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Idris Assani

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Editor
Prof. Dr. Idris Assani
The University of North Carolina at Chapel Hill
Department of Mathematics
CB 3250 Phillips Hall
CHAPEL HILL NC 27599-3250
United States of America
E-Mail: assani@email.unc.edu

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Preface

The present volume contains contributions from participants to the 2011 and 2012 Chapel Hill Ergodic Theory Workshops. These workshops were held on March 17–21 2011 and March 22–25 2012 at the University of North Carolina at Chapel Hill. These workshops have been a yearly event since the summer 2002. The keynote speakers were Prof. Y. Sinai (2011) from Princeton University and Prof. J.-C. Yoccoz (2012) from the College de France.

The list of all participants to these workshops can be seen on the website https://ergwork.web.unc.edu/.

The purpose of these workshops is to get together young researchers (Graduate students, Post-Doctoral students, Assistant Professors) and senior researchers to foster collaborations, exchange ideas, consolidate progress in this very active research area that is Ergodic Theory and Dynamical Systems. Attention was paid to the participation of members from under-represented groups and particularly women.

Most of the papers in this volume are the results of discussions of open problems during these workshops. We hope to further extend the objectives of these workshops with the publication of these proceedings.

It is a pleasure to acknowledge the institutions whose support made these events possible. First we thank the National Science for their continued support. Thanks also to the Department of Mathematics at UNC Chapel Hill and the energetic support of its staff for hosting these events.

Finally, we thank Walter de Gruyter for publishing these proceedings.

I. Assani – Editor

Contents

Ethan Akin

Furstenberg Fractals

Abstract: Hillel Furstenberg has introduced a dynamical systems interpretation of self-similarity for fractals. While he has concentrated primarily upon measures, we present here the foundations of the topological version of his construction.

Keywords: Fractal, Furstenberg Fractal, Nonwandering Point, Recurrent Point, Symbolic Dynamics

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Ethan Akin: Mathematics Department, The City College, 137 Street and Convent Avenue, New York City, NY 10031, USA, e-mail: ethanakin@earthlink.net

1 Introduction

While the name "fractal" refers to fractional dimension, the salient characteristic of many fractals is self-similarity. When you blow up the scale, what you see is similar to the original figure. For a general reference see Falconer [2]. In several recent lectures Hillel Furstenberg [3] has introduced a dynamical systems model of this phenomenon. Bryna Kra [6] is preparing an elaboration of more recent work by Furstenberg.

He begins with a square in the plane, subdivided into four subsquares. For each of these there is an expansion which map onto the original figure by translation and dilation. The dynamics operates on pairs (H, x) where H is a closed subset of the square and x is a point of H. Select the subsquare which contains the point, intersect with H and then apply the appropriate expansion to get a new pair. We regard H as self-similar at x when the pair (H, x) is a recurrent point for this dynamical system.

Furstenberg focusses on a measure version of this system, as do the subsequent works by Gavish [4] and Hochman [5]. In particular, the ambiguity which occurs when the point x is on the boundary between neighboring subsquares causes no problem. For our topological system we replace the square by the symbol space X consisting of all infinite sequences on a finite alphabet $\mathcal{A}$. X is partitioned by $\{X_a : a \in \mathcal{A}\}$ where X_a consists of those sequences whose first term is a. These pieces play the role of the subsquares. The shift map $\sigma : X \to X$ is defined by $\sigma(x)_i = x_{i+1}$ for all i. The shift restricts to a homeomorphism of each X_a onto X. These correspond to the homothetic maps from the subsquares back to the original large square.

Our state space is $\mathcal{E}_X$ consisting of all pairs (H, x) with H a closed subset of X and $x \in H$. For the dynamical system Φ we intersect H with the X_a which contains x, i.e.

with $a = x_1$, and then apply the shift. That is,

$$\Phi(H, x) \quad =_{def} \quad (\sigma(H \cap X_{x_1}), \sigma(x)). \tag{1.1}$$

Because $\mathcal{E}_X$ is compact the omega limit set

$$\omega\Phi(H, x) = \bigcap_n \overline{\{\Phi^k(H, x) : k \geq n\}} \tag{1.2}$$

is never empty. However,

$$\omega_H\Phi(x) \quad =_{def} \quad \{y \in H : (H, y) \in \omega\Phi(H, x)\} \tag{1.3}$$

might be.

We extend by analogy some standard dynamic language. We call x a *nonwandering point* for H if $\omega_H\Phi(x)$ is nonempty. We call x a *recurrent point* for H when $x \in \omega_H\Phi(x)$. This just says that (H, x) is a recurrent point for the dynamical system Φ in the usual sense. We call x a *transitive point* for H when $\omega_H\Phi(x) = H$. H is called a *Furstenberg Fractal* when the set of nonwandering points for H is dense in H and a *Uniform Furstenberg Fractal* when every point of H is a nonwandering point. In the next section we show:

Theorem 1.1. *If H is a Furstenberg Fractal, i.e. the set of nonwandering points is dense in H, then the set of transitive points is a dense, G_δ subset of H.*

In Section 3, we provide a construction which yields all Furstenberg Fractals and all Uniform Furstenberg Fractals. From the construction we obtain:

Theorem 1.2. *The set of Uniform Furstenberg Fractals is a dense, G_δ subset of the space of nonempty closed subsets of X.*

Thus, there are many sets H such that every point of H is a nonwandering point for H. What about the analogue of minimality? Under what circumstances is every point of H a transitive point for H? The answer is, except for the trivial case when H is a singleton, never. In fact, in Section 4 we prove:

Theorem 1.3. *If H is a Furstenberg Fractal containing more than one point then the set of points of H which are not recurrent for H is a dense subset of H.*

Finally, in Section 5 we weaken the notion of nonwandering point, calling $x \in H$ a *weak nonwandering point* for H when the first coordinates of the orbit $\{\Phi^n(H, x)\}$ are repeatedly close to H only up to isometry. We obtain:

Theorem 1.4. *Let H be a closed subset of X.*

Every point of H is a weak nonwandering point for H iff there exists a bijective isometry J on the metric space X such that $J(H)$ is a Uniform Furstenberg Fractal.

The set of weak nonwandering points for H is dense in H iff there exists a bijective isometry J on X such that $J(H)$ is a Furstenberg Fractal.

2 Furstenberg Fractals

We begin by setting up notation for and reviewing the elementary properties of symbol spaces on a finite alphabet.

Let $\mathcal{A}$ be a finite alphabet and $\mathbb{Z}_+$ denote the set of positive integers. Let $X = \mathcal{A}^{\mathbb{Z}_+}$ denote the space of infinite sequences in $\mathcal{A}$ and $X_k = \mathcal{A}^{\{1,\dots,k\}}$ be the set of words of length k. For $a \in X_k$ let $\ell(a) = k$ and for $x \in X$ let $\ell(x) = \infty$ so that ℓ is the length function. We denote by θ the empty word. It is the unique word of length zero and $X_0 = \{\theta\}$ is the singleton consisting of the empty word. By convention we define $x_i = \theta$ for $i > \ell(x)$.

Define the disjoint unions:

$$X^* = \bigcup_{i=0}^{\infty} X_i, \quad X_k^* = \bigcup_{i=0}^{k} X_i \text{ and } \overline{X^*} = X \cup X^*. \tag{2.1}$$

For any $x \in \overline{X^*}$ and $k = 0, 1, \dots$ we let $x_{[1..k]} = x$ if $\ell(x) \le k$ and otherwise $x_{[1..k]} = x_1 \dots x_k$ is the word of length k consisting of the initial k letters of x. Let $\rho_k : X \to X_k$ denote the projection and $\rho_k : \overline{X^*} \to X_k^*$ the retraction, both defined by $x \mapsto x_{[1:k]}$. In general, we will write $x_{[n+1:n+k]}$ for the word $x_{n+1} \dots x_{n+k}$, which has length k when $\ell(x) \ge n + k$. By convention, $\rho_0(x) = x_{[1:0]} = \theta$ and for $i > \ell(x)$, $x_i = \theta$.

