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Analysis and Applications in Turbulence and Porous Media

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Preface

This book presents advanced stochastic models and simulation methods for random flows and transport of particles by turbulent velocity fields and flows in porous media. Two main classes of models are constructed: (1) turbulent flows are modeled as synthetic random fields which have certain statistics and features mimicking those of turbulent fluid in the regime of interest, and (2) the models are constructed in the form of stochastic differential equations for stochastic Lagrangian trajectories of particles carried by turbulent flows. In both these classes, we develop Random flight models for the trajectories of tracer particles in turbulence and in flows through porous media. The boundary value problems in stochastic formulation for high-dimensional PDEs present a powerful research instrument in many modern branches of science and technology, in particular, in the turbulence simulation, transport in porous media, random load analysis in mechanical systems, geodesy, composite materials, elastography for biological tissues, acoustic scattering from rough surfaces, defects in metals, X-ray diffraction analysis of epitaxial layers, dislocations in crystals, etc. Interesting example is related to the coagulation of particles carried in turbulent flows governed by Smoluchowski nonlinear systems of coagulation equations with random coefficients. We present detailed results of numerical simulations for these applied problems and discuss stochastic interpretations related to the physics of the relevant problems.

A considerable amount of material is devoted to the random field description and different stochastic simulation methods, mainly the stochastic spectral and Fourier-wavelet methods for homogeneous vector Gaussian random fields, and the Karhunen–Loève expansions for inhomogeneous random fields. Of special interest are the so-called partially homogeneous random fields which we use in the development of stochastic models for boundary value problems with random boundary conditions, in particular, for Stokes flows, which are presented in the last chapter.

The book is written for mathematicians, physicists, and engineers studying processes associated with probabilistic interpretation, researchers in applied and computational mathematics, in environmental and engineering sciences dealing with turbulent transport and flows in porous media, as well as nucleation, coagulation, and chemical reaction analysis under fluctuation conditions. It can be of interest for students and post-graduates studying numerical methods for solving stochastic boundary value problems of mathematical physics and dispersion of particles by turbulent flows and flows in porous media.

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Berlin, Novosibirsk, June 2012 *Karl K. Sabelfeld*

Contents

Chapter 1

Introduction

1.1 Why random fields?

Probabilistic approach and stochastic simulation are becoming more and more popular in all branches of science and technology, especially in problems where the data are randomly fluctuating, or they are highly irregular in a deterministic sense. As a rule, in such problems it is very difficult and expensive to carry out measurements to extract the desired data. Important examples we deal with in this book are the turbulent flow simulation [146], and construction of flows through porous media [34, 65]. The temporal and spatial scales of the input parameters in this class of problems vary enormously, and their behavior is very complicated, so that there is no chance of describing it deterministically. In the stochastic approach, the input parameters are considered as random fields, and one needs to know only a few of their functions, such as the mean and correlation tensor, whose behavior in time and space is much more regular, so that usually it is considerably easier to extract them through measurements.

In most applications, it is assumed that the random fields are Gaussian, or that they can be obtained by a functional transformation of Gaussian fields. Generally, it is very difficult to construct efficient simulation methods for inhomogeneous random fields, even if they are Gaussian. Therefore, the most developed methods deal with homogeneous or quasihomogeneous random fields where the characteristic scales of the variations of the means of the field are considerably larger than the correlation scale. There are highly intensive studies and literature concerned with the simulation of homogeneous random fields.

The most important class of simulation methods seeks to construct stochastic models based on spectral representations. We shall consider general real-valued Gaussian homogenous random fields $u(\mathbf{x})$ defined on the multidimensional Euclidean space $\mathbb{R}^d$. Under quite general conditions, a real-valued Gaussian homogenous random field $u(\mathbf{x})$ can be represented through a stochastic Fourier integral [146]:

$$u(\mathbf{x}) = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i}\mathbf{k}\cdot\mathbf{x}} E^{1/2}(\mathbf{k}) \tilde{W}(\mathrm{d}\mathbf{k}), \tag{1.1}$$

where $\tilde{W}(\mathrm{d}\mathbf{k})$ is a complex-valued white noise random measure on $\mathbb{R}^d$, with $\tilde{W}(B) = \overline{\tilde{W}(-B)}$, $\langle \tilde{W}(B) \rangle = 0$, and $\langle \tilde{W}(B)\overline{\tilde{W}(B')} \rangle = \mu(B \cap B')$ for the Lebesgue measure μ and all Lebesgue-measurable sets B, B'. We use angle brackets $\langle \cdot \rangle$ to denote statistical (ensemble) averages. The spectral density $E(\mathbf{k})$ is a nonnegative even function

representing the strength (energy) of the random field associated to the wavenumber $\mathbf{k}$, meaning the length scale $1/|\mathbf{k}|$ and direction $\mathbf{k}/|\mathbf{k}|$.

We mention several simulation methods based on the spectral representation: (i) the discrete spectral method (DSM) [226] which is simply a deterministic discrete approximation of the Fourier Stieltjes integral; (ii) the randomized spectral method (RSM) [102, 142, 191]) which is based on a randomized approximation of the same Fourier Stieltjes integral; (iii) the Fourier wavelet method (FWM) [49,50,104,118] is a different approximation of the Fourier Stieltjes integral based on reexpansion in a special family of orthogonal functions, and is obtained by an expansion of the Gaussian white noise in a wavelet basis.

