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Dedicated to the late Professor Heinz Bauer, whose kind advice and warm encouragements were the origin of this work.

Preface

The organization of this book follows the lines in the author's unpublished lecture notes at University of Erlangen-Nürnberg [123] partly combined with other lecture notes [127]. But the basic setting has been changed from non-symmetric Dirichlet forms to lower bounded semi-Dirichlet forms. In this book, we intend to extend those results for symmetric Dirichlet forms given in [55] to lower bounded semi-Dirichlet forms and to time dependent semi-Dirichlet forms as well. In Chapter 1 and Chapter 2, we present basic analytic properties related to the lower bounded semi-Dirichlet forms. In particular, the Markov property and the related potential theory similar to the symmetric Dirichlet forms are formulated. Although the dual semigroups are only positivity preserving and not Markovian in general, many of the results can be obtained with minor modifications of the arguments for the non-symmetric Dirichlet forms presented in [123] and [104]. Furthermore, by changing the basic measure using a coexcessive function, we can obtain a dual Markov resolvent relative to the measure. For this dual pair of the Markov resolvents, we introduce an auxiliary bilinear form. Although it does not satisfy the sector condition in general, it works efficiently in carrying out the corresponding stochastic calculus.

From Chapter 3 to Chapter 5, properties of the Markov processes associated with regular lower bounded semi-Dirichlet forms are studied. In particular, stochastic calculus related to the associated Hunt process $\mathbb{M}$ is investigated. Usually the stochastic calculus for Markov processes is developed in connection with their dual Markov processes. For a regular semi-Dirichlet form, by changing the basic measure using a suitable family of coexcessive functions, we can obtain a family of dual Hunt processes of $\mathbb{M}$ relative to the changed measures. Furthermore, we can construct a pseudo Hunt process which is in duality with $\mathbb{M}$ relative to the original basic measure. Since the pseudo Hunt process can be treated as if it is an ordinary Hunt process, the stochastic calculus related to the semi-Dirichlet forms can be performed by modifying the calculus for symmetric Dirichlet forms. But, due to the lack of the excessiveness of the basic measure, some results need to be changed from the symmetric or non-symmetric Dirichlet forms. The essential difference is to use the weak sense energy instead of the energy. The symmetric or non-symmetric cases correspond to the case that the constant function 1 can be taken as a coexcessive function.

The contents of Chapter 6 are essentially taken from Chapter 5 of the author's note [127]. For a given time dependent family of semi-Dirichlet forms possessing a common domain and a common basic measure, an associated space-time Markov process is constructed and a related parabolic potential theory is developed via stochastic calculus.

Contrary to the time independent case, the space-time Markov process involves non-exceptional semipolar sets so that a partially different approach to the related stochastic calculus is required. Although only certain basic parts of it will be presented in this section, it is possible to modify most of the arguments of Chapter 5 to get the parallel results in the time dependent cases.

We intend this to be a self-contained textbook. Most results are stated accompanied by their proof. References are given for those statements without proof.

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Contents

Chapter 1

Dirichlet forms

In this chapter, basic settings throughout this book are presented. The semi-Dirichlet forms $(\mathcal{E}, \mathcal{F})$ that this book concerns itself with are bilinear forms satisfying $(\mathcal{E}.1) \sim (\mathcal{E}.4)$. For a semi-Dirichlet form, associated semigroups and resolvents are constructed and their Markov property is established in Section 1.1. Since we mainly consider the semi-Dirichlet forms, we call them Dirichlet forms for short. Closability to generate regular Dirichlet forms is explained in Section 1.2. Irreducibility, transience and their related properties are studied in Section 1.3. As a preparation for the stochastic calculus developed after Chapter 3, we consider in Section 1.4 a dual Markov resolvent relative to a basic measure being changed by a suitable coexcessive function $\widehat{h}_\delta$ together with an associated auxiliary bilinear form $\mathcal{A}^{(\delta)}$. We introduce condition $(\mathcal{E}.5)$ on the original Dirichlet form $\mathcal{E}$ so that $\mathcal{A}^{(\delta)}$ is well controlled by $\mathcal{E}$. Furthermore, an additional condition $(\mathcal{E}.6)$ and its consequence are stated also in Section 1.4. Condition $(\mathcal{E}.5)$ will be assumed for Dirichlet form $\mathcal{E}$ in most parts after Chapter 3, while $(\mathcal{E}.6)$ will only be used in Theorem 5.1.4 and its consequences. In the final Section 1.5, we give typical examples of diffusion type as well as jump type Dirichlet forms satisfying $(\mathcal{E}.5)$ and $(\mathcal{E}.6)$.

1.1 Semi-Dirichlet forms and resolvents

Let X be a locally compact separable metric space and $\mathcal{B}(X)$ the Borel σ-algebra on X. We shall fix an everywhere dense positive Radon measure m on X and denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and the norm in $L^2(X;m)$, respectively. Let $\mathcal{C}(X)$ be the space of all continuous functions on X. For a family of functions $\mathcal{D}$, let us denote by $\mathcal{D}_0$, $\mathcal{D}^+$ and $\mathcal{D}_b$ the sub-family of compact support, non-negative and bounded functions of $\mathcal{D}$, respectively. For a dense linear subspace $\mathcal{F}$ of $L^2(X;m)$, if a bilinear form $\mathcal{E}$ defined on $\mathcal{F} \times \mathcal{F}$ satisfies the following conditions $(\mathcal{E}.1)$, $(\mathcal{E}.2)$ and $(\mathcal{E}.3)$, then we call $(\mathcal{E}, \mathcal{F})$ *a closed form on* $L^2(X;m)$.

$(\mathcal{E}.1)$ $\mathcal{E}$ is *lower bounded*: There exists a non-negative constant α_0 such that

$$\mathcal{E}_{\alpha_0}(u,u) \geq 0 \quad \text{for all } u \in \mathcal{F}, \tag{1.1.1}$$

where $\mathcal{E}_\alpha(u,v) = \mathcal{E}(u,v) + \alpha(u,v)$.

$(\mathcal{E}.2)$ $\mathcal{E}$ satisfies the *sector condition*: There exists a constant $K \geq 1$ such that

$$|\mathcal{E}(u,v)| \leq K\mathcal{E}_{\alpha_0}(u,u)^{1/2}\mathcal{E}_{\alpha_0}(v,v)^{1/2} \text{ for all } u,v \in \mathcal{F}. \tag{1.1.2}$$

($\mathcal{E}$.3) $\mathcal{F}$ is a Hilbert space relative to the inner product

$$\mathcal{E}_\alpha^{(s)}(u,v) = \frac{1}{2}\left(\mathcal{E}_\alpha(u,v) + \mathcal{E}_\alpha(v,u)\right) \text{ for all } \alpha > \alpha_0.$$

For $\alpha > \alpha_0$, put $K_\alpha = K + \alpha/(\alpha - \alpha_0)$. Since $(u,u) \le \frac{1}{\alpha-\alpha_0}\mathcal{E}_\alpha(u,u)$, (1.1.2) implies

$$|\mathcal{E}_\alpha(u,v)| \le K_\alpha \mathcal{E}_\alpha(u,u)^{1/2}\mathcal{E}_\alpha(v,v)^{1/2} \text{ for all } u,v \in \mathcal{F}. \tag{1.1.3}$$

If equation (1.1.1) holds for $\alpha_0 = 0$, then $(\mathcal{E},\mathcal{F})$ is called *non-negative.* $(\mathcal{E},\mathcal{F})$ is called *symmetric* if $\mathcal{E}(u,v) = \mathcal{E}(v,u)$ for all $u,v \in \mathcal{F}$. If $(\mathcal{E},\mathcal{F})$ is non-negative and symmetric, then equation (1.1.2) holds for $\alpha_0 = 0$ and $K = 1$ by the Schwarz inequality.

For later use, we shall first show the following Stampacchia's theorem.

Theorem 1.1.1. *Let Γ be a non-empty closed convex subset of $\mathcal{F}$. If J is a continuous linear functional on $\mathcal{F}$ with respect to $\mathcal{E}_\alpha$ for $\alpha > \alpha_0$, then there exists a unique function $v \in \Gamma$ satisfying*

$$\mathcal{E}_\alpha(v, w-v) \ge J(w-v) \textit{ for all } w \in \Gamma. \tag{1.1.4}$$

Proof. Let us fix $\alpha > \alpha_0$.
Uniqueness: Assume that v_1 and v_2 satisfy equation (1.1.4). Then

$$\mathcal{E}_\alpha(v_1, v_2 - v_1) \ge J(v_2 - v_1) \text{ and } \mathcal{E}_\alpha(v_2, v_1 - v_2) \ge J(v_1 - v_2).$$

Hence

$$\begin{aligned}\mathcal{E}_\alpha(v_2 - v_1, v_2 - v_1) &= \mathcal{E}_\alpha(v_2, v_2 - v_1) - \mathcal{E}_\alpha(v_1, v_2 - v_1)\\ &\le J(v_2 - v_1) - J(v_2 - v_1) = 0,\end{aligned}$$

which implies that $v_2 = v_1$ m-a.e.
Existence: We shall first prove equation (1.1.4) under the assumption that $\mathcal{E}$ is symmetric.

Let $I(v) = \mathcal{E}_\alpha(v,v) - 2J(v)$ and let $d = \inf_{v\in\Gamma} I(v)$. Since

$$\begin{aligned}I(v) &\ge \mathcal{E}_\alpha(v,v) - 2\|J\|\mathcal{E}_\alpha(v,v)^{1/2}\\ &= (\mathcal{E}_\alpha(v,v)^{1/2} - \|J\|)^2 - \|J\|^2 \ge -\|J\|^2,\end{aligned}$$

it follows that $d > -\infty$. Let $v_n \in \Gamma$ be a sequence satisfying $d \le I(v_n) < d + \frac{1}{n}$. Then

$$\begin{aligned}\mathcal{E}_\alpha(v_n - v_m, v_n - v_m) &= 2\mathcal{E}_\alpha(v_n, v_n) + 2\mathcal{E}_\alpha(v_m, v_m)\\ &\quad - 4\mathcal{E}_\alpha\left(\frac{v_n + v_m}{2}, \frac{v_n + v_m}{2}\right)\\ &= 2I(v_n) + 2I(v_m) - 4I\left(\frac{v_n + v_m}{2}\right)\\ &< 2\left(d + \frac{1}{n}\right) + 2\left(d + \frac{1}{m}\right) - 4d\\ &\to 0, \quad n,m \to \infty.\end{aligned}$$

Hence $\{v_n\}$ converges to some $v \in \Gamma$ relative to $\mathcal{E}_\alpha$ and $\lim_{n\to\infty} I(v_n) = I(v) = d$. For any $w \in \Gamma$ and $0 < \epsilon < 1$, since $(1-\epsilon)v + \epsilon w = v + \epsilon(w - v) \in \Gamma$,

$$\begin{aligned} 0 &\le I(v + \epsilon(w-v)) - I(v) \\ &= 2\epsilon\mathcal{E}_\alpha(v, w-v) - 2\epsilon J(w-v) + \epsilon^2\mathcal{E}_\alpha(w-v, w-v). \end{aligned}$$

