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Moshe Marcus
Laurent Véron

Nonlinear Second Order Elliptic Equations Involving Measures

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Authors
Prof. Dr. Moshe Marcus
Technion – Israel Institute of Technology
Dept. of Mathematics
Technion City
32000 Haifa
Israel
marcusm@math.technion.ac.il

Prof. Dr. Laurent Véron
Laboratoire de Mathematiques
CNRS UMR 6083
Faculte des Sciences et Techniques
Universite Francois Rabelais
Parc de Grandmont
37200 Tours
France
veronl@univ-tours.fr

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Preface

In the last 40 years semilinear elliptic equations became a central subject of study in the theory of nonlinear partial differential equations. On the one hand, the interest in this area is of a theoretical nature, due to its deep relations to other branches of mathematics, especially linear and nonlinear harmonic analysis and probability. On the other hand, this study is of interest because of its applications. Equations of this type come up in various areas such as: problems of physics and astrophysics, problems of differential geometry, logistic problems related for instance to population models and, most importantly, the study of branching processes and superdiffusions.

An important family of such equations is that involving an absorption term, the model of which is $-\Delta u + g(x, u) = 0$ where $ug(x, u) \geq 0$. Such equations are of particular interest because in them we have two competing effects: the diffusion expressed by the linear differential part and the absorption produced by the nonlinearity g. Furthermore, equations of this type with power nonlinearities play a crucial role in the study of superdiffusions.

Naturally, the study of semilinear problems is based on linear theory and in particular on the theory of boundary value problems with L^1 and, more generally, measure data. In addition to the classical theory of the Laplace equation, this study requires certain ideas of harmonic analysis such as the Herglotz theorem on boundary trace of positive harmonic functions and the resulting integral representation, Kato's lemma and the boundary Harnack principle. These topics and their application to boundary value problems are treated in the first chapter.

In the second chapter we turn to the main topic of this monograph: boundary value problems for the semilinear problem

$$\begin{aligned} -\Delta u + g(x, u) &= f \quad \text{in } \Omega \\ u &= h \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

where f and h are L^1 functions or more generally measures. Generally we assume that $t \mapsto g(\cdot, t)$ is a continuous mapping from $\mathbb{R}$ into $L^1(\Omega; \rho)$, where $\rho(x) = dist(x, \partial\Omega)$, that $g(x, \cdot)$ is non-decreasing for every $x \in \Omega$ and $g(x, 0) = 0$. ($L^1(\Omega; \rho)$ denotes the weighted Lebesgue space with weight ρ.) In addition we assume that

$$\lim_{t \to \infty} g(\cdot, t)/t = \infty \tag{2}$$

uniformly with respect to x in compact subsets of Ω. Two standard examples:

$$g(x, t) = \rho(x)^{\beta} |t|^{q-1} t, \quad g(x, t) = \exp t - 1. \tag{3}$$

The problem (1) is understood in a weak sense; we require that $u \in L^1(\Omega)$ and $g \circ u \in L^1(\Omega; \rho)$, that the equation holds in the distribution sense and that the data is attained in a weak sense, related to weak convergence of measures. In addition it is assumed that $f \in L^1(\Omega; \rho)$ or, more generally, $f = \mu \in \mathfrak{M}(\Omega; \rho)$, i.e., μ is a Borel measure in Ω such that

$$\int_\Omega \rho \, d|\mu| < \infty.$$

For the boundary data, it is assumed that $h \in L^1(\partial\Omega)$ or, more generally, $h = \nu \in \mathfrak{M}(\partial\Omega)$, i.e., ν is a finite Borel measure on $\partial\Omega$.

Problems with L^1 data are discussed in Section 2.1. In this case the boundary value problem possesses a unique solution $u \in L^1(\Omega)$ such that $g \circ u \in L^1(\Omega; \rho)$ for every $f \in L^1(\Omega; \rho)$ and $h \in L^1(\partial\Omega)$.

An interesting feature of boundary value problems with measure data is that, in general, the problem is not solvable for every measure. If (1) has a solution for $h = 0$ and a measure $f = \mu \in \mathfrak{M}(\Omega; \rho)$, we say that μ is g-good in Ω. The space of such measures is denoted by $\mathfrak{M}^g(\Omega; \rho)$. Similarly, if (1) has a solution for $f = 0$ and a measure $h = \nu \in \mathfrak{M}(\partial\Omega)$, we say that ν is g-good on $\partial\Omega$. The space of such measures is denoted by $\mathfrak{M}^g(\partial\Omega)$. If $\mathfrak{M}^g(\Omega; \rho) = \mathfrak{M}(\Omega; \rho)$ we say that the nonlinearity g is subcritical in the interior. Similarly, if $\mathfrak{M}^g(\partial\Omega) = \mathfrak{M}(\partial\Omega)$ we say that g is subcritical relative to the boundary.

In Section 2.2 we present basic results on boundary value problems with measures. For instance, assuming that μ and ν are g-good, we show that (1) with $f = \mu$, $h = \nu$ has a unique weak solution u and derive estimates for $\|u\|_{L^1(\Omega)}$ and $\|g \circ u\|_{L^1(\Omega;\rho)}$ in terms of the norms of μ and ν in their respective spaces. In particular we find that, if a solution exists it is unique.

An important tool in our study is an extension of the method of sub- and supersolutions to the case of weak solutions and a general class of nonlinearities. This too is presented in Section 2.2

In Section 2.3 we present a sufficient condition for interior and boundary subcriticality. It is shown that this condition also implies stability with respect to weak convergence of data. Further, in Section 2.4, we discuss the structure of the space of good measures when the nonlinearity g is supercritical in the interior (resp. on the boundary), i.e., $\mathfrak{M}^g(\Omega; \rho) \subsetneq \mathfrak{M}(\Omega; \rho)$ (resp. $\mathfrak{M}^g(\partial\Omega) \subsetneq \mathfrak{M}(\partial\Omega)$).

Chapter 3 is devoted to a study of the boundary trace problem for positive solutions of the equation

$$-\Delta u + g(x, u) = 0, \tag{4}$$

with g as in (1), and related boundary value problems. The basic model for our study is the boundary trace theory for positive harmonic functions due to Herglotz.

By Herglotz's theorem any positive harmonic function in a bounded Lipschitz domain admits a boundary trace expressed by a bounded measure and the harmonic function is uniquely determined by this trace via an integral representation.

The notion of a boundary trace of a function u in Ω depends on the regularity properties of the function. For instance, if $u \in C(\bar{\Omega})$ then it has a boundary trace in $C(\partial\Omega)$, namely, $u\lfloor_{\partial\Omega}$. If u belongs to a Sobolev space $W^{1,p}(\Omega)$ for some $p > 1$ then it has a boundary trace in $L^p(\partial\Omega)$ (and even in a more regular space, namely, $W^{1-\frac{1}{p},p}(\partial\Omega)$). The measure boundary trace of a positive harmonic function is defined as follows: let $\{\Omega_n\}$ be an increasing sequence of domains converging to Ω; under some restrictions on this sequence it can be shown that the sequence of measures $\{u\lfloor_{\partial\Omega_n} dS\}$ converges weakly in $\mathfrak{M}(\bar{\Omega})$ (= the space of finite Borel measures in $\bar{\Omega}$) to a measure $\nu \in \mathfrak{M}(\partial\Omega)$ that is independent of $\{\Omega_n\}$. This limiting measure is the measure boundary trace of u. If Ω is of class C^2 the harmonic function u can be recovered from its measure boundary trace via the Poisson integral. If the domain is merely Lipschitz, the Poisson kernel must be replaced by the Martin kernel. (For more details see Section 1.3.)