If a has finite length and $\ell(b) \le \infty$ we define the *concatenation* $c = ab$ by

$$c_i =_{def} \begin{cases} a_i & \text{for } i \le \ell(a) \\ b_{i-\ell(a)} & \text{for } i > \ell(a). \end{cases} \tag{2.2}$$

We then call c an *extension* of a and a a *restriction* or *initial string* of c. We call b the *follower* of a in c. When c is finite we also call b a *terminal string* of c. For example, with $x \in X$ and $k \in \mathbb{Z}_+$, $\rho_k(x)$ is the initial sting of x with length k. If $a = \theta$ then $c = ab = b$.

For a a finite word and $C \subset X^*$ a set of – possibly infinite length – words we define the set of *followers of a in C* to be

$$F_C(a) =_{def} \{b \in \overline{X^*} : ab \in C\}. \tag{2.3}$$

In particular, $F_C(\theta) = C$.

If A is a set of words of finite length and B is a set of words of possibly infinite length then we let

$$AB =_{def} \{ab \in \overline{X^*} : a \in A \text{ and } b \in B\}. \tag{2.4}$$

We call a set A a *same length set* or an SL set if A is a nonempty set of words all of the same finite length. We call $A \subset \overline{X^*}$ an *extension regular set* or an ER set if for distinct elements $a, b \in A$ it is never true that b is an extension of a. Notice that a and b may restrict to the same proper initial string but neither is an initial string of the other. Clearly any SL set is an ER set.

Let A and C be nonempty sets of words. We say that C is an *extension* of A or A *extends to* C when every $a \in A$ extends to some $c \in C$ and every $c \in C$ restricts to some $a \in A$. We also say A is a *restriction* of C or C *restricts to* A. In the case of a single finite word a we will say that C extends a etc. when C extends $A = \{a\}$.

If $a \in X$, i.e. a is an infinite word, then a itself is the only word to which a extends.

Observe that if A is an ER set which extends to C then the $a \in A$ to which $c \in C$ restricts is unique. In that case we define $\rho_{(C,A)} : C \to A$ by $c \mapsto a$ where a is the restriction of c. In any case $a \in A$ may extend to many elements of C. If $C = AB$ for some set of words B then C is an extension of A. If A is an ER set then $C = AB$ iff C is an extension of A such that the set of followers $F_C(a) = B$ for all $a \in A$.

On $\overline{X^*}$ we define the usual metric d by $d(x, y) = 0$ when $x = y$ and otherwise

$$d(x, y) =_{def} \min\{2^{-k} : k \geq 0 \text{ and } \rho_k(x) = \rho_k(y)\}. \tag{2.5}$$

Equivalently, $d(x, y) = 2^{-i+1}$ where i is the smallest index such that $x_i \neq y_i$. In particular, if $\ell(x) = k$ and y is an extension of x then $d(x, y) = 2^{-k}$.

The space $\overline{X^*}$ is compact and the closed subspace X has the compact product topology. As the notation notation suggests, X^* is a dense, open subset of $\overline{X^*}$ consisting of isolated points. Regarded as words of length at most 1 the set $\{\theta\} \cup \mathcal{A}$ receives the usual zero-one metric.

The metric d is an ultrametric. That is, $d(x, y) \leq \max(d(x, z), d(z, y))$ for all $x, y, z \in \overline{X^*}$. It follows that for every $\epsilon > 0$ the set

$$\bar{V}_\epsilon =_{def} \{(x, y) \in \overline{X^*} \times \overline{X^*} : d(x, y) \leq \epsilon\} \tag{2.6}$$

is a clopen equivalence relation and for $x, y \in \overline{X^*}$

$$d(x, y) \leq 2^{-k} \quad \Longleftrightarrow \quad y \in \bar{V}_{2^{-k}}(x) \quad \Longleftrightarrow \quad x_{[1:k]} = y_{[1:k]}. \tag{2.7}$$

On $\overline{X^*}$ define the surjective shift map σ as usual by

$$\sigma(x)_i = x_{i+1} \quad \text{for all } i \in \mathbb{Z}_+. \tag{2.8}$$

In particular, $\sigma(\theta) = \theta$ and $\ell(\sigma(x)) = \max(0, \ell(x) - 1)$. The Lipschitz constant of σ with respect to the metric d is 2. Clearly, for positive integers n, k

$$\rho_k(\sigma^n(x)) = x_{[n+1:n+k]}. \tag{2.9}$$

For a any finite word, define the injective map τ_a on $\overline{X^*}$ by

$$\tau_a(x) = ax. \tag{2.10}$$

Clearly, τ_a is a contraction with Lipschitz constant 2^{-k} where $k = \ell(a)$. Furthermore,

$$\sigma^k \circ \tau_a = 1_{\overline{X^*}}. \tag{2.11}$$

The image of τ_a is the clopen ball $\bar{V}_{2^{-k}}(y)$ for any $y \in \tau_a(\overline{X^*})$.

The compact set X of infinite words is preserved by σ and each τ_a. In fact, $\sigma^{-1}(X) = X$ and $\tau_a^{-1}(X) = X$ for all $a \in X^*$.

Let $2^{\overline{X^*}}$ denote the space of closed subsets of $\overline{X^*}$. Let d denote the Hausdorff metric induced from the metric on $\overline{X^*}$. It is easy to check that for H, K distinct elements of $2^{\overline{X^*}}$

$$d(H,K) \ =_{def} \ \min\{2^{-k} : k \geq 0 \text{ and } \rho_k(H) = \rho_k(K)\}, \tag{2.12}$$

and so

$$d(H,K) \leq 2^{-k} \quad \Longleftrightarrow \quad \rho_k(H) = \rho_k(K), \tag{2.13}$$

i.e. iff H and K have exactly the same set of initial strings of length at most k. By convention, $\rho_0(H) = \{\theta\}$ for the empty set H as well as for every nonempty set. In particular, the empty set has distance 1 from any nonempty set in $2^{\overline{X^*}}$.

It is easy to check that if K is any clopen subset of $\overline{X^*}$ that the map $\wedge K$ on $2^{\overline{X^*}}$ defined by $H \mapsto H \cap K$ is continuous. Also if f is any continuous map on X then continuous as well is the induced map, also denoted f on $2^{\overline{X^*}}$ and defined by associating to H its image $f(H)$. In particular, we have continuous maps defined on $2^{\overline{X^*}}$ induced from σ and the τ_a's.

Since X is a closed subset of $\overline{X^*}$ the space 2^X of closed subsets of X is closed in $2^{\overline{X^*}}$.

Let $\mathcal{E}_X \subset 2^X \times X$ be the closed set

$$\mathcal{E}_X \quad = \quad \{(H,x) : x \in H\}. \tag{2.14}$$

On the product we use the max metric:

$$d((H,x),(K,y)) =_{def} \max(d(H,K), d(x,y)). \tag{2.15}$$

On $\mathcal{E}_X$ define the *Furstenberg Fractal Map* Φ with $\Phi(H,x) = (\phi(H,x), \sigma(x))$ where

$$\phi(H,x) =_{def} \sigma(H \cap \tau_{x_1}(X)). \tag{2.16}$$

Thus, it is the image of the shift on that portion of H whose elements begin with the same letter as x.

It follows that $\Phi^n(H,x) = (\phi_n(H,x), \sigma^n(x))$ where

$$\phi_n(H,x) =_{def} \sigma^n(H \cap \tau_{x_{[1:n]}}(X)) = F_H(x_{[1:n]}) \tag{2.17}$$

where $x_{[1:n]} = \rho_n(x)$ is the initial string of x with length n. In particular, for $x, y \in H$

$$x_{[1:n]} \ = \ y_{[1:n]} \quad \Longrightarrow \quad \phi_n(H,x) = \phi_n(H,y). \tag{2.18}$$

Recall also that

$$\tau_{x_{[1:n]}}(X) = \bar{V}_{2^{-n}}(x). \tag{2.19}$$

Because the projection map $x \mapsto x_1$, i.e. ρ_1, is locally constant and because $\wedge K$ is continuous for a clopen K, it follows that Φ is continuous.