This implies equation (1.1.4). In the general case, put

$$\begin{aligned} \mathcal{E}_\alpha^{(s)}(u,v) &= \frac{1}{2}\left(\mathcal{E}_\alpha(u,v) + \mathcal{E}_\alpha(v,u)\right) \\ \mathcal{E}_\alpha^{(a)}(u,v) &= \frac{1}{2}\left(\mathcal{E}_\alpha(u,v) - \mathcal{E}_\alpha(v,u)\right). \end{aligned}$$

Further put $B_\alpha^{(\beta)}(u,v) = \mathcal{E}_\alpha^{(s)}(u,v) + \beta\mathcal{E}_\alpha^{(a)}(u,v)$. Then the theorem holds for $\beta = 0$. Hence, assuming that there exists a function $v \in \Gamma$ such that $B_\alpha^{(\beta_0)}(v, w-v) \ge J(w-v)$ for all $w \in J$, it is enough to show the same inequality replaced β_0 by β satisfying $\beta_0 \le \beta < \beta_0 + 1/K_\alpha$. By the assumption, for any fixed $u \in \mathcal{F}$ and $\beta_0 \le \beta < \beta_0 + 1/K_\alpha$, since $J(v) - (\beta - \beta_0)\mathcal{E}_\alpha^{(a)}(u,v)$ is a continuous linear functional relative to $B_\alpha^{(\beta_0)}$, there exists unique function $T(u) \in \Gamma$ such that $B_\alpha^{(\beta_0)}(T(u), w - T(u)) \ge J(w - T(u)) - (\beta-\beta_0)\mathcal{E}_\alpha^{(a)}(u, w - T(u))$ for all $w \in \Gamma$. In particular, by putting $v_1 = T(u_1), v_2 = T(u_2)$ for $u_1, u_2 \in \mathcal{F}$, it holds that

$$\begin{aligned} B_\alpha^{(\beta_0)}(v_1, v_1 - v_2) &\le J(v_1 - v_2) - (\beta-\beta_0)\mathcal{E}_\alpha^{(a)}(u_1, v_1 - v_2) \\ B_\alpha^{(\beta_0)}(v_2, v_1 - v_2) &\ge J(v_1 - v_2) - (\beta-\beta_0)\mathcal{E}_\alpha^{(a)}(u_2, v_1 - v_2). \end{aligned}$$

Therefore, noting that $\mathcal{E}^{(a)}(v_1 - v_2, v_1 - v_2) = 0$ and using equation (1.1.3) we obtain that

$$\begin{aligned} \mathcal{E}_\alpha(v_1 - v_2, v_1 - v_2) &= B_\alpha^{(\beta_0)}(v_1 - v_2, v_1 - v_2) \\ &\le (\beta - \beta_0)\mathcal{E}_\alpha^{(a)}(u_2 - u_1, v_1 - v_2) \\ &\le (\beta - \beta_0)K_\alpha\mathcal{E}_\alpha(u_2 - u_1, u_2 - u_1)^{1/2} \\ &\quad \times \mathcal{E}_\alpha(v_1 - v_2, v_1 - v_2)^{1/2}. \end{aligned}$$

This implies that

$$\mathcal{E}_\alpha(v_1 - v_2, v_1 - v_2) \le (K_\alpha(\beta - \beta_0))^2\mathcal{E}_\alpha(u_1 - u_2, u_1 - u_2),$$

that is, T is a contraction operator relative to $\mathcal{E}_\alpha$ if $K_\alpha(\beta - \beta_0) < 1$. Hence there exists a fixed point $v \in \Gamma$ such that $T(v) = v$, that is

$$B_\alpha^{(\beta_0)}(v, w - v) \ge J(w - v) - (\beta - \beta_0)\mathcal{E}_\alpha^{(a)}(v, w - v).$$

This yields the desired relation

$$B_\alpha^{(\beta)}(v, w - v) \ge J(w - v) \quad \text{for all } w \in \Gamma. \qquad \square$$

Theorem 1.1.2. *Suppose that $(\mathcal{E}, \mathcal{F})$ is a closed form on $L^2(X;m)$. Then there exist strongly continuous semigroups $\{T_t\}_{t>0}$ and $\{\widehat{T}_t\}_{t>0}$ on $L^2(X;m)$ such that $\|T_t\| \le e^{\alpha_0 t}$, $\|\widehat{T}_t\| \le e^{\alpha_0 t}$, $(T_t f, g) = (f, \widehat{T}_t g)$ and whose resolvents G_α and $\widehat{G}_\alpha$ given by $G_\alpha f = \int_0^\infty e^{-\alpha t} T_t f\, dt$ and $\widehat{G}_\alpha f = \int_0^\infty e^{-\alpha t} \widehat{T}_t f\, dt$ satisfy*

$$\mathcal{E}_\alpha(G_\alpha f, u) = (f, u) = \mathcal{E}_\alpha(u, \widehat{G}_\alpha f), \tag{1.1.5}$$

for all $f \in L^2(X;m)$, $u \in \mathcal{F}$ and $\alpha > \alpha_0$.

Proof. For any $\beta > 0$ and $f \in L^2(X;m)$, applying Theorem 1.1.1 to $\mathcal{E}_{\alpha_0+\beta}$, $J(v) = (f, v)$ and $\Gamma = \mathcal{F}$, we obtain a unique function $G_{\alpha_0+\beta} f \in \mathcal{F}$ satisfying $\mathcal{E}_{\alpha_0+\beta}\left(G_{\alpha_0+\beta} f, w - G_{\alpha_0+\beta} f\right) \ge (f, w - G_{\alpha_0+\beta} f)$ for all $w \in \mathcal{F}$. By putting $w = G_{\alpha_0+\beta} f \pm u$ for any $u \in \mathcal{F}$, it holds that $\mathcal{E}_{\alpha_0+\beta}(G_{\alpha_0+\beta} f, u) = (f, u)$. Similarly, for any $g \in L^2(X;m)$ and $\beta > 0$, there exists $\widehat{G}_{\alpha_0+\beta} g \in \mathcal{F}$ satisfying $\mathcal{E}_{\alpha_0+\beta}(u, \widehat{G}_{\alpha_0+\beta} g) = (u, g)$ for all $u \in \mathcal{F}$. Obviously

$$(G_{\alpha_0+\beta} f, g) = \mathcal{E}_{\alpha_0+\beta}(G_{\alpha_0+\beta} f, \widehat{G}_{\alpha_0+\beta} g) = (f, \widehat{G}_{\alpha_0+\beta} g), \tag{1.1.6}$$

for all $f, g \in L^2(X;m)$. Furthermore, for any $\beta, \gamma > 0$, since

$$\begin{aligned}
&\mathcal{E}_{\alpha_0+\beta}(G_{\alpha_0+\gamma} f - (\beta-\gamma) G_{\alpha_0+\beta} G_{\alpha_0+\gamma} f, v) \\
&\quad = \mathcal{E}_{\alpha_0+\gamma}(G_{\alpha_0+\gamma} f, v) + (\beta-\gamma)(G_{\alpha_0+\gamma} f, v) - (\beta-\gamma)(G_{\alpha_0+\gamma} f, v) \\
&\quad = (f, v),
\end{aligned}$$

for any $v \in \mathcal{F}$, it follows that $G_{\alpha_0+\gamma} f - (\beta-\gamma) G_{\alpha_0+\beta} G_{\alpha_0+\gamma} f = G_{\alpha_0+\beta} f$, that is $\{G_{\alpha_0+\alpha}\}$ satisfies the resolvent equation

$$G_{\alpha_0+\beta} f - G_{\alpha_0+\gamma} f + (\beta-\gamma) G_{\alpha_0+\beta} G_{\alpha_0+\gamma} f = 0. \tag{1.1.7}$$

Since

$$\beta \|G_{\alpha_0+\beta} f\|^2 \le \mathcal{E}_{\alpha_0+\beta}(G_{\alpha_0+\beta} f, G_{\alpha_0+\beta} f) = (f, G_{\alpha_0+\beta} f) \le \|f\| \cdot \|G_{\alpha_0+\beta} f\|,$$

it follows that $\|G_{\alpha_0+\beta} f\| \le \frac{1}{\beta}\|f\|$.

Similarly, there exists a resolvent $\{\widehat{G}_{\alpha_0+\beta}\}_{\beta>0}$ such that $\beta\|\widehat{G}_{\alpha_0+\beta} f\| \le \|f\|$ and

$$\mathcal{E}_{\alpha_0+\beta}(G_{\alpha_0+\beta} f, u) = (f, u) = \mathcal{E}_{\alpha_0+\beta}(u, \widehat{G}_{\alpha_0+\beta} f)$$

for all $f \in L^2(X;m)$ and $u \in \mathcal{F}$.

Define the generator L of $\{G_{\alpha_0+\beta}\}$ by $\mathcal{D}(L) = \{G_{\alpha_0+\beta} f : f \in L^2(X;m)\}$ and $Lu = \beta u - f$ for $u = G_{\alpha_0+\beta} f \in \mathcal{D}(L)$ and $\beta > 0$. Since $G_{\alpha_0+\gamma} f = G_{\alpha_0+\beta}\left(f + (\beta-\gamma) G_{\alpha_0+\gamma} f\right)$ by equation (1.1.7), $\mathcal{D}(L)$ is independent of the choice of β.

If a function $g \in L^2(X;m)$ satisfies $(G_{\alpha_0+\beta} f, g) = 0$ for all $f \in L^2(X;m)$, then $\widehat{G}_{\alpha_0+\beta} g = 0$. This implies that $(u, g) = \mathcal{E}_{\alpha_0+\beta}(u, \widehat{G}_{\alpha_0+\beta} g) = 0$ for all $u \in \mathcal{F}$. Hence $g = 0$ by the denseness of $\mathcal{F}$ in $L^2(X;m)$. Therefore, $\{G_{\alpha_0+\beta} f : f \in L^2(X;m)\}$ is dense in $L^2(X;m)$. In particular, since $\|\gamma G_\gamma G_\beta f - G_\beta f\| = \|G_\gamma f - \beta G_\gamma G_\beta f\| \le \frac{1}{\gamma-\alpha_0}\|f - \beta G_\beta f\|$ and the right-hand side converges to zero as γ increases to infinity, it follows that $\lim_{\gamma\to\infty} \gamma G_\gamma u = u$ in $L^2(X;m)$ for all $u = G_\beta f \in \mathcal{D}(L^{(\alpha_0)})$ and hence for all $u \in L^2(X;m)$. Furthermore, since the domain $\mathcal{D}(L^{(\alpha_0)})$ of the generator $L^{(\alpha_0)}$ is dense in $L^2(X;m)$, the Hille–Yoshida theorem implies that there exists a strongly continuous contraction semigroup $T_t^{(\alpha_0)}$ on $L^2(X;m)$ with resolvent $\{G_{\alpha_0+\beta}\}$. Similarly, $\{\widehat{G}_{\alpha_0+\beta}\}$ is also a resolvent of a strongly continuous contraction semigroup $\widehat{T}_t^{(\alpha_0)}$. Put $T_t f = e^{\alpha_0 t} T_t^{(\alpha_0)} f$ and $\widehat{T}_t f = e^{\alpha_0 t} \widehat{T}_t^{(\alpha_0)} f$. Then they are strongly continuous semigroups such that $\|T_t\| \le e^{\alpha_0 t}$ and $\|\widehat{T}_t\| \le e^{\alpha_0 t}$. Furthermore, their resolvents are respectively given by $G_\alpha f = G_{\alpha_0+(\alpha-\alpha_0)} f$ and $\widehat{G}_\alpha f = \widehat{G}_{\alpha_0+(\alpha-\alpha_0)} f$ for $\alpha > \alpha_0$. Equation (1.1.5) is clear from this. □

Define the *approximating form* $\mathcal{E}^\alpha$ of $\mathcal{E}$ by $\mathcal{E}^\alpha(u,v) = \alpha(u - \alpha G_\alpha u, v)$ for $u, v \in L^2(X;m)$ and put $\mathcal{E}^\alpha_\beta(u,v) = \mathcal{E}^\alpha(u,v) + \beta(u,v)$.

