Gerrit van Dijk

Distribution Theory

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Convolution, Fourier Transform, and Laplace Transform

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Preface

The mathematical concept of a distribution originates from physics. It was first used by O. Heaviside, a British engineer, in his theory of symbolic calculus and then by P. A. M. Dirac around 1920 in his research on quantum mechanics, in which he introduced the delta-function (or delta-distribution). The foundations of the mathematical theory of distributions were laid by S. L. Sobolev in 1936, while in the 1950s L. Schwartz gave a systematic account of the theory. The theory of distributions has numerous applications and is extensively used in mathematics, physics and engineering. In the early stages of the theory one used the term generalized function rather than distribution, as is still reflected in the term delta-function and the title of some textbooks on the subject.

This book is intended as an introduction to distribution theory, as developed by Laurent Schwartz. It is aimed at an audience of advanced undergraduates or beginning graduate students. It is based on lectures I have given at Utrecht and Leiden University. Student input has strongly influenced the writing, and I hope that this book will help students to share my enthusiasm for the beautiful topics discussed.

Starting with the elementary theory of distributions, I proceed to convolution products of distributions, Fourier and Laplace transforms, tempered distributions, summable distributions and applications. The theory is illustrated by several examples, mostly beginning with the case of the real line and then followed by examples in higher dimensions. This is a justified and practical approach in our view, it helps the reader to become familiar with the subject. A moderate number of exercises are added with hints to their solutions.

There is relatively little expository literature on distribution theory compared to other topics in mathematics, but there is a standard reference [10], and also [6]. I have mainly drawn on [9] and [5].

The main prerequisites for the book are elementary real, complex and functional analysis and Lebesgue integration. In the later chapters we shall assume familiarity with some more advanced measure theory and functional analysis, in particular with the Banach–Steinhaus theorem. The emphasis is however on applications, rather than on the theory.

For terminology and notations we generally follow N. Bourbaki. Sections with a star may be omitted at first reading. The index will be helpful to trace important notions defined in the text.

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Leiden, November 2012 Gerrit van Dijk

Contents

1 Introduction

Differential equations appear in several forms. One has ordinary differential equations and partial differential equations, equations with constant coefficients and with variable coefficients. Equations with constant coefficients are relatively well understood. If the coefficients are variable, much less is known. Let us consider the singular differential equation of the first order

$$x\,u' = 0\,. \tag{$*$}$$

Though the equation is defined everywhere on the real line, classically a solution is only given for $x > 0$ and $x < 0$. In both cases $u(x) = c$ with c a constant, different for $x > 0$ and $x < 0$ eventually. In order to find a global solution, we consider a weak form of the differential equation. Let φ be a C^1 function on the real line, vanishing outside some bounded interval. Then equation $(*)$ can be rephrased as

$$\langle x\,u', \varphi\rangle = \int_{-\infty}^{\infty} x\,u'(x)\,\varphi(x)\,\mathrm{d}x = 0\,.$$

Applying partial integration, we get

$$\langle x\,u', \varphi\rangle = -\langle u, \varphi + x\,\varphi'\rangle = 0\,.$$

Take $u(x) = 1$ for $x \geq 0$, $u(x) = 0$ for $x < 0$. Then we obtain

$$\langle x\,u', \varphi\rangle = -\int_0^{\infty} [\varphi(x) + x\,\varphi'(x)]\,\mathrm{d}x = -\int_0^{\infty} \varphi(x)\,\mathrm{d}x + \int_0^{\infty} \varphi(x)\,\mathrm{d}x = 0\,.$$

Call this function H, known as the Heaviside function. We see that we obtain the following (weak) global solutions of the equation:

$$u(x) = c_1\,H(x) + c_2\,,$$

with c_1 and c_2 constants. Observe that we get a two-dimensional solution space. One can show that these are all weak solutions of the equation.

The functions φ are called test functions. Of course one can narrow the class of test functions to C^k functions with $k > 1$, vanishing outside a bounded interval. This is certainly useful if the order of the differential equation is greater than one. It would be nice if we could assume that the test functions are C^∞ functions, vanishing outside a bounded interval. But then there is really something to show: do such functions exist? The answer is yes (see Chapter 2). Therefore we can set up a nice theory of global solutions. This is important in several branches of mathematics and physics. Consider, for example, a point mass in $\mathbb{R}^3$ with force field having potential $V = 1/r$,

r being the distance function. It satisfies the partial differential equation $\Delta V = 0$ outside 0. To include the origin in the equation, one writes it as

$$\Delta V = -4\pi\delta$$

with δ the functional given by $\langle \delta, \varphi \rangle = \varphi(0)$. So it is the desire to go to global equations and global solutions to develop a theory of (weak) solutions. This theory is known as distribution theory. It has several applications also outside the theory of differential equations. To mention one, in representation theory of groups, a well-known concept is the character of a representation. This is perfectly defined for finite-dimensional representations. If the dimension of the space is infinite, the concept of distribution character can take over that role.

2 Definition and First Properties of Distributions

Summary

In this chapter we show the existence of test functions, define distributions, give some examples and prove their elementary properties. Similar to the notion of support of a function we define the support of a distribution. This is a rather technical part, but it is important because it has applications in several other branches of mathematics, such as differential geometry and the theory of Lie groups.