As a first step in our study of the trace problem for positive solutions of (4) we consider *moderate* solutions. A positive solution of (4) is moderate if it is dominated by a harmonic function. The following result is a consequence of the Herglotz theorem.

A positive solution u is moderate if and only if $g \circ u \in L^1(\Omega; \rho)$. Every positive moderate solution possesses a boundary trace represented by a bounded measure.

So far the trace problem for positive solutions of the nonlinear equation appears to be similar to the trace problem for positive harmonic functions. However, beyond this similarity, the nonlinear problem presents two essentially new aspects. The first is a fact already mentioned before: in general, there exist positive finite measures on $\partial\Omega$ that are not boundary traces of any solution of (4). The second: the equation may have positive solutions that do not have a boundary trace in $\mathfrak{M}(\partial\Omega)$.

Both aspects are present in the basic examples (3). In the case of power nonlinearities $g(t) = |t|^q \operatorname{sign} t$, if $q \geq (N+1)/(N-1)$ and $N \geq 2$ there is no solution with boundary trace given by a Dirac measure. In fact in this case there is no solution with an isolated singularity. In other words, isolated point singularities are *removable*. (For details see Subsection 3.4.3 and 4.2.1.)

The second aspect occurs whenever g satisfies the Keller–Osserman condition discussed below. This condition is satisfied by power nonlinearities for every $q > 1$ and by the exponential nonlinearity.

J.B. Keller [60] and R. Osserman [96] provided a sharp condition on the growth of g at infinity which guarantees that the set of solutions of (4) is *uniformly bounded from above* in compact subsets of Ω. Qualitatively the condition means that the superlinearity of g at infinity is sufficiently strong. Assuming that this condition holds uniformly with respect to $x \in \Omega$, they derived an a priori estimate for solutions of (4) in terms of $\rho(x) = \operatorname{dist}(x, \partial\Omega)$. This estimate implies that equation (4), in bounded domains, possesses a *maximal solution*. If, in addition, Ω satisfies the classical Wiener condition then the maximal solution blows up everywhere on the boundary. (If $g(x, 0) = 0$ the boundedness assumption on the domain is not needed.) A solution that blows up everywhere on the boundary is called a *large solution*. Evidently, large solutions do not posses a boundary trace in $\mathfrak{M}(\partial\Omega)$.

In Section 3.1 we show that every positive solution has a boundary trace that is given by an outer regular Borel measure; however this measure need not be finite. If the solution is moderate this reduces to the boundary trace previously mentioned. The boundary trace $\bar{\mu}$ of a positive solution u has a singular set F (possibly empty) such that $\bar{\mu}$ is infinite on F while $\bar{\mu}$ is a Radon measure on $\partial\Omega \setminus F$. The singular set is closed. A point $y \in \partial\Omega$ is singular (relative to u) if $y \in F$ and regular otherwise. The singular and regular boundary points are determined by a local integral condition.

A boundary trace $\bar{\mu}$ can also be represented by a couple (F, μ) where F is the singular set of the trace and μ is a Radon measure on $\partial\Omega \setminus F$. The set of all positive measures that can be represented in this manner is denoted by $\mathcal{B}_{reg}$. A solution whose boundary trace is of the form $(F, 0)$ is called a purely singular solution.

Assumimg that the Keller–Osserman condition holds uniformly in Ω, for every compact set $F \subset \partial\Omega$ there exists a solution U_F that is maximal in the set of solutions vanishing on $\partial\Omega \setminus F$ (see Section 3.2). U_F is called the *maximal solution relative to* F. In the subcritical case, the boundary trace of U_F is $(F, 0)$. In the supercritical case, the singular set of U_F – denoted by $k_g(F)$ – may be smaller than F. The maximal solutions U_F play a crucial role in the study of the boundary value problem

$$\begin{aligned} -\Delta u + g(x,u) &= 0 \quad \text{in } \Omega \\ u &= \bar{\mu} \quad \text{on } \partial\Omega \end{aligned} \tag{5}$$

when $g \in \mathcal{G}_0$ and $\bar{\mu} \in \mathcal{B}_{reg}$.

In Section 3.3 we present a general result providing necessary and sufficient conditions for existence and uniqueness of solutions of (5) assuming that g satisfies the local Keller–Osserman condition and the global barrier condition and that, for every $x \in \Omega$, $g(x, \cdot)$ is convex. (See definitions 3.1.9 and 3.1.10.) These conditions are sufficient for the existence of the maximal solution U_F.

In Section 3.4 we study problem (5) when g is given by

$$g(x,t) = \rho(x)^{\beta} |t|^{q-1} t, \quad q > 1,\ \beta > -2. \tag{6}$$

Assuming that Ω is a smooth domain we show: (i) A g-barrier exists at every boundary point and the global barrier condition holds and (ii) g is subcritical if and only if

$$1 < q < q_c(\alpha) := (N + \beta + 1)/(N - 1).$$

Next we apply the result of Section 3.3 to problem (5) with g as above assuming that q is in the subcritical range. We show that, under these assumptions:

Problem (5) *possesses a unique solution for every* $\bar{\mu} \in \mathcal{B}_{reg}$.

There follows a description of the main steps in the proof of this result:

I. *For every* $y \in \partial\Omega$ *there is a unique solution with boundary trace* $(\{y\}, 0)$ *denoted by* $u_{\infty,y}$.

II. *If u is a solution with singular boundary set F then for every $y \in F$,*

$$u \geq u_{\infty,y}.$$

Using these two results we show that:

III. *For every compact $F \subset \partial\Omega$, the maximal solution U_F is the unique solution with trace* $(F, 0)$.

The proof is completed by establishing the following:

IV. *If, for every compact set $F \subset \partial\Omega$,* (5) *has a unique solution with boundary trace* $(F, 0)$ *then the boundary value problem has a unique solution for every measure* $\bar{\mu} \in \mathcal{B}_{reg}$.

Two particular cases of the boundary value problem (5) have received special attention in the literature.

The first is the case of large solutions already mentioned above. In the language of boundary traces, the singular boundary set of a large solution is the whole boundary. In the case of Lipschitz domains, the global Keller–Osserman condition implies the existence of a large solution. However, in more general domains, the maximal solution may not blow up everywhere on the boundary. Therefore, in such a case a large solution does not exist.

The question of existence and uniqueness of a large solution under various assumptions on g and Ω has been a subject of intense study. In addition to its intrinsic interest, this topic is useful in delineating the limitations that are naturally imposed on the goals of our study of general boundary value problems.

The subject of large solutions is discussed in detail in Chapter 5.