Lemma 1.1.3. *For any $\alpha > \alpha_0$ and $\beta \ge \alpha_0\,(\alpha/(\alpha-\alpha_0))^2$, the following inequalities hold:*

$$\mathcal{E}(\alpha G_\alpha u, \alpha G_\alpha u) \le \mathcal{E}^\alpha(u,u); \tag{1.1.8}$$

$$|\mathcal{E}^\alpha(u,v)| \le K \mathcal{E}^\alpha_\beta(u,u)^{1/2} \mathcal{E}_{\alpha_0}(v,v)^{1/2}; \tag{1.1.9}$$

$$|\mathcal{E}^\alpha(u,u)| \le K^2 \mathcal{E}_{\alpha_0}(u,u) + K\sqrt{\beta}\|u\| \mathcal{E}_{\alpha_0}(u,u)^{1/2}. \tag{1.1.10}$$

In particular, if $\alpha_0 = 0$ then, for any $\alpha > 0$ and $u \in \mathcal{F}$,

$$\mathcal{E}(\alpha G_\alpha u, \alpha G_\alpha u) \le \mathcal{E}^\alpha(u,u) \le K^2 \mathcal{E}(u,u). \tag{1.1.11}$$

Proof. Let $\alpha > \alpha_0$. Then equation (1.1.8) follows from

$$\begin{aligned}\mathcal{E}^\alpha(u, \alpha G_\alpha u) &= \alpha(u - \alpha G_\alpha u, \alpha G_\alpha u) = \mathcal{E}(\alpha G_\alpha u, \alpha G_\alpha u)\\ &= \mathcal{E}^\alpha(u,u) - \alpha(u - \alpha G_\alpha u, u - \alpha G_\alpha u)\\ &\le \mathcal{E}^\alpha(u,u).\end{aligned}$$

To show equation (1.1.9), by using ($\mathcal{E}$.2) we have

$$|\mathcal{E}^\alpha(u,v)| = |\mathcal{E}(\alpha G_\alpha u, v)| \le K \mathcal{E}_{\alpha_0}(\alpha G_\alpha u, \alpha G_\alpha u)^{1/2} \mathcal{E}_{\alpha_0}(v,v)^{1/2}.$$

In the right-hand side, using equation (1.1.8), it holds that

$$0 \le \mathcal{E}_{\alpha_0}(\alpha G_\alpha u, \alpha G_\alpha u) \le \mathcal{E}^\alpha(u,u) + \alpha_0 \left(\frac{\alpha}{\alpha-\alpha_0}\right)^2 \|u\|^2 \le \mathcal{E}^\alpha_\beta(u,u),$$

which shows equation (1.1.9). By putting $v = u$ in equation (1.1.9) it holds that

$$\mathcal{E}^\alpha(u,u)^2 \le K^2 \left(|\mathcal{E}^\alpha(u,u)| + \beta(u,u)\right) \mathcal{E}_{\alpha_0}(u,u).$$

Hence

$$|\mathcal{E}^\alpha(u,u)| \le \frac{K^2}{2}\mathcal{E}_{\alpha_0}(u,u) + \left(K^2\beta\|u\|^2 \mathcal{E}_{\alpha_0}(u,u) + \frac{K^4}{4}\mathcal{E}_{\alpha_0}(u,u)^2\right)^{1/2}.$$

By noting the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ holding for $a, b \geq 0$, equation (1.1.10) follows easily from this. Equation (1.1.11) follows from equation (1.1.10) by putting $\alpha_0 = 0$ and $\beta = 0$. □

Since $\mathcal{E}^\alpha(u, v) = \alpha(u, v - \alpha\widehat{G}_\alpha v)$, by a similar argument to the proof of Lemma 1.1.3, we have

$$\mathcal{E}(\alpha\widehat{G}_\alpha u, \alpha\widehat{G}_\alpha u) \leq \mathcal{E}^\alpha(u, u), \tag{1.1.8$'$}$$

$$|\mathcal{E}^\alpha(u, v)| \leq K\mathcal{E}_{\alpha_0}(u, u)^{1/2}\mathcal{E}^\alpha_\beta(v, v)^{1/2}. \tag{1.1.9$'$}$$

Theorem 1.1.4. (i) *A function $u \in L^2(X; m)$ belongs to $\mathcal{F}$ if and only if $\overline{\lim}_{\alpha\to\infty}\mathcal{E}^\alpha(u, u) < \infty$.*

(ii) *If $u, v \in \mathcal{F}$, then $\lim_{\alpha\to\infty} \mathcal{E}^\alpha(u, v) = \mathcal{E}(u, v)$.*

(iii) *If $u \in \mathcal{F}$, then for any $\beta > \alpha_0$, $\lim_{\alpha\to\infty} \mathcal{E}_\beta(\alpha G_\alpha u - u, \alpha G_\alpha u - u) = 0$.*

Proof. Since $(u - \alpha G_{\alpha+\alpha_0}u, u) \geq 0$,

$$\mathcal{E}^\alpha(u, u) = \alpha(u - \alpha G_{\alpha+\alpha_0}u, u) - \alpha_0\alpha^2(G_\alpha G_{\alpha+\alpha_0}u, u) \geq -\frac{\alpha_0\alpha}{\alpha - \alpha_0}(u, u).$$

Hence $\underline{\lim}_{\alpha\to\infty}\mathcal{E}^\alpha(u, u) \geq -\alpha_0(u, u)$. To show the assertion in (i), assume that $u \in \mathcal{F}$. Then $\overline{\lim}_{\alpha\to\infty}\mathcal{E}^\alpha(u, u) < \infty$ by equation (1.1.10). Suppose conversely that $\overline{\lim}_{\alpha\to\infty}\mathcal{E}^\alpha(u, u) < \infty$. Then $\overline{\lim}_{\alpha\to\infty}\mathcal{E}_\beta(\alpha G_\alpha u, \alpha G_\alpha u) < \infty$ for any $\beta > \alpha_0$. Hence there exists a subsequence $\{\alpha_n G_{\alpha_n} u\}$ converging weakly to some $v \in \mathcal{F}$ relative to the inner product $\mathcal{E}^{(s)}_\beta$. By the continuity of the resolvent, $\lim_{n\to\infty} \alpha_n G_{\alpha_n} u = u$ in $L^2(X; m)$ and consequently $u = v \in \mathcal{F}$ which gives the if part of (i). Since

$$\lim_{\alpha\to\infty} \mathcal{E}^\alpha(u, v) = \lim_{\alpha\to\infty} \big(\mathcal{E}_\beta(\alpha G_\alpha u, v) - \beta(\alpha G_\alpha u, v)\big) = \mathcal{E}(u, v),$$

(ii) follows from (iii).

To prove (iii), as in the proof of Theorem 1.1.2, let us introduce the generator L of $\{G_\beta\}$ with domain $\mathcal{D}(L) = \{G_\beta f : f \in L^2(X; m)\}$ which is independent of $\beta > \alpha_0$. If it is shown that $\mathcal{D}(L)$ is dense in $\mathcal{F}$, then any function $u \in \mathcal{F}$ can be approximated by a sequence of functions of $\mathcal{D}(L)$ relative to $\mathcal{E}_\beta$. Then, by virtue of equations (1.1.8) and (1.1.10), $\alpha G_\alpha u$ is also approximated by functions of the form $\alpha G_\alpha u_n$ with $u_n \in \mathcal{D}(L)$ uniformly for α relative to $\mathcal{E}_\beta$. If $u = G_\beta f \in \mathcal{D}(L)$ for $f \in L^2(X; m)$, then (iii) is obvious from $\lim_{\alpha\to\infty} \mathcal{E}_\beta(\alpha G_\alpha u - u, \alpha G_\alpha u - u) = \lim_{\alpha\to\infty}(\alpha G_\alpha f - f, \alpha G_\alpha u - u) = 0$.

To show the denseness of $\mathcal{D}(L)$ in $\mathcal{F}$, take any function $u \in \mathcal{F}$. By equation (1.1.8) and (i), $\mathcal{E}_\beta(nG_n u, nG_n u)$ is bounded relative to $n \geq 1$ for $\beta > \alpha_0$. Hence we can apply the Banach–Saks theorem to obtain a sequence of Cesàro means of $\{nG_n u\}$ which converges to u relative to $\mathcal{E}_\beta$. More precisely, choose a subsequence $u_k = n_k G_{n_k} u$ which converges weakly to u as $k \to \infty$ relative to $\mathcal{E}^{(s)}_\beta$. By choosing a subsequence, we may assume that $\max\{|\mathcal{E}^{(s)}_\beta(u - u_\ell, u - u_k)| : 1 \leq \ell \leq k - 1\} < 1/k$ for any k.

Then

$$\begin{aligned}
&\mathcal{E}_\beta\left(u - \frac{1}{k}(u_1 + \cdots + u_k), u - \frac{1}{k}(u_1 + \cdots + u_k)\right) \\
&\quad = \frac{1}{k^2}\sum_{i=1}^{k} \mathcal{E}_\beta\,(u - u_i, u - u_i) + \frac{1}{k^2}\sum_{i \neq j}^{k} \mathcal{E}_\beta^{(s)}(u - u_i, u - u_j) \\
&\quad \leq \frac{4}{k}\sup_n \mathcal{E}_\beta(u_n, u_n) + \frac{2}{k^2}\sum_{i=2}^{k}\frac{i-1}{i}.
\end{aligned}$$

This shows that $\{\frac{1}{k}\sum_{i=1}^{k} u_i\}$ is an $\mathcal{E}_\beta$-Cauchy sequence converging to u in $L^2(X;m)$. Hence it converges strongly in $(\mathcal{E},\mathcal{F})$ to u. Since $(1/k)\sum_{i=1}^{k} u_i$ belongs to $\mathcal{D}(L)$, this shows the denseness of $\mathcal{D}(L)$ in $\mathcal{F}$. $\square$

Theorem 1.1.5. *Suppose that $(\mathcal{E},\mathcal{F})$ is a closed form and let $\{T_t\}_{t>0}$ be the semi-group corresponding to $(\mathcal{E},\mathcal{F})$ by Theorem 1.1.2. Then the following conditions are mutually equivalent.*

($\mathcal{E}$.4) *For all $u \in \mathcal{F}$ and $a \geq 0$, $u \wedge a \in \mathcal{F}$ and $\mathcal{E}(u \wedge a, u - u \wedge a) \geq 0$.*

($\mathcal{E}$.4a) *For all $u \in \mathcal{F}$, $u^+ \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(u^+ \wedge 1, u - u^+ \wedge 1) \geq 0$.*