Another class of methods includes methods which deal with the expansions in the physical space, in the relevant system of orthonormal functions: (i) methods based on expansions in the wavelet basis (WM) [231,264]; (ii) the Karhunen–Loève expansion method (K–L) [230,249] based on the expansions in eigenfunctions of the correlation operator – note that this also works for inhomogeneous random fields; (iii) the moving averages mMethod (MAM) [140], based on the representation of the random field in the form of a convolution of a deterministic function (more precisely, a Fourier transform of a square root of the spectral function) with the Gaussian white noise in the physical space. We mention the the fast Fourier transform spectral method (FFTSM) (e. g., see [39, 168]) which is a particular case of the discrete spectral method whose nodes are chosen as a diadic mesh to apply further the fast Fourier method. The matrix factorization method (MFM) [40, 224] and the circulant embedding method (CEM) [41] are based on the Holessky decomposition of the covariance matrix.

The methods listed above all have their advantages as well as their disadvantages. For example, DSM, RSM, and MAM are simple and convenient for implementation; they provide the possibility to calculate the values of the random field at some points on demand. But in multidimensional cases, DSM and MAM are less efficient. FFTSM is also simple for implementation, but it calculates the random field only on a diadic mesh and has therefore a disadvantage that the samples are periodic. Furthermore, FWM and WM models are efficient for simulating multiscale processes but they are difficult in implementation.

The K–L model is highly efficient but is not universal, since it is necessary to solve the eigenvalue problem for the correlation operator.

More details about the above mentioned methods can be found in [25, 47, 54, 103, 104, 165] where a comparative analysis of some methods is also given. In particular, in [25, 47], RSM and FWM are compared by analyzing a fractal random field with the spectral function $F(k) = k^{-\alpha}$ $(1 < \alpha < 3)$, where the calculated structure function was compared with the exact result. The main conclusion is that to construct the samples of a multiscale random field with a fixed desired accuracy, the cost of RSM is considerably lower than that of FWM if $\lg(l_{\max}/l_{\min}) \leq 4$ where $l_{\min}$ and $l_{\max}$ are the minimal and maximal spatial scales of the random field, respectively. In [104] we have shown that a logarithmically uniform subdivision of the spectral space (we have

introduced such a subdivision in [119]) when calculating two- and a few-point statistical characteristics of the fractal random field, the RSM is more efficient than FWM for all values of $l_{\max}/l_{\min}$. In particular, when calculating the structure function of a multiscale random field with $\alpha = -5/3$, $l_{\max}/l_{\min} = 10^{12}$ it was found that the cost of FWM was 12 times larger than that of RSM; results were obtained for 9 decades, with a fixed accuracy.

Up to now, we discussed the calculation of statistical characteristics by ensemble averaging over the samples constructed by the relevant method. In many practical problems (e. g., in underground hydrology) only data obtained through spatial averaging is at hand, for instance, statistical characteristics obtained by spatial averages, or over a family of Lagrangian trajectories generated in one fixed sample of the field (e. g., see [35, 65]). If the random field is ergodic, then the ensemble averages can be well approximated by the appropriate space averages. This is very important when a boundary value problem with random parameters is solved: then, in contrast to the ensemble averaging, we have to solve the problem only once and then make the relevant space averaging. In practical calculations, to increase the efficiency, it is sometimes reasonable to combine both the space and ensemble averaging, e. g., see [99, 120]. The same technique is used also in simulation of turbulent transport [63, 240].

And so we stress that good ergodic properties of the constructed random field model are very important and highly desired in practical problems. In [104] we studied the ergodic properties of RSM and FWM. Calculations of structure functions through ensemble and space averaging have shown that the ergodic properties of FWM are much better than those of RSM. Therefore, to obtain a good approximation through space averaging in RSM, it is necessary to take many thousands of harmonics per each decade. However, this conclusion was made only for random processes (i. e., random fields depending on one scalar variable).

Thus, as discussed above, random fields provide a useful mathematical framework for representing disordered heterogeneous media in theoretical and computational studies. The random fields may appear in a very simple form in the problem, and then the focus is only on a detailed and accurate simulation of the samples. For instance, in many optic problems there is a need to simulate a random surface with a given correlation function, while the reflected light intensity is then easily calculated as in integral over a certain angle region.

In more sophisticated models, random fields enter differential or integral equations in the form of a coefficient, a kernel, a right-hand side, a boundary condition, or even the boundary itself.

1.2 Some examples

Let us give some examples. First we mention turbulent transport, where the velocity field representing the turbulent flow is modeled as a random field $\vec{v}(\mathbf{x}, t)$ with statistics

encoding important empirical features, and the temporal dynamics of the position $\vec{X}(t)$ and velocity $\vec{V}(t) = \frac{\mathrm{d}\vec{X}}{\mathrm{d}t}$ of immersed particles is then governed by equations involving this random field such as

$$m\ \mathrm{d}\vec{V}(t) = -\gamma\big(\vec{V}(t) - \vec{v}(\vec{X}(t), t)\big)\,\mathrm{d}\,t + \sqrt{2k_{\mathrm{B}}T\gamma}\,\mathrm{d}\,\mathbf{W}(t), \tag{1.2}$$

where m is particle mass, γ is its friction coefficient, k_{B} is Boltzmann's constant, T is the absolute temperature, and $\mathbf{W}(t)$ is a random Wiener process representing molecular collisions.

