93; 154]). The advantages of the integral formulation are well known: the solution is obtained on the whole, the boundary conditions are automatically taken into account, and the questions of existence and uniqueness are resolved on the basis of the well-developed Fredholm theory.

In this book, we deal with integral formulation of PDEs in the form of local integral equations whose kernels are defined on disks, spheres, planes, and other standard domains.

Let us recall the famous mean value relation for harmonic functions. Assume for simplicity that G is a bounded domain in $\mathbb{R}^n$ and let $S(x,r) \subset \bar{G}$ be a sphere of radius r centered at a point x. Let $u(x)$ be a regular harmonic function in G, i.e.

$$\Delta u(x) = 0\,, \quad x \in G\,, \quad u \in C^2(G)\,. \tag{1.1}$$

This means that the function $u(x)$ satisfies the Laplace equation in G and the second derivatives of u are continuous in G. Then for all $x \in G$ and all $S(x,r) \subset \bar{G}$ the following spherical mean value relation is true:

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int\limits_{S(x,r)} u(y)\,dS(y)\,. \tag{1.2}$$

On the right-hand side is the spherical mean of u, i.e. the integral of u over the sphere $S(x,r)$ with respect to the surface element measure $dS(y)$; ω_n is the surface area of a unit sphere in $\mathbb{R}^n$.

The following converse mean value relation is well known [33].

Theorem 1.1 (Weak converse mean value theorem). *If a function $u(x) \in C^2(G)$ satisfies the mean value relation* (1.2) *for all $x \in G$ and for all $S(x,r) \subset \bar{G}$, then $u(x)$ is harmonic in G.*

Less known is the following remarkable converse mean value theorem.

Theorem 1.2 (The strong converse mean value theorem). *Assume that the Dirichlet problem*

$$\Delta u(x) = 0\,, \quad u|_\Gamma = \varphi\,, \quad u \in C^2(G) \cap C(\bar{G})\,, \tag{1.3}$$

G bounded, has a solution for any continuous function φ. Then if a function $v \in C(\bar{G})$ satisfies the mean value relation for at least one sphere $S(x,r) \subset \bar{G}$, for all $x \in G$, then $v(x)$ is harmonic in G.

This exciting statement can be found in the book of Courant and Hilbert [33] where examples were given showing that all the conditions of the theorem are necessary. However, in [33], there was no attempt to derive an integral formulation of the Dirichlet problem from this approach.

In this book, we deal with the integral formulations of the boundary value problems for various PDEs which we derive from the strong converse spherical mean value relations. In Chapter 2, we give the results for simple equations such as the Laplace equation, the diffusion and Helmholtz equations, and the heat equation. Chapter 3 includes some high-order equations (biharmonic, polyharmonic, and metaharmonic equations), and Chapter 4 deals with some elliptic systems. Chapters 5, 6, and 7 are devoted to the Lamé equation, pseudovibration elastic, and thermoelastic equations, respectively.

It should be noted that different mean value relations can be derived: we can relate the spherical means of the solution and its derivatives to the values of these functions at the center of the sphere. There are many relations of this kind (e.g. see [189]) and we will use some of them. However, the most interesting are the mean value relations which provide equivalent integral formulations of the original differential equation such as the strong converse mean value theorem described above for the Laplace equation. Therefore, our general objective is to give equivalent integral formulations of the corresponding PDEs.

In Chapter 8, we give some applications: these are probabilistic numerical algorithms for solving PDEs (the so-called Random Walk on Spheres algorithms) which we construct on the basis of the integral formulations. The solution in such methods is represented in the form of expectations over Markov processes generated in some sense by the spherical means. The algorithms are simple enough and provide effective numerical solutions to the boundary value problems for complicated domains of high dimension. In addition, the implementation of the algorithms can easily be carried out for parallel computers.

The spherical mean value relations can also be applied to solving different problems of mathematical physics, for example, getting information about eigenvalues, constructing high-order finite difference schemes, finding asymptotics of the solutions, solving inverse problems of the potential theory, etc.

The probabilistic representations described in Chapter 8 assume minimum a priori information about the smoothness of the solution (e.g. the integral equation equivalent to the Laplace equation assumes only that the solution is continuous in $\bar{G}$). Actually an approach based on spherical mean value relations enables us to construct generalized solutions (see, e.g. [141]). Another interesting advantage of these representations is the possibility of solving numerically the exterior boundary value problems in complicated high-dimensional domains.

It should be noted that there is not much literature concerning spherical mean value relations for PDEs. First, there is the classical book of Courant and Hilbert [33]. In [189] (see also the references in this book) a short review of the works in this field has been carried out. The book [124] contains a set of mean value relations for metaharmonic equations. A series of articles by Diaz and Payne [39] deal with the mean value relations for elasticity problems. In [97], the mean value relations for some parabolic equations, and in [67], the mean value relations for telegraph equations were derived. We also mention the mean value relations where the integration is taken over other domains, for instance, ellipsoids [51].

There is also an interesting class of approximate mean value relations [58; 141; 132] which characterize the relevant equations. Special mean value relations are used in the Monte Carlo Random Walk on Spheres methods and can be found in the corresponding literature [51], [160].

In Chapters 9 and 10, we present further applications of the spherical and plane integral operators described in Chapters 1–7. The approach presented in Chapters 9 and 10 are based on the Poisson-type integral representations for disks, spheres, half-planes, half-spaces, and other standard domains. The original differential boundary value problem is equivalently reformulated in the form of a system of integral equations defined on the intersection surfaces of these standard domains. Then, we invert the system of integral equations by a spectral expansion of the kernels.

In Chapter 11, we present a stochastic boundary method which can be considered as a randomized version of the method of fundamental solutions (MFS). We suggest to solve the large system of linear equations for the weights in the expansion over the fundamental solutions by a randomized singular value decomposition method we introduced in [178]. In addition, we also deal with solving inhomogeneous problems where we use the integral representation through the Green integral formula. The relevant volume integrals are calculated by a Monte Carlo integration technique which uses the symmetry of the Green function. We also construct a stochastic boundary method based on the spectral inversion of the Poisson formula representing the solution in a disk. This is done for the Laplace equation, and the system of elasticity equations. We stress that the stochastic boundary method proposed is of high generality, and it can be applied to any bounded and unbounded domain with any boundary condition provided the existence and uniqueness of the solution are proven. We present a series of numerical results that illustrate the performance of the suggested methods.

Chapter 12 deals with an elasticity problem with random loads.

In Chapter 13, we study boundary value problems with stochastic boundary conditions. We construct exact proper orthogonal decomposition for some classical boundary value problems, for a disk, ball, half-plane, and a half-space, with a Dirichlet and Neumann boundary conditions, where the boundary functions are white noise or homogeneous (2π-periodic) random processes. In case the boundary function is a white noise, the solutions are treated as generalized random fields with the convergence in the proper spaces and relevant generalized treatment of boundary conditions. In the last section of this chapter we study a response of an elastic half-space to random excitations of displacements on the boundary under the condition of no shearing forces. We analyze the white noise excitations and general random fluctuations of displacements prescribed on the boundary. We consider the case of partially ordered defects on the boundary whose positions are governed by an exponential–cosine-type correlation function. The analysis is based on a Poisson-type integral formula which we derive here for the case of zero shearing forces on the boundary. We obtain exact representations for the displacement correlation tensor and the Karhunen–Loève expansion for the solution of the Lamé equation itself and analyze some features of the correlation structure of the displacements. The Monte Carlo technique developed can be applied to a wide class of differential equations with random boundary conditions.