($\mathcal{E}$.4b) *For all $u \in \mathcal{F}$, $u^+ \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq -\alpha_0\|u - u^+ \wedge 1\|^2$.*

($\mathcal{E}$.4c) *$\{T_t\}$ is sub-Markov: If $f \in L^2(X;m)$ satisfies $0 \leq f \leq 1$ m-a.e., then $0 \leq T_t f \leq 1$ m-a.e.*

($\mathcal{E}$.4d) *$\{\widehat{T}_t\}$ is positivity preserving and contractive in $L^1(X;m)$: If $f \in L^1(X;m)$ satisfies $f \geq 0$ m-a.e., then $\widehat{T}_t f \geq 0$ m-a.e. and $\|\widehat{T}_t f\|_{L^1} \leq \|f\|_{L^1}$.*

Proof. ($\mathcal{E}$.4) $\Rightarrow$ ($\mathcal{E}$.4a): For any $u \in \mathcal{F}$, $u^+ = (-u) \wedge 0 \in \mathcal{F}$. Hence by noting that $u^+ \wedge 1 = (u \wedge 1)^+$ and $u^- = (u \wedge 1)^-$, we obtain ($\mathcal{E}$.4a) by

$$\begin{aligned}
&\mathcal{E}(u^+ \wedge 1, u - u^+ \wedge 1) \\
&\quad = \mathcal{E}((u \wedge 1)^+, u \wedge 1 - (u \wedge 1)^+) + \mathcal{E}(u^+ \wedge 1, u - u \wedge 1) \\
&\quad = \mathcal{E}((u \wedge 1)^+, u \wedge 1 - (u \wedge 1)^+) + \mathcal{E}(u^+ \wedge 1, u^+ - u^+ \wedge 1) \geq 0.
\end{aligned}$$

($\mathcal{E}$.4a) $\Rightarrow$ ($\mathcal{E}$.4b) follows from

$$\begin{aligned}
\mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) &= \mathcal{E}(u - u^+ \wedge 1, u - u^+ \wedge 1) \\
&\quad + 2\mathcal{E}(u^+ \wedge 1, u - u^+ \wedge 1) \\
&\geq -\alpha_0\|u - u^+ \wedge 1\|^2.
\end{aligned}$$

$(\mathcal{E}.4b) \Rightarrow (\mathcal{E}.4c)$: Put $u = \alpha G_\alpha f$ for $f \in L^2(X;m)$ such that $0 \le f \le 1$ m-a.e. and $\alpha > \alpha_0$. Then

$$\begin{aligned}\alpha_0\|u - u^+ \wedge 1\|^2 &\ge -\frac{1}{2}\{\mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) + \mathcal{E}(u - u^+ \wedge 1, u - u^+ \wedge 1)\}\\ &= -\mathcal{E}(u, u - u^+ \wedge 1) = \alpha(u - f, u - u^+ \wedge 1)\\ &= \alpha\|u - u^+ \wedge 1\|^2 + \alpha(u^+ \wedge 1 - f, u - u^+ \wedge 1).\end{aligned}$$

Since $0 \le f \le 1$,

$$(u^+ \wedge 1 - f, u - u^+ \wedge 1) \ge 0,$$

which implies that $(\alpha - \alpha_0)\|u - u^+ \wedge 1\| \le 0$. Hence $u - u^+ \wedge 1 = 0$, which implies that $0 \le \alpha G_\alpha f \le 1$. The sub-Markov property of the associated semigroup is clear from this.

$(\mathcal{E}.4c) \Rightarrow (\mathcal{E}.4)$: Suppose that $(\mathcal{E}.4c)$ holds. For any $u \in \mathcal{F}$ and $a \ge 0$, since $\alpha G_\alpha(u \wedge a) \le a$, $(u - a)^+ \times (\alpha G_\alpha - I)(u \wedge a) \le 0$. Hence

$$\mathcal{E}^\alpha(u \wedge a, u - u \wedge a) = -\alpha\left((\alpha G_\alpha - I)(u \wedge a), (u - a)^+\right) \ge 0.$$

Hence, using (1.1.9)$'$ we get that

$$\begin{aligned}\mathcal{E}^\alpha\left((u - a)^+, (u - a)^+\right) &= \mathcal{E}^\alpha(u, (u - a)^+) - \mathcal{E}^\alpha(u \wedge a, (u - a)^+)\\ &\le \mathcal{E}^\alpha(u, (u - a)^+)\\ &\le K\mathcal{E}_{\alpha_0}(u, u)^{1/2}\mathcal{E}^\alpha_\beta((u - a)^+, (u - a)^+)^{1/2}\end{aligned}$$

for $\beta = \alpha_0(\alpha/(\alpha - \alpha_0))^2$. This yields that $\overline{\lim}_{\alpha\to\infty}\mathcal{E}^\alpha((u - a)^+, (u - a)^+) < \infty$ and hence $(u - a)^+ \in \mathcal{F}$. Since $u \wedge a = u - (u - a)^+$, $u \wedge a \in \mathcal{F}$ and

$$\mathcal{E}(u \wedge a, u - u \wedge a) = \lim_{\alpha\to\infty}\mathcal{E}^\alpha(u \wedge a, u - u \wedge a) \ge 0.$$

$(\mathcal{E}.4c) \Leftrightarrow (\mathcal{E}.4d)$: If $(\mathcal{E}.4c)$ holds. Then for any $f \in L^2(X;m)$ such that $0 \le f \le 1$ and $g \in L^1_+(X;m) \cap L^2(X;m)$, since $0 \le (T_t f, g) \le \|g\|_{L^1(X;m)}$, it follows that $0 \le (f, \widehat{T}_t g) \le \|g\|_{L^1(X;m)}$. This implies $(\mathcal{E}.4d)$. Similarly $(\mathcal{E}.4d)$ implies $(\mathcal{E}.4c)$. □

By virtue of $(\mathcal{E}.4c)$, since $T_t f_1 \le T_t f_2$ for any $f_1, f_2 \in L^2(X;m)$ such that $f_1 \le f_2$, for any $f \in L^\infty_+(X;m)$, by taking a sequence $f_n \in L^2(X;m)$ such that $f_n \uparrow f$, $T_t f$ is well defined by $T_t f = \lim_{n\to\infty} T_t f_n$. In fact, if $\{f^1_n\}, \{f^2_n\} \subset L^2(X;m)$ are two increasing sequences converging to f a.e., then the limits $T^1_t f = \lim_{n\to\infty} T_t f^1_n$ and $T^2_t f = \lim_{n\to\infty} T_t f^2_n$ exist. For any $g \in L^2_+(X;m)$,

$$(g, T^i_t f) = \lim_{n\to\infty}(g, T_t f^i_n) = \lim_{n\to\infty}(\widehat{T}_t g, f^i_n) = (\widehat{T}_t g, f)$$

which implies $T^1_t f = T^2_t f$. For any $f \in L^\infty(X;m)$, by considering $f = f^+ - f^-$, $T_t f$ is well defined by $T_t f = T_t f^+ - T_t f^-$. By the sub-Markov property, the extended resolvent on $L^\infty(X;m)$ satisfies $\|T_t f\|_\infty \le \|f\|_\infty$. Also, its resolvent $\{G_\alpha\}$ can be extended to a sub-Markov resolvent on $L^\infty(X;m)$ satisfying $\|\alpha G_\alpha f\|_\infty \le \|f\|_\infty$.

By the duality relation, since $(T_t f, g) = (f, \widehat{T}_t g) \geq 0$ for any non-negative functions $f, g \in L^2_+(X;m)$, $\widehat{T}_t g \geq 0$ for any $g \in L^2_+(X;m)$. Hence $\widehat{T}_t$ and $\widehat{G}_\alpha$ are extended to a semigroup and a resolvent on $L^1(X;m)$. By the duality relation, the extended operators satisfy $\|\widehat{T}_t f\|_{L^1(m)} \leq \|f\|_{L^1(m)}$ and $\|\alpha \widehat{G}_\alpha f\|_{L^1(m)} \leq \|f\|_{L^1(m)}$ for any $f \in L^1(X;m)$.

Let $\widehat{T}_t$ and $\widehat{G}_\alpha$ be those in Theorem 1.1.2. Then, similarly to Theorem 1.1.5, they are sub-Markov if and only if

$$(\hat{\mathcal{E}}.4) \qquad \mathcal{E}(u - u \wedge a, u \wedge a) \geq 0$$

for all $u \in \mathcal{F}$ and $a \geq 0$.

Definition 1. A bilinear form $(\mathcal{E}, \mathcal{F})$ is called a *lower-bounded semi-Dirichlet form* if it satisfies $(\mathcal{E}.1)$, $(\mathcal{E}.2)$, $(\mathcal{E}.3)$, and $(\mathcal{E}.4)$. In this monograph, we call the lower bounded semi-Dirichlet form simply as *Dirichlet form*. In particular, if $\alpha_0 = 0$, then $(\mathcal{E}, \mathcal{F})$ is called *non-negative Dirichlet form*. Furthermore, if a non-negative Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies the dual sub-Markov property $(\hat{\mathcal{E}}.4)$, then $(\mathcal{E}, \mathcal{F})$ is called a *non-symmetric Dirichlet form*. A non-symmetric Dirichlet form is called a *symmetric Dirichlet form* if $\mathcal{E}$ is symmetric.

For a Dirichlet form $(\mathcal{E}, \mathcal{F})$, since

$$\alpha\left(|u| - \alpha G_\alpha |u|, |u|\right) \leq \alpha\left(u - \alpha G_\alpha u, u\right),$$

Theorem 1.1.4 implies that $|u| \in \mathcal{F}$ and

$$\mathcal{E}(|u|, |u|) \leq \mathcal{E}(u, u) \tag{1.1.12}$$

for any $u \in \mathcal{F}$. In particular, if $u, v \in \mathcal{F}$, then $u \wedge v = \frac{1}{2}(u + v - |u - v|) \in \mathcal{F}$.

1.2 Closability and regular Dirichlet forms

Let $\mathcal{E}$ be a bilinear form defined on $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$ satisfying $(\mathcal{E}.1)$ and $(\mathcal{E}.2)$ for a dense linear subspace $\mathcal{D}(\mathcal{E})$ of $L^2(X;m)$. We say that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is *closable* on $L^2(X;m)$ if the following condition holds:

$$\text{If } u_n \in \mathcal{D}(\mathcal{E}) \text{ satisfies } \lim_{m,n\to\infty} \mathcal{E}(u_n - u_m, u_n - u_m) = 0$$
$$\text{and } \lim_{n\to\infty} (u_n, u_n) = 0, \text{ then } \lim_{n\to\infty} \mathcal{E}(u_n, u_n) = 0. \tag{1.2.1}$$

Suppose that $\mathcal{E}$ is closable on $L^2(X;m)$. Denote by $\mathcal{F}$ the family of functions $u \in L^2(X;m)$ for which there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}$ such that $\lim_{n\to\infty} u_n = u$ in $L^2(X;m)$. In this case, we call the sequence $\{u_n\}$ an *approximating sequence* of u and $(\mathcal{E}, \mathcal{F})$ the *smallest closed extension* of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Theorem 1.2.1. *Suppose that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is closable on $L^2(X; m)$. For $u, v \in \mathcal{F}$, and its approximating sequences $\{u_n\}$ and $\{v_n\}$ of u and v, respectively,*

$$\mathcal{E}(u, v) = \lim_{n\to\infty} \mathcal{E}(u_n, v_n) \tag{1.2.2}$$

exists independently of the choice of the approximating sequences. Furthermore, the smallest closed extension $(\mathcal{E}, \mathcal{F})$ of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a closed form.