Learning Targets

- ✓ Understanding the definition of a distribution.
- ✓ Getting acquainted with the notion of support of a distribution.

2.1 Test Functions

We consider the Euclidean space $\mathbb{R}^n$, $n \geq 1$, with elements $x = (x_1, \dots, x_n)$. One defines $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$, the *length* of x.

Let φ be a complex-valued function on $\mathbb{R}^n$. The closure of the set of points $\{x \in \mathbb{R}^n : \varphi(x) \neq 0\}$ is called the *support* of φ and is denoted by $\operatorname{Supp} \varphi$.

For any n-tuple $k = (k_1, \dots, k_n)$ of nonnegative integers k_i one defines the *partial differential operator* D^k as

$$D^k = \left(\frac{\partial}{\partial x_1}\right)^{k_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{k_n} = \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} .$$

The symbol $|k| = k_1 + \cdots + k_n$ is called the *order* of the partial differential operator. Note that order 0 corresponds to the identity operator. Of course, in the special case $n = 1$ we simply have the differential operators $\mathrm{d}^k/\mathrm{d}x^k$ $(k \geq 0)$.

A function $\varphi : \mathbb{R}^n \to \mathbb{C}$ is called a C^m function if all partial derivatives $D^k\varphi$ of order $|k| \leq m$ exist and are continuous. The space of all C^m functions on $\mathbb{R}^n$ will be denoted by $\mathcal{E}^m(\mathbb{R}^n)$. In practice, once a value of $n \geq 1$ is fixed, this space will be simply denoted by $\mathcal{E}^m$.

A function $\varphi : \mathbb{R}^n \to \mathbb{C}$ is called a C^∞ function if all its partial derivatives $D^k\varphi$ exist and are continuous. A C^∞ function with compact support is called a *test function*. The space of all C^∞ functions on $\mathbb{R}^n$ will be denoted by $\mathcal{E}(\mathbb{R}^n)$, the space of test functions on $\mathbb{R}^n$ by $\mathcal{D}(\mathbb{R}^n)$. In practice, once a value of $n \geq 1$ is fixed, these spaces will be simply denoted by $\mathcal{E}$ and $\mathcal{D}$ respectively.

It is not immediately clear that nontrivial test functions exist. The requirement of being C^∞ is easy (for example every polynomial function is), but the requirement of also having compact support is difficult. See the following example however.

Example of a Test Function

First take $n = 1$, the real line. Let φ be defined by

$$\varphi(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1\,. \end{cases}$$

Then $\varphi \in \mathcal{D}(\mathbb{R})$. To see this, it is sufficient to show that φ is infinitely many times differentiable at the points $x = \pm 1$ and that all derivatives at $x = \pm 1$ vanish. After performing a translation, this amounts to showing that the function f defined by $f(x) = \mathrm{e}^{-1/x}$ $(x > 0)$, $f(x) = 0$ $(x \leq 0)$ is C^∞ at $x = 0$ and that $f^{(m)}(0) = 0$ for all $m = 0, 1, 2, \ldots$. This easily follows from the fact that $\lim_{x \downarrow 0} \mathrm{e}^{-1/x}/x^k = 0$ for all $k = 0, 1, 2, \ldots$.

For arbitrary $n \geq 1$ we denote $r = \|x\|$ and take

$$\varphi(x) = \begin{cases} \exp\left(-\frac{1}{1-r^2}\right) & \text{if } r < 1 \\ 0 & \text{if } r \geq 1\,. \end{cases}$$

Then $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

The space $\mathcal{D}$ is a complex linear space, even an algebra. And even more generally, if $\varphi \in \mathcal{D}$ and ψ a C^∞ function, then $\psi\varphi \in \mathcal{D}$. It is easily verified that $\operatorname{Supp}(\psi\varphi) \subset \operatorname{Supp}\varphi \cap \operatorname{Supp}\psi$.

We define an important *convergence principle* in $\mathcal{D}$

Definition 2.1. *A sequence of functions $\varphi_j \in \mathcal{D}$ $(j = 1, 2, \ldots)$ converges to $\varphi \in \mathcal{D}$ if the following two conditions are satisfied:*

(i) *The supports of all φ_j are contained in a compact set, not depending on j,*

(ii) *For any n-tuple k of nonnegative integers the functions $D^k\varphi_j$ converge uniformly to $D^k\varphi$ $(j \to \infty)$.*

2.2 Distributions

We can now define the notion of a distribution. A *distribution* on the Euclidean space $\mathbb{R}^n$ (with $n \geq 1$) is a continuous complex-valued linear function defined on $\mathcal{D}(\mathbb{R}^n)$, the linear space of test functions on $\mathbb{R}^n$. Explicitly, a function $T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$ is a distribution if it has the following properties:

a. $T(\varphi_1 + \varphi_2) = T(\varphi_1) + T(\varphi_2)$ for all $\varphi_1, \varphi_2 \in \mathcal{D}$,

b. $T(\lambda\varphi) = \lambda\, T(\varphi)$ for all $\varphi \in \mathcal{D}$ and $\lambda \in \mathbb{C}$,

c. If φ_j tends to φ in $\mathcal{D}$ then $T(\varphi_j)$ tends $T(\varphi)$.

Instead of the term linear function, the term *linear form* is often used in the literature.