The second case to receive special attention is that of solutions with isolated singularities. If the nonlinearity is subcritical then, for every $y \in \partial\Omega$ there exist *moderate* solutions with isolated singularity at y. If, in addition, a g-barrier exists at y then there exist non-moderate solutions with an isolated singularity at y. Such a solution is called a ‘very singular solution’. Alternatively we say that the solution has a ‘strong isolated singularity’ at y.

Assume that g is subcritical and that a g-barrier exists at $y \in \partial\Omega$. Let $u_{k,y}$ denote the solution with boundary trace $k\delta_y$. For $k > 0$ this solution is dominated by $kP(\cdot, y)$ (where P denotes the Poisson kernel); therefore it is a moderate solution. However, the existence of a barrier at y implies that

$$u_{\infty,y} = \lim_{k\to\infty} u_{k,y} \tag{7}$$

is a solution of the equation which vanishes on $\partial\Omega \setminus \{y\}$. Evidently this solution has a strong singularity at y. The analysis of the set of solutions with strong isolated singularities plays an important role in the study of boundary value problems in the subcritical case. A question of special interest is the uniqueness of the very singular solution at

y. A related question is that of the asymptotic behavior of such solutions. These questions are studied in Section 3.4 and Chapter 4 for various families of nonlinearities, applying different methods.

Several other problems associated to singular and large solutions are considered in Chapter 6. These include: the limit of fundamental solutions when the mass goes to infinity; symmetry of large solutions; higher order terms in the asymptotics of large solutions and their dependence on the geometry of the domain.

This monograph was conceived and planned jointly by the two authors. However, it falls into two essentially independent parts. The first part, consisting of Chapters 1–3, was written by the first author and is an outgrowth of a set of notes [73] originally intended for inclusion in a handbook planned by North Holland Ltd. (The handbook project was terminated before the completion of the notes.) The second part, consisting of Chapters 4–6, was written by the second author.

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Contents

Chapter 1

Linear second order elliptic equations with measure data

1.1 Linear boundary value problems with L^1 data

We begin with linear boundary value problems with L^1 data of the form

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= \eta \quad \text{on } \partial\Omega \end{aligned} \tag{1.1.1}$$

where Ω is a domain in $\mathbb{R}^N$. To simplify the presentation we shall assume that $N \geq 3$. However, with slight modifications, most of the results apply as well to $N = 2$. Unless otherwise stated, we assume that Ω is a bounded domain of class C^2.

Definition 1.1.1. A bounded domain $\Omega \subset \mathbb{R}^N$ is of class C^2 if there exists a positive number r_0 such that, for every $X \in \partial\Omega$, there exists a set of Cartesian coordinates $\xi = \xi^X$, centered at X, and a function $F_X \in C^2(\mathbb{R}^{N-1})$ such that $F_X(0) = 0$, $\nabla F_X(0) = 0$ and

$$\Omega \cap B_{r_0}(X) = \{\xi : |\xi| < r_0,\ \xi_1 > F_X(\xi_2, \ldots, \xi_N)\}. \tag{1.1.2}$$

The set of coordinates ξ^X is called a *normal set of coordinates* at X and F_X is called the *local defining function* at X.

The normal set of coordinates at X is not uniquely defined. However, the direction of the positive ξ_1^X axis coincides with the direction of the unit normal at X pointing into the domain and two sets of normal coordinates at X are related by a rotation around the ξ_1^X axis. As Ω is bounded, $\partial\Omega$ can be covered by a finite number of balls $\{B_{r_0}(X_i)\}_{i=1}^k$, $X_1, \ldots, X_k \in \partial\Omega$. Therefore,

$$\|\partial\Omega\|_{C^2} := \sup\{\|F_X\|_{C^2(\bar{B}_{r_0}(0))} : x_0 \in \partial\Omega\} < \infty$$

and there exists $\kappa \in C(0,1)$ such that $D^2 F_X$ has modulus of continuity κ for every $X \in \partial\Omega$. The pair $(r_0, \|\partial\Omega\|_{C^2})$ is called a C^2 *characteristic of* Ω.

We denote by G and P the Green and Poisson kernels respectively of $-\Delta$ in Ω. If $f \in C^1(\bar{\Omega})$ and $\eta \in C(\partial\Omega)$ then a classical result states that the boundary value problem (1.1.1) possesses a unique solution in $C^2(\Omega) \cap C(\bar{\Omega})$. The solution u is given by

$$u(x) = \int_\Omega G(x,y) f(y) dy + \int_{\partial\Omega} P(x,y)\eta(y) dS_y. \tag{1.1.3}$$

Put

$$C_0^2(\bar{\Omega}) := \{\phi \in C^2(\bar{\Omega}) : \phi = 0 \text{ on } \partial\Omega\}.$$

Assume that $f \in C^1(\bar{\Omega})$ and $\eta \in C^1(\partial\Omega)$ and let $\phi \in C_0^2(\bar{\Omega})$. Multiplying the equation in (1.1.1) by ϕ and integrating by parts we obtain

$$\int_\Omega f\phi \, dx = -\int_\Omega u\Delta\phi dx + \int_{\partial\Omega} \eta\partial_{\mathbf{n}}\phi dS \tag{1.1.4}$$

where dS denotes the surface element on $\partial\Omega$ and $\partial_{\mathbf{n}}$ denotes differentiation in the outer normal direction on $\partial\Omega$.

Denote by $L^1(\Omega;\phi)$ the weighted Lebesgue space with weight ϕ, where ϕ is a positive measurable function in Ω. Let ρ be the function given by

$$\rho(x) = \begin{cases} \operatorname{dist}(x, \partial\Omega) & \forall x \in \Omega \\ 0 & \text{otherwise.} \end{cases}$$

We note that the integral on the left-hand side of (1.1.4) is well defined for every $f \in L^1(\Omega;\rho)$ and the last integral on the right-hand side is well defined for every $\eta \in L^1(\partial\Omega)$. Accordingly we define a weak solution of (1.1.1) with L^1 data as follows:

Definition 1.1.2. Assume that

$$f \in L^1(\Omega;\rho), \quad \eta \in L^1(\partial\Omega). \tag{1.1.5}$$

A function $u \in L^1(\Omega)$ is a weak solution of (1.1.1) if it satisfies (1.1.4) for every $\phi \in C_0^2(\bar{\Omega})$.

Recall the following estimates for G and P (see e.g. [53]):

$$G(x,y) \sim \min(\rho(x),\rho(y))\,|x-y|^{1-N}, \quad \left|\nabla_x^k G(x,y)\right| \le C\,|x-y|^{2-k-N}, \tag{1.1.6}$$

for every $x, y \in \Omega$ and

$$P(x,y) \sim \rho(x)\,|x-y|^{-N}, \quad \left|\nabla_x^k P(x,y)\right| \le C\,|x-y|^{1-k-N} \tag{1.1.7}$$

for every $x \in \Omega$, $y \in \partial\Omega$ and $k = 0, 1, 2, \ldots$.