Proof. Let $\{u_n\}$ be an approximating sequence of $u \in \mathcal{F}$. Since $\mathcal{E}_{\alpha_0}(u_n, u_n) \geq 0$ for any $u_n \in \mathcal{D}(\mathcal{E})$, it satisfies the triangle inequality:

$$|\mathcal{E}_{\alpha_0}(u_n, u_n)^{1/2} - \mathcal{E}_{\alpha_0}(u_m, u_m)^{1/2}| \leq \mathcal{E}_{\alpha_0}(u_n - u_m, u_n - u_m)^{1/2}. \tag{1.2.3}$$

Hence $\lim_{n\to\infty} \mathcal{E}_{\alpha_0}(u_n, u_n)$ and hence $\lim_{n\to\infty} \mathcal{E}(u_n, u_n)$ exists. Since

$$\begin{aligned}|\mathcal{E}(u_n, v_n) - \mathcal{E}(u_m, v_m)| \leq\ & K\mathcal{E}_{\alpha_0}(u_n, u_n)^{1/2}\mathcal{E}_{\alpha_0}(v_n - v_m, v_n - v_m)^{1/2} \\ &+ K\mathcal{E}_{\alpha_0}(v_m, v_m)^{1/2}\mathcal{E}_{\alpha_0}(u_n - u_m, u_n - u_m)^{1/2},\end{aligned}$$

the uniform boundedness of $\{\mathcal{E}_{\alpha_0}(u_n, u_n)\}$ and $\{\mathcal{E}_{\alpha_0}(v_m, v_m)\}$ yields the existence of equation (1.2.2). $(\mathcal{E}.1)$ and $(\mathcal{E}.2)$ of $(\mathcal{E}, \mathcal{F})$ follows easily from the corresponding properties of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. If $\{w_n\} \subset \mathcal{F}$ is an $\mathcal{E}_\alpha$-Cauchy sequence, then for the approximating sequences $\{w_{n,k}\} \subset \mathcal{D}(\mathcal{E})$ of w_n, the diagonal sequence $\{w_{n,n}\}$ converges to $w \in \mathcal{F}$. This implies $(\mathcal{E}.3)$ for $(\mathcal{E}, \mathcal{F})$. □

Theorem 1.2.2. *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a closable form satisfying $(\mathcal{E}.4)$ for $\mathcal{D}(\mathcal{E})$ instead of $\mathcal{F}$. Then the smallest closed extension $(\mathcal{E}, \mathcal{F})$ also satisfies $(\mathcal{E}.4)$, that is, $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form on $L^2(X; m)$.*

Proof. Let $\{G_\alpha\}$ be the resolvent associated with $(\mathcal{E}, \mathcal{F})$. Then it is enough to show that the resolvent $\{G_\alpha\}$ is sub-Markov.

Take a function $f \in L^2(X; m)$ such that $0 \leq f \leq 1$ m-a.e. and put $u = \alpha G_\alpha f$ for $\alpha > \alpha_0$. Since $u \in \mathcal{F}$, there exists an approximating sequence $\{u_n\} \subset \mathcal{D}(\mathcal{E})$ such that $\lim_{n\to\infty} u_n = u$ relative to $\mathcal{E}_\alpha$. By virtue of Theorem 1.1.5, if $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ satisfies $(\mathcal{E}.4)$, then it satisfies $(\mathcal{E}.4a)$ and $(\mathcal{E}.4b)$. In particular $u_n^+ \wedge 1 \in \mathcal{D}(\mathcal{E})$ and

$$\begin{aligned}-\alpha_0\|u_n^+ \wedge 1\|^2 &\leq \mathcal{E}(u_n^+ \wedge 1, u_n^+ \wedge 1) \leq \mathcal{E}(u_n^+ \wedge 1, u_n) \\ &\leq K\mathcal{E}_{\alpha_0}(u_n^+ \wedge 1, u_n^+ \wedge 1)^{1/2}\mathcal{E}_{\alpha_0}(u_n, u_n)^{1/2}.\end{aligned}$$

This implies that $\mathcal{E}(u_n^+ \wedge 1, u_n^+ \wedge 1)$ is bounded relative to $n \geq 1$. Since equation (1.1.2) holds for $u_n, v_n \in \mathcal{D}(\mathcal{E})$, it also holds for $u, v \in \mathcal{F}$ by the definition of $(\mathcal{E}, \mathcal{F})$. Hence the $\mathcal{E}_{\alpha_0}$-boundedness of $\{u_n - u_n^+ \wedge 1\}$ implies that $\lim_{n\to\infty} \mathcal{E}(u_n - u, u_n - u_n^+ \wedge 1) = 0$.

Therefore, by using ($\mathcal{E}$.4a) and equation (1.1.1),

$$\begin{aligned}
\alpha_0\|u-u^+\wedge 1\|^2 &= \alpha_0 \lim_{n\to\infty} \|u_n-u_n^+\wedge 1\|^2 \\
&\geq -\lim_{n\to\infty} \mathcal{E}(u_n-u_n^+\wedge 1, u_n-u_n^+\wedge 1) \\
&\geq -\lim_{n\to\infty} \left\{\mathcal{E}(u_n-u_n^+\wedge 1, u_n-u_n^+\wedge 1)+\mathcal{E}(u_n^+\wedge 1, u_n-u_n^+\wedge 1)\right\} \\
&= -\lim_{n\to\infty} \mathcal{E}(u_n, u_n-u_n^+\wedge 1) = -\lim_{n\to\infty} \mathcal{E}(u, u_n-u_n^+\wedge 1) \\
&= \lim_{n\to\infty} \alpha(u-f, u_n-u_n^+\wedge 1) = \alpha(u-f, u-u^+\wedge 1) \\
&= \alpha\|u-u^+\wedge 1\|^2+(u^+\wedge 1-f, u-u^+\wedge 1).
\end{aligned}$$

Since $(u^+\wedge 1-f, u-u^+\wedge 1)\geq 0$, this shows that $u=u^+\wedge 1$. □

Definition 2. A Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(X;m)$ is called a *regular* Dirichlet form with *core* $\mathcal{C}_1$ if $\mathcal{C}_1$ is a sub-family of $\mathcal{C}_0(X)$ such that $\mathcal{F}\cap\mathcal{C}_1$ is $\mathcal{E}_\alpha$-dense in $\mathcal{F}$ and uniformly dense in $\mathcal{C}_0(X)$.

For a regular Dirichlet form $(\mathcal{E},\mathcal{F})$ and its associated resolvent $\{G_\alpha\}$, since $\int_X G_\alpha g(x)f(x)m(dx)$ is a non-negative bilinear form relative to $f,g\in\mathcal{C}_0(X)$, it is represented as

$$\int_X G_\alpha g(x)f(x)m(dx)=\int_X\int_X f(x)g(y)G_\alpha(dxdy) \tag{1.2.4}$$

for a positive Radon measure $G_\alpha(dxdy)$ on $X\times X$. Note that $G_\alpha(dxdy)$ does not charge any set of zero $m\otimes m$-measure.

As a particular case, suppose that $(\mathcal{E},\mathcal{F})$ is a symmetric regular Dirichlet form. Then the approximating form $\mathcal{E}^\alpha$ can be written as

$$\begin{aligned}
\mathcal{E}^\alpha(u,u) &= \frac{\alpha^2}{2}\int_X\int_X (u(x)-u(y))^2\, G_\alpha(dxdy) \\
&\quad + \alpha\int_X u^2(x)\,(1-\alpha G_\alpha 1(x))\, m(dx).
\end{aligned} \tag{1.2.5}$$

By virtue of Theorem 1.1.4, $u\in L^2(X;m)$ belongs to $\mathcal{F}$ if and only if $\mathcal{E}^\alpha(u,u)$ remains bounded as α increases to infinity. In particular, for any $u,v\in\mathcal{F}_b$, it follows from equation (1.2.5) and the inequalities $|uv(x)|\leq|u(x)||v(x)|$ and

$$|uv(x)-uv(y)|\leq\|v\|_\infty|u(x)-u(y)|+\|u\|_\infty|v(x)-v(y)|$$

m-a.e. x,y, that $uv\in\mathcal{F}$ and

$$\mathcal{E}(uv,uv)^{1/2}\leq\|v\|_\infty\mathcal{E}(u,u)^{1/2}+\|u\|_\infty\mathcal{E}(v,v)^{1/2}. \tag{1.2.6}$$

1.3 Transience and recurrence of Dirichlet forms

For a given Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X;m)$, let $\{T_t\}_{t>0}$ and $\{\widehat{T}_t\}_{t>0}$ be the semigroups on $L^2(X;m)$ associated with $(\mathcal{E}, \mathcal{F})$ by Theorem 1.1.2. A Borel measurable subset B of X is called an *invariant set* relative to $\{T_t\}$ if, for all $t > 0$ and $f \in L^2(X;m)$,

$$1_B T_t f = 1_B T_t(1_B f) \quad m\text{-a.e.} \tag{1.3.1}$$

An invariant set relative to $\{\widehat{T}_t\}$ is defined similarly. If B is an invariant set relative to $\{T_t\}$, then for any $g \in L^2(X;m)$,

$$(f, \widehat{T}_t(1_B g)) = (1_B T_t f, g) = (1_B T_t(1_B f), g) = (1_B f, \widehat{T}_t(1_B g)).$$

Hence $\widehat{T}_t(1_B g) = 1_B \widehat{T}_t(1_B g)$ and hence

$$\widehat{T}_t g - \widehat{T}_t(1_{X\setminus B} g) = \widehat{T}_t(1_B g) = 1_B \left(\widehat{T}_t g - \widehat{T}_t(1_{X\setminus B} g)\right).$$

This implies

$$1_{X\setminus B}\widehat{T}_t g = 1_{X\setminus B}\widehat{T}_t(1_{X\setminus B} g) \quad \text{for all } t > 0 \text{ and } g \in L^2(X;m), \tag{1.3.2}$$

that is $X \setminus B$ is an invariant set relative to $\{\widehat{T}_t\}$. Since the converse assertion also holds, B is an invariant set relative to $\{T_t\}$ if and only if $X \setminus B$ is an invariant set relative to $\{\widehat{T}_t\}$.