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2 Scalar second-order PDEs

2.1 Spherical mean value relations for the Laplace equation and integral formulation of the Dirichlet problem

We consider equations in a domain G whose boundary ∂G is denoted by Γ and the closure by $\bar{G}$. Note that the domain G may be unbounded. Throughout the whole book, we use the following notations (partially already introduced in the previous chapter):

- $S(x,r)$ – a sphere of radius r centered at the point x,
- $B(x,r)$ – a ball of radius r centered at the point x,
- Ω – the unit sphere $S(0,1)$,
- $d\Omega$ – the surface element of Ω,
- $dS = do$ – the surface element of $S(0,r)$.
- ω_m – the area of the surface of the unit sphere in $\mathbb{R}^m$.
- $N_r u(x) = \frac{1}{\omega_m}\int_\Omega u(x+rs)d\Omega(s)$ – the spherical mean of the function $u(x)$.

Let us start with a simple case, the Dirichlet problem for the Laplace equation in a domain $G \subset \mathbb{R}^m$:

$$\Delta u(x) = 0\,, \quad x \in G\,, \tag{2.1}$$

$$u(y) = \varphi(y)\,, \quad y \in \Gamma\,. \tag{2.2}$$

We seek a regular solution to (2.1) and (2.2), i.e. $u \in C^2(G)\bigcap C(\bar{G})$.

2.1.1 Direct spherical mean value relation

It is well known that every regular solution to (2.1) satisfies the spherical mean value relation

$$u(x) = N_r u(x) := \frac{1}{\omega_m}\int\limits_\Omega u(x+rs)\; d\Omega(s) \tag{2.3}$$

for each $x \in G$ and for all spheres $S(x,r)$ contained in $\bar{G} := G\bigcup\Gamma$. The same is true for the volume mean value relation (it can be obtained directly from (2.3) by integrating)

$$u(x) = \frac{m}{\omega_m r^m}\int\limits_{B(x,r)} u(y)\; dy. \tag{2.4}$$

The mean value relation (2.3) can be derived by different methods. For small r, it is possible to use the method based on the power expansion of the integrand. We present this method here and we will use it later to derive the mean value relations for different equations. The following statement is very useful, in particular, to get power expansions of the spherical means.

We denote by D the differential operator $D = \left(-i\frac{\partial}{\partial x_k}\right)_{1\le k\le m}$ and $D^\alpha = D_1^{\alpha_1}\cdots D_m^{\alpha_m}$, where α is the multiindex:

$$\alpha = (\alpha_1, \dots, \alpha_m)\,, \quad \alpha! = \alpha_1!\dots\alpha_m!\,, \quad D_k = -i\frac{\partial}{\partial x_k}\,, \quad |\alpha| = \alpha_1 + \cdots + \alpha_m\,.$$

In [189], the following result is presented.

Lemma 2.1. *Let η be a measure in $\mathbb{R}^m$ with a compact support and let $h(y) = \hat{\eta}$ be the Fourier transform of η:*

$$h(y) = \hat{\eta} = \int_{\mathbb{R}^m} \exp\{-i(y,s)\}d\eta(s)\,.$$

Then

$$\int u(x+ry)\,d\eta(y) = \{h(-rD)\,u\}\,(x) \tag{2.5}$$

for x, r, and u for which the left-hand side exists and the right-hand side converges.

Proof. The integral exists for sufficiently small $r > 0$ and as a function of r is analytic; the same for the right-hand side. Thus it is sufficient to prove the statement for sufficiently small r. The expansion of the integrand and the integration yields

$$\begin{aligned}
\int_{\mathbb{R}^m} e^{-i(y,s)}d\eta(s) &= \sum_{k=0}^{\infty}\frac{1}{k!}\int[-i(y,s)]^k d\eta(s)\\
&= \sum_{k=0}^{\infty}\frac{(-i)^k}{k!}\left\{\int\sum_{|\alpha|=k}\binom{k}{\alpha}y^\alpha s^\alpha d\eta(s)\right\}\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left\{\sum_{|\alpha|=k}\binom{k}{\alpha}y^\alpha\int s^\alpha d\eta(s)\right\}\\
&= \sum_{\alpha}\frac{1}{\alpha!}(-iy)^\alpha\left\{\int s^\alpha d\eta(s)\right\}.
\end{aligned}$$

Here

$$\binom{k}{\alpha} = \frac{k!}{\alpha!(k-|\alpha|)!}\,.$$

Let

$$u(x+z) = \sum_\alpha a_\alpha(x)z^\alpha, \quad a_\alpha(x) = \frac{1}{\alpha!}\frac{\partial^{|\alpha|}u(x)}{\partial x_1^{\alpha_1}\cdots\partial x_m^{\alpha_m}}\,.$$

Then

$$\begin{aligned}
\int u(x+rs)d\eta(s) &= \int\left\{\sum_\alpha a_\alpha(x)r^{|\alpha|}s^\alpha\right\}d\eta(s) = \sum_\alpha a_\alpha(x)r^{|\alpha|}\left\{\int s^\alpha d\eta(s)\right\}\\
&= \sum_\alpha\frac{1}{\alpha!}(iD^\alpha)u(x)r^{|\alpha|}\left\{\int s^\alpha d\eta(s)\right\}.
\end{aligned}$$

□

We consider the case when η is the uniform measure $d\Omega$. To formulate the result we define a function $W_m(\cdot)$ of the dimension m. Let $J_\nu(z)$ be the Bessel function

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+\nu+1)\,\Gamma(k+1)} \left(\frac{z}{2}\right)^{2k+\nu}. \tag{2.6}$$

Here $\Gamma(\cdot)$ is the Euler Gamma-function. Let us introduce the function

$$W_m(z) = \Gamma\left(\frac{m}{2}\right)\left(\frac{2}{z}\right)^{m/2-1} J_{m/2-1}(z)\,. \tag{2.7}$$

Theorem 2.1. *Let $u(x)$ be a real-valued analytic function. Then*

$$N_r u(x) = \left\{W_m\left(ir\sqrt{\Delta}\right)\right\} u(x) \tag{2.8}$$

for x, r, and u for which the left-hand side is defined and the right-hand side exists.

Proof. We get (2.8) from (2.5) by choosing the measure η as $d\Omega/\omega_m$ and taking into account that the Fourier measure of $d\Omega/\omega_m$ is $W_m(|y|i)$, and $|D| = (-\Delta)^{1/2}$. □

For example, in $\mathbb{R}^3$

$$h(iy) = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} \sin(\theta)\, d\theta \exp(|y|\cos\theta) = \frac{\mathrm{sh}(|y|)}{|y|}\,;$$

hence (2.5) has the form

$$N_r u(x) = \left\{\frac{\mathrm{sh}\,(r\sqrt{\Delta})}{r\sqrt{\Delta}}\right\} u(x)\,. \tag{2.9}$$

Note that the last relation is the expansion in powers of the Laplace operator given in [33]

$$u(x) = N_r u(x) - \sum_{k=1}^{\infty} \frac{r^{2k}}{(2k+1)!}\,\Delta^k u(x) \tag{2.10}$$

since

$$W_m\left(ir\sqrt{\lambda}\right) = \sum_{k=0}^{\infty} \frac{\lambda^k\, r^{2k}}{2^k\, k!\, m(m+2)\cdots(m+2k-2)}\,.$$

In $\mathbb{R}^2$ the “spherical” (a circular) mean has the representation

$$\frac{1}{2\pi}\int_0^{2\pi} u(x+re^{i\varphi})\,d\varphi = \left\{J_0\left(r\sqrt{-\Delta}\right)\right\} u(x) = \sum_{k=0}^{\infty} (k!)^{-2}\left(\frac{r}{2}\right)^{2k} \Delta^k u(x)\,.$$

Here $J_0(|x|)$ is the Bessel function that is obtained as the Fourier transform of the uniform measure $d\varphi/2\pi$.