If f, h are non-negative functions on a domain D, the notation $h \sim f$ means

$$\exists c > 0 \quad \text{such that} \quad c^{-1}h \le f \le ch.$$

In (1.1.6) and (1.1.7) the constants depend only on the C^2 characteristic of the boundary and the diameter of the domain.

Employing these estimates one can establish the existence and uniqueness of weak solutions.

Proposition 1.1.3. For every $f \in L^1(\Omega;\rho)$ and $\eta \in L^1(\partial\Omega)$ problem (1.1.1) possesses a unique weak solution. The solution is given by (1.1.3) and satisfies,

$$\|u\|_{L^1(\Omega)} \leq C\big(\|f\|_{L^1(\Omega;\rho)} + \|\eta\|_{L^1(\partial\Omega)}\big) \tag{1.1.8}$$

and

$$\|u_+\|_{L^1(\Omega)} \leq C\big(\|f_+\|_{L^1(\Omega;\rho)} + \|\eta_+\|_{L^1(\partial\Omega)}\big) \tag{1.1.9}$$

where C is a constant depending only on Ω.

Proof. First observe that, by virtue of estimates (1.1.6) and (1.1.7), the function u defined by (1.1.3) is in $L^1(\Omega)$ and satisfies (1.1.8).

Approximate f and η by sequences $\{f_n\} \subset C_c^\infty(\Omega)$ and $\{\eta_n\} \in C_c^\infty(\partial\Omega)$ in $L^1(\Omega;\rho)$ and $L^1(\partial\Omega)$ respectively. Denote by u_n the solution of (1.1.1) with f, η replaced by f_n, η_n. By (1.1.8) $\{u_n\}$ converges in $L^1(\Omega)$ to u. Furthermore, as u_n satisfies

$$-\int_\Omega u_n \Delta\phi\, dx = \int_\Omega f_n\phi\, dx - \int_{\partial\Omega} \eta_n \partial_{\mathbf{n}}\phi\, dS,$$

we conclude that u satisfies (1.1.4).

If f, η are non-negative then they can be approximated by sequences $\{f_n\} \subset C_c^\infty(\Omega)$ and $\{\eta_n\} \in C_c^\infty(\partial\Omega)$ consisting of non-negative functions. By the maximum principle, $u_n \geq 0$ and consequently $u \geq 0$. In the general case let v_1 and v_2 be the weak solutions of (1.1.1) with f, η replaced by f_+, η_+ and f_-, η_- respectively, where

$$f_+ = \max(f, 0), \quad f_- = \max(-f, 0).$$

Then $v_i \geq 0$ and $u = v_1 - v_2$. Therefore $u_+ \leq v_1$ and (1.1.9) follows from (1.1.8) applied to v_1. $\square$

1.2 Measure data

The previous result can be fully extended to the case where the functions f, η are replaced by measures. Recall that a positive Borel measure on Ω is called a Radon measure if it is bounded on compact sets. A Borel measure with possibly changing signs is called a *signed Radon measure* if it is the difference of two positive Radon measures, at least one of which is finite. If μ is a signed Radon measure, denote by μ_+ and μ_- its positive and negative parts and by $|\mu|$ the total variation measure

$$|\mu| = \mu_+ + \mu_-.$$

Denote by $\mathfrak{M}(\Omega)$ the space of finite Borel measures endowed with the norm

$$\|\mu\|_{\mathfrak{M}(\Omega)} = |\mu|(\Omega) \tag{1.2.1}$$

and by $\mathfrak{M}_\rho(\Omega)$ (or $\mathfrak{M}(\Omega;\rho)$) the space of signed Radon measures μ such that

$$\|\mu\|_{\mathfrak{M}_\rho(\Omega)} := \int_\Omega \rho\, d\,|\mu| < \infty. \tag{1.2.2}$$

Finally denote by $\mathfrak{M}_{\mathrm{loc}}(\Omega)$ the space of set functions μ on

$$\mathcal{B}_c(\Omega) = \{E \Subset \Omega : E \text{ Borel}\}$$

such that $\mu \mathbf{1}_K$ is a finite measure for every compact $K \subset \Omega$. Following Bourbaki, such a set function is called a *real valued Radon measure*. A set function μ belongs to this space if and only if it is the difference of two positive Radon measures μ_1, μ_2. If at least one of these two is finite then μ is a signed Radon measure. However, if both are unbounded then μ is not a measure on Ω.

The space $\mathfrak{M}_{\mathrm{loc}}(\Omega)$ can be characterized as the set of continuous linear functionals on $C_c(\Omega)$ endowed with the inductive limit. A functional ℓ on $C_c(\Omega)$ is continuous in this sense if and only if, for every compact set $K \subset \Omega$, ℓ is continuous on

$$\mathcal{C}_K = \{f \in C_c(\Omega) : \operatorname{supp} f \subset K\}.$$

Definition 1.2.1. Assume that

$$\mu \in \mathfrak{M}_\rho(\Omega), \quad \nu \in \mathfrak{M}(\partial\Omega). \tag{1.2.3}$$

A function $u \in L^1(\Omega)$ is a weak solution of the problem

$$\begin{aligned} -\Delta u &= \mu \quad \text{in } \Omega, \\ u &= \nu \quad \text{on } \partial\Omega \end{aligned} \tag{1.2.4}$$

if it satisfies

$$\int_\Omega \phi\, d\mu = -\int_\Omega u \Delta\phi dx + \int_{\partial\Omega} \partial_{\mathbf{n}}\phi d\nu \tag{1.2.5}$$

for every $\phi \in C_0^2(\bar{\Omega})$.

Theorem 1.2.2. Assume $\mu \in \mathfrak{M}_\rho(\Omega)$ and $\nu \in \mathfrak{M}(\partial\Omega)$.

(i) Problem (1.2.4) has a unique weak solution u given by

$$u(x) = \int_\Omega G(x,y) d\mu(y) + \int_{\partial\Omega} P(x,y) d\nu(y). \tag{1.2.6}$$

Furthermore

$$\|u\|_{L^p(\Omega)} \le C(p)\Big(\|\mu\|_{\mathfrak{M}_\rho(\Omega)} + \|\nu\|_{\mathfrak{M}(\partial\Omega)} \Big), \quad 1 \le p < \frac{N}{N-1}, \tag{1.2.7}$$

and

$$\|u_+\|_{L^p(\Omega)} \le C(p)\Big(\|\mu_+\|_{\mathfrak{M}_\rho(\Omega)} + \|\nu_+\|_{\mathfrak{M}(\partial\Omega)} \Big), \quad 1 \le p < \frac{N}{N-1}, \tag{1.2.8}$$

where $C(p)$ is a constant depending only on p and Ω.

(ii) For every $p \in [1, \frac{N}{N-1})$, $u \in W^{1,p}_{\text{loc}}(\Omega)$ and, if $\Omega' \Subset \Omega$,

$$\|u\|_{W^{1,p}(\Omega')} \le C(p, \Omega')\Big(\|\mu\|_{\mathfrak{M}(\Omega')} + \|\nu\|_{\mathfrak{M}(\partial\Omega)} \Big). \tag{1.2.9}$$

(iii) If $\nu = 0$ then (1.2.7) and (1.2.8) hold for every $p \in [1, N/(N-2)$. If, in addition, $\mu \in \mathfrak{M}(\Omega)$ then, for every $p \in [1, \frac{N}{N-1})$, $u \in W^{1,p}(\Omega)$ and

$$\|u\|_{W^{1,p}(\Omega)} \le C(p) \|\mu\|_{\mathfrak{M}(\Omega)}, \tag{1.2.10}$$

where $C(p)$ is a constant depending only on p and Ω.