Lemma 1.3.1. *For any non-negative Borel function f and $\alpha > 0$, $\{x : G_\alpha f(x) > 0\}$ is an invariant set relative to $\{T_t\}$.*

Proof. Put $B = \{x : G_\alpha f(x) = 0\}$. For any non-negative function $g \in L^1(X;m)$ such that $g = 0$ m-a.e. on $X \setminus B$,

$$\begin{aligned} 0 = (G_\alpha f, g) &\geq e^{-\alpha t}(T_t G_\alpha f, g) = e^{-\alpha t}\left(G_\alpha f, \widehat{T}_t(1_B g)\right) \\ &\geq e^{-\alpha t}\left(1_{X\setminus B}\cdot G_\alpha f, \widehat{T}_t(1_B g)\right) \end{aligned}$$

Hence $1_{X\setminus B}\widehat{T}_t(1_B g) = 0$ m-a.e. and hence $\widehat{T}_t(1_B g) = 1_B\widehat{T}_t(1_B g)$. This implies that B is an invariant set of $\{\widehat{T}_t\}$ and hence $X \setminus B$ is an invariant set of $\{T_t\}$. □

Theorem 1.3.2. *B is an invariant set relative to both $\{T_t\}$ and $\{\widehat{T}_t\}$ if and only if $I_B u, I_B v \in \mathcal{F}$ for all $u, v \in \mathcal{F}$ and satisfies*

$$\mathcal{E}(u, v) = \mathcal{E}(1_B u, 1_B v) + \mathcal{E}(1_{X\setminus B} u, 1_{X\setminus B} v). \tag{1.3.3}$$

Proof. Suppose that B is an invariant set relative to both $\{T_t\}$ and $\{\widehat{T}_t\}$. Then, as we noted before Lemma 1.3.1, $X \setminus B$ is also an invariant set relative to $\{T_t\}$ and $\{\widehat{T}_t\}$. Hence, for all $u \in \mathcal{F}$, $1_B G_\alpha u = 1_B G_\alpha(1_B u)$ and $1_{X\setminus B} G_\alpha u = 1_{X\setminus B} G_\alpha(1_{X\setminus B} u)$. Therefore,

$$(u, u - \alpha G_\alpha u) = (1_B u, 1_B u - \alpha G_\alpha(1_B u)) + \left(1_{X\setminus B} u, 1_{X\setminus B} - \alpha G_\alpha(1_{X\setminus B} u)\right).$$

Hence, by Theorem 1.1.4 (i) and (ii), $1_B u, 1_{X\setminus B} u \in \mathcal{F}$ and equation (1.3.3) holds.

Conversely, suppose that $1_B u, 1_B v \in \mathcal{F}$ and equation (1.3.3) hold for all $u, v \in \mathcal{F}$. By putting $1_B u$ and $1_{X\setminus B} v$ instead of u, v, respectively, it holds that

$$\mathcal{E}(1_B u, 1_{X\setminus B} v) = 0 \quad u, v \in \mathcal{F}. \tag{1.3.4}$$

For any $\alpha > \alpha_0$, since

$$\mathcal{E}_\alpha\left(G_\alpha f, 1_{X\setminus B} v\right) = (f, 1_{X\setminus B} v) = \mathcal{E}_\alpha\left(G_\alpha(1_{X\setminus B} f), 1_{X\setminus B} v\right),$$

it follows that $\mathcal{E}_\alpha\left(G_\alpha f - G_\alpha(1_{X\setminus B} f), 1_{X\setminus B} v\right) = 0$. In particular, by putting $v = G_\alpha f - G_\alpha(1_{X\setminus B} f)$, we get that $\mathcal{E}_\alpha\left(v, 1_{X\setminus B} v\right) = 0$. Hence, by equation (1.3.4) we get that

$$\mathcal{E}_\alpha\left(1_{X\setminus B} v, 1_{X\setminus B} v\right) = 0,$$

which shows that $1_{X\setminus B} v = 0$, that is $1_{X\setminus B} G_\alpha f = 1_{X\setminus B} G_\alpha(1_{X\setminus B} f)$. Similarly, by taking B in place of $X \setminus B$, we have $1_B G_\alpha f = 1_B G_\alpha(1_B f)$. □

A Dirichlet form $(\mathcal{E}, \mathcal{F})$ (or semigroup $\{T_t\}$) is called *irreducible* if any invariant set B relative to $\{T_t\}$ satisfies $m(B) = 0$ or $m(X \setminus B) = 0$. As noted before Lemma 1.3.1, if B is a $\{T_t\}$-invariant set, then $X \setminus B$ is an invariant set relative to $\{\widehat{T}_t\}$. Hence $(\mathcal{E}, \mathcal{F})$ is irreducible if and only if any invariant set B relative to $\{\widehat{T}_t\}$ satisfies $m(B) = 0$ or $m(X \setminus B) = 0$.

As we have seen after Theorem 1.1.5, $\{G_\alpha\}$ can be considered as a sub-Markov $L^\infty(X; m)$-resolvent. For any $f \in L^\infty_+(X; m)$, since $G_\alpha f$ is increasing as α decreases, the *potential operator*

$$Gf = \lim_{n\to\infty} G_{1/n} f \tag{1.3.5}$$

is well defined. Similarly, $\{\widehat{G}_\alpha\}$ can be considered as a contractive $L^1(X; m)$-resolvent. If $f \in L^1_+(X; m)$, $\widehat{G}_\alpha f$ is also increasing as α decreases. Hence we can define the *copotential operator*

$$\widehat{G} f = \lim_{n\to\infty} \widehat{G}_{1/n} f. \tag{1.3.6}$$

A Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *transient* if there exists a strictly positive function $f \in L^\infty(X; m)$ such that

$$Gf < \infty \qquad m\text{-a.e.} \tag{1.3.7}$$

An irreducible Dirichlet form is called *recurrent* if it is non-transient.

If $(\mathcal{E}, \mathcal{F})$ is transient, then there exists a strictly positive function $f \in L^\infty(X;m)$ satisfying equation (1.3.7). Let $\{B_n\}$ be an increasing sequence of compact sets such that $\cup B_n = X$ and $Gf(x) \le n$ for a.e. $x \in B_n$. Then the function g defined by $g(x) = \sum_{n=1}^\infty (1/nm(B_n))2^{-n} 1_{B_n}(x)$ is a strictly positive function satisfying $g \in L^1(X;m) \cap L^\infty(X;m)$ and

$$(f, \widehat{G}g) = \sum_{n=1}^\infty \frac{1}{nm(B_n)2^n}(Gf, 1_{B_n}) < \infty.$$

In particular,

$$\widehat{G}g < \infty \qquad m\text{-a.e.} \tag{1.3.8}$$

Similarly, if there exists a strictly positive function $g \in L^1(X;m)$ satisfying equation (1.3.8), then there exists a strictly positive function $f \in L^\infty(X;m)$ such that equation (1.3.7) holds. We shall show in Theorem 1.3.4 that this is also equivalent to the condition that equation (1.3.8) holds for all $g \in L^1(X;m)$. To show it, we need the following Hopf's maximal ergodic inequality.

Lemma 1.3.3. *For a given function $f \in L^1(X;m)$ and a positive number α, let E_α be the set defined by $E_\alpha = \{x \in X : \sup_n \widehat{G}_{\alpha/n} f(x) > 0\}$. Then*

$$\int_{E_\alpha} \widehat{G}_\alpha f(x) m(dx) \ge 0. \tag{1.3.9}$$

Proof. Let $E_\alpha^n = \{x \in X : \max_{1\le k\le n} \widehat{G}_{\alpha/k} f(x) \ge 0\}$. If $x \in E_\alpha^n$, then

$$\begin{aligned}
&\widehat{G}_\alpha f(x) + \max_{1\le k\le n}(\widehat{G}_{\alpha/k} f - \widehat{G}_\alpha f)^+(x) \\
&\ge \max_{1\le k\le n} \widehat{G}_{\alpha/k} f(x) = \max_{1\le k\le n}(\widehat{G}_{\alpha/k} f)^+(x).
\end{aligned}$$

Noting that

$$\begin{aligned}
\max_{1\le k\le n}(\widehat{G}_{\alpha/k} f - \widehat{G}_\alpha f)^+(x) &= \max_{1\le k\le n}\left(\frac{k-1}{k}\right)\alpha(\widehat{G}_\alpha \widehat{G}_{\alpha/k} f)^+(x) \\
&\le \alpha \widehat{G}_\alpha \max_{1\le k\le n}(\widehat{G}_{\alpha/k} f)^+(x),
\end{aligned}$$

we have

$$\begin{aligned}
&\int_{E_\alpha^n} \widehat{G}_\alpha f(x) m(dx) \\
&\ge \int_{E_\alpha^n}\left(\max_{1\le k\le n}(\widehat{G}_{\alpha/k} f)^+(x) - \alpha\widehat{G}_\alpha(\max_{1\le k\le n}(\widehat{G}_{\alpha/k} f)^+)(x)\right) m(dx) \\
&= \int_X \max_{1\le k\le n}(\widehat{G}_{\alpha/k} f)^+(x) m(dx) - \int_{E_\alpha^n} \alpha\widehat{G}_\alpha(\max_{1\le k\le n}(\widehat{G}_{\alpha/k} f)^+)(x) m(dx) \\
&\ge \int_X \max_{1\le k\le n}(\widehat{G}_{\alpha/k} f)^+(x) m(dx) - \int_X \alpha\widehat{G}_\alpha(\max_{1\le k\le n}(\widehat{G}_{\alpha/k} f)^+)(x) m(dx) \\
&\ge 0.
\end{aligned}$$

□

Theorem 1.3.4. $(\mathcal{E}, \mathcal{F})$ *is transient if and only if there exists a strictly positive function* $g \in L^1(X;m)$ *such that* $\widehat{G}g < \infty$ *m-a.e.*

Proof. Suppose that $g \in L^1(X;m)$ satisfies $g > 0$ m-a.e. and $\widehat{G}g < \infty$ m-a.e. For any $f \in L^1_+(X;m)$, define E_α as in Lemma 1.3.3 by taking $f - cg$ in place of f. Then it holds that $\int_{E_\alpha} \widehat{G}_\alpha(f - cg)(x)m(dx) \geq 0$. Since the set $B = \{\widehat{G}f = \infty\}$ is contained in E_α up to a negligible set,

$$\frac{1}{\alpha}\int_X f dm \geq \int_{E_\alpha} \widehat{G}_\alpha f dm \geq c \int_{E_\alpha} \widehat{G}_\alpha g dm.$$

Hence, for any compact set K and integer N, $(1/c)\int_X f dm \geq \alpha \int_{B\cap K} \widehat{G}_\alpha(g \wedge N)dm$. Since $g \wedge N \in L^2(X;m)$, letting α tend to infinity, we have $(1/c)\int_X f dm \geq \int_{B\cap K}(g \wedge N)dm$. Then let $K \uparrow X$, $N \uparrow \infty$ and $c \uparrow \infty$ to get $\int_B g dm = 0$ which implies $m(B) = 0$. □

For a non-negative measurable function $g \in L^\infty(X;m) \cap L^1(X;m)$, define a bilinear form $\mathcal{E}^g$ by

$$\mathcal{E}^g(u,v) = \mathcal{E}(u,v) + (u,v)_{g\cdot m}, \tag{1.3.10}$$

where $(u,v)_{g\cdot m} = \int_X u(x)v(x)g(x)dm(x)$. Since $\mathcal{E}(u,u) \leq \mathcal{E}^g(u,u) \leq \mathcal{E}_\alpha(u,u)$ for $\alpha \geq \|g\|_\infty$, $(\mathcal{E}^g, \mathcal{F})$ satisfies $(\mathcal{E}.1)$, $(\mathcal{E}.2)$ and $(\mathcal{E}.3)$. Hence there corresponds a semigroup $\{T^g_t\}$ and its resolvent $\{G^g_\alpha\}$ on $L^2(X;m)$ satisfying $\|T^g_t f\| \leq e^{\alpha_0 t}\|f\|$ and

$$\mathcal{E}^g_\alpha(G^g_\alpha f, u) = (f,u) \tag{1.3.11}$$

for all $f \in L^2(X;m), u \in \mathcal{F}$ and $\alpha > \alpha_0$. Furthermore, if $\mathcal{E}$ satisfies $(\mathcal{E}.4)$, then so does $\mathcal{E}^g$. In particular, $\{G^g_\alpha\}$ can be extended to $\alpha > 0$ as a sub-Markov resolvent on $L^\infty(X;m)$. Denote by $g \cdot G_\alpha$ the operator defined by $(g \cdot G_\alpha)f(x) = g(x)G_\alpha f(x)$.