Note that if we choose the uniform measure in the disk whose Fourier transform is $2J_1(|x|)/|x|$, we get

$$\frac{1}{\pi}\int_0^{2\pi}\int_0^{1} u(x+rte^{i\alpha})\,t\,dt\,d\alpha = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!}\left(\frac{r}{2}\right)^{2k} \Delta^k u(x)\,.$$

This relation can be generalized by choosing a general distribution ρ on the interval $[0,1]$ which leads to

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 u(x+rte^{i\alpha})\,d\rho(t)\,d\alpha = \sum_{k=0}^{\infty}(k!)^{-2}\left(\frac{r}{2}\right)^{2k} a_k\Delta^k u(x)\,,$$

where

$$a_k = \int_0^1 (t/2)^{2k}\,d\rho(t)\,.$$

Remark 2.1. Note that the power expansions were given by Pizetti [137] (see also [33]): if a function $u(x) = u(x_1,\ldots,x_m)$ belongs to $C^{2p+2}(G)$, $G \subset \mathbb{R}^m$, then for all $x \in G$ and all sufficiently small r

$$N_r u(x) = \sum_{k=0}^{p} r^{2k} C_k u(x) + Q_p(r)u(x)\,,$$

and

$$\begin{aligned} N_B u(x) &\equiv \frac{m}{\omega_m r^m}\int_{B(x,r)} u(x+y)\,dy \\ &= \sum_{k=0}^{p} r^{2k} C'_k u(x) + Q'_p(r)u(x)\,, \end{aligned}$$

where C_k and C'_k are the differential operators of the form

$$C_k = \frac{\Gamma(m/2)}{2^{2k}k!\Gamma(m/2+k)}\Delta^k\,,$$

and

$$C'_k = \frac{m}{m+2k}C_k\,, \quad k \ge 0\,.$$

The remainders can be estimated: for instance, in 3D

$$Q_p(r)u(x) = \frac{1}{4\pi(2p+1)!}\int_{B(x,r)} \frac{(r-|y|)^{2p+1}}{r|y|}\Delta^{p+1}u(x+y)\,dy\,,$$

hence

$$|Q_p(r)u(x)| \le c(p)r^{2p+2}\|u^{(2p+2)}\|_B\,,$$

where the norm $\|u^{(2p+2)}\|_B$ is defined by

$$\|u^{(2p+2)}\|_B = \sum_{i_1,i_2,\ldots,i_\nu=1}^{m} \sup_{x\in B(x,r)}\left|\frac{\partial}{\partial x_{i_1}}\frac{\partial}{\partial x_{i_2}}\cdots\frac{\partial}{\partial x_{i_\nu}}u(x)\right|\,.$$

The spherical mean value relation for small radii for the harmonic functions follows from the obtained expansion (2.10). Generalization for all $S(x,r) \subset \bar{G}$ can be obtained by the analytical continuation or by using the Green formula. Let us illustrate this in the case $\mathbb{R}^m$, $m \geq 3$. The Green formula reads

$$u(x) = \int\limits_{\partial G} \left\{ \mathcal{E}(x,y) \frac{\partial}{\partial \nu} u(y) - u(y) \frac{\partial}{\partial \nu} \mathcal{E}(x,y) \right\} do(y)$$
$$- \int\limits_{G} \Delta u(z) \mathcal{E}(x,z) dz \, .$$

Here

$$\mathcal{E}(x,y) = \frac{|x-y|^{2-m}}{(m-2)\omega_m} \, ,$$

ν is the outward normal vector to the boundary ∂G and

$$do(y) = r^{m-1} d\omega_m = \omega_m r^{m-1} d\Omega \, .$$

From this formula we immediately get the desired mean value relation, since $\partial/\partial\nu = \partial/\partial r$ for $G = B(x,r)$, $\Delta u = 0$ and

$$\int\limits_{S(x,r)} \frac{\partial}{\partial \nu} u do = 0 \, .$$

The last equality follows from the well-known Green formula

$$\int\limits_{G} (u \Delta v - v \Delta u) dx = \int\limits_{\partial G} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) do \, ,$$

which is true for arbitrary functions $u, v \in C^2(G)$.

We would now like to answer the questions:

Does the spherical mean value relation uniquely characterize the harmonic functions?

Is it possible to use the spherical mean value relation (in one form or another) to give an equivalent integral formulation of the Dirichlet problem for the Laplace equation?

How do we solve numerically the integral equation generated by the relevant spherical mean value relation?

In the next section, we shall give answers to these questions.

In [224], the following general problem was studied: does the weak spherical mean value relation uniquely characterize the corresponding differential equation? We now present the relevant statements.

Assume that u is a real-valued analytic function which satisfies the mean value relation

$$u(x_0) = \int u(x_0 + ry) d\sigma(y) \, , \tag{2.11}$$

where σ is a measure with compact support. Then, taking in Lemma 2.1 the measure $\eta = \sigma - \delta_{x_0}$, δ_{x_0} being the Dirac measure concentrated at the point x_0 and

$$h(z) = \sum_{\alpha} b_\alpha z^\alpha ,$$

we get

$$\sum_{\alpha} (-1)^{|\alpha|} r^{|\alpha|} b_\alpha D^\alpha u(x) = 0 . \tag{2.12}$$

From this, for sufficiently small $r > 0$, we get that

$$Q_j(D)u = 0 , \quad j = 0, 1, 2, \dots , \tag{2.13}$$

where

$$Q_j(D) = \sum_{|\alpha|=j} b_\alpha D^\alpha .$$

Thus the spherical mean value relation (2.11) for a real-valued analytic function holds if and only if the function u satisfies the infinite system of differential equations (2.13).

More generally, for continuous functions, the following statement holds (see [224]).

Theorem 2.2. *A function $u \in C(G)$, G bounded, satisfies the mean value relation* (2.11) *for all $x \in G$ and all $0 < r < d(x)$ if and only if the function u is a weak solution to the system* (2.13).

It is interesting to find out when the system (2.13) is equivalent to a single equation $L(D)u = 0$. In this case σ is a distribution characterizing the operator $L(D)$.

Theorem 2.3. *The system* (2.13) *is equivalent to a single equation of the type*

$$L(D)u = 0 ,$$

where $L(\xi)$ is a homogeneous polynomial if and only if the polynomial $L(\xi)$ is divisible through all Q_j and there is an integer k such that $L = cQ_k$, c being a constant.

Finally, the form of the characterizing distribution is given in the following statement.

Theorem 2.4. *For each homogeneous polynomial L there exists a mass with a compact support such that a function $u \in C(G)$ is a weak solution to $L(D)u = 0$ if and only if the spherical mean value relation* (2.11) *holds for all $x \in G$ and all* $0 < r < d(x)$*. Each mass of the form $L(D)m$ (m is a distribution with a compact support and $\hat{m}(0) \neq 0$) has this property.*

In the next section, we deal with the strong converse mean value theorem which uniquely characterizes not only the equation but also the solution of the Dirichlet problem and provides an equivalent integral equation.

2.1.2 Converse mean value theorem

The mean value relation (2.3) characterizes the solutions to (2.1). We start with the weak converse mean value theorem, whose proof is well known (e.g. see [33]).

Proposition 2.1. *Suppose that a function $u(\cdot)$ is continuous in G and for every sphere $S(x,r)$ contained in G, $u(\cdot)$ satisfies the mean value relation* (2.3). *Then $u(\cdot)$ is harmonic in G.*

The proof follows from expansion (2.10).

We formulate a stronger result which presents an equivalent formulation of the problem (2.1), (2.2).