Proposition 2.1.
(a) *The second coordinate projection map* $p_2 : \mathcal{E}_X \to X$ *maps* Φ *to* σ. *The coordinate projection maps* p_2 *and* $p_1 : \mathcal{E}_X \to 2^X \setminus \{\emptyset\}$ *are open, continuous surjections.*
(b) *The isometric embedding* $\iota : X \to \mathcal{E}_X$ *defined by*

$$\iota(x) =_{def} (\{x\}, x) \tag{2.20}$$

maps σ *to* Φ.
(c) *If* Y *is a closed subset of* X *which is* σ *+invariant, i.e.* $\sigma(Y) \subset Y$, *then* $\mathcal{E}_Y = \mathcal{E}_X \cap (2^Y \times Y)$ *is closed and* Φ *+invariant.*
(d) *For* $(H, x), (K, y) \in E$ *we write* $(H, x) < (K, y)$ *if* $H \subset K$ *and* $x = y$.

$$(H, x) \ < \ (K, y) \quad \Longrightarrow \quad \Phi(H, x) \ < \ \Phi(K, y). \tag{2.21}$$

Proof. (a) It is clear that p_2 maps Φ to σ. If $(H, x) \in \mathcal{E}_X$ and $d(H, K) < \epsilon$ then there exists $y \in K$ with $d(x, y) < \epsilon$ and so $d((H, x), (K, y)) < \epsilon$. On the other hand, if $d(x, y) < \epsilon$ then $d((H, x), (K, y)) < \epsilon$ if $K = H \cup \{y\}$. Thus, p_1 and p_2 map the ϵ ball centered at (H, x) in $\mathcal{E}_X$ onto the ϵ balls centered at x in X and centered at H in 2^X. Hence, p_1 and p_2 are open surjections.

(b), (c) and (d) are easy and are left to the reader. □

Lemma 2.2. *Let* $(H, x) \in \mathcal{E}_X, K \in 2^X$ *and* $y \in X$. *For positive integers* n, k

$$\begin{aligned} d(\phi_n(H, x), K) \ \le \ 2^{-k} \quad &\Longleftrightarrow \quad \rho_k(F_H(x_{[1:n]})) \ = \ \rho_k(K) \\ d(\sigma^n(x), y) \ \le \ 2^{-k} \quad &\Longleftrightarrow \quad x_{[n+1:n+k]} \ = \ y_{[1:k]}. \end{aligned} \tag{2.22}$$

If $y \in K$ *then* $d(\Phi^n(H, x), (K, y)) \ \le \ 2^{-k}$ *iff both of these conditions hold.*

Proof. The equivalences in (2.22) follow from (2.17), (2.13) and (2.7). Then the result for Φ follows from (2.15). □

Definition 2.3. For $(H, x) \in \mathcal{E}_X$ and $k \in \mathbb{Z}_+$ define the sets of integers $\mathcal{N}_H(x, k)$ and $\tilde{\mathcal{N}}_H(x, k)$ to be the sets of nonnegative integers such that

$$\begin{aligned} n \ \in \ \mathcal{N}_H(x, k) \quad &\Longleftrightarrow \quad \rho_k(F_H(x_{[1:n]})) \ = \ \rho_k(H) \\ n \ \in \ \tilde{\mathcal{N}}_H(x, k) \quad &\Longleftrightarrow \quad n \ \in \ \mathcal{N}_H(x, k) \text{ and } x_{[n+1:n+k]} \ = \ x_{[1:k]}. \end{aligned} \tag{2.23}$$

Define $\nu_H(x, k)$ and $\tilde{\nu}_H(x, k)$ to be the smallest positive integers in $\mathcal{N}_H(x, k)$ and $\tilde{\mathcal{N}}_H(x, k)$, respectively, with the value ∞ when the corresponding subset of $\mathbb{Z}_+$ is empty. We will omit the subscript when the set H is understood.

Recall that $x_{[1:0]} = \theta$ and $F_H(\theta) = H$. Hence, $0 \in \tilde{\mathcal{N}}_H(x, k) \subset \mathcal{N}_H(x, k)$.
It is useful to observe that for $x, y \in H$

$$\begin{aligned} x_{[1:n]} \ = \ y_{[1:n]} \quad &\Longrightarrow \quad \mathcal{N}_H(x, k) \cap [1, n] \ = \ \mathcal{N}_H(y, k) \cap [1, n], \\ x_{[1:n+k]} \ = \ y_{[1:n+k]} \quad &\Longrightarrow \quad \tilde{\mathcal{N}}_H(x, k) \cap [1, n] \ = \ \tilde{\mathcal{N}}_H(y, k) \cap [1, n]. \end{aligned} \tag{2.24}$$

Now for $(H, x) \in \mathcal{E}_X$ let

$$\omega_H \Phi(x) =_{def} \{y \in H : (H, y) \in \omega\Phi(H, x)\}. \tag{2.25}$$

We will say that x is a *nonwandering point* for H when $\omega_H \Phi(x)$ is nonempty. We say that x is a *recurrent point* for H when x lies in $\omega_H \Phi(x)$. Finally, we say that x is a *transitive point* for H when $\omega_H \Phi(x) = H$. Of course, a transitive point for H is a recurrent point for H and a recurrent point for H is a nonwandering point for H. We denote by $Trans_H$ the – possibly empty – set of transitive points for H.

Lemma 2.4. *Let* $(H, x) \in \mathcal{E}_X$.

(a) *The following are equivalent:*
   (i) *The point x is a nonwandering point for H, i.e.* $\omega_H \Phi(x) \neq \emptyset$.
   (ii) *For every $k \in \mathbb{Z}_+$ the set $\mathcal{N}_H(x, k) \cap \mathbb{Z}_+$ is nonempty.*
   (iii) *For every $k \in \mathbb{Z}_+$ the set $\mathcal{N}_H(x, k)$ is infinite.*
   (iv) *For every $k \in \mathbb{Z}_+$ the number $\nu_H(x, k)$ is a finite integer.*
   (v) *H is a limit point of the sequence $\{\phi_n(H, x)\}$ in $2^{\overline{X^*}}$.*

(b) *The following are equivalent:*
   (i) *The point x is a recurrent point for H, i.e.* $x \in \omega_H \Phi(x)$.
   (ii) *For every $k \in \mathbb{Z}_+$ the set $\tilde{\mathcal{N}}_H(x, k) \cap \mathbb{Z}_+$ is nonempty.*
   (iii) *For every $k \in \mathbb{Z}_+$ the set $\tilde{\mathcal{N}}_H(x, k)$ is infinite.*
   (iv) *For every $k \in \mathbb{Z}_+$ the number $\tilde{\nu}_H(x, k)$ is a finite integer.*
   (v) *(H, x) is a recurrent point for Φ.*

Proof. In (a) (i)–(iv) are equivalent by Lemma 2.2 and Definition 2.3 and they imply (v). If $\phi_{n_i}(H, x)$ converges to H then by going to a subsequence we can assume that $\sigma^{n_i}(x)$ converges to a point y. Hence, $\Phi^{n_i}(H, x)$ converges to (H, y). Since $\mathcal{E}_X$ is closed, $(H, y) \in \mathcal{E}_X$ and so $y \in H$.

In (b) (i)–(v) are equivalent by Lemma 2.2 and Definition 2.3. □

Definition 2.5. A closed set $H \subset X$ is called a *Furstenberg Fractal* when the set of nonwandering points for H is dense in H, that is, $\{x \in H : \omega_H \Phi(x) \neq \emptyset\}$ is dense in H. Equivalently, H is a Furstenberg Fractal when for every $k \in \mathbb{Z}_+$ and every word $a \in \rho_k(H)$ there exists a nonwandering point x for H such that $x_{[1:k]} = a$.