In more complicated stochastic models of turbulent flows, both the drift and the dispersion terms are constructed as some functions using data extracted from physical and statistical laws.

Let us consider a passive scalar dispersed by the turbulent velocity field. The passive scalar is assumed to follow the streamlines of the flow. We assume that the source of particles is quite arbitrary; for instance, it might be situated on the surface or in the space, or even at given points. Let us denote by $q(\mathbf{x}, t)$ the spatial-temporal density distribution function of the source, i. e,, the number of emitted particles per unit volume in a unit time interval at the phase point $(\mathbf{x}, t)$. Initially, the spatial density of particles is given by $q_0(\mathbf{x})$. The particles are transported by a 3D turbulent velocity field $\vec{u}(\mathbf{x}, t) = (u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), u_3(\mathbf{x}, t))$. Let us denote by $\mathbf{X}(t; \mathbf{x}_0, t_0)$ and $\mathbf{V}(t; \mathbf{x}_0, t_0) = \vec{u}(\mathbf{X}(t; \mathbf{x}_0, t_0), t)$ the Lagrangian spatial coordinates and the velocity, respectively. Then, neglecting molecular diffusion, the instantaneous concentration $c(\mathbf{x}, t)$ is governed by

$$\frac{\partial c(\mathbf{x}, t)}{\partial t} + \sum_{i=1}^{3} u_i(\mathbf{x}, t)\frac{\partial c}{\partial x_i} = q(\mathbf{x}, t), \quad t > 0, \quad c(\mathbf{x}, 0) = q_0(\mathbf{x}). \tag{1.3}$$

The turbulent velocity field $\vec{u}(\mathbf{x}, t)$ is considered to be an incompressible 3D random field. Accordingly, the concentration $c(\mathbf{x}, t)$, satisfying equation (1.3) with random coefficients u_i, is a scalar random field, and we are interested in calculating the mean concentration $\langle c(\mathbf{x}, t)\rangle$ and the mean fluxes $\langle u_i(\mathbf{x}, t)c(\mathbf{x}, t)\rangle$, $i = 1, 2, 3$. These functions can be evaluated by tracking Lagrangian trajectories, which are obtained in turn by solving the system of stochastic differential equations

$$\begin{aligned} d\mathbf{X}(t) &= \mathbf{V}(t)dt, \\ d\mathbf{V}(t) &= \mathbf{a}(t, \mathbf{X}(t), \mathbf{V}(t))dt + \sqrt{C_0\bar{\varepsilon}(\mathbf{X}(t), t)}\,d\mathbf{W}(t), \end{aligned}$$

where the function $\mathbf{a}$ is to be defined in each specific situation, C_0 is the universal Kolmogorov constant ($C_0 \approx 4 \div 6$), $\bar{\varepsilon}(\mathbf{x}, t)$ is the mean dissipation rate of the kinetic energy of turbulence, and $\mathbf{W}(t)$ is a standard 3D Wiener process.

A second example concerns transport through porous media, such as groundwater aquifers, in which the hydraulic conductivity $\mathsf{K}(\mathbf{x})$ is modeled as a random field reflecting the empirical variability of the porous medium. The Darcy flow rate $\vec{q}(\mathbf{x})$ in

response to pressure applied at the boundary is governed by the Darcy equation

$$\begin{aligned} \vec{q}(\mathbf{x}) &= -\mathsf{K}(\mathbf{x}) \operatorname{grad} \phi(\mathbf{x}), \\ \operatorname{div} \vec{q} &= 0, \end{aligned} \tag{1.4}$$

in which the random hydraulic conductivity function appears as a coefficient, and the applied pressure is represented in the boundary conditions for the internal pressure head ϕ. Our concern is with the computational simulation of random fields for applications such as these. Interesting insights into the dynamics of transport in disordered media can be achieved already through relatively simple random models for the velocity field, such a finite superposition of Fourier modes, with each amplitude independently evolving according to an Ornstein–Uhlenbeck process. Here efficient and accurate numerical simulations of the flow can be achieved through application of the well-developed literature on simulating stochastic ordinary differential equations.

In conventional deterministic numerical methods, boundary value problems for random PDEs are solved as follows. First, one constructs a synthesized sample of the input random parameter. Then the obtained deterministic equation is solved numerically, say, by the finite element method, and gives the solution in all points of the grid domain. These two steps are repeated many times, so that the obtained statistics are sufficient for calculation of the desired sufficiently accurate averages. This approach is used in stochastic finite element methods (e. g., see [2, 150, 230, 249]). Obviously, this technique is generally time consuming, and to solve problems of practical interest one needs supercomputers to extract sufficient statistical information.