Theorem 2.5 (Strong converse mean value theorem). *Let $\varphi(\cdot)$ be a given continuous and bounded function on Γ; if G is unbounded, we suppose*

(H) *the solution $u(\cdot)$ to the Dirichlet problem associated with $\varphi(\cdot)$ tends to 0 at infinity for the dimensions $m \geq 3$ and is bounded in the case $m = 2$.*

Assume that G is a domain for which the problem (2.1), (2.2) *has a unique solution for any continuous and bounded function φ. Suppose that there exists a function $v \in C(G \bigcup \Gamma)$, $v|_\Gamma = \varphi$, such that the mean value relation* (2.3) *holds at every point $x \in G$ for at least one sphere $S(x,r) \subset G \bigcup \Gamma$; if G is unbounded, we suppose that $v(\cdot)$ tends to 0 at infinity in dimensions $m \geq 3$ and is bounded when $m = 2$.*

Then $v(\cdot)$ is the unique regular solution to the problem (2.1), (2.2). *The same conclusion holds if the volume mean value relation*

$$v(x) = \frac{m}{\sigma_m r^m} \int\limits_{B(x,r)} v(y)\, dy$$

holds for every point $x \in G$ and at least one ball $B(x,r) \subset G \bigcup \Gamma$.

Proof. We use the same arguments as those given in [33], where the converse mean value theorem for the harmonic functions for bounded domains was given.

We shall now give the proof for the volume mean value relation (the same arguments work in the case of the spherical mean value relation).

Let u be the solution to the Dirichlet problem and x be fixed in G. Since u satisfies the volume mean value relation for every ball $B(y,r)$ contained in G, we conclude that the function $u - v$ satisfies the volume mean value relation for $B(x,r)$. Let F be the set of points of G where $u - v$ attains its maximum M: even if G is unbounded, as $u - v$ tends to 0 at infinity (or is bounded, in dimensions $m = 2$), F is a compact set; let x_0 be a point of F whose distance to Γ is minimum. If x_0 were an interior point of G, we could find a ball $B(x_0, r_0) \subset G$ for which the volume mean value relation holds and then $u - v$ would be equal to M inside $B(x_0, r_0)$, which is a contradiction with the definition of x_0. Therefore, x_0 belongs to Γ. We repeat the same argument for the minimum of $u - v$. Since, by hypothesis, $(u - v)|_\Gamma = 0$, we conclude $u \equiv v$ in G. □

Using this converse mean value relation we can formulate an integral equation equivalent to the Dirichlet problem.

2.1.3 Integral equation equivalent to the Dirichlet problem

In this section, we use the *strong converse mean value relation* to derive an equivalent integral equation of the second kind.

Let us denote by $d(x)$ the distance from a point $x \in G$ to the boundary Γ and let

$$d^* = \sup_{x \in G} d(x)\,.$$

We also introduce an "ε-boundary":

$$\Gamma_\varepsilon = \{x \in G:\ d(x) \leq \varepsilon\}\,.$$

Let

$$\delta_x(y) = \delta(|x-y| - d(x))\,.$$

For simplicity, we use this notation to indicate that $\delta_x(y)$ is equal to $1/\omega_m$ if $|x-y| = d(x)$ and to 0 otherwise. We define a kernel function k_ε by

$$k_\varepsilon(x,y) := \begin{cases} \delta_x(y) & \text{if } x \in G \setminus \Gamma_\varepsilon\,, \\ 0 & \text{if } x \in \Gamma_\varepsilon\,, \end{cases} \tag{2.14}$$

and define the integral operator K_ε by

$$K_\varepsilon \psi(x) := \int_G k_\varepsilon(x,y)\psi(y)\,dy \tag{2.15}$$

for each $\psi(\cdot) \in C(G)$.

We now fix the boundary function φ and suppose that the conditions of the converse mean value relation are satisfied. Denote by u the solution to the Dirichlet problem corresponding to φ and by $f_\varepsilon(x)$ the function

$$f_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in G \setminus \Gamma_\varepsilon, \\ u(x) & \text{if } x \in \Gamma_\varepsilon. \end{cases}$$

Consider the integral equation

$$v(x) = K_\varepsilon v(x) + f_\varepsilon(x)\,. \tag{2.16}$$

Note that the mean value relation implies (2.16). Thus if we assume that $u(x)$ is known in Γ_ε, (2.16) presents the desired integral equation of the second kind, and the converse mean value theorem states that this equation has a unique solution which coincides with the solution to the Dirichlet problem for the Laplace equation.

It remains to propose a method of calculation of the solution to (2.16). We show that the conventional successive iteration method applied to (2.16) is convergent.

Theorem 2.6. *For any* $\varepsilon > 0$, *the integral equation* (2.16) *has a unique solution given by the Neumann series*

$$v(x) = f_\varepsilon(x) + \sum_{i=1}^{\infty} K_\varepsilon^i f_\varepsilon(x) \tag{2.17}$$

and it coincides with the solution to (2.1) *and* (2.2) *if the assumptions of the converse mean value relation are satisfied.*

Proof. Let ε be fixed. It is simple to show the convergence of the series

$$f_\varepsilon(x) + \sum_{i=1}^{N} K_\varepsilon^i f_\varepsilon(x) \ , \tag{2.18}$$

if $d^* < \infty$. It is then sufficient to prove the existence of $0 < \lambda < 1$ such that for any continuous and bounded function g

$$\|K_\varepsilon^2 g\|_{L_\infty} < \lambda \|g\|_{L_\infty}$$

(this fact also implies the uniqueness of the solution to (2.17)). Let

$$\nu(\varepsilon) = \varepsilon^2/(4d^{*2}) \, . \tag{2.19}$$

For $x \in G \setminus \Gamma_\varepsilon$ we have

$$\begin{aligned}\int_G k_\varepsilon(x,y) \int_G k_\varepsilon(y,y')dy' \ dy &= \int_{G\setminus\Gamma_\varepsilon} \delta_x(y) \left(\int_G \delta_y(y')dy' \right) dy \\ &= \int_{G\setminus\Gamma_\varepsilon} \delta_x(y)dy \le 1 - \nu(\varepsilon) < 1 \, .\end{aligned}$$

Let $v(x) := f_\varepsilon(x) + \sum_{i=1}^{\infty} K_\varepsilon^i f_\varepsilon(x)$. It is clear that v satisfies

$$v(x) = K_\varepsilon v(x) + f_\varepsilon(x) \, .$$

If x belongs to Γ_ε, then, for any y, $k_\varepsilon(x,y) = 0$ and thus $v(x) = f_\varepsilon(x) = u(x)$. On the other hand, if x belongs to $G \setminus \Gamma_\varepsilon$, then the definition of k_ε implies that v satisfies the mean value relation at x with the sphere of radius $d(x)$. We conclude by using Theorem 2.5. □

Remark 2.2. Note that the condition $d^* < \infty$ is not necessary for the convergence of the iteration method. However, it is difficult to study the convergence of the iteration method in this general case. In the last chapter, we prove the convergence for the half-space $\mathbb{R}_+^3$ and the exterior of a sphere.

In the integral equation (2.16) and in the iteration method we used the spheres $S(x, d(x))$. However, the *strong converse mean value relation* implies that in each step of the iteration method we can use a sphere of an arbitrary radius, say, r_x, such that $S(x, r_x) \subset \bar{G}$. In the last chapter, we will see that this scheme with a special choice of r_x is reasonable when solving the Poisson equation by the Random Walk method.

2.1.4 Poisson–Jensen formula

More generally, a local integral equation can be derived if we replace the mean value relation by the Poisson formula representing the solution at an arbitrary point $x \in S(x_0, r) \subset \mathbb{R}^3$:

$$u(x) = \int\limits_{S(x_0,r)} H(x,y)u(y)\, dS(y)\,, \tag{2.20}$$

where

$$H(x,y) = \frac{r^2 - x^2}{4\pi r|x-y|^3}\,,$$

if we suppose that this equality holds for every $x_0 \in G$ at least for one sphere $S(x_0, r) \subset \bar{G}$, then $u(\cdot)$ is the unique regular solution to the Dirichlet problem.