Theorem 2.6. *If H is a Furstenberg Fractal then $Trans_H$ is a dense G_δ subset of H.*

Proof. For positive integers n, k and for $a \in \rho_k(H)$ let

$$O(n, k, a) =_{def} \{z \in H : \rho_k(F_H(z_{[1:n]})) = \rho_k(H) \text{ and } z_{[n+1:n+k]} = a\}. \tag{2.26}$$

Observe that if $y \in H$ with $y_{[1:k]} = a$ then

$$z \in O(n, k, a) \iff d(\Phi^n(H, z), (H, y)) < 2^{-k+1}. \tag{2.27}$$

Furthermore, if $z, z' \in H$ with $z_{[1:n+k]} = z'_{[1:n+k]}$ then $z \in O(n,k,a)$ iff $z' \in O(n,k,a)$. Thus, $O(n,k,a)$ is clopen in the relative topology on H. Thus, $O(k,a) = \bigcup_{n \in \mathbb{Z}_+} O(n,k,a)$ is open in H.

Now if $x \in H$ and $p, k \in \mathbb{Z}_+$ then there exists a nonwandering point x' with $x_{[1:p]} = x'_{[1:p]}$. Since $\mathcal{N}(x',k)$ is infinite it contains some integer $n > p$. That is $\rho_k(F_H(x'_{[1:n]})) = \rho_k(H)$. So there exists $z \in H$ with $z_{[1:n]} = x'_{[1:n]}$ and $z_{[n+1:n+k]} = a$. Thus, $z \in O(k,a)$ and since $n > p$, $x_{[1:p]} = z_{[1:p]}$. It follows that for every k and a the set $O(k,a)$ is open and dense in H. From (2.26) we see that

$$Trans_H = \bigcap \{O(k,a) : k \in \mathbb{Z}_+ \text{ and } a \in \rho_k(H)\}. \tag{2.28}$$

From the Baire Category Theorem it follows that $Trans_H$ is a dense G_δ subset of H. □

A closed subset A of Y is called a *transitive subset* for a continuous map f on Y when A is nonempty and $\{x \in A : A \subset \omega f(x)\}$ is dense in A. This means that for $[A]$, the smallest closed + invariant subset of X which contains A, the restriction $f|[A]$ is topologically transitive and $Trans_{f|[A]} \cap A$ is dense in A (and so is a residual subset of A). Thus, we have:

Corollary 2.7. *If $H \in 2^X$ then H is a Furstenberg Fractal iff $\mathcal{E}_H$ is a transitive subset of $\mathcal{E}_X$ for the map Φ.*

For a continuous map f on Y and A a closed subset of Y we define $\omega f[A] = \bigcap_n \overline{\bigcup_{k \ge n} f^k(A)}$. This is the topological lim sup of the sequence $\{f^n(A)\}$ of closed sets.

Lemma 2.8. *If $A \in 2^X$ and $B \in p_1(\omega\Phi[\mathcal{E}_A])$ then $\mathcal{E}_B \subset \omega\Phi[\mathcal{E}_A]$.*

Proof. Let $y \in B$ and let k be an arbitrary positive integer. To say that $B \in p_1(\omega\Phi[\mathcal{E}_A])$ says that for arbitrarily large n there exists $x \in A$ so that $\rho_k(F_A(x_{[1:n]})) = \rho_k(B)$. In particular, there exists $z \in A$ such that $z_{[1:n]} = x_{[1:n]}$ and $z_{[n+1:n+k]} = y_{[1:k]}$. Thus, $d(\Phi^n(A,z),(B,y)) \le 2^{-k}$. It follows that $(B,y) \in \omega\Phi[\mathcal{E}_A]$. □

Theorem 2.9. *Assume that $H \in 2^X$ is a Furstenberg Fractal. Let $[\mathcal{E}_H]$ be the smallest closed subset of $\mathcal{E}_X$ which contains $\mathcal{E}_H$ and is + invariant for Φ. For every $B \in p_1([\mathcal{E}_H])$, $\mathcal{E}_B \subset [\mathcal{E}_H]$ and $\{B \in p_1([\mathcal{E}_H]) : B$ is a Furstenberg Fractal$\}$ is a residual subset of $p_1([\mathcal{E}_H])$.*

Proof. The first result only needs that the set $Trans_H$ is nonempty. That is, there exists $x \in H$ such that $\mathcal{E}_H \subset \omega\Phi(H,x)$. Notice that always $\omega\Phi(H,x) \subset \omega\Phi[\mathcal{E}_H] \subset [\mathcal{E}_H]$. If $\mathcal{E}_H$ is contained in the invariant set $\omega\Phi(H,x)$ then these three sets are equal. So the first result follows from Lemma 2.8.

Hence, $[\mathcal{E}_H] = p_1^{-1}(p_1([\mathcal{E}_H]))$ and so the restriction $p_1|[\mathcal{E}_H] : [\mathcal{E}_H] \to p_1([\mathcal{E}_H])$ of the open map $p_1 : \mathcal{E}_X \to 2^X$ is itself open.

Recall the theorem of Ulam, see, e.g., [1, Theorem 1.2].

Theorem 2.10. *If $h : Y_1 \to Y_2$ is an open surjection between compact metric spaces and A is a residual subset of Y_1 then $\{y \in Y_2 : A \cap h^{-1}(y)$ is dense in $h^{-1}(y)\}$ is a residual subset of Y_2.*

By the Ulam Theorem $\{B \in p_1([\mathcal{E}_H]) : \mathcal{E}_B \cap Trans_{\Phi|[\mathcal{E}_H]}$ is dense in $\mathcal{E}_B\}$ is a residual subset of $p_1([\mathcal{E}_H])$. If $(B, y) \in Trans_{\Phi|[\mathcal{E}_H]}$ then y is a transitive point for B. If such points are dense in $\{B\} \times B$ then B is a Furstenberg Fractal. □

We saw in Theorem 2.6 that if the nonwandering points for H are dense in H, then the transitive points are residual in H. We can sharpen the demand upon H:

Definition 2.11. A closed set $H \subset X$ is called a *Uniform Furstenberg Fractal* when every point of H is a nonwandering point for H, that is, for all $x \in H$ $\omega_H\Phi(x) \neq \emptyset$. Equivalently, H is a Uniform Furstenberg Fractal when for every $x \in H$ and every $k \in \mathbb{Z}_+$ the set $\mathcal{N}_H(x, k) \cap \mathbb{Z}_+$ is nonempty (and so every such $\mathcal{N}_H(x, k)$ is infinite by Lemma 2.4).

3 The Fractal Constructions

Our major result is to characterize Furstenberg Fractals in a way which allows us to construct all of them. The Uniform Furstenberg Fractals are easier to describe and so we begin with them.

Theorem 3.1. *A closed subset H of X is a Uniform Furstenberg Fractal iff*

$$\nu_H(x, k) =_{def} \min \mathbb{Z}_+ \cap \mathcal{N}_H(x, k), \tag{3.1}$$

defines a function $\nu_H : H \times \mathbb{Z}_+ \to \mathbb{Z}_+$.

In that case ν_H is continuous, or, equivalently, locally constant, and for every positive integer k the set

$$\pi_k(H) =_{def} \{x_{[1:n]} : x \in H \text{ and } n = \nu_H(x, k)\} \tag{3.2}$$

is a finite, extension regular set of words of finite length.

Proof. From Lemma 2.4 it follows that H is a Uniform Furstenberg Fractal iff for every $(x, k) \in H \times \mathbb{Z}_+$ the set $\mathcal{N}_H(x, k) \cap \mathbb{Z}_+$ is nonempty and it then follows that the set is infinite. Thus, the function ν_H is well defined on $H \times \mathbb{Z}_+$ when H is a Uniform Furstenberg Fractal. On the other hand, when $\nu_H(x, k)$ is finite for all $x \in H$ and $k \in \mathbb{Z}_+$ then H is a Uniform Furstenberg Fractal. Assume now that H is such.