In the Monte Carlo approach, the algorithms are designed so that the solution is calculated only in the desired set of points without constructing the solution in the whole domain (e. g., see [191, 203, 204]). To evaluate different statistical characteristics of random boundary value problems we use the double randomization technique (e. g., see [191]).

This approach is possible if the desired statistical characteristics (e. g., the mean or the correlation tensor) can be represented in the form of a double expectation over the input random parameters, and over the trajectories of a Markov process used in a stochastic estimator for solving the deterministic equation. The advantage of this method is that there is no need to solve the equation many times, hence, the cost of this method is drastically decreased compared to the stochastic finite element method. The well-known drawback of stochastic simulation methods should be mentioned: the error behaves like $\varepsilon \sim O(N^{-1/2})$, where N is the number of samples; hence, it is reasonable to apply the Monte Carlo methods if the desired accuracy is not too high. So, for realistic applied problems the typical Monte Carlo accuracy lies in the range of 0.1 % to several percent.

The basic idea behind double randomization can be explained by the following very simple example. Assume we have to evaluate an integral

$$J(x;\omega) = \int_D f(x, y; \omega) dy, \tag{1.5}$$

where $f(x, y; \omega)$ is a random function indexed through x, y, defined on a probability space, ω being the relevant random element. Obviously of interest are statistical characteristics of the random process $J(x; \omega)$, like the expectation $\langle J(x; \omega) \rangle$ and the covariance $\langle J(x_1; \omega) J(x_2; \omega) \rangle$.

In deterministic methods, to calculate the expectation $\langle J(x; \omega) \rangle$, one has first to construct a sample of the random function f, say, $f(x, y; \omega_1)$, and then calculate the integral $J(x; \omega_1)$ by one of the quadrature formulas. This is then repeated N times, N large enough to guarantee that the average over N samples provides a good approximation to $\langle J(x; \omega) \rangle$. Thus, in short, one must solve a deterministic problem (in this case, evaluation of the integral) N times, N being the number of samples.

Double randomization is based on the representation of the desired functional as a double expectation. Indeed, we choose an arbitrary probability density function $p(y)$, $y \in D$, arbitrary enough but so that $p(y) \neq 0$ for y, where $f(x, y; \omega) \neq 0$ for all x and ω. Then we can write

$$J(x; \omega) = E_p\big[f(x, \xi; \omega)/p(\xi) \big] \tag{1.6}$$

where E_p stands for the average over the random points ξ distributed in D according to the density p. Therefore,

$$\langle J(x; \omega) \rangle = \big\langle E_p\big[f(x, \xi; \omega)/p(\xi) \big] \big\rangle = E_{(\omega, p)} \big[f(x, \xi; \omega)/p(\xi) \big], \tag{1.7}$$

where $E_{(\omega, \xi)}$ stands for averaging over random elements (ω, ξ). This statement is exactly the Fubini theorem, and it shows that the desired result can be obtained by averaging over random samples of ω and ξ.

This approach works also when the deterministic problem is not simply an integral evaluation, but a PDE, or an integral equation, or nonlinear system of equations, e. g., like the Smoluchowski equations. The main challenging problem here is to transform the solution of the original problem to the evaluation of an expectation over relevant stochastic elements, including averaging in functional spaces. In our case the expectations are often constructed over Markov chains and Gaussian random fields. Note that in applied problems, we deal also with generalized random fields, but in Monte Carlo simulations they do not complicate the situation, but, quite the contrary, the stochastic simulation algorithms can efficiently use this feature.

Let us illustrate this by the following simple but important example when the input randomness enters the problem as a generalized 2D random field $u(x, \boldsymbol{\omega})$, $x \in [0, L]$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_m)$, in the form:

$$u(x, \boldsymbol{\omega}) = \sum_{j=1}^{m} \delta(x - \omega_j), \tag{1.8}$$

where δ is the Dyrac delta function, and the random points $\omega_1, \omega_2, \ldots, \omega_m$ are distributed on $[0, L]$ with a density $p(\boldsymbol{\omega})$ independent of the spatial coordinate x. The

random points $\omega_1, \omega_2, \ldots, \omega_m$ may have a quite different distribution on $[0, L]$, i. e., they may be all independent of each other, they may form a Markov chain with a certain transition probability density, or they may be placed almost periodically, with small but correlated random shifts ω_j from a mean fixed step $\langle \delta\omega_j \rangle$: $\omega_{j+1} = \omega_j + \delta\omega_j$ where $\langle \delta\omega_j \rangle = \rho^{-1} = m/L$ is the mean density of the points on $[0, L]$. Thus the random field $u(x, \boldsymbol{\omega})$ is stationary with the mean $\langle u(x, y; \boldsymbol{\omega}) \rangle = \rho$, as can be found by direct calculation.

This kind of random process enters the boundary elastic displacements produced by the nets of misfit dislocations in crystals [87], in many problems related to renewal processes, in the analysis of statistic of neuronal spike trains (for instance, see [52, 68, 180]), etc.