In 2D, there is a further generalization of formula (2.20) – the so-called Poisson–Jensen formula [108]. Let $u(z) = u(r,\theta)$ be a univalent function harmonic in a disk $K(0,R)$ except for a set of singularities where u has logarithmic poles. We denote the nonzero poles by $\{\zeta_i\}$, $i = 1, \ldots,$ where the indexing is chosen so that $|\zeta_1| < |\zeta_2| < \cdots < |\zeta_i| < \cdots$. Let $\mu_j \ln|z - \zeta_j|$ be the principal part corresponding to ζ_j. It is also convenient to introduce the principal part $\mu_0 \ln|z|$ corresponding to the point $z = 0$ taking $\mu_0 = 0$ if this point is not a logarithmic pole of $u(z)$.

Let us consider the conform mapping of the disk $|z| \le \rho$ on itself by

$$l_\zeta(z) = \rho^2 \frac{z - \zeta}{\rho^2 - \bar{\zeta} z}\,,$$

where $|z| \le \rho < R$. This transformation maps the point ζ into the center of the disk. Since $l_\zeta(z)$ is an analytic function in the disk $|z| < \rho^2/|\zeta|$, having a single simple zero point at $z = \zeta$ and $|l_\zeta(z)| = \rho$ at all points of the circle $|z| = \rho$, then the function

$$u_\zeta(z) = \ln\left[\frac{1}{\rho}|l_\zeta(z)|\right]$$

is harmonic if $|z| < \rho^2/|\zeta|$, $z \ne \zeta$ and has in this disk a logarithmic pole at the point $z = \zeta$ with the principal part $\ln|z - \zeta|$ and $u = 0$ on the boundary $z = \rho$. If one part of the logarithmic poles of $u(z)$, namely, the points $0, \zeta_1, \ldots, \zeta_{\nu(\rho)}$ lies inside this disk and the rest is outside of it then

$$v(z) = u(z) - \sum_{k=0}^{\nu(\rho)} \mu_k u_{\zeta_k}(z) = u(z) - \sum_{k=0}^{\nu(\rho)} \mu_k \ln\left|\frac{\rho(z - \zeta_k)}{\rho^2 - \bar{\zeta}_k z}\right|$$

is harmonic in all the points of a disk whose radius is larger than ρ (the radius is equal to the minimum of the two values: $|\zeta_{\nu(\rho)+1}|$ and $\rho^2/|\zeta_\nu(\rho)|$), while on the boundary $|z| = \rho$ this function coincides with $u(z)$. Therefore, for all $|z| = r < \rho$ we get

$$v(r,\theta) = \frac{1}{2\pi}\int\limits_0^{2\pi} v(\rho,\alpha)\frac{r^2 - \rho^2}{r^2 + \rho^2 - 2r\rho\cos(\theta - \alpha)}\,d\alpha\,;$$

hence

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} v(\rho,\alpha)\frac{r^2-\rho^2}{r^2+\rho^2-2r\rho\cos(\theta-\alpha)}\,d\alpha + \sum_{k=0}^{\nu(\rho)}\mu_k \ln\left|\frac{\rho(z-\zeta_k)}{\rho^2-\bar{\zeta}_k z}\right| \tag{2.21}$$

which is known as the Poisson–Jensen formula. If the function has no logarithmic poles, then all the values μ_j are zeros, and we come to the Poisson formula.

2.2 The diffusion and Helmholtz equations

The aim of this section is to treat the diffusion equation

$$\Delta u(x) - \lambda u(x) = 0\,, \quad u|_\Gamma = \varphi\,, \tag{2.22}$$

where λ is a nonnegative constant and the Helmholtz equation

$$\Delta u(x) + \lambda u(x) = 0\,, \ \ u|_\Gamma = \varphi\,. \tag{2.23}$$

Using the function $W_m(z)$ introduced by (2.7) in the previous section we define a new function

$$w_m(r,\lambda) = W_m(ir\sqrt{\lambda})\,.$$

Since $I_\nu(z) = J_\nu(iz)/i^\nu$, we get the expansion

$$\begin{aligned} w_m(r,\lambda) &= \Gamma\left(\frac{m}{2}\right)\left(\frac{2}{r\sqrt{\lambda}}\right)^{m/2-1} I_{m/2-1}(r\sqrt{\lambda}) \\ &= \Gamma\left(\frac{m}{2}\right)\sum_{k=0}^{\infty}\frac{\lambda^k r^{2k}}{2^{2k}k!\Gamma(k+m/2)} \\ &= \sum_{k=0}^{\infty}\frac{\lambda^k r^{2k}}{2^k k! m(m+2)\cdots(m+2k-2)}\,, \end{aligned} \tag{2.24}$$

since

$$\Gamma\left(k+\frac{m}{2}\right) = \Gamma\left(\frac{m}{2}\right) m(m+2)\cdots(m+2k-2)/2^k\,.$$

2.2.1 Diffusion equation

From (2.8), we deduce the mean value relation for the solution to (2.22):

$$u(x) = w_m^{-1}(r,\lambda)N_r u(x)\,. \tag{2.25}$$

We obtain this mean value relation for arbitrary $S(x,r) \subset \bar{G}$ using a different approach.

Let

$$v(x,r) = N_r u = \frac{1}{\omega_m} \int\limits_{\Omega} u(x+sr)\,d\Omega(s)\,.$$

Then the function $v(x,r)$ satisfies the Darboux equation [33]

$$\frac{\partial^2 v}{\partial r^2} + \frac{m-1}{r}\frac{\partial v}{\partial r} - \Delta v = 0 \tag{2.26}$$

with the initial conditions

$$v(x,0) = u(x), \quad \frac{\partial v}{\partial r}(x,0) = 0\,.$$

Indeed, using the Gauss theorem we get

$$\begin{aligned} \frac{\partial v(x,r)}{\partial r} &= \frac{1}{\omega_m} \int\limits_{\Omega} \left(\sum_{i=1}^{m} s_i \frac{\partial u}{\partial x_i} \right) d\Omega \\ &= \frac{1}{\omega_m r^{m-1}} \int\limits_{S(x,r)} \frac{\partial u}{\partial \nu} do = \frac{1}{\omega_m r^{m-1}} \int\limits_{B(x,r)} \Delta u dy\,. \end{aligned} \tag{2.27}$$

Here

$$\frac{\partial}{\partial \nu} = \sum_{i=1}^{m} s_i \frac{\partial}{\partial x_i}$$

denotes the differentiation with respect to the outward normal vector to the sphere Ω and dy is the volume element.

Differentiating once more yields the desired result

$$\frac{\partial^2 v}{\partial r^2} = -\frac{m-1}{\omega_m r^m} \int\limits_{B(x,r)} \Delta u dy + \frac{1}{\omega_m r^{m-1}} \int\limits_{S(x,r)} \Delta u do = -\frac{m-1}{r}\frac{\partial v}{\partial r} + \Delta v\,.$$

The operator Δ and the spherical mean operator N_r are permutable [33]; therefore, from (2.22), we get $\Delta v = \lambda v$. Thus substituting $\Delta v = \lambda v$ in the Darboux equation we find that $v = N_r u(x)$ solves the problem

$$\begin{gathered} \frac{\partial^2 v}{\partial r^2} + \frac{m-1}{r}\frac{\partial v}{\partial r} - \lambda v = 0\,, \\ v(x,0) = u(x)\,, \quad \frac{\partial v}{\partial r}(x,0) = 0\,. \end{gathered}$$

The solution to this problem is

$$v(x,r) = v(x,0) \cdot w_m(\lambda, r)\,,$$

which is the desired spherical mean value relation. In $G \subset \mathbb{R}^3$, we have from (2.9)

$$u(x) = \frac{\sqrt{\lambda} r}{\sinh(\sqrt{\lambda} r)} N_r u(x)\,. \tag{2.28}$$

Note that (2.25) and (2.28) are true for complex λ such that $\mathrm{Re}\sqrt{\lambda}r < \pi$.