Proof. The uniqueness of solutions of the classical Dirichlet problem implies that problem (1.2.4) has at most one solution.

Let u be the function defined by (1.2.6). In the first part of the proof we show that this function satisfies estimates (1.2.7)–(1.2.10); in the second part we show that u is the weak solution of (1.2.4).

If $\nu = 0$ and $\mu \in \mathfrak{M}(\Omega, \rho)$ then, by (1.1.6),

$$|u(x)| \le C \int_\Omega |x-y|^{2-N} \rho(y) d\, |\mu|(y). \tag{1.2.11}$$

Let Γ_α denote the function $\Gamma_\alpha(x) = |x|^{\alpha - N}$. The measure $\rho|\mu|$ is bounded with compact support in $\mathbb{R}^N$ and $\Gamma_2 \in L^p_{loc}(\mathbb{R}^N)$ for $1 \le p < N/(N-2)$. Hence $\Gamma_2 * (\rho|\mu|) \in L^p(\mathbb{R}^N)$ and (1.2.7) holds.

If $\mu = 0$, $\nu \in \mathfrak{M}(\partial\Omega)$ then, by (1.1.7),

$$|u(x)| \le C \int_{\partial\Omega} |x-y|^{1-N} d\, |\nu|(y). \tag{1.2.12}$$

Hence, for $1 \le p < N/(N-1)$,

$$\|u\|_{L^p(\Omega)} \le C \int_{\partial\Omega} \Big(\int_\Omega |x-y|^{p(1-N)} dx \Big)^{1/p} d\, |\nu|(y) \le C(p) \|\nu\|_{\mathfrak{M}(\partial\Omega)}.$$

This completes the proof of (1.2.7).

Estimate (1.2.8) follows by the same argument as in the proof of Proposition 1.1.3.

If $\nu = 0$ and $\mu \in \mathfrak{M}(\Omega)$ then, by the second inequality in (1.1.6),

$$|\nabla u(x)| \le C \int_\Omega |x-y|^{1-N} d\, |\mu|(y).$$

Therefore, for $1 \le p < N/(N-1)$, $u \in W^{1,p}(\Omega)$ and (1.2.10) holds.

Next we verify (1.2.9). Put

$$u_1(x) = \int_\Omega G(x,y) d\mu(y), \quad u_2 = \int_{\partial\Omega} P(x,y) d\nu(y).$$

If $\Omega' \Subset \Omega$ then $\mu \mathbf{1}_{\Omega'} \in \mathfrak{M}(\Omega')$ so that (1.2.10) implies that u_1 satisfies (1.2.9). On the other hand u_2 is harmonic in Ω and for every compact subdomain $\Omega' \subset \Omega$ we have

$$\sup_{\Omega'} |u_2| \leq C \operatorname{dist}(\Omega', \partial\Omega)^{1-N} |\nu|.$$

This proves (1.2.9).

We turn to the second part of the proof: to show that u is a weak solution of (1.2.4).

First we prove this statement in the case that $\nu = 0$ and $\mu \in \mathfrak{M}(\Omega)$.

Let $\{f_n\}$ be a sequence of functions in $C_c^\infty(\Omega)$ such that $f_n \rightharpoonup \mu$ weakly relative to $C_0(\bar{\Omega})$. Denote by u_n the solution of (1.2.4) with μ replaced by f_n and $\nu = 0$. In this case we know that

$$u_n = \int_\Omega G(x, y) f_n(x) dx.$$

By (1.2.10), $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$, $1 \leq p < N/(N-1)$. Therefore there exists a subsequence $\{u_{n_k}\}$ which converges in $L^p(\Omega)$. Since

$$-\int_\Omega u_{n_k} \Delta\phi \, dx = \int_\Omega f_{n_k} \phi \, dx$$

for every $\phi \in C_0^2(\bar{\Omega})$ we conclude that $w = \lim u_{n_k}$ satisfies

$$-\int_\Omega w \Delta\phi \, dx = \int_\Omega \phi \, d\mu$$

for every $\phi \in C_0^2(\bar{\Omega})$. It follows that w is the unique solution of the problem

$$-\Delta w = \mu \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega.$$

Since the limit does not depend on the subsequence it follows that $w = \lim u_n$.

Next we show that $w = u$ in the case when $\mu \in \mathfrak{M}(\Omega)$ and $\mu \geq 0$. In this case we may choose the sequence $\{f_n\}$ so that $f_n \geq 0$.

Given $\epsilon > 0$, let $\varphi_\epsilon \in C_c^\infty(\mathbb{R}^N)$ be a function such that

$$0 \leq \varphi_\epsilon \leq 1, \quad \varphi_\epsilon = 0 \text{ in } B_{\epsilon/2}(0), \quad \varphi_\epsilon = 1 \text{ in } \mathbb{R}^N \setminus B_\epsilon(0).$$

Then

$$\begin{aligned} u_n(x) &= \int_\Omega G(x, y) f_n(y) dy \\ &= \int_\Omega G(x, y)\varphi_\epsilon(|x - y|) f_n(y) dy + \int_\Omega G(x, y)(1 - \varphi_\epsilon(|x - y|)) f_n(y) dy \\ &=: u_{n,1}(x) + u_{n,2}(x). \end{aligned}$$

For every $x \in \Omega$ the function $y \mapsto G(x, y)\varphi_\epsilon(|x - y|)$ is continuous in $\bar{\Omega}$; therefore, the weak convergence of $\{f_n\}$ implies

$$u_{n,1}(x) \to \int_\Omega G(x, y)\varphi_\epsilon(|x - y|) d\mu(y) \quad \forall x \in \Omega.$$

Thus

$$w(x) - u(x) = \int_\Omega G(x,y)(1 - \varphi_\epsilon(|x-y|))d\mu(y) + \lim_{n\to\infty} u_{n,2}(x).$$

Let F be a compact subset of Ω, $\epsilon < \frac{1}{4}\mathrm{dist}\,(F, \partial\Omega)$ and

$$(F)_\epsilon := \{x \in \mathbb{R}^N : \mathrm{dist}\,(x, F) < \epsilon\}.$$

Then

$$\begin{aligned}\int_F u_{n,2} dx &= \int_\Omega \int_F G(x,y)(1 - \varphi_\epsilon(|x-y|))dx f_n(y)dy \\ &\le \int_\Omega f_n dy \sup_{y\in(F)_\epsilon} \int_{|x-y|<\epsilon} G(x,y)dx.\end{aligned}$$

Hence

$$\limsup_{n\to\infty} \int_F u_{n,2}\, dx \le \mu(\Omega) \sup_{y\in(F)_\epsilon} \int_{|x-y|<\epsilon} G(x,y)dx$$

and the last term tends to zero as $\epsilon \to 0$. Similarly we obtain,

$$\lim_{\epsilon\to 0} \int_F \int_\Omega G(x,y)(1 - \varphi_\epsilon(|x-y|))d\mu(y) = 0.$$

Consequently (using the lemma of Fatou):

$$\begin{aligned}0 &\le \int_F (w-u)dx \\ &\le \int_F \int_\Omega G(x,y)(1 - \varphi_\epsilon(|x-y|))d\mu(y) + \liminf_{n\to\infty} \int_F u_{n,2}(x)dx\end{aligned}$$

and the right-hand side tends to zero as $\epsilon \to 0$. It follows that $u = w$ in F and (as F is an arbitrary compact subset of Ω) $u = w$ in Ω.