In the analysis of dislocations in crystals, the goal is to evaluate the x-ray diffraction peak profiles from distributions of misfit and threading dislocations [87]. The x-ray scattering amplitude from a film of thickness d is given by an integral of the form

$$A(q_x, q_z) = \int_{-\infty}^{\infty} dx \int_0^d dz \ \exp\left\{\mathrm{i}[q_x x + q_z z + V(x, z)]\right\}, \tag{1.9}$$

where

$$V(x, z) = \int K(x - x', z) u(x', \boldsymbol{\omega}) dx' \tag{1.10}$$

is the resulting displacement field due to all misfit dislocations, the kernel $K(x - x', z)$ is the Green function given explicitly, and the random field $u(x', \omega)$ is defined by (1.8). The scattered intensity $I(q_x, q_z) = \left\langle |A(q_x, q_z)|^2 \right\rangle$ can be directly calculated by the Monte Carlo double randomization method, as explained above, by a randomized evaluation of the integral representation:

$$I(q_x, q_z) = \int_{-\infty}^{\infty} dx \int_0^d \int_0^d dz_1 \, dz_2 \, e^{\mathrm{i}[q_x x + q_z (z_1 - z_2)]} \left\langle e^{\mathrm{i}[V(x_1, z_1) - V(x_2, z_2)]} \right\rangle. \tag{1.11}$$

Note that in the case where the random field u is Gaussian, it is possible to evaluate the expectation explicitly (see [87]).

Thus the boundary value problems with random coefficients, parameters, random source terms, stochastically distributed boundary functions, or even with randomly moving boundaries are used as a powerful instrument in modern science and technology. We mention here applied fields such as structural mechanics, composite materials [2], porous media and soils [34, 99, 196, 260], biological tissues [258], geodesy [182, 212], turbulence, [13, 103, 104, 146, 191], etc. In engineering-related stochastic boundary value problems, the common computational techniques include Monte Carlo methods, stochastic finite elements, finite difference, and spectral methods. Among these methods, the finite volume and boundary element techniques are the methods most adaptable to problems in solid and structural mechanics characterized with highly irregular and complex structures [2, 230, 249]. We mention also classical potential

problems dealing with random boundary conditions and sources [31] where the Monte Carlo methods are very efficient (e. g., see [191, 206–208]).

The book is organized as follows. Chapter 2 presents different simulation methods for Gaussian random fields. Chapter 3 deals with the stochastic Lagrangian models of relative dispersion of a pair of fluid particles. In Chapter 4 a new version of the 2-particle relative turbulent dispersion model is developed. A combined Eulerian–Lagrangian model is presented in Chapter 5. In Chapter 6 we describe a stochastic Lagrangian model extended to the intermittent turbulence. Chapter 7 presents results of numerical experiments. A 1-particle stochastic Lagrangian model for a horizonatlly homogeneous turbulent flow is described in Chapter 8. Formulation of the footprint problem and the methods based on backward Lagrangian trajectories are presented in Chapter 9. Applications of the stochastic Lagrangian models to evaluate the particle transport in the boundary layer of the atmosphere are given in Chapter 10. Comparisons of different 2-particle models are described in Chapter 11. Algorithms for concentration and fluxes in turbulent flows by stochastic Lagrangian models are presented in Chapter 12. Application to the footprint problem for the case of an abrupt change of roughness is given in Chapter 13. A Lagrangian stochastic model for the transport in porous medium is presented in Chapters 14 and 15. Chapter 16 deals with the analysis of the coagulation of aerosol particles in intermittent turbulent flows. Finally, in Chapter 17 we give an example of a Stokes flow which is governed by Stokes equations with random boundary conditions.

1.3 Fundamental concepts

We introduce two main classes of stochastic simulation models presented in this book: an *Eulerian* class which is based on a random velocity field model defined on a fixed coordinate system. The velocity field is generated over a prescribed spatial domain, but by a direct stochastic construction rather than the much more expensive simulation of the nonlinear Navier–Stokes PDE's.

In the *Lagrangian* class the motion of fluid particles is stochastically modeled as a random trajectory $X(t)$ of any immersed particle representing, for example, a tracer, pollutant, or chemical reactant. It is computed using the local value of the stochastically constructed velocity. The random trajectory thus can be defined as a random process determined from the random ordinary differential equation $\frac{dX(t)}{dt} = u(t; X(t)$ where $u(\tau, r)$ is a random field, or, alternatively, it can be determined from a stochastic Ito-type differential equation $dX(t) = A\,dt + B\,dW(t)$.

These two classes of models use different mathematical apparatus: the Eulerian stochastic models focus on the simulation of spatial-temporal random fields with a desired statistical characteristics, while the stochastic Lagrangian models are based on the so-called Ito-type stochastic differential equations, known in physics as Langevin-type equations.

As mentioned in [103], the primary challenge in most Eulerian fluid Monte Carlo simulations is the generation of a synthetic random velocity $u(x;t)$ which has certain statistics and features mimicking those of a turbulent fluid in the regime of interest. Thus a fully developed turbulent flow at a sufficiently high Reynolds number should possess a wide inertial range of scales over which the statistics of the velocity field assume a self-similar fractal structure. A quantitative way to express this criterion is

$$\langle |u(x+r;t) - u(x;t)|^2 \rangle = S_v^I |r|^{2H} \quad \text{for } L_K \ll |r| \ll L_0,$$

where $\langle \cdot \rangle$ denotes a statistical average, $0 < H < 1$ is the Hurst exponent which takes the Kolmogorov value $H = 1/3$ for fully developed turbulence, L_K is the dissipation length scale and L_0 is the integral length scale, which define the extent of the inertial scaling range, and S_v^I is a (dimensional) scaling prefactor. Other desired properties in turbulence simulations are the incompressibility of the fluid and appropriate geometric symmetries such as isotropy.