The set $\pi_k(H)$ clearly consists of words of finite length. Next, we show it is extension regular. Suppose that with $n = \nu(x, k), a = x_{[1:n]}$ and $m = \nu_H(y, k), b = y_{[1:m]}$ we have $n \le m$ and $a = b_{[1:n]}$. Then $a = y_{[1:n]}$ and by definition of $\nu_H(x, k)$, $\rho_k(F_H(a)) = \rho_k(H)$. This implies that $n \in \mathcal{N}_H(y, k)$ and so $\nu_H(y, k) = m \le n$. It follows that $\nu_H(y, k) =$

$\nu_H(x,k)$ and $a = b$. So if b is an extension of a in $\pi_k(H)$ then $b = a$. We have also shown that for $x, y \in H$

$$n = \nu_H(x,k) \quad \text{and} \quad y_{[1:n]} = x_{[1:n]} \quad \Longrightarrow \quad \nu_H(y,k) = n. \tag{3.3}$$

In particular, $\{y : \nu(y,k) = \nu(x,k)\}$ contains the set of $y \in H$ with $y_{[1:n]} = x_{[1:n]}$ which is a neighborhood of x in H. Hence, ν_H is a locally constant function and so is continuous. In particular, with k fixed $\{\{x : \nu_H(x,k) = n\} : n \in \mathbb{Z}_+\}$ is a partition of H by clopen sets. By compactness, all but finitely many members are empty. Thus, for each k

$$\nu^*(k) =_{def} \max\{\nu_H(x,k) : x \in H\} \tag{3.4}$$

is finite. Since the words of $\pi_k(H)$ have length bounded by $\nu^*(k)$ it follows that the ER set $\pi_k(H)$ is finite. □

From Theorem 3.1 we can perform an inductive construction to yield an arbitrary Uniform Furstenberg Fractal.

Construction 3.2. *Begin with a finite alphabet $\mathcal{A}$.*
- *Let A_0 be an arbitrary SL set, words all of length $\ell_0 \geq 1$.*
- *Assume the SL set A_n has been defined with words of length ℓ_n.*
- *Let B_n be an arbitrary finite ER set which is an extension of A_n.*
- *Let $C_n = B_n A_n$, concatenations of B_n words with A_n words.*
- *Let A_{n+1} be an arbitrary SL set which is an extension of C_n, and let ℓ_{n+1} denote the length of the words in A_{n+1}.*

Define H by

$$x \in H \quad \Longleftrightarrow \quad x_{[1:n]} \in A_n \quad \textit{for all} \quad n \in \mathbb{Z}_+. \tag{3.5}$$

Remark 3.3. Suppose that $\{0 = p_0 \leq q_0 < p_1 \leq q_1 < \ldots\}$ is an infinite sequence, then we can define

$$\hat{A}_n = A_{p_n}, \quad \hat{B}_n = B_{q_n}, \quad \text{and} \quad \hat{C}_n = \hat{B}_n \hat{A}_n, \tag{3.6}$$

and it is easy to check that the subset constructed using this sequence is exactly H.

Observe that A_{n+1} is an extension of A_n with $\ell_{n+1} \geq 2\ell_n \geq 2^n$.

Theorem 3.4. *Every subset H obtained from Construction 3.2 is a Uniform Furstenberg Fractal and every Uniform Furstenberg Fractal can be obtained via such a construction.*

Proof. Clearly, the results of the construction satisfy Theorem 1.11. On the other hand, given $A_n = \rho_{\ell_n}(H)$ let $B_n = \pi_{\ell_n}(H)$. Then H is an extension of $C_n = B_n A_n$. With ℓ_{n+1} at least the maximum length of a word in C_n let $A_{n+1} = \rho_{\ell_{n+1}}(H)$. □

Corollary 3.5. *The set of Uniform Furstenberg Fractals is a residual subset of $2^X \setminus \{\emptyset\}$.*

Proof. If $K \in 2^X$ is nonempty and k is an arbitrary positive integer we can perform Construction 3.2 with $A_0 = \rho_k(K)$ and obtain a Uniform Furstenberg Fractal H with $\rho_k(H) = \rho_k(K)$. Thus, the set of Uniform Furstenberg Fractals is dense in $2^X \setminus \{\emptyset\}$.

Given $k \in \mathbb{Z}_+$ and $n > 2k$ call a set B of words of length n a k-regular set if for every $a \in W$ there exists r with $k < r < n - k$ such that every element of $a_{[1:r]}A$ extends to an element of B where $A = \rho_k(B)$. From Theorem 3.1 it follows that H is a Uniform Furstenberg Fractal iff for every positive integer k there exists $n > 2k$ such that $\rho_n(H)$ is k-regular. For a fixed n and k the condition that $\rho_n(H)$ be k regular is a clopen condition and so that $\rho_n(H)$ is k-regular for some n is an open condition. It follows that the set of Uniform Furstenberg Fractals is G_δ. □

We can similarly characterize Furstenberg Fractals.

Definition 3.6. For H a Furstenberg Fractal, $k, \ell \in \mathbb{Z}_+$ and $a \in \rho_\ell(H)$, let

$$\nu_H(a,k) = \min\{n : n \geq \ell \text{ and } n \in \mathcal{N}_H(x,k) \quad \text{with } a = x[1:\ell]\}, \tag{3.7}$$

which is finite because H is a Furstenberg Fractal. With $n = \nu_H(a,k)$ let

$$\begin{aligned} W(a,k) &=_{def} \rho_n(H \cap \tau_a(X)) \qquad \text{and} \\ W'(a,k) &=_{def} \{\rho_n(x) : x \in H \cap \tau_a(X) \text{ and } n \in \mathcal{N}(x,k)\}. \end{aligned} \tag{3.8}$$

Thus, $W'(a,k)$ consists of all initial strings in H which extend a and of length n after which the follower sets of length k in H equal to $\rho_k(H)$. Here n is the smallest positive integer such that this set is nonempty. $W(a,k)$ consists of all initial strings in H which extend a and have length this same n.

Lemma 3.7. *For H a Furstenberg Fractal, $k, \ell \in \mathbb{Z}_+$ let $A = \rho_\ell(H)$, $B = \bigcup\{W(a,k) : a \in A\}$, and $B' = \bigcup\{W'(a,k) : a \in A\}$. Then B is an ER set and $B' \subset B$. Furthermore, every element of A extends to an element of B' and every element of B restricts to an element of A.*

Proof. Assume $b \in W(a,k), c \in W(d,k)$. If $a = d$ then b and c have the same length and so neither is a proper extension of the other. If $a \neq d$ then since b extends a and c extends d, neither can be a proper extension of the other. Hence, B is an ER set. Every word in B is an initial string of an element of H with length at least ℓ. Since A lists all initial strings in H of length ℓ, everything in B restricts to an element of A. Each $W'(a,k)$ is nonempty because H is a Furstenberg Fractal. This says that every element of A extends to some element of B'. □

Construction 3.8. *Begin with a finite alphabet $\mathcal{A}$.*
- *Let A_0 be an arbitrary SL set, words all of length $\ell_0 \geq 1$.*
- *Assume the SL set A_n has been defined with words of length ℓ_n.*
- *Let B_n be an arbitrary finite ER set and $B'_n \subset B_n$ such that every $a \in A_n$ extends to at least one element of B'_n and every $b \in B_n$ restricts to a – necessarily unique – element of A_n.*

- *Let* $C_n = (B_n \setminus B'_n) \cup B'_n A_n$*, concatenations of* B'_n *words with* A_n *words together with the remaining words in* B_n*.*
- *Let* A_{n+1} *be an arbitrary SL set which is an extension of* C_n*, and let* ℓ_{n+1} *denote the length of the words in* A_{n+1}*.*

Define H by

$$x \in H \quad \Longleftrightarrow \quad x_{[1:n]} \in A_n \quad \text{for all} \quad n \in \mathbb{Z}_+. \tag{3.9}$$

Theorem 3.9. *Every subset H obtained from Construction 3.8 is a Furstenberg Fractal and every Furstenberg Fractal can be obtained via such a construction.*

Proof. In the construction it is clear that every $a \in A_n$ extends to some element $x \in H$. From this it follows from the original definition that H is a Furstenberg Fractal. The converse follows from Lemma 3.7. □

4 Density of Non-Recurrent Points

From Construction 3.2 we see that there is a rich supply of subsets H such that every point of H is a nonwandering point and by Theorem 2.6 most of these points are transitive points. It is natural to ask if every point can be transitive. The answer is no. Except for the case where $H = \{x\}$ with x a recurrent point for the map σ, it is not even possible that every point be a recurrent point for H. In fact, while the set of transitive points for H is residual in H, the set of non-recurrent points is dense.