Lemma 1.3.5. *Let g be a strictly positive function of $L^1(X;m) \cap L^\infty(X;m)$. Then, for all $f \in L^2(X;m)$ and $\alpha > \alpha_0 + \|g\|_\infty$,*

$$G^g_\alpha f = \sum_{n=0}^\infty (-1)^n G_\alpha (g \cdot G_\alpha)^n f. \tag{1.3.12}$$

If $f \in L^\infty(X;m) \cap L^2(X;m)$, then for any $\alpha > 0$,

$$G_\alpha f = G^g_\alpha f + G^g_\alpha(g \cdot G_\alpha)f = G^g_\alpha f + G_\alpha(g \cdot G^g_\alpha)f. \tag{1.3.13}$$

Moreover, $G^g g \leq 1$ m-a.e and $(\mathcal{E}^g, \mathcal{F})$ is transient.

If $(\mathcal{E}, \mathcal{F})$ is irreducible, then $(\mathcal{E}^g, \mathcal{F})$ is also irreducible. In particular, if $(\mathcal{E}, \mathcal{F})$ is recurrent, then $G^g g = 1$ m-a.e. for any non-negative function $g \in L^1(X;m) \cap L^\infty(X;m)$ such that $\int_X g(x)m(dx) > 0$.

Proof. Since $\|(g \cdot G_\alpha) f\| \le \|g\|_\infty \|f\|/(\alpha - \alpha_0)$, the right-hand side of equation (1.3.12) is well defined and belongs to $\mathcal{F}$ because it can be written as $G_\alpha h$ for $h = \sum_{n=0}^\infty (-1)^n (g \cdot G_\alpha)^n f \in L^2(X;m)$. Let us denote the right-hand side of (1.3.12) by $\bar{G}_\alpha^g f$. Then, for any $u \in \mathcal{F}$,

$$\begin{aligned}
\mathcal{E}_\alpha^g(\bar{G}_\alpha^g f, u) &= \mathcal{E}_\alpha\left(\sum_{n=0}^\infty (-1)^n G_\alpha (g \cdot G_\alpha)^n f, u\right) \\
&\quad + \left(\sum_{n=0}^\infty (-1)^n G_\alpha (g \cdot G_\alpha)^n f, u\right)_{g\cdot m} \\
&= \sum_{n=0}^\infty (-1)^n \left((g \cdot G_\alpha)^n f, u\right) - \sum_{n=1}^\infty (-1)^n \left((g \cdot G_\alpha)^n f, u\right) \\
&= (f, u).
\end{aligned}$$

This implies $\bar{G}_\alpha^g f = G_\alpha^g f$ for $\alpha > \alpha_0 + \|g\|_\infty$. By the sub-Markov property, G_α^g and $\bar{G}_\alpha^g$ can be extended to the operators on $L^\infty(X;m)$ and $\bar{G}_\alpha^g f = G_\alpha^g f$ for any $f \in L^\infty(X;m)$ and $\alpha > 0$. Equation (1.3.13) is clear from equation (1.3.12).

To show the inequality $G^g g \le 1$, for $\beta > \alpha_0$, note that $(\mathcal{E}, \mathcal{F})$ can be considered as a Dirichlet form on $L^2(X; (\beta + g) \cdot m)$ for which $(\mathcal{E}.1)$, $(\mathcal{E}.2)$ and $(\mathcal{E}.3)$ hold by taking $\alpha_0/\beta < 1$ instead of α_0. Let us denote by $\{K_\alpha^\beta\}_{\alpha>0}$ the resolvent of $(\mathcal{E}, \mathcal{F})$ on $L^2(X; (\beta + g) \cdot m)$ given by

$$\mathcal{E}(K_\alpha^\beta f, u) + \alpha (K_\alpha^\beta f, u)_{(\beta+g)\cdot m} = (f, u)_{(\beta+g)\cdot m}$$

for all $u \in \mathcal{F}$. Since

$$\begin{aligned}
&\mathcal{E}\left(G_\beta^g(\beta f + g f), u\right) + \left(G_\beta^g(\beta f + g f), u\right)_{(\beta+g)\cdot m} \\
&\quad = \mathcal{E}_\beta^g\left(G_\beta^g(\beta f + g f), u\right) = (\beta f + g f, u) \\
&\quad = (f, u)_{(\beta+g)\cdot m}
\end{aligned}$$

for all $u \in \mathcal{F}$, $G_\beta^g(\beta f + g f) = K_1^\beta f$. Since $\{K_\alpha^\beta\}$ is a sub-Markov resolvent by Theorem 1.1.5, $0 \le K_1^\beta f \le 1$ for all $\beta > \alpha_0$ and f satisfying $0 \le f \le 1$. Furthermore, since $\{G_\beta^g\}$ can be extended to $\beta > 0$ as a resolvent on $L^\infty(X;m)$, K_1^β can be extended to $\beta > 0$ as a bounded linear operator on $L^\infty(X;m)$. If $\gamma > \beta > \alpha_0$ and $f \in L^\infty(X;m) \cap L^2(X;m)$, since

$$\begin{aligned}
&\mathcal{E}\left(K_1^\beta f - K_1^\gamma f, u\right) + \left(K_1^\beta f - K_1^\gamma f, u\right)_{(\beta+g)\cdot m} \\
&\quad = (\gamma - \beta)\left(K_1^\gamma f - f, u\right)_m \\
&\quad = (\gamma - \beta)\left(\frac{1}{\beta + g}(K_1^\gamma f - f), u\right)_{(\beta+g)\cdot m},
\end{aligned}$$

it follows that

$$K_1^\beta f - K_1^\gamma f = (\gamma - \beta) K_1^\beta \left(\frac{1}{\beta + g} (K_1^\gamma f - f) \right).$$

By putting $(\beta + g) f / (\gamma + g)$ instead of f, it follows that

$$\begin{aligned} K_1^\beta f &= K_1^\gamma \left(\frac{\beta + g}{\gamma + g} f \right) + K_1^\beta \left(\frac{\gamma - \beta}{\beta + g} K_1^\gamma \left(\frac{\beta + g}{\gamma + g} f \right) \right) \\ &= K_1^\gamma \sum_{n=0}^{k-1} \left(\frac{\gamma - \beta}{\beta + g} K_1^\gamma \right)^n \left(\frac{\beta + g}{\gamma + g} f \right) \\ &\quad + K_1^\beta \left(\frac{\gamma - \beta}{\beta + g} K_1^\gamma \right)^k \left(\frac{\beta + g}{\gamma + g} f \right). \end{aligned}$$

This relation also holds for $0 < \beta \le \alpha_0 < \gamma$. Hence, it holds that

$$K_1^\beta f = K_1^\gamma \left(\sum_{n=0}^{\infty} \left(\frac{\gamma - \beta}{\gamma + g} K_1^\gamma \right)^n \left(\frac{\beta + g}{\gamma + g} f \right) \right).$$

Since $K_1^\gamma f \le 1$ for any $\gamma > \alpha_0$ and $f \in L^2(X; m)$ with $f \le 1$, for $0 < \beta \le \alpha_0$, K_1^β can be considered as an operator on $L^\infty(X; m)$ satisfying $K_1^\beta 1 \le 1$. Therefore,

$$G^g g = \lim_{\beta \to 0} G_\beta^g g \le \lim_{\beta \to 0} K_1^\beta 1 \le 1.$$

This implies the transience of $(\mathcal{E}^g, \mathcal{F})$.

If $\{G_\alpha\}$ is irreducible, then equation (1.3.12) implies the irreducibility of $\{G_\alpha^g\}$. If $(\mathcal{E}, \mathcal{F})$ is recurrent, then for any non-negative function g such that $\int_X g\,dm > 0$, $G^g g \ge G_\alpha g$ for any $\alpha \ge \|g\|_\infty$. Put $B = \{x : G_\alpha f(x) > 0\}$ for a non-negative function f. For a.e. $x \in X \setminus B$, since $0 = G_\alpha f(x) \ge (\beta - \alpha) G_\beta (1_B G_\alpha f)(x)$ for $\beta > \alpha$, it follows that $G_\beta(1_B G_\alpha f) = 0$ and hence $G_\beta 1_B = 0$ a.e. on $X \setminus B$. Hence B is an invariant set of $\{T_t\}$ and hence $B = X$ a.e., that is $G^g g > 0$ by irreducibility. Furthermore, it holds that $G^g g \le 1$ for non-negative function g. Then by letting α tend to 0 in equation (1.3.13), we obtain that

$$G\left(g(1 - G^g g)\right) = G^g g \le 1.$$

Hence, the recurrence of $(\mathcal{E}, \mathcal{F})$ yields that $G^g g = 1$ a.e. □

By using Lemma 1.3.5, we have the following maximum principle.

Corollary 1.3.6. *Suppose that $(\mathcal{E}, \mathcal{F})$ is transient and let $f \in L^1(X; m) \cap L^\infty(X; m)$ be a non-negative function such that $Gf(x) = \lim_{\alpha \to 0} G_\alpha f(x) < \infty$ a.e. For any Borel set B, if $f(x) = 0$ for a.e. $x \in X \setminus B$, then*

$$\|Gf\|_\infty = \text{ess.sup}\,\{Gf(x) : x \in B\}.$$

In particular, there exists a strictly positive m-integrable function g such that $\|Gg\|_\infty < \infty$.