1.3.1 Random functions in a broad sense

Let us start with an informal description of random functions. A random function, as well as a deterministic function, is defined by some dependence $\xi(\theta)$ which describes a mapping from the space of parameters $\theta \in \Theta$ to the space of values of $\xi(\theta)$, thus it can be real- or complex-valued, or generally vector-valued. But in the stochastic case, $\xi(\theta)$ is a random value for each fixed θ. Thus, a random function is a family of random variables $\xi(\theta) = \xi(\theta, \omega)$ defined on a probability space $(\Omega, \mathcal{F}, P)$ depending on a parameter $\theta \in \Theta$. The deterministic function $\xi(\theta, \omega_0)$ for a fixed value $\omega = \omega_0$ is called a sample function (or a sample trajectory). Here we recall that a finite set of random variables $\xi_1, \xi_2, \ldots \xi_n$ is fully defined by the mutual distribution function

$$F(x_1, x_2, \ldots, x_n) = P\{\xi_1 < x_1, \xi_2 < x_2, \ldots, \xi_n < x_n\}. \tag{1.12}$$

In the case of random functions, we have to characterize an infinite family of random variables. So we can say that the infinite family of random variables $\xi(\theta)$ is defined if statistical characteristics of any finite sets of random variables

$$\xi(\theta_1), \xi(\theta_2), \ldots \xi(\theta_n), \quad \theta_i \in \Theta, \quad i = 1, 2 \ldots, n; \quad n = 1, 2 \ldots \ , \tag{1.13}$$

i. e., any finite-dimensional distributions are defined, which means, the random function $\xi(\theta)$ is defined by its finite-dimensional distributions

$$F_{\theta_1, \theta_2, \ldots, \theta_n}(x_1, x_2, \ldots, x_n), \quad \theta_i \in \Theta; \quad i = 1, 2 \ldots, n; \quad n = 1, 2 \ldots \ , \tag{1.14}$$

and any function $F_{\theta_1, \theta_2, \ldots, \theta_n}(x_1, x_2, \ldots, x_n)$ is interpreted as a mutual distribution function of the set of random variables (1.13).

To make this interpretation correct, the family of distributions should satisfy some assumptions. These assumptions are quite natural:

$$F_{\theta_1,\theta_2,\ldots,\theta_n,\theta_{n+1},\ldots,\theta_{n+m}}(x_1, x_2, \ldots, x_n, +\infty, \ldots, +\infty) = F_{\theta_1,\theta_2,\ldots,\theta_n}(x_1, x_2, \ldots, x_n), \quad (1.15)$$

$$F_{\theta_1,\theta_2,\ldots,\theta_n}(x_1, x_2, \ldots, x_n) = F_{\theta_1,\theta_2,\ldots,\theta_n}(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), \quad (1.16)$$

where $i_1, i_2, \ldots, i_n$ is any permutation of the indices $1, 2 \ldots, n$.

Thus we are in a position to give the following definition.

Random function on a set of parameters $\theta \in \Theta$ having real values $\xi(\theta)$ is defined as a family of distributions (1.14) which satisfies the conditions (1.15) and 1.16). The functions $F_{\theta_1,\theta_2,\ldots,\theta_n}(x_1, x_2, \ldots, x_n)$ are called finite-dimensional distributions of the random function $\xi(\theta)$.

This definition is clear and simple, and is sufficient when we are interested in statistical characteristics for a finite set of the parameter's values. But it is not satisfactory when we are trying to characterize the function in its entirety, for all infinite values of the parameter. For instance, this definition cannot provide us with a graph of the random function. With this definition, we cannot even answer such an important question as whether the sample of the random function is continuous or differentiable. A different definition considers the random function as an element of an appropriate functional space and will be given later. In this section we deal with the definition of the random function in athewise sense given above by the family of distribution functions.

Generalization to vector random functions is obvious: a vector random function $\xi(\theta)$ is defined as a vector with n components, scalar random functions: $\xi(\theta) = (\xi_1(\theta), \xi_2(\theta), \ldots \xi_n(\theta))$. The distribution function of this vector random function is the function of nm variables

$$F_{\theta_1,\theta_2,\ldots,\theta_n}(x_{11}, x_{12}, \ldots, x_{nm}) = P\{\xi_1(\theta_1) < x_{11}, \xi_1(\theta_2) < x_{12}, \ldots, \xi_n(\theta_m) < x_{mn}\}.$$

The distribution functions can be represented through the probability density $f_{\theta_1,\ldots,\theta_n}(x_1, \ldots, x_n)$:

$$F_{\theta_1,\ldots,\theta_n}(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{\theta_1,\ldots,\theta_n}(y_1, \ldots, y_n)\, dy_1 \ldots y_n.$$