Theorem 4.1. *Let* NR_H *denote the set of points of H which are not recurrent points for H. If H is a Furstenberg Fractal containing more than one point, then* NR_H *is dense in H.*

From this we immediately obtain:

Corollary 4.2. *Let H be a nonempty closed subset of X. If every point of H is a recurrent point for H then H is a singleton set.*

The proof of Theorem 4.1 will require some preliminary work.

Assume that $H \in 2^X$ is a Furstenberg Fractal containing more than one point and for an arbitrary positive integer k let $a \in \rho_k(H)$. For the remainder of this section, let $H_a = \{x \in H : x_{[1:k]} = a\}$. We will construct a point in $H_a \cap NR_H$. Since k and a were arbitrary, this will prove that NR_H is dense in H.

Since H is a Furstenberg Fractal, H_a contains a nonwandering point y. Because H contains more than one point, $\rho_n(H)$ contains more than one word for n greater than or equal to some positive n_1. If $m \in \mathcal{N}_H(y, n_1)$ with $m > k$ then $y_{[1:m]}$ is followed in H by more than one word in H, i.e. there are at least two distinct elements of H with initial segment $y_{[1:m]}$. As $m > k$ both of these points of H lie in H_a. So for sufficiently large n

$\rho_n(H_a)$ contains more than one word. Let k^* denote the smallest positive integer such that $\rho_{k^*}(H_a)$ contains more than one word. By definition of H_a we have $k^* > k$. Let b, c be distinct words in $\rho_{k^*}(H_a)$. Thus we have

$$x \in H_a \quad \Longrightarrow \quad x_{[1:k^*-1]} = b_{[1:k^*-1]} = c_{[1:k^*-1]} \tag{4.1}$$

and since $b \neq c$ we have $b_{k^*} \neq c_{k^*}$.

Lemma 4.3. *Let $x \in H$ with $x_{[1:k^*]} = b$ and assume that $m = \tilde{v}(x, k^*)$ is finite. If $y \in H$ with $y_{[1:m]} = x_{[1:m]}$ and $y_{[m+1:m+k^*]} = c$, then $y_{[1:m+k^*-1]} = x_{[1:m+k^*-1]}$ and $m < \tilde{v}(y, k^*)$.*

Proof. By definition of $\tilde{v}(x, k^*)$ we have $x_{[m+1:m+k^*]} = x_{[1:k^*]}$ which equals b. Since b and c agree until the last place, x and y agree up to the $m + k^* - 1$. Since m is the first positive member of $\tilde{\mathcal{N}}(x, k^*)$ it follows from (2.24) that no positive integer less than m is in $\tilde{\mathcal{N}}(y, k^*)$. Since, $m+k^*-1 \geq k^*$, y agrees with x in the first k^* places and so $y_{[1:k^*]} = b$. Since $y_{[m+1:m+k^*]} = c \neq b$, we have $m \notin \tilde{\mathcal{N}}(y, k^*)$ either. Thus, $m < \tilde{v}(y, k^*)$. □

Proof of Theorem 4.1. Now we construct the required point of $H_a \cap NR_H$. In fact, we will construct a point $y \in H_a$ such that $\tilde{\mathcal{N}}(y, k^*)$ contains no positive integers and so $\tilde{v}(y, k^*) = \infty$.

Begin with $y^{(0)} \in H$ with $y^{(0)}[1 : k^*] = b$. If $\tilde{v}(y^{(0)}, k^*) = \infty$, then let $y = y^{(0)}$. Otherwise, let $m^0 = \tilde{v}(y^{(0)}, k^*)$. By definition of $\tilde{\mathcal{N}}(y^{(0)}, k^*)$ there exists $y^{(1)} \in H$ with $y^{(1)}_{[1:m^0]} = y^{(0)}_{[1:m^0]}$ and with $y^{(1)}_{[m^0+1:m^0+k^*]} = c$. By Lemma 4.3 we have

$$y^{(1)}_{[1:m^0+k^*-1]} = y^{(0)}_{[1:m^0+k^*-1]} \quad \text{and} \quad m^0 < \tilde{v}(y^{(1)}, k^*). \tag{4.2}$$

Inductively, we construct a sequence $y^{(0)}, y^{(1)}, \dots y^{(n)}$ of points in H with for $i = 0, \dots, n-1$, $m^i = \tilde{v}(y^{(i)}, k^*)$ is an increasing sequence of positive integers and with

$$y^{(i+1)}_{[1:m^i+k^*-1]} = y^{(i)}_{[1:m^i+k^*-1]} \quad \text{and} \quad y^{(i+1)}_{[m^i+1:m^i+k^*]} = c. \tag{4.3}$$

By Lemma 4.3 $m^{n-1} < \tilde{v}(y^{(n)}, k^*)$. If $\tilde{v}(y^{(n)}, k^*) = \infty$ then the process terminates and we let $y = y^{(n)}$ otherwise let $m^n = \tilde{v}(y^{(n)}, k^*)$ and continue to the next step.

If the process does not terminate then we obtain an infinite sequence $\{y^{(0)}, y^{(1)}, \dots\}$ in H which converges to $y \in X$ with

$$y_{[1:m^n+k^*-1]} = y^{(n)}_{[1:m^n+k^*-1]} \quad \text{for} \quad n = 0, 1, \dots. \tag{4.4}$$

Notice that the sequence m^n of positive integers is strictly increasing and so is unbounded. Since H is closed, $y \in H$. If j is any positive integer then for some n, $j + k^* < m^n$. By definition of $m^n = \tilde{v}(y^{(n)}, k^*)$ we have that $j \notin \tilde{\mathcal{N}}(y^{(n)}, k^*)$. Since y agrees with $y^{(n)}$ through the $j + k^*$ place it follows from (2.24) that $j \notin \tilde{\mathcal{N}}(y, k^*)$. That is, $\tilde{v}(y, k^*) = \infty$.

Thus, whether the process terminates or not we have $y \in NR_H$. Finally, $y_{[1:m^0+k^*-1]} = y^{(0)}_{[1:m^0+k^*-1]}$. Since m^0 is a positive integer, or is infinite, we have $y_{[1:k^*]} = y^{(0)}_{[1:k^*]} = b$. As a is an initial string in b, it follows that $y \in H_a$. □

5 Isometries and Furstenberg Fractals

We have seen that a singleton set $H = \{x\}$ is a Furstenberg fractal iff x is a recurrent point for the shift map σ. However, while we are using a dynamic definition, being a fractal should really be a geometric concept. If any singleton is a fractal then any other singleton should be one as well. We observe that any two points are isometric and this suggests that we weaken the notion of nonwandering point by allowing variation by isometries. As we will see in Lemma 5.1 below we get the same results whether we use isometries between subsets of X or restrictions of isometric automorphisms of X. By an *isometry* we mean a map which preserves distance, and so is necessarily injective. By an isometry between two subsets A, B of a metric space we mean a bijective isometry from one to the other. Finally, an isometric automorphism of a metric space M is a bijective isometry from M onto M. So the inverse is an isometric automorphism as well.

Recall that with $X = \mathcal{A}^{\mathbb{Z}_+}$ and with $\overline{X^*}$ equal to X together with the set of finite words $X^* = \bigcup_{k=0}^{\infty} X_k$ the usual metric on X was extended to a metric on $\overline{X^*}$ by (2.5). This definition implies that if $x_1, x_2 \in \overline{X^*}$ with $k = \ell(x_1) \leq \ell(x_2)$ then

$$d(x_1, x_2) = d(x_1, \rho_k(x_2)). \tag{5.1}$$

Now let A be a subset of X or X_k and let j be an isometric injection of A into X or X_k. By definition of the distance we have for $x_1, x_2 \in A$ that for all i less than or equal to the length of x_1 (= length x_2) that

$$\rho_i(x_1) = \rho_i(x_2) \quad \Longrightarrow \quad \rho_i(j(x_1)) = \rho_i(j(x_2)). \tag{5.2}$$

Hence, $\rho_i(j(x))$ depends only on $\rho_i(x)$.