The diffusion equation is treated exactly according to the scheme described for the Laplace equation, since the integral formulation can also be given for (2.22).

Proposition 2.2. *We suppose that the Dirichlet problem for the diffusion equation has a unique solution for any continuous function φ; if G is unbounded, we suppose* (H)*. Let $\varphi(\cdot)$ be a given continuous and bounded function on Γ. Assume that $u \in C(G \bigcup \Gamma)$ satisfies the mean value relation* (2.25) *for each $x \in G$ at least for one sphere $S(x, r_x) \subset G$ and $v|_\Gamma = \varphi$. Then $u(\cdot)$ solves the Dirichlet problem* (2.22).

Proof. The proof repeats the arguments used in the case of the Laplace equation, since the strong maximum principle holds (this follows from the inequality $|w_m(r, \lambda)| \geq 1$ for nonnegative λ). □

2.2.2 Helmholtz equation

If $\lambda < 0$, we come to the Helmholtz equation.

We suppose in this case that $G \subset \mathbb{R}^3$ is a bounded domain and $0 > \lambda > \lambda_0(G)$ where $\lambda_0(G)$ is the principal eigenvalue of the Laplace operator in G.

Note that $\sqrt{\lambda} < \frac{\pi}{d^*}$; indeed, it is well known that if $G_2 \subset G_1$, then $\lambda_0(G_2) \geq \lambda_0(G_1)$; in our case, for all $x \in G$, $S(x, d^*) \subset G$, and the eigenvalue $\lambda_0(S(x, d^*))$ is equal to $\frac{\pi^2}{d^{*2}}$.

Then the solution to (2.23) can be written in $\mathbb{R}^3$ in the form

$$u(x) = \frac{\sqrt{\lambda}r}{\sin(\sqrt{\lambda}r)} N_r u(x)\,, \tag{2.29}$$

for any r such that $S(x, r)$ is included in G.

Under these restrictions the maximum principle is true (see [56]) and we conclude that the converse spherical mean value theorem is true.

Note that for the Helmholtz equation we can derive the volume mean value relation in a ball $B(x, r) \subset \mathbb{R}^n$

$$u(x) = \frac{1}{V_r 2^{n/2}\Gamma(\frac{n}{2}+1)\tau_{n/2}(\sqrt{\lambda}r)} \int\limits_{B(x,r)} u(y)\,dV(y)\,,$$

where $V_r = \pi^{n/2}r^n/\Gamma(n/2+1)$ is the volume of the ball $B(x, r)$ and $\tau_\alpha(z) = z^{-\alpha}J_\alpha(z)$. We notice that the function $\tau_\alpha(z)$ is related to the function $W_\alpha(z)$ by

$$W_\alpha(z) = \Gamma(\alpha/2)2^{\alpha/2-1}\tau_{\alpha/2-1}(z)\,.$$

The above volume mean value relation is obtained by integrating the spherical mean value relation for the Helmholtz equation

$$u(x) = \frac{1}{W_n\omega_n} \int\limits_{\Omega} u(x + rs)\,d\Omega\,.$$

Remark 2.3. The weak spherical mean value relations characterize the high-order elliptic equations with constant coefficients. Let us mention the following result (see, e.g. [55] and [56]) which we briefly mentioned above.

Let μ be a nonnegative Borel measure with total mass equal to 1, such that the support of μ is contained in the unit sphere of $\mathbb{R}^m$ and not contained in any hyperplane. If u is a continuous function on some open set $G \subset \mathbb{R}^m$ having the mean value property

$$u(x) = \int u(x + ry)\mu(dy)$$

for every $x \in G$ and every positive $r < d(x)$, then $u \in C^\infty(G)$ and

$$\sum_{i_1+\cdots+i_m=n} A_{i_1\ldots i_m} \frac{\partial^n u}{\partial x_1^{i_1} \cdots \partial x_m^{i_m}} = 0, \quad n = 1, 2, \ldots ,$$

where the coefficients are the moments

$$A_{i_1\ldots i_m} = \int x_1^{i_1} \cdots x_m^{i_m} \mu(dx) .$$

Conversely, every infinitely differentiable solution of the last system of differential equations has the above mean-value property.

2.3 Generalized second-order elliptic equations

The expansions of the spherical mean of the type (2.10), (see also (3.10) below) suggest a device for defining a generalized Laplacian of nondifferentiable functions. For example, if in

$$\Delta v(x) = f$$

the functions v and f are merely continuous, one can say that $\Delta v = f$ in a generalized sense if

$$\lim_{r\to 0} \frac{1}{r^2} [N_r v - v] = \frac{f}{2m} .$$

An analogous point of view is reported in [141] and [187].

From an expansion that we will obtain in Chapter 5 it is possible to define generalized solutions to the Lamé equation

$$\mu\Delta\mathbf{u}(x) + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u}(x) = \mathbf{f}(x) ,$$

where $x \in \mathbb{R}^n$, $\mathbf{u} = (u_1, \ldots, u_n)$, as continuous functions satisfying the relation

$$\lim_{r\to 0} \frac{1}{r^2} [N_r^1 \mathbf{u} - \mathbf{u}] = \frac{\mathbf{f}}{2(\lambda + \mu(n+1))} . \qquad (2.30)$$

Here, the averaging matrix operator N_r^1 is defined by

$$(N_r^1\mathbf{u})_i(x) = \frac{1}{\omega_n}\int_\Omega (a + bs_i^2)u_i(x+rs)\,d\Omega + \frac{b}{\omega_n}\sum_{j\neq i}^{n}\int_\Omega s_is_ju_j(x+rs)\,d\Omega\,,\quad i=1,\dots,n\,,$$

where $a = 1-\beta$, $b = n\beta$, and

$$\beta = \frac{(n+2)(\lambda+\mu)}{2(\lambda+\mu(n+1))}\,.$$

In [58], elliptic equations with nonconstant coefficients were considered:

$$Lu = 0\,,$$

where

$$L = \sum_{i,k=1}^{m} a_{ik}\frac{\partial^2}{\partial x_i\partial x_k}\,,$$

and the coefficients are supposed merely measurable and bounded and the matrix $a(x) = (a_{ik}(x))$ is symmetric and positive definite. Fulks [58] (see also [141]) proved that

$$\frac{1}{\omega_m}\int_\Omega u(x + ra(x)^{1/2}s)H_{m-1}(ds) - u(x) = \frac{r^2}{2m}Lu(x) + O(r^2) \tag{2.31}$$

as $r \to 0$, u being any twice continuously differentiable function. Here, $a(x)^{1/2}$ denotes the positive square root of the matrix $a(x)$ and H_{m-1} the $(m-1)$-dimensional measure. From (2.31) it is convenient to pass to the mean over an ellipsoid. Multiplying (2.31) by mr^{m-1} and integrating with respect to r, we obtain, using an obvious change of variables in the m-fold repeated integral

$$\frac{1}{\operatorname{meas}\mathcal{E}(x,r)}\int_{\mathcal{E}(x,r)} u(y)dy - u(x) = \frac{r^2}{2(m+2)}Lu(x) + O(r^2)\,, \tag{2.32}$$

where $\mathcal{E}(x,r)$ is the ellipsoid

$$\mathcal{E}(x,r) = \{y \in \mathbb{R}^m : (a^{-1}(x)(y-x), y-x) < r^2\}\,,$$

with the center at x and axes on the eigenvectors of the matrix $a(x)$, while

$$\operatorname{meas}\mathcal{E}(x,r) = (\omega_m/m)r^m[\det a(x)]^{1/2}\,.$$

Note that the boundaries $\partial\mathcal{E}(x,r)$ of the ellipsoids $\mathcal{E}(x,r)$ are the level surfaces of the Levi function

$$\mathcal{L}(x,y) = \frac{1}{(m-2)\omega_m[\det a(x)]^{1/2}}\left(a^{-1}(x)(y-x), y-x\right)^{1-m/2}\,,$$

the fundamental solution of the constant elliptic operator L, x fixed.