From this the following well-known property follows

$$f_{\theta_1,\ldots,\theta_n}(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{\theta_1,\ldots,\theta_n,\theta_{n+1},\ldots\theta_{n+m}}(x_1, \ldots x_n, y_1, \ldots, y_m)\, dy_1 \ldots y_m.$$

A characteristic function of a finite-dimensional distribution is defined by

$$\varphi_{\theta_1,\dots,\theta_n} = \mathbf{E}\exp\left\{\mathrm{i}\sum_{k=1}^{n}\xi(\theta_k)u_k\right\},$$

where $\mathbf{E}$ stands for the mathematical expectation, and $u_1,\dots u_n$ are real numbers. If the density $f_{\theta_1,\dots,\theta_n}$ exists, then

$$\varphi_{\theta_1,\dots,\theta_n} = \int_{\boldsymbol{R}^n} \mathrm{e}^{\sum\limits_{i=1}^{n} x_i u_i} f_{\theta_1,\dots,\theta_n}(x_1,\dots,x_n)\,dx_1\dots dx_n,$$

i. e., the characteristic function is a Fourier transform of the probability density.

Moment functions of $\xi(\theta)$ are defined by

$$m_{j_1,\dots,j_s}(\theta_1,\dots,\theta_s) = \mathbf{E}[\xi(\theta_1)]^{j_1}\dots[\xi(\theta_s)]^{j_s}, \quad j_k \geq 0, \quad (k = 1,2,\dots,s)$$

if the expectation on the right-hand side exists for all $\theta_i \in \Theta,\ i = 1,2,\dots,s$.

A random function $\xi(\theta)$ belongs to the class $L_p(\Theta)$, $(\xi \in L_p(\Theta))$ if $\mathbf{E}|\xi(\theta)|^p < \infty$ for all $\theta \in \Theta$. So if $(\xi \in L_p(\Theta)$, then all moments of the order $q \leq p$ are finite.

If the characteristic functions of the finite-dimensional distributions are given, then the moments of any integer order can be obtained by taking derivatives. Indeed, if $(\xi \in L_p(\Theta)$, then

$$m_{j_1,\dots,j_s}(\theta_1,\dots,\theta_s) = (-1)^q\,\frac{\partial^q \varphi(u_1,\dots,u_s)}{\partial u_1^{j_1}\dots u_s^{j_s}}$$

for $q \leq p$ where $q = j_1 + \dots + j_s$.

Centered moments are defined by

$$\bar{m}_{j_1,\dots,j_s}(\theta_1,\dots,\theta_s) = \mathbf{E}\big[\xi(\theta_1) - m_1(\theta_1)\big]^{j_1}\dots\big[\xi(\theta_s) - m_1(\theta_s)\big]^{j_s}.$$

Here $m_i(\theta)$ is the expectation of $\xi_i(\theta)$: $m_i(\theta) = \mathbf{E}\xi_i(\theta)$.

The *correlation function* $R(\theta_1,\theta_2)$ is defined by

$$R(\theta_1,\theta_2) = \mathbf{E}\big[\xi(\theta_1) - m(\theta_1)\big]\big[\xi(\theta_2) - m(\theta_2)\big].$$

The variance is the quantity $\sigma^2(\theta) = R(\theta,\theta)$, and the correlation coefficient is just the normalized correlation function:

$$r(\theta_1,\theta_2) = \frac{R(\theta_1,\theta_2)}{\sigma(\theta_1)\,\sigma(\theta_2)}.$$

If the random variables $\xi(\theta_1)$ and $\xi(\theta_2)$ are independent, then the correlation coefficient is zero. Note that the reverse is not true: zero correlation does not imply that the variables are independent. However, if the mutual 2D distribution of $\xi(\theta_1)$ and $\xi(\theta_2)$ is Gaussian, then the zero correlation implies that $\xi(\theta_1)$ and $\xi(\theta_2)$ are independent.

Generalization to complex-valued random functions is straightforward. A complex-valued random function is defined by $\zeta(\theta) = \xi(\theta) + \mathrm{i}\eta(\theta)$ and can be considered as a 2-dimensional vector of real-valued random functions. For a complex-valued function $\zeta(\theta) \in L_p(\Theta)$ means that

$$\mathbf{E}|\zeta(\theta)|^p < \infty, \quad \theta \in \Theta, i.e., \quad \xi \in L_p(\Theta) \quad \text{and} \quad \eta \in L_p(\Theta).$$

The correlation function for a complex-valued random function is defined by

$$R(\theta_1, \theta_2) = \mathbf{E}\big[\xi(\theta_1) - m(\theta_1)\big]\overline{\big[\xi(\theta_2) - m(\theta_2)\big]},$$

where $\overline{\xi}$ stands for the complex conjugate of ξ.

The following properties of the correlation functions can be easily checked:

1. $R(\theta, \theta) \geq 0$, where the equality appears if and only if the random function is constant with probability one.
2. $R(\theta_1, \theta_2) = \overline{R(\theta_2, \theta_1)}$.
3. $|R(\theta_1, \theta_2)|^2 \leq R(\theta_1, \theta_1)R(\theta_2, \theta_2)$.
4. For any integer n, $\theta_1, \ldots \theta_n$ and complex numbers $\lambda_1, \ldots, \lambda_n$,

$$\sum_{j,k=1}^{n} R(\theta_j, \theta_k)\, \lambda_j \bar{\lambda}_k \geq 0.$$

Note that properties 1–3 follow from property 4.