Proof. Put $g = \beta 1_B$ and $\alpha = 0$ in equation (1.3.13). Then, by the last result of Lemma 1.3.5,

$$Gf(x) = G^{\beta I_B} f(x) + G^{\beta I_B} (\beta I_B \cdot Gf)(x) \le G^{\beta I_B} f(x) + \text{ess.} \sup_{y \in B} Gf(y).$$

Furthermore, since $G^{\beta 1_B}(\beta 1_B) \le 1$, it follows that

$$\lim_{\beta \to \infty} G^{\beta I_B} f = \lim_{\beta \to \infty} G^{\beta I_B}(I_B f) = 0,$$

which yields the first assertion. To show the latter assertion, take a strictly positive bounded m-integrable function h such that $Gh < \infty$ a.e. Put $B_n = \{x : Gh(x) \le n\}$ and $g_n(x) = 2^{-n}(1/n)1_{B_n}h(x)$. Then $Gg_n \le 2^{-n}(1/n)G(1_{B_n}h) \le 2^{-n}$. Hence it is enough to put $g(x) = \sum_{n=1}^{\infty} g_n(x)$. □

Theorem 1.3.7. *$(\mathcal{E}, \mathcal{F})$ is recurrent if and only if $Gf = \infty$ m-a.e. for all $f \in L^1_+(X;m) \cap L^\infty(X;m)$ such that $\int_X f dm > 0$.*

Proof. Suppose that $(\mathcal{E}, \mathcal{F})$ is recurrent and let $B = \{Gf = \infty\}$. Then $X \setminus B$ is an invariant set of $\{T_t\}$. In fact, for all $t > 0$ and $g \in L^1_+(X;m)$ such that $g = 0$ outside of $\{Gf \le n\}$,

$$\infty > (Gf, g) \ge (T_t Gf, g) \ge (T_t(I_B Gf), g) = (I_B Gf, \widehat{T}_t g),$$

which implies that $\widehat{T}_t g = 0$ m-a.e. on B. Hence $(T_t(I_B u), g) = (I_B u, \widehat{T}_t g) = 0$, that is $I_{X \setminus B} T_t(I_B u) = 0$. This implies $1_{X \setminus B} T_t u = 1_{X \setminus B} T_t(1_{X \setminus B} u)$, that is $X \setminus B$ is an invariant set of $\{T_t\}$. According to irreducibility, this implies that $m(B) = 0$ or $m(X \setminus B) = 0$. Suppose that $m(B) = 0$. By virtue of Lemma 1.3.1, since $\{x : Gf(x) > 0\}$ is a non-trivial invariant set of $\{T_t\}$, $Gf > 0$ m-a.e. and hence $G_\alpha f > 0$ m-a.e. for any $\alpha > 0$. Then, by the resolvent equation, $\alpha G G_\alpha f = Gf - G_\alpha f \le Gf < \infty$ m-a.e. which contradicts the hypothesis of recurrence. □

If $\alpha G_\alpha 1 = 1$ m-a.e. for all $\alpha > 0$, then $(\mathcal{E}, \mathcal{F})$ is called *conservative*.

Corollary 1.3.8. *If $(\mathcal{E}, \mathcal{F})$ is recurrent, then it is conservative.*

Proof. Letting β tend to 0 in $G_\beta(1 - (\alpha - \beta)G_\alpha 1) = G_\alpha 1 \le \frac{1}{\alpha}$, we obtain that $G(1 - \alpha G_\alpha 1) \le \frac{1}{\alpha}$ m-a.e. Since $1 - \alpha G_\alpha 1 \ge 0$, Theorem 1.3.7 implies that $\alpha G_\alpha 1 = 1$. □

Let us define the *extended Dirichlet form* $(\mathcal{E}, \mathcal{F}_e)$ of $(\mathcal{E}, \mathcal{F})$ as follows: $\mathcal{F}_e$ is the family of functions u for which there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\} \subset \mathcal{F}$ such that $\lim_{n \to \infty} u_n = u$ a.e. and $\mathcal{E}(u, u) = \lim_{n \to \infty} \mathcal{E}(u_n, u_n)$ exists.

The sequence $\{u_n\}$ is called an *approximating sequence* of u. Generally, $\mathcal{E}(u,u)$ depends on the choice of the approximating sequence. But, if $\alpha_0 = 0$, then, for any function $u \in \mathcal{F}_e$, $\mathcal{E}(u,u) = \lim_{n\to\infty} \mathcal{E}(u_n,u_n)$ is well defined independently of the choice of the approximating sequence $\{u_n\}$. In fact, the symmetric part $\mathcal{E}^{(s)}$ becomes non-negative and hence the triangle inequality holds. Hence, by

$$\left|\mathcal{E}(u_n,u_n)^{1/2} - \mathcal{E}(u_m,u_m)^{1/2}\right| \le \mathcal{E}(u_n-u_m,u_n-u_m)^{1/2},$$

$\mathcal{E}(u,u)$ exists. Furthermore, by the sector condition, for any $u, v \in \mathcal{F}_e$ and their approximating sequences $\{u_n\}$, $\{v_n\}$ respectively, $\mathcal{E}(u,v) = \lim_{n,m\to\infty} \mathcal{E}(u_n,v_m)$ is well defined.

Theorem 1.3.9. *Suppose that $(\mathcal{E},\mathcal{F})$ is transient and $\alpha_0 = 0$. If $(|f|, G|f|) < \infty$, then $Gf \in \mathcal{F}_e$ and satisfies*

$$\mathcal{E}(Gf,u) = \int_X f(x)u(x)m(dx) \tag{1.3.14}$$

for all $u \in \mathcal{F}_e$. In particular, $(\mathcal{E}^{(s)},\mathcal{F}_e)$ is a Hilbert space and there exists a strictly positive bounded integrable function g and a constant K_g depending on g such that

$$\int_X |u|(x)g(x)m(dx) \le K_g\mathcal{E}(u,u)^{1/2} \tag{1.3.15}$$

for all $u \in \mathcal{F}_e$.

Proof. Let $(\bar{\mathcal{E}}^{(s)},\bar{\mathcal{F}})$ be the Hilbert space determined as the abstract completion of $(\mathcal{E}^{(s)},\mathcal{F})$, that is $\bar{u} \in \bar{\mathcal{F}}$ is an equivalence class of $\mathcal{E}$-Cauchy sequences $\{u_n\}$ and $\bar{\mathcal{E}}(\bar{u},\bar{u}) = \lim_{n\to\infty}\mathcal{E}(u_n,u_n)$. By the sector condition ($\mathcal{E}$.2), $\bar{\mathcal{E}}(\bar{u},\bar{v}) = \lim_{n,m\to\infty}\mathcal{E}(u_n,v_n)$ is well defined for $\bar{u} = \{u_n\}$ and $\bar{v} = \{v_n\}$. As in the last part of the proof of Theorem 1.1.4, for any $\bar{\mathcal{E}}^{(s)}$-bounded sequence $\{u_n\} \subset \mathcal{F}$, there exists an element $\bar{u} \in \bar{\mathcal{F}}$ such that a subsequence of Cesàro means of $\{u_n\}$ converges to $\bar{u}$ relative to $\bar{\mathcal{E}}$.

Assume that $g \in L^1(X;m) \cap L^\infty(X;m)$ is a strictly positive function satisfying $\|Gg\|_\infty < \infty$. Since $\mathcal{E}(G_\alpha g, G_\alpha g) = (g - \alpha G_\alpha g, G_\alpha g) \le (g, Gg)$, there exists a sequence $\{u_n\} \subset \mathcal{F}$ constituting a Cesàro sum of $G_{\alpha_n}g$ for $\alpha_n \downarrow 0$ which is $\mathcal{E}$-Cauchy. Put $\bar{u} \in \bar{\mathcal{F}}$ the equivalence class containing $\{u_n\}$. For any $v \in \mathcal{F}_b$, $\lim_{n\to\infty}\mathcal{E}(u_n,v) = \bar{\mathcal{E}}(\bar{u},v)$. On the other hand, since $\{u_n\}$ is a convex combination of $G_{\alpha_n}g$, $\lim_{n\to\infty}\mathcal{E}(u_n,v) = \lim_{n\to\infty}\mathcal{E}(G_{\alpha_n}g,v) = (g,v)$ and hence $\bar{\mathcal{E}}(\bar{u},v) = (g,v)$. In particular,

$$\int_X |v|(x)g(x)m(dx) \le K_g\mathcal{E}(v,v)^{1/2} \tag{1.3.16}$$

for any $v \in \mathcal{F}_b$ and a constant $K_g = K_0\bar{\mathcal{E}}(\bar{u},\bar{u})^{1/2}$. Since any function of $\mathcal{F}$ can be approximated by the functions of $\mathcal{F}_b$, equation (1.3.16) holds for any $v \in \mathcal{F}$. If $\{v_n\}$

is an $\mathcal{E}$-Cauchy sequence corresponding to $\bar{v} \in \bar{\mathcal{F}}$, then equation (1.3.16) implies that it converges to a function v in $L^1(X; g \cdot m)$. By the definition of $\mathcal{F}_e$, this implies that $v \in \mathcal{F}_e$. Hence we can identify any element $\bar{v} \in \bar{\mathcal{F}}$ with a function of $\mathcal{F}_e$. In particular, $(\mathcal{E}, \mathcal{F}_e)$ is a Hilbert space and $Gg \in \mathcal{F}_e$. Assume that a non-negative function f satisfies $(f, Gf) < \infty$. Then, for the function g given above, $Gf_n \in \mathcal{F}_e$ and satisfies $\mathcal{E}(Gf_n - Gf_m, Gf_n - Gf_m) = (f_n - f_m, Gf_n - Gf_m)$ for $f_n = f \wedge (ng)$. Hence $\lim_{n,m\to\infty} \mathcal{E}(Gf_n - Gf_m, Gf_n - Gf_m) = 0$ which yields that $Gf \in \mathcal{F}_e$ and that the relation (1.3.14) holds. For a general function f, it is enough to consider f^+ and f^- separately. □

1.4 An auxiliary bilinear form

As we have seen after Theorem 1.1.5, for any $\alpha > 0$, G_α can be extended to a sub-Markov resolvent on $L^\infty(X; m)$. Hence, $G_\alpha f$ can be further extended to all non-negative measurable functions f by $G_\alpha f = \lim_{k\to\infty} G_\alpha(f \wedge kg)$ by using a strictly positive function $g \in L^\infty(X; m) \cap L^1(X; m)$. Similarly $\widehat{G}_\alpha f$ is well defined for any $f \in L^1(X; m)$. Under this extension, a non-negative function u (resp. $\widehat{u}$) is called *α-excessive* (resp. *α-coexcessive*) if

$$\beta G_{\alpha+\beta} u \le u \quad \text{(resp. } \beta \widehat{G}_{\alpha+\beta} \widehat{u} \le \widehat{u}) \qquad m\text{-a.e.} \tag{1.4.1}$$

for all β. The 0-excessive function and 0-coexcessive function are called *excessive* function and *coexcessive* function, respectively.

Theorem 1.4.1. *The following conditions are equivalent to each other for $u \in \mathcal{F}$ (resp. $\widehat{u} \in \mathcal{F}$) and $\alpha > \alpha_0$.*

(i) *u is α-excessive (resp. $\widehat{u}$ is α-coexcessive).*

(ii) *$\mathcal{E}_\alpha(u, v) \ge 0$ (resp. $\widehat{u} \ge 0$ and $\mathcal{E}_\alpha(v, \widehat{u}) \ge 0$) for all $v \in \mathcal{F}^+$.*

(iii) *$\mathcal{E}_\alpha(u, v) \ge 0$ (resp. $\widehat{u} \ge 0$ and $\mathcal{E}_\alpha(v, \widehat{u}) \ge 0$) for all $v \in \mathcal{F}^+ \cap \mathcal{C}_0(X)$.*

Proof. Since $\lim_{\beta\to\infty} \beta G_\beta u = u$ in $L^2(X; m)$, (i) ⇒ (ii) follows from

$$\begin{aligned}
\mathcal{E}_\alpha(u, v) &= \lim_{\beta\to\infty} \{\beta(u - \beta G_\beta u, v) + \alpha(u, v)\} \\
&= \lim_{\beta\to\infty} \{\beta(u - \beta G_{\alpha+\beta} u, v) + \alpha(u - \beta G_\beta G_{\alpha+\beta} u, v)\} \\
&= \lim_{\beta\to\infty} \{\beta(u - \beta G_{\alpha+\beta} u, v) + \alpha(u - \beta G_\beta u, v)\} \\
&\ge 0.
\end{aligned}$$

The equivalence of (ii) and (iii) is obvious.