Thus, if J is an isometric automorphism of X then for each positive integer k, J induces an isometric automorphism J_k on X_k well defined by

$$J_k(\rho_k(x)) = \rho_k(J(x)), \tag{5.3}$$

for every $x \in X$.

Furthermore, we can define for each word a of length ℓ the isometric automorphism J_a of X by

$$\begin{aligned} J_a(x) &=_{def} \sigma^{\ell}(J(\tau_a(x))), \\ \text{i.e.} \quad J_{\ell}(a)J_a(x) &= J(ax). \end{aligned} \tag{5.4}$$

That is, J_a maps the successors of a to the successors of $J_{\ell}(a)$.

Putting these together we obtain an isometry $J_{a,k}$ on X_k by

$$J_{\ell}(a)J_{a,k}(w) = J_{\ell+k}(aw). \tag{5.5}$$

Lemma 5.1. *Let A be a set of finite or infinite words for the alphabet $\mathcal{A}$ and let j map A into X^*, the set of finite and infinite words for $\mathcal{A}$. Assume that*

(i) *The set A is extension regular. That is, no finite word in A is the initial segment of any other word in A.*
(ii) *The length of $j(x)$ equals the length of x for every $x \in A$.*
(iii) *If $x_1, x_2 \in A$ with the same length then $d(j(x_1), j(x_2)) = d(x_1, x_2)$.*

There exists an isometric automorphism J of X which extends j in the sense that for every $x \in X$ and positive integer i

$$\begin{aligned} x \in A \quad &\Longrightarrow \quad J(x) = j(x); \\ \rho_i(x) \in A \quad &\Longrightarrow \quad \rho_i(J(x)) = j(\rho_i(x)). \end{aligned} \tag{5.6}$$

Proof. Let $A_0 = A$ and $j_0 = j$. For $k = 1, 2, \ldots$ we define a set of finite and infinite words A_k and a map j_k which satisfy conditions (i)–(iii) and in addition
(iv) Every word of A_k has length at least k and $\rho_k(A_k) = X_k$.
(v) If $x \in A_{k-1}$ has length greater than or equal to k then $x \in A_k$ and $j_k(x) = j_{k-1}(x)$.
(vi) If $x \in A_k \setminus A_{k-1}$ then x is a word of length k and $\rho_{k-1}(j_k(x)) = j_{k-1}(\rho_{k-1}(x))$.

Assuming that A_{k-1} and j_{k-1} have been defined we let A_k be the union of all words of A_{k-1} with length greater than or equal to k (= all words of A with length at least k) together with all words of length k which are not the initial segments of any word in A.

As above j_{k-1} induces an isometric automorphism $\tilde{j}_{k-1}$ of X_{k-1} by

$$\tilde{j}_{k-1}(y) = \rho_{k-1}(j_{k-1}(x)) \quad \text{for} \quad y = \rho_{k-1}(x) \text{ with } x \in A_{k-1}. \tag{5.7}$$

Fix $y \in X_{k-1}$. If for $a \in \mathcal{A}$ the length k word $ya = \rho_k(x)$ for some $x \in A_{k-1}$ then $\rho_k(j_{k-1}(x)) = \tilde{j}_{k-1}(y)b$ for some $b \in \mathcal{A}$ and with y fixed the association $a \mapsto b$ defines an injective map from a (possibly empty) subset of $\mathcal{A}$ into $\mathcal{A}$. Extend this association to a bijection of $\mathcal{A}$. Then $ya \mapsto \tilde{j}_{k-1}(y)b$ defines j_k on all the new words in A_k of length k which extend y. On the words x of length greater than k we use $j_k(x) = j_{k-1}(x) = j(x)$. This completes the inductive construction.

The isometry J is well defined by the equations

$$\rho_k(J(x)) \;=\; j_k(\rho_k(x)) \quad \text{for} \quad x \in X, k \in \mathbb{Z}_+. \tag{5.8}$$

□

Recall (2.13) which says that for $H, K \in 2^X$ the Hausdorff distance $d(H,K) < 2^{-k+1}$ iff $\rho_k(H) = \rho_k(K)$. If J is an isometry of X then the induced map on 2^X is an isometry of the Hausdorff distance.

Proposition 5.2. *Let $H, K \subset X$. For any positive integer k the following are equivalent:*
(a) *$\rho_k(H)$ and $\rho_k(K)$ are isometric subsets of X_k.*
(b) *There exists an isometric automorphism J of X such that $d(J(H), K) < 2^{-k+1}$.*
(c) *There exists an isometric automorphism J of X such that $\rho_k(J(H)) = \rho_k(K)$.*

Proof. (a) ⇒ (c): If $j : \rho_k(H) \to \rho_k(K)$ is an isometric bijection then by Lemma 5.1 j extends to an isometric automorphism J of X and by (5.6), $\rho_k(J(H)) = j(\rho_k(H)) = \rho_k(K)$.

(c) ⇒ (a): J induces an isometry j on X_k by $j(\rho_k(x)) = \rho_k(J(x))$. In particular, $j(\rho_k(H)) = \rho_k(J(H))$ which equals $\rho_k(K)$ by assumption.

(b) ⇔ (c): This follows from (2.13) as above. □

Definition 5.3. Let $H \in 2^X$. A point $x \in H$ is called a *weak nonwandering point* for H when for every positive integer k there exist arbitrarily large integers n such that $\rho_k(H)$ and $\rho_k(\phi_n(H,x))$ are isometric subsets of X_k.

Recall that x is a nonwandering point for H when for every positive integer k there exist arbitrarily large integers n such that $\rho_k(H) = \rho_k(\phi_n(H,x))$. Thus, a nonwandering point is a weak nonwandering point.

Lemma 5.4. *Let J be an isometric automorphism of X. For $H \in 2^X$ a point $x \in H$ is a weak nonwandering point for H iff $J(x)$ is a weak nonwandering point for $J(H)$.*

Proof. An isometric automorphism J induces for each k an isometric automorphism J_k on X_k given by (5.3). Let $a = \rho_n(x)$. The induced map given by (5.5) satisfies

$$J_{a,k}(\rho_k(\phi_n(H,x))) = \rho_k(\phi_n(J(H),J(x)). \tag{5.9}$$

If $J(x)$ is a weak nonwandering point of $J(H)$ then there exists arbitrarily large integers n such that $\rho_k(\phi_n(J(H),J(x))$ is isometric to $\rho_k(J(H)) = J_k(\rho_k(H))$. Hence, for the same integers n, $\rho_k(\phi_n(H,x))$ is isometric to $\rho_k(H)$ and so x is a weak nonwandering point for H. □

Definition 5.5. A closed subset H of X is called a *Weak Furstenberg Fractal* when the set of weak nonwandering points for H is dense in H. H is called a *Uniform Weak Furstenberg Fractal* when every point of H is a weak nonwandering point for H.

Theorem 5.6. *Let $H \in 2^X$.*

H is a Uniform Weak Furstenberg Fractal iff there exists an isometric automorphism J of X such that $J(H)$ is a Uniform Furstenberg Fractal.

H is a Weak Furstenberg Fractal iff there exists an isometric automorphism J of X such that $J(H)$ is a Furstenberg Fractal.

Proof. Weak nonwandering points are preserved by isometries and nonwandering points are weak nonwandering points. Hence, if H is isometric to a Uniform Furstenberg Fractal then every point of H is a weak nonwandering point and if H is isometric to a Furstenberg Fractal then H has a dense set of weak nonwandering points. We prove the converse results.

For any positive integers k and any $x \in H, a \in \rho_k(H)$ let

$$\begin{aligned} n_H(x,k) &= \min\{\, n > k : \rho_k(H) \text{ and } \rho_k(\phi_n(H,x)) \\ &\qquad\qquad \text{are isometric subsets of } X_k\}, \qquad (5.10) \\ n_H(a,k) &= \min\{\, n_H(x,k) : x \in H \text{ and } \rho_k(x) = a\,\}. \end{aligned}$$

Thus, $n_H(x,k)$ is finite for all x and k iff every point is a weak nonwandering point, while $n_H(a,k)$ is finite for all k and $a \in \rho_k(H)$ iff the weak nonwandering points are dense in H. Compare (3.1) and (3.7)

Following the proof of (5.9) in Lemma 5.4 it follows that $n_{J(H)}(J(x),k) = n_H(x,k)$ and $n_{J(H)}(J_k(a),k) = n_H(a,k)$ when J is an isometry.