Now we consider the case when $\mu \in \mathfrak{M}_\rho(\Omega)$ and $\mu \ge 0$. We approximate Ω by a sequence $\{\Omega_k\}$ of smooth domains such that $\Omega_k \uparrow \Omega$ and put $\mu_k = \mu \mathbf{1}_{\Omega_k}$. Let v_k be the solution of

$$-\Delta v = \mu_k \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega.$$

By the previous part of the proof

$$v_k = \int_\Omega G(x,y)d\mu_k(y)$$

and $v_k \uparrow u$. Since v_k satisfies

$$\int_\Omega \phi\, d\mu_k = -\int_\Omega v_k \Delta\phi dx,$$

for every $\phi \in C_0^2(\bar{\Omega})$ and $u \in L^1(\Omega)$ we conclude that

$$\int_\Omega \phi \, d\mu = -\int_\Omega u \Delta \phi dx$$

for every ϕ as above. Thus u is the weak solution of (1.2.4). To show that this result remains valid when μ is not necessarily positive we apply the last statement to μ_+ and μ_- separately.

Finally we prove that

$$u(x) = \int_{\partial\Omega} P(x, y) d\nu(y)$$

is a weak solution of (1.2.4) with $\mu = 0$. Let $\{h_n\}$ be a sequence of smooth functions converging weakly to ν relative to $C(\partial\Omega)$. Then

$$w_n(x) = \int_{\partial\Omega} P(x, y) h_n dS_y$$

is the classical solution of

$$-\Delta w = 0 \text{ in } \Omega, \quad w = h_n \text{ on } \partial\Omega.$$

Thus

$$0 = \int_\Omega w_n \Delta \phi dx - \int_{\partial\Omega} h_n \partial_{\mathbf{n}} \phi \, dS \tag{1.2.13}$$

for every $\phi \in C_0^2(\bar{\Omega})$. As $P(x, \cdot) \in C(\partial\Omega)$ for every $x \in \Omega$, it follows that $w_n \to u$ everywhere in Ω. By (1.2.7) $\{w_n\}$ is bounded in $L^p(\Omega)$ for some $p > 1$. These two facts imply that $w_n \to u$ in $L^1(\Omega)$. Therefore (1.2.13) implies

$$0 = \int_\Omega u \Delta \phi dx - \int_{\partial\Omega} \partial_{\mathbf{n}} \phi \, d\nu$$

for every $\phi \in C_0^2(\bar{\Omega})$.

Thus u is a weak solution of (1.2.4) if either $\mu = 0$ or $\nu = 0$. By linearity this implies the result in the general case. □

We mention the following useful corollary.

Corollary 1.2.3. Let μ be a real valued Radon measure in Ω and suppose that $u \in L^1_{\text{loc}}(\Omega)$ satisfies $-\Delta u = \mu$ in Ω, i.e.,

$$-\int_D u \Delta \phi \, dx = \int_D \phi \, d\mu \quad \forall \phi \in C_c^\infty(\Omega). \tag{1.2.14}$$

Then, $u \in W^{1,p}_{\text{loc}}(\Omega)$ for every $p \in [1, N/(N-1))$ and, for every domain $D \Subset \Omega$ of class C^2,

$$-\int_D u \Delta \phi \, dx = \int_D \phi \, d\mu - \int_{\partial D} u\lfloor_{\partial D} \, \partial_{\mathbf{n}} \phi \, dS, \tag{1.2.15}$$

for every $\phi \in C_0^2(\bar{D})$. Here $u\lfloor_{\partial D}$ denotes the L^1 Sobolev trace of u on ∂D.

Proof. Put

$$v_D(x) = \int_D G(x,y)\,d\mu(y) \quad \forall x \in D.$$

By Theorem 1.2.2, applied to the measure $\mu_D := \mu \mathbf{1}_D$ in Ω,

$$\int_\Omega \varphi\,d\mu_D = \int_\Omega v_D \Delta\varphi\,dx \quad \forall \varphi \in C_0^2(\bar{\Omega})$$

and $v_D \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Thus

$$v_D \in W^{1,p}(D), \quad -\Delta v_D = \mu \text{ in } D$$

for every $p \in [1, N/(N-1))$. It follows that $u - v_D$ is harmonic in D and consequently, $u \in W^{1,p}_{\mathrm{loc}}(D)$. As D is any C^2 domain strongly contained in Ω, it follows that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Consequently, by the Sobolev trace theorem, u possesses an L^1 trace on every compact $N-1$-dimensional C^1 manifold contained in Ω.

Let $\epsilon_0 := \operatorname{dist}(\bar{D}, \partial\Omega)$ and let $\psi \in C_c^\infty(\Omega)$ be a function such that $0 \le \psi \le 1$ and

$$\psi(x) = \begin{cases} 1 & \text{if } \operatorname{dist}(x, D) < \epsilon_0/2 \\ 0 & \text{if } \operatorname{dist}(x, D) > 3\epsilon_0/4. \end{cases}$$

For $0 < \epsilon < \epsilon_0/2$ let $u_\epsilon := J_\epsilon(u\psi)$ and $\mu_\epsilon = J_\epsilon(\mu\psi)$. Then $-\Delta u_\epsilon = \mu_\epsilon$ in D (in the classical sense) and $u_\epsilon \in C(\bar{D})$. Therefore,

$$-\int_D u_\epsilon \Delta\phi\,dx = \int_D \phi\mu_\epsilon dx - \int_{\partial D} u_\epsilon \partial_{\mathbf{n}}\phi\,dS,$$

for every $\phi \in C_0^2(\bar{D})$. Letting $\epsilon \to 0$ we obtain (1.2.15). □

Remark 1.2.2.A. Let $\{\mu_n\} \subset \mathfrak{M}_\rho(\Omega)$, $\{\nu_n\} \subset \mathfrak{M}(\partial\Omega)$ and assume that $\mu_n \to \mu$ strongly in $\mathfrak{M}_\rho(\Omega)$ and $\nu_n \to \nu$ strongly in $\mathfrak{M}(\partial\Omega)$. Let u (resp. u_n) be the weak solution of (1.2.4) with data μ, ν (resp. with data μ_n, ν_n). By Theorem 1.2.2 (i), $u_n \to u$ strongly in $L^p(\Omega)$ for $p \in [1, N/(N-1))$. Briefly:

The L^1 weak solution of (1.2.4) *is stable with respect to strong convergence of the data.*

The next result shows that the solution is also stable (in a weaker sense) with respect to an appropriate type of weak convergence of the data.