Note that the same approach was used in [51] when constructing the Random Walk on ellipsoids.

Remark 2.4. Let $C = (c_{jk})$ be a symmetric $n \times n$-matrix and $r \neq 0$, and let

$$\sigma_x^{r,C}(u) = \int_{\mathcal{E}_r} u(x+y)\,do(y)\,,$$

where the integration is taken over the surface of the ellipsoid

$$\mathcal{E}_r = \left\{ y : \sum c_{jk} y_j y_k = r^2 \right\}\,.$$

It can be shown (e.g. see [73]) that if u is the solution to the following equation with constant matrix of coefficients $A = (a_{jk})$:

$$\sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} = 0\,,$$

then the ellipsoidal mean value relation

$$\sigma_x^{r,C}(u) = \sigma_x^{r,C+tA}(u)$$

is true for all t.

For the Laplace equation it means that the integrals of harmonic functions over the confocal ellipsoids do not depend on the parameter t (if the function is considered in a domain G, then t is such that the convex hull of supp $\sigma_x^{r,C+tA}(u)$ lies in G).

2.4 Parabolic equations

2.4.1 Heat equation

Let Q be a domain in $\mathbb{R}^{n+1}$ whose points will be denoted by $(x,t) = (x_1, x_2, \ldots, x_n, t)$. We deal in this section with the heat equation

$$\frac{\partial u}{\partial t} = \Delta u(x,t) + f(x,t), \quad (x,t) \in Q\,. \tag{2.33}$$

Let

$$Z(x,t) = \theta(t)[4\pi t]^{-n/2} \exp\left\{ \frac{|x|^2}{4t} \right\}$$

be the fundamental solution to (2.33), where $\theta(t)$ is the Heaviside step function: $\theta(t) = 0$ if $t \leq 0$ and $\theta(t) = 1$ if $t > 0$. Introduce the function

$$Z^{(\alpha)} = Z(x,t) - (4\pi\alpha)^{-n/2}\,,$$

depending on a positive parameter α and define a family of domains $B_\alpha(x,t)$:

$$B_\alpha(x,t) = \left\{ (x',t') \in \mathbb{R}^{n+1} : \ t' < t,\ Z^{(\alpha)}(x-x', t-t') > 0 \right\}$$

with the boundaries

$$\partial B_\alpha(x,t) = \left\{(x',t') \in \mathbb{R}^{n+1} :\ t' \le t,\ Z^{(\alpha)}(x-x',t-t') = 0\right\}.$$

We also define a family of domains $B_\alpha^{(\beta)}(x,t)$ depending on the parameter β, $0 < \beta \le \alpha$:

$$B_\alpha^{(\beta)}(x,t) = B_\alpha(x,t) \cap \mathbb{R}^n \times (t-\beta,t),$$

and let

$$\partial B_\alpha^{(\beta)}(x,t) = \partial B_\alpha(x,t) \cap \mathbb{R}^n \times [t-\beta,t].$$

We call the domain $B_\alpha(x,t)$ a balloid and the set $\partial B_\alpha(x,t)$, a spheroid of radius α with the center at (x,t). The domains $B_\alpha^{(\beta)}(x,t)$ and $\partial B_\alpha^{(\beta)}(x,t)$ are called truncated balloid and truncated spheroid, respectively.

We now obtain a mean value theorem where the solution of (2.33) at the point (x,t) is expressed in terms of its values integrated over a spheroid or a truncated spheroid with the center at (x,t). Then by averaging we derive the relations where the solution of (2.33) at the point (x,t) is expressed in terms of its values integrated over the spheroid and balloid with the center at (x,t).

Note that the balloid $B_\alpha(x,t)$ is situated between the planes $t' = t$ and $t' = t-\alpha$. The intersection of $B_\alpha(x,t)$ and the plane $t' = t-\tau$ $(0 < \tau < \alpha)$ is an n-dimensional ball $B(x,R(\tau))$ where

$$R(\tau) = (2\tau n \ln(\alpha/\tau))^{1/2}.$$

The maximum of the radius $R(\tau)$ is attained at $\tau = \alpha/e$; its value is $R_{\max} = (2n\alpha/e)^{1/2}$. Hence,

$$B_\alpha(x,t) \subset B(x,R_{\max}) \times (t-\alpha,t), \tag{2.34}$$

and

$$\partial B_\alpha^{(\beta)}(x,t) \subset B(x,R_{\max}) \times (t-\beta,t).$$

Relation (2.34) shows that the balloid $B_\alpha(x,t)$ tends to the point (x,t) as $\alpha \to 0$. Therefore, for each point $(x,t) \in Q$, there exists α such that

$$B_\alpha(x,t) \subset Q. \tag{2.35}$$

Thus let $(x,t) \in Q$, and we suppose that α is chosen so that condition (2.35) is satisfied. Obviously,

$$\frac{\partial}{\partial t'} Z^{(\alpha)}(x-x',t-t') + \Delta_{x'} Z^{(\alpha)}(x-x',t-t') = 0, \quad t < t', \tag{2.36}$$

where Δ_y is the Laplace operator (differentiation with respect to the variable y). From (2.33) and (2.36) we get

$$\begin{aligned} &Z^{(\alpha)}(x-x',t-t')\Delta_{x'}u(x',t') - u(x',t')\Delta_{x'}Z^{(\alpha)}(x-x',t-t') \\ &\qquad = \frac{\partial}{\partial t'}\left\{Z^{(\alpha)}(x-x',t-t')u(x',t')\right\} - Z^{(\alpha)}(x-x',t-t')f(x',t'). \end{aligned}$$

Integrating this equality in the ball $B(x, R(t-t'))$ we get by the Green formula

$$\begin{aligned}
&\int\limits_{B(x,R(t-t'))} \frac{\partial}{\partial t'} \left\{ Z^{(\alpha)}(x-x', t-t') u(x', t') \right\} dx' \\
&\quad = \int\limits_{\partial B(x,R(t-t'))} \left\{ Z^{(\alpha)} \frac{\partial u}{\partial n_{x'}} - u(x', t') \frac{\partial}{\partial n_{x'}} Z^{(\alpha)} \right\} dS_{x'} \\
&\quad + \int\limits_{B(x,R(t-t'))} \left\{ Z^{(\alpha)}(x-x', t-t') f(x', t') \right\} dx' ,
\end{aligned} \tag{2.37}$$

where $Z^{(\alpha)} = Z^{(\alpha)}(x-x', t-t')$, $n_{x'}$ is the exterior normal vector at x', and $dS_{x'}$ is the surface element of the sphere $\partial B(x, R(t-t'))$ at the point x'. Notice that $Z^{(\alpha)}(x - x', t-t') = 0$ for $x' \in \partial B(x, R(t-t'))$ (by definition); hence we can change the order of the operator $\frac{\partial}{\partial t'}$ and the integral on the left-hand side of (2.37). Thus we get from (2.37)

$$\begin{aligned}
&\frac{\partial}{\partial t'} \left\{ \int\limits_{B(x,R(t-t'))} Z^{(\alpha)} u(x', t') dx' \right\} \\
&\quad = \int\limits_{\partial B(x,R(t-t'))} -\frac{\partial Z^{(\alpha)}}{\partial n_{x'}} u(x', t') dS_{x'} + \int\limits_{B(x,R(t-t'))} Z^{(\alpha)} f(x', t') dx' .
\end{aligned} \tag{2.38}$$