First assume that every point is a weak nonwandering point.

As in Theorem 3.1, the function $n_H(x,k)$ is locally constant and so for each k the values remain bounded as x varies over H. Let $N_H(k) = \max\{n_H(x,k) : x \in H\}$.

We begin with a sequence of positive integers $\{k_i : i = 0,1,\dots\}$ with $k_0 = 0$ and $k_{i+1} > k_i + N_H(k_i)$. We construct a sequence of isometric automorphisms $J^{(i)}$ on X such that with $H_i = J^{(i)}(H)$ we have

- On X_{k_i} the induced isometries $J^{(i)}_{k_i}$ and $J^{(i-1)}_{k_i}$ agree.
- For each $y \in H_i$, with $n = n_{H_i}(y,k_i)$, so that $k_i < n < k_{i+1} - k_i$,

$$\rho_{k_i}(H_i) = \rho_{k_i}(\phi_n(H_i,y)). \tag{5.11}$$

Let $J^{(0)} = 1_X$ so that $H_0 = H$. Assume that $J^{(i-1)}$ has been constructed. We construct an isometry Z on X and define $J^{(i)}$ to be $Z \circ J^{(i-1)}$.

On each word w in $X_{k_i} \setminus \rho_{k_i}(H_{i-1})$ define $j(w) = w$.

Now for each $x \in H_{i-1}$, $n = n_{H_{i-1}}(x,k_i)$ satisfies $k_i < n < k_{i+1} - k_i$ and there is an isometry Z_x from $\rho_{k_i}(\phi_n(H_{i-1},x))$ onto $\rho_{k_i}(H_{i-1})$. While we write this isometry as Z_x, it really is chosen to depend just on $x_{[1:n]}$.

Map each word of length $n + k_i$ of the form $x_{[1:n+k_i]}$ to $x_{[1:n]}Z_x(x_{[n+1:n+k_i]})$.

Thus, we have defined an isometry j on the words of the form $x_{[n+k_i]}$ as well as on all words of length k_i which are not initial segments of an element of H_{i-1}. Apply Lemma 5.1 to extend j to an isometric isomorphism Z on X. Since $k_i < n_{H_{i-1}}(x,k_i)$, Z induces the identity on words of length k_i. Thus, with $J^{(i)} = Z \circ J^{(i-1)}$ and $H_i = Z(H_{i-1})$ satisfy the required properties.

The sequence of isometries $\{J^{(i)}\}$ stabilizes on finite words and so we define J so that

$$J_k = J^{(i)}_k \quad \text{for all} \quad i \quad \text{with} \quad k_i > k. \tag{5.12}$$

From (5.11) we see that $J(H)$ is a Uniform Furstenberg Fractal.

For the case of dense weak nonwandering points we define $N_H(k)$ to be the maximum on the finite set $n_H(a,k)$ with $a \in \rho_k(H)$ and again choose a sequence $\{k_i\}$ with $k_0 = 0$ and $k_{i+1} > N_H(k_i) + k_i$.

We alter the inductive construction so that
- On X_{k_i} the induced isometries $J^{(i)}_{k_i}$ and $J^{(i-1)}_{k_i}$ agree.
- For every $a \in \rho_{k_i}(H_i)$, there exists $y \in H_i$ with $a = y_{[1:k_i]}$ and $n = n_{H_i}(a, k_i) = n_{H_i}(y, k_i)$, so that $k_i < n < k_{i+1} - k_i$,

$$\rho_{k_i}(H_i) = \rho_{k_i}(\phi_n(H_i, y)). \tag{5.13}$$

For the inductive step we choose for each $a \in \rho_{k_i}(H_{i-1})$, $x(a) \in H_{i-1}$ with $x(a)_{[1:k_i]} = a$ and with $n_{H_{i-1}}(a, k_i) = n_{H_{i-1}}(x(a), k_i)$. This time, in addition to being the identity on $X_{k_i} \setminus \rho_{k_i}(H_{i-1})$, j is defined only on those $x_{[1:n+k_i]}$ with $x \in H_{i-1}$ and $x_{[1:n]} = x(a)_{[1:n]}$ for some $a \in \rho_{k_i}(H_{i-1})$.

Extend j to Z, let $J_i = Z \circ J_{i-1}$ and define J using (5.12) as before. Then $J(H)$ is a Furstenberg Fractal by (5.13). □

Recall that the topology of 2^X depends only on the topology of X. It is independent of the choice of compatible metric on X. The associated Hausdorff metrics all induce the same topology on 2^X. All of the dynamics in the first four sections are thus independent of the choice of metric. However, once we introduce isometries the choice of metric becomes important. If for the ultrametric defined using (2.5) we replace the sequence $\{2^{-k}\}$ by any strictly decreasing sequence $\{r_k\}$ with limit 0 and with $r_0 = 1$ then there exists a continuous increasing function ϕ on $\mathbb{R}$ such that the new metric is $\phi(d)$. It follows that the new metric has the same isometries as the old one and so all the results of this section continue to hold. However, something like this ultrametric structure is required so that cylinder sets defined by subsets of the same cardinality of X_k are isometric and so that the extension lemma Lemma 5.1 will hold.

Bibliography

[1] E. Akin. Lectures on Cantor and Mycielski sets for dynamical systems, in: *Chapel Hill Ergodic Theory Workshops*, ed. I. Assani, pp. 21–79, Contemp. Math. 356, Amer. Math. Soc., 2004.

[2] K. Falconer. *Fractal geometry – mathematical foundations and applications*, 2^{nd} ed., John Wiley & Sons, NY, 2003.

[3] H. Furstenberg. Ergodic fractal measures and dimension conservation, *Ergod. Th. & Dynam. Sys.* **28** (2008), 405–422.

[4] M. Gavish. Measures with uniform scaling scenery, *Ergod. Th. & Dynam. Sys.* **31:** (2011), 33–48.

[5] M. Hochman. (2013) *Dynamics on fractals and fractal distributions*, arXiv:1008.3731v2.

[6] B. Kra. (2013) *Furstenberg Fractals: Lectures by Hillel Furstenberg* (in preparation).

Idris Assani and Kimberly Presser

A Survey of the Return Times Theorem

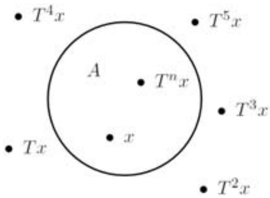
Abstract: The goal of this paper is to survey the history, development and current status of the Return Times Theorem and its many extensions and variations. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and let $T : X \to X$ be a measure-preserving transformation. Perhaps the oldest result in ergodic theory is that of *Poincaré's Recurrence Principle* [73] which states:

Theorem 1. *For any set $A \in \mathcal{F}$, the set of points x of A such that $T^n x \notin A$ for all $n > 0$ has zero measure. This says that almost every point of A returns to A. In fact, almost every point of A returns to A infinitely often.*

The *return time* for a given element $x \in A$,

$$r_A(x) = \inf\{k \geq 1 : T^k x \in A\},$$

is the first time that the element x returns to the set A. This is visualized in Figure 2.1.



By Theorem 1, there is set of full measure in A such that all elements of this set have a finite return time. Our study of the return times theorem asks how we can further generalize this notion.

Keywords: Return times, Wiener Wintner dynamical systems, weighted averages, generic point, unique ergodicity, pointwise characteristic factors.

Classification: 37A05, 37A30, 37A45, 37A50, 37B20

Idris Assani: UNC Chapel Hill, Department of Mathematics, USA, e-mail: assani@math.unc.edu
Kimberly Presser: Shippensburg University, Department of Mathematics, USA,
e-mail: kjpres@ship.edu

1 Origins

As stated above in Theorem 1, μ-a.e. x returns to A infinitely often. One question to ask is how frequently this occurs. Consider the time average

$$\frac{1}{N}\sum_{n=1}^{N} \chi_A(T^n x).$$