First recall the standard definition of 'weak convergence' in $\mathfrak{M}(K)$, the space of finite Borel measures on a compact set $K \subset \mathbb{R}^N$. We say that the sequence $\{\mu_k\}$ converges 'weakly' to μ if

$$\int_K f d\mu_k \to \int_K f d\mu \quad \forall f \in C(K).$$

This is in fact weak* convergence in the dual, $\mathfrak{M}(K)$, of $C(K)$. The topology of weak convergence is metrizable, a bounded sequence is pre-compact, i.e. contains a weakly convergent subsequence and every weakly convergent sequence is bounded. For this and other properties of weak convergence of measures we refer the reader to any standard measure theory textbook.

When Ω is a bounded domain, $\mathfrak{M}(\Omega)$ is the dual of

$$C_0(\bar{\Omega}) = \{f \in C(\bar{\Omega});\ f = 0 \text{ on } \partial\Omega\}.$$

Note that $C_0(\bar{\Omega})$ is the closure of $C_c(\Omega)$ in $C(\bar{\Omega})$. In this case we say that $\{\mu_k\}$ converges 'weakly' to μ if

$$\int_\Omega f d\mu_k \to \int_\Omega f d\mu \quad \forall f \in C_0(\bar{\Omega}).$$

As before, the topology of weak convergence is metrizable and the properties mentioned above persist.

Finally, consider the space $\mathfrak{M}_\rho(\Omega)$ when Ω is a bounded C^1 domain. This space is the dual of

$$C_0(\bar{\Omega};\rho) = \{h : h/\rho \in C_0(\bar{\Omega})\}.$$

Here $h/\rho \in C_0(\bar{\Omega})$ means that h/ρ has a continuous extension to $\bar{\Omega}$, which is zero on $\partial\Omega$. Therefore we define:

A sequence $\{\mu_k\} \subset \mathfrak{M}_\rho(\Omega)$ converges weakly to $\mu \in \mathfrak{M}_\rho(\Omega)$ if

$$\int_\Omega f d\mu_k \to \int_\Omega f d\mu \quad \forall f \in C_0(\bar{\Omega};\rho). \tag{1.2.16}$$

Thus the weak convergence in the sense of (1.2.16) is equivalent to the weak convergence $\rho\mu_n \rightharpoonup \rho\mu$ in $\mathfrak{M}(\Omega)$, i.e. with respect to $C_0(\bar{\Omega})$. Again, the topology of weak convergence is metrizable, a bounded sequence is pre-compact and every weakly convergent sequence is bounded.

Definition 1.2.4. A sequence $\{\mu_n\} \subset \mathfrak{M}(\Omega)$ is tight if for every $\epsilon > 0$ there exists a neighborhood U_ϵ of $\partial\Omega$ such that $|\mu_n|(U_\epsilon \cap \Omega) < \epsilon$. Similarly, a sequence $\{\mu_n\} \subset \mathfrak{M}(\Omega;\rho)$ is tight in this space if $\{\rho\mu_n\}$ is tight in $\mathfrak{M}(\Omega)$.

Remark. If a sequence in $\mathfrak{M}(\Omega)$ is weakly convergent but not tight, it might have a weak limit in $\mathfrak{M}(\bar{\Omega})$ that is different from the weak limit in $\mathfrak{M}(\Omega)$. Here is a simple example. Let $\{A_n\}$ be a sequence of points in Ω such that $A_n \to A \in \partial\Omega$. Denote by μ_n (resp. μ) the Dirac measure of mass 1 concentrated at A_n (resp. A). Then, in $\mathfrak{M}(\bar{\Omega})$, $\{\mu_n\}$ converges weakly to μ but, in $\mathfrak{M}(\Omega)$, it converges weakly to 0. Evidently this sequence is not tight in $\mathfrak{M}(\Omega)$.

Theorem 1.2.5. (i) Let $\{\mu_n\} \subset \mathfrak{M}(\Omega)$ and $\{\nu_n\} \subset \mathfrak{M}(\partial\Omega)$. Assume that $\mu_n \rightharpoonup \mu$ relative to $C_0(\bar{\Omega})$ while $\nu_n \rightharpoonup \nu$ relative to $C(\partial\Omega)$. Let u be the weak solution of (1.2.4) and let u_n be the weak solution of (1.2.4) with μ, ν replaced by μ_n, ν_n. Then:

$$u_n \rightharpoonup u \text{ weakly in } W^{1,p}_{\text{loc}}(\Omega), \quad u_n \to u \text{ strongly in } L^p(\Omega), \tag{1.2.17}$$

for every $p \in [1, N/(N-1)$.

If in addition $\nu_n = 0$ for all n then

$$u_n \to u \text{ strongly in } W^{1,p}(\Omega), \quad u_n \to u \text{ strongly in } L^q(\Omega), \tag{1.2.18}$$

for every $p \in [1, N/(N-1), q \in [1, N/(N-2))$.

(ii) Let $\{\mu_n\}$ be a bounded and tight sequence in $\mathfrak{M}(\Omega; \rho)$ such that $\mu_n \rightharpoonup \mu$ relative to $C_0(\bar{\Omega}; \rho)$. Let $\{\nu_n\} \subset \mathfrak{M}(\partial\Omega)$ and assume that $\nu_n \rightharpoonup \nu$ relative to $C(\partial\Omega)$. Then (1.2.17) holds.

Remark 1.1. Note that in part (i) we do not assume 'tightness' but in part (ii) this assumption is needed. The following example shows that the conclusion of Theorem 1.2.5 (ii) may fail in the absence of tightness. Let $\{A_n\}$ be a sequence of points in Ω converging to a point $A \in \partial\Omega$. Put $\mu_n = \frac{1}{a_n}\delta_{A_n}$ where $a_n = \operatorname{dist}(A_n, \partial\Omega)$. ($\delta_A$ denotes the Dirac measure of mass 1 concentrated at A.) Then $\{\mu_n\}$ is bounded in $\mathfrak{M}(\Omega; \rho)$ but it is not tight. Furthermore $\mu_n \rightharpoonup 0$ weakly in $\mathfrak{M}(\Omega; \rho)$. But, if u_n is the solution of (1.2.4) with boundary data 0 then $u_n \to P(\cdot, A)$ pointwise in Ω.

Note also that every bounded sequence in $\mathfrak{M}(\Omega)$ is tight in $\mathfrak{M}(\Omega; \rho)$ although it may not be tight in $\mathfrak{M}(\Omega)$.

Proof. (i) By Theorem 1.2.2 (ii), for every $\Omega' \Subset \Omega$, $\{u_n\}$ is bounded in $W^{1,p}(\Omega')$, for every $p \in [1, N/(N-1))$. Consequently there exists a subsequence $\{u_{n_k}\}$ and $v \in W^{1,p}_{\text{loc}}(\Omega)$ such that

$$u_{n_k} \rightharpoonup v \quad \text{weakly in } W^{1,p}_{\text{loc}}(\Omega)$$

for all p as above. By the Sobolev imbedding theorem

$$u_{n_k} \to v \quad \text{in } L^q_{\text{loc}}(\Omega), \quad 1 \le q < N/(N-2).$$

By taking a further subsequence we may assume that $u_{n_k} \to v$ a.e. in Ω.