We now integrate (2.38) with respect to t' over $[t-\beta, t]$, $(0 < \beta \le \alpha)$:

$$\begin{aligned}
&\lim_{t' \to t} \int\limits_{B(x,R(t-t'))} Z^{(\alpha)}(x-x', t-t') u(x', t') dx' \\
&\quad - \int\limits_{B(x,R(\beta))} Z^{(\alpha)}(x-x', \beta) u(x', t-\beta) dx' \\
&\quad = \int\limits_{\partial B_\alpha^{(\beta)}(x,t)} -\frac{\partial Z^{(\alpha)}}{\partial n_{x'}} (x-x', t-t') u(x', t') dS_{x'} dt' \\
&\quad + \int\limits_{B_\alpha^{(\beta)}(x,t)} Z^{(\alpha)}(x-x', t-t') f(x', t') dx' dt' .
\end{aligned} \tag{2.39}$$

We prove that

$$\lim_{t' \to t} \int\limits_{B(x,R(t-t'))} Z^{(\alpha)}(x-x', t-t') u(x', t') dx' = u(x, t) . \tag{2.40}$$

Indeed, let

$$\gamma(\rho, \tau) = (4\pi\tau)^{-n/2} \exp\left\{ -\frac{\rho^2}{4\tau} \right\} - (4\pi\alpha)^{-n/2}$$

and let $\tau = t - t'$. Then

$$
\begin{aligned}
&\int\limits_{B(x,R(\tau))} Z^{(\alpha)}(x - x', \tau)u(x', t - \tau)dx' \\
&= \int\limits_0^{R(\tau)} \rho^{n-1}d\rho \int\limits_{S(0,1)} \gamma(\rho,\tau)u(x + \rho\omega, t - \tau)d\Omega(\omega) \\
&= \int\limits_0^{R(\tau)} \rho^{n-1}d\rho \int\limits_{S(0,1)} \gamma(\rho,\tau)u(x, t - \tau)d\Omega(\omega) \\
&\quad + \int\limits_0^{R(\tau)} \rho^{n-1}d\rho \int\limits_{S(0,1)} \gamma(\rho,\tau)[u(x + \rho\omega, t - \tau) - u(x, t - \tau)]d\Omega(\omega)\,,
\end{aligned}
\tag{2.41}
$$

where $d\Omega(\omega)$ is the surface element of the unit sphere $S(0,1)$. Now,

$$
\begin{aligned}
&\int\limits_0^{R(\tau)} \rho^{n-1}\gamma(\rho,\tau)\omega_n d\rho \\
&= \int\limits_0^{R(\tau)} \frac{2\rho^{n-1}}{\Gamma(n/2)} \left\{(4\tau)^{-n/2}\exp\{-\rho^2/4\tau\} - (4\alpha)^{-n/2}\right\} d\rho \\
&= \int\limits_0^{\frac{n}{2}\ln\frac{\alpha}{\tau}} \frac{r^{n/2-1}}{\Gamma(n/2)} \left(e^{-\tau} - \left(\frac{\tau}{\alpha}\right)^{n/2}\right) dr \to 1
\end{aligned}
\tag{2.42}
$$

as $\tau \to 0$. We used here the fact that $\omega_m = 2\pi^{n/2}/\Gamma(n/2)$. Analogously, we can prove that

$$
\int\limits_0^{R(\tau)} \rho^n\gamma(\rho,\tau)d\rho \to 0
\tag{2.43}
$$

as $\tau \to 0$.

We get from (2.41) by (2.42) and (2.43) the desired relation (2.40). Thus, from (2.39) and in view of (2.40), we get the following mean value relation:

Theorem 2.7. *If the parameters α and β, $0 < \beta \le \alpha$, are chosen so that $B_\alpha^{(\beta)}(x,t) \subset Q$, then the regular solutions to the heat equation* (2.33) *satisfy the relation*

$$
\begin{aligned}
u(x,t) = &\int\limits_{B(x,R(\beta))} Z^{(\alpha)}(x - x', \beta)u(x', t - \beta)dx' \\
&+ \int\limits_{\partial B_\alpha^{(\beta)}(x,t)} \left(-\frac{\partial Z^{(\alpha)}}{\partial n_{x'}}\right) u(x', t')dS_{x'}dt' \\
&+ \int\limits_{B_\alpha^{(\beta)}(x,t)} Z^{(\alpha)}(x - x', t - t')f(x', t')dx'dt'\,.
\end{aligned}
\tag{2.44}
$$

An important particular case of (2.44) is $\beta = \alpha$:

$$u(x,t) = \int\limits_{\partial B_\alpha(x,t)} -\frac{\partial Z^{(\alpha)}}{\partial n_{x'}}(x-x',t-t')u(x',t')dS_{x'}dt' + \int\limits_{B_\alpha(x,t)} Z^{(\alpha)}(x-x',t-t')f(x',t')dx'dt' ,$$

which was derived in [97]. Further mean value relations can be obtained by the integration of (2.44) with respect to β over the interval $[0,\alpha]$. This yields

Theorem 2.8. *Under the assumptions of the previous theorem, the following mean value relation holds:*

$$\begin{aligned} u(x,t) = & \int\limits_{\partial B_\alpha(x,t)} \left(1-\frac{t-t'}{\alpha}\right)\left(-\frac{\partial Z^{(\alpha)}}{\partial n_{x'}}(x-x',t-t')\right)u(x',t')dS_{x'}dt' \\ & + \frac{1}{\alpha}\int\limits_{B_\alpha(x,t)} Z^{(\alpha)}(x-x',t-t')u(x',t')dx'dt' + F_\alpha(x,t) , \end{aligned} \tag{2.45}$$

where

$$F_\alpha(x,t) = \int\limits_{B_\alpha(x,t)} \left(1-\frac{t-t'}{\alpha}\right) Z^{(\alpha)}(x-x',t-t')f(x',t')dx'dt' . \tag{2.46}$$

Here we used the relations

$$\begin{aligned} \int\limits_0^\alpha d\beta \int\limits_{B(x,R(\beta))} Z^{(\alpha)}(x-x',\beta)u(x',t-\beta)dx' & = \int\limits_{B_\alpha(x,t)} Z^{(\alpha)}(x-x',t-t')u(x',t')dx'dt' , \\ \int\limits_0^\alpha d\beta \int\limits_{\partial B_\alpha^{(\beta)}(x,t)} -\frac{\partial Z^{(\alpha)}}{\partial n_{x'}}(x-x',t-t')u(x',t')dS_{x'}dt' & = \int\limits_{\partial B_\alpha(x,t)} (\alpha-(t-t'))\left(-\frac{\partial Z^{(\alpha)}}{\partial n_{x'}}\right)u(x',t')dS_{x'}dt' , \\ \int\limits_0^\alpha d\beta \int\limits_{B_\alpha^{(\beta)}(x,t)} Z^{(\alpha)}(x-x',t-t')f(x',t')dx'dt' & = \int\limits_{B_\alpha(x,t)} (\alpha-(t-t'))Z^{(\alpha)}(x-x',t-t')f(x',t')dx'dt' . \end{aligned}$$