By (1.2.7) $\{u_n\}$ is uniformly bounded in $L^p(\Omega)$, $1 \le p < N/(N-1)$. Therefore $\{u_n\}$ is uniformly integrable in $L^r(\Omega)$, $1 \le r < N/(N-1)$. Since $u_{n_k} \to v$ a.e. in Ω we conclude that

$$u_{n_k} \to v \quad \text{in } L^p(\Omega), \quad 1 \le p < N/(N-1).$$

Now, for every n,

$$-\int_\Omega u_n \Delta\varphi \, dx = \int_\Omega \varphi \, d\mu_n - \int_{\partial\Omega} \partial_{\mathbf{n}}\varphi \, d\nu_n \quad \forall \varphi \in C^2_0(\bar{\Omega}).$$

Replacing n by n_k and taking the limit as $k \to \infty$ we obtain

$$-\int_\Omega v\Delta\varphi\,dx = \int_\Omega \varphi\,d\mu - \int_{\partial\Omega} \partial_{\mathbf{n}}\varphi\,d\nu \quad \forall\varphi \in C_0^2(\bar{\Omega}).$$

Thus v is the weak solution of (1.2.5) and, by uniqueness, $v = u$. Since the limit does not depend on the subsequence we obtain (1.2.17).

If in addition, $\nu_n = 0$ for all n, (1.2.18) is obtained by the same argument, using Theorem 1.2.2 (iii).

(ii) Let ψ_k be a function in $C_c^\infty(\Omega)$ such that $0 \le \psi_k \le 1$, and

$$\psi_k(x) = \begin{cases} 1 & \text{if } \rho(x) > 2^{-k} \\ 0 & \text{if } \rho(x) < 2^{-k-1}. \end{cases}$$

Note that $\psi_k \uparrow 1$ in Ω.

Let $u'_{k,n}$ (resp. u'_k) denote the weak solution of (1.2.4) with μ replaced by $\mu_n\psi_k$ (resp. by $\mu\psi_k$). Put

$$v_{k,n} = u_n - u'_{k,n}, \quad v_k = u - u'_k.$$

Thus

$$v_{k,n}(x) = \int_\Omega G(x,y)(1-\psi_k)d\mu_n(y)$$

and

$$v_k(x) = \int_\Omega G(x,y)(1-\psi_k)d\mu(y).$$

The tightness assumption implies that

$$\lim_{k\to\infty} \|\mu_n(1-\psi_k)\|_{\mathfrak{M}(\Omega;\rho)} = 0 \tag{1.2.19}$$

uniformly with respect to n. Therefore, by Theorem 1.2.2,

$$\begin{aligned} &\|v_k\|_{L^p(\Omega)} \to 0 \quad \text{as } k \to \infty, \\ &\left\|v_{k,n}\right\|_{L^p(\Omega)} \to 0 \quad \text{as } k \to \infty, \qquad \text{uniformly with respect to } n, \end{aligned} \tag{1.2.20}$$

for $1 \le p < N/(N-1)$).

For fixed k, $\{\psi_k\mu_n\}$ converges strongly to $\psi_k\mu$. Therefore, by part (i),

$$u'_{k,n} \rightharpoonup u'_k \text{ weakly in } W^{1,p}_{\mathrm{loc}}(\Omega), \quad u'_{k,n} \to u'_k \text{ strongly in } L^p(\Omega), \tag{1.2.21}$$

for every $p \in [1, N/(N-1))$.

Combining (1.2.20) and (1.2.21) we obtain (1.2.17). □

1.3 M-boundary trace

Definition 1.3.1. A sequence $\{D_n\}$ is an *exhaustion* of Ω if $\bar{D}_n \subset D_{n+1}$ and $D_n \uparrow \Omega$. We say that an exhaustion $\{D_n\}$ is of class C^α if each domain D_n is of this class. If, in addition, Ω is a C^α domain, $\alpha > 0$, and the sequence of domains $\{D_n\}$ is *uniformly* of class C^α we say that $\{D_n\}$ is a uniform C^α exhaustion.

Note. $\{D_n\}$ is uniformly of class C^α if there exists r_0, γ_0, n_0 such that, for every $X \in \partial D$:

There exists a system of Cartesian coordinates ξ centered at X, a sequence $\{f_n\} \subset C^\alpha(B^{N-1}_{r_0}(0))$ and $f \in C^\alpha(B^{N-1}_{r_0}(0))$ such that the following statement holds. Let

$$Q_0 := \{\xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{N-1} : |\xi'| < r_0, |\xi_N| < \gamma_0\}.$$

Then the surfaces $\partial D_n \cap Q_0$, $n > n_0$ and $\partial\Omega \cap Q_0$ can be represented by $\xi_1 = f_n(\xi')$ and $\xi_1 = f(\xi')$ respectively and

$$f_n \to f \quad \text{in } C^\alpha(B^{N-1}_{r_0}(0)).$$

At this point we introduce some additional notation and a few related technical remarks.

Recall our basic assumption: Ω is a bounded domain in $\mathbb{R}^N$ whose boundary Σ is a C^2 manifold. We use the notation:

$$\begin{aligned} &\rho(x) = \operatorname{dist}(x, \partial\Omega), && \Sigma_\beta = \{x \in \Omega : \rho(x) = \beta\}, \\ &D_\beta = \{x \in \Omega : \rho(x) > \beta\}, && \Omega_\beta = \Omega \setminus \bar{D}_\beta. \end{aligned} \tag{1.3.1}$$

The outward, unit normal vector to $\partial\Omega$ at x_0 is denoted by $\mathbf{n}_{x_0}$.

Proposition 1.3.2. There exists a positive number β_0 such that:

(a) For every point $x \in \bar{\Omega}_{\beta_0}$, there exists a unique point $\sigma(x) \in \partial\Omega$ such that $|x - \sigma(x)| = \rho(x)$. This implies,

$$x = \sigma(x) - \rho(x)\mathbf{n}_{\sigma(x)}.$$

(b) The mappings $x \mapsto \rho(x)$ and $x \mapsto \sigma(x)$ belong to $C^2(\bar{\Omega}_{\beta_0})$ and $C^1(\bar{\Omega}_{\beta_0})$ respectively. Furthermore,

$$\lim_{x \to \sigma(x)} \nabla\rho(x) = -\mathbf{n}_{\sigma(x)}.$$

(c) Denote by $\Pi : \bar{\Omega}_{\beta_0} \mapsto [0, \beta_0] \times \Sigma$ the mapping given by $\Pi(x) = (\rho(x), \sigma(x))$. Then Π is a C^1- diffeomorphism.

For the proof we refer the reader to [53] and [82]. In view of this result, (ρ, σ) may serve as a set of coordinates in a strip around the boundary. These are called *the flow coordinates of* Ω.

In the following lemmas we state some consequences of the proposition.