For two random functions $\zeta_1(\theta)$ and $\zeta_2(\theta)$ (belonging to $L_p(\Theta)$) one defines the cross-correlation function

$$R_{\zeta_1\zeta_2}(\theta_1, \theta_2) = \mathbf{E}\big[\zeta_1(\theta_1) - \mathbf{E}\zeta_1(\theta_1)\big]\ \overline{\big[\zeta_2(\theta_2) - \mathbf{E}(\theta_2)\big]}$$

Extension to multidimensional complex-values random functions. Let $\zeta_1(\theta), \zeta_2(\theta), \ldots, \zeta_r(\theta)$ be a set of random complex-valued functions. It is considered as a complex-valued random function $\zeta(\theta) = (\zeta_1(\theta), \zeta_2(\theta), \ldots, \zeta_r(\theta))^T$, $\theta \in \Theta$. Here $(\cdot)^T$ stands for the transpose, so $\zeta(\theta) = (\zeta_1(\theta), \zeta_2(\theta), \ldots, \zeta_r(\theta))^T$, $\theta \in \Theta$ is a column.

For two columns, $\xi = (\xi_1, \xi_2, \ldots, \xi_r)^T$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_r)^T$, we define a matrix $\xi\eta^*$ by

$$\xi\eta^* = \begin{pmatrix} \xi_1\bar{\eta}_1 & \xi_1\bar{\eta}_2 & \ldots & \xi_1\bar{\eta}_m \\ \xi_2\bar{\eta}_1 & \xi_2\bar{\eta}_2 & \ldots & \xi_2\bar{\eta}_m \\ \ldots & \ldots & \ldots & \ldots \\ \xi_r\bar{\eta}_1 & \xi_r\bar{\eta}_2 & \ldots & \xi_r\bar{\eta}_m \end{pmatrix}.$$

For a random vector function $\zeta(\theta) = (\zeta_1(\theta), \dots \zeta_1(\theta_r))^T$ we put

$$\mathbf{m}(\theta) = (m_1(\theta), \dots, m_r(\theta))^T = \mathbf{E}\zeta(\theta) = (\mathbf{E}\zeta_1(\theta), \dots, \mathbf{E}\zeta_r(\theta))^T,$$
$$R(\theta_1, \theta_2) = \Big(R_{ij}(\theta_1, \theta_2)\Big)_{i,j=1,\dots r} = \mathbf{E}\Big([\zeta(\theta_1) - \mathbf{m}(\theta_1)]\,[\zeta(\theta_2) - \mathbf{m}(\theta_2)]^*\Big)_{ij}$$
$$= \Big(\mathbf{E}[\zeta_i(\theta_1) - m_i(\theta_2)]\,\overline{[\zeta_j(\theta_2) - m_j(\theta_2)]}\Big)_{ij=1,\dots,r}.$$

The vector $\mathbf{m}(\theta)$ is an r-dimensional complex-valued vector function; it is the expectation, called also a mean of the random function $\zeta(\theta)$. The matrix $R(\theta_1, \theta_2)$ is called a correlation matrix.

1.3.2 Gaussian random vectors

There is an important class of random functions completely defined by their first two moments, the expectation and correlation function. They are called Gaussian random functions and by definition have a Gaussian form of the finite-dimensional distributions. So we recall here first the case of random vectors.

The *Gaussian* random vector $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T$ is defined by its characteristic function

$$\varphi(u) = \mathbf{E}\,\mathrm{e}^{\mathrm{i}(\mathbf{u},\xi)} = \exp\left\{\mathrm{i}\,(\mathbf{m}, \mathbf{u}) - \frac{1}{2}(R\mathbf{u}, \mathbf{u})\right\}, \tag{1.17}$$

where $\mathbf{m} = (m_1, \dots, m_n)$, $\mathbf{u} = (u_1, \dots, u_n)$, R is a nonnegative definite real-valued symmetric matrix $R = (r_{ik})$, $\quad i, k = 1, 2, \dots, n$. Here we use the conventional definition of a scalar product so that $(\mathbf{m}, \mathbf{u}) = \sum_{k=1}^{n} m_k u_k$ and $(R\mathbf{u}, \mathbf{u}) = \sum_{j,k=1}^{n} r_{jk}\, u_j\, u_k$.

The following statement explains the role of the above definition.

The function

$$\psi(\mathbf{u}) = \exp\left\{\mathrm{i}(\mathbf{m}, \mathbf{u}) - \frac{1}{2}(R\mathbf{u}, \mathbf{u})\right\}$$

is a characteristic function of a random vector ξ if and only if the real-valued matrix R is non-negative definite and symmetric. The rank of the matrix R equals the dimension of a subspace where the distribution of the vector ξ is concentrated.

If r, the rank of the matrix R is less than n, then the random vector is concentrated in an r-dimensional hyperplane; hence it has no density. Such a distribution is called a singular Gaussian distribution. If $r = n$, the random vector ξ has the density

$$f(\mathbf{x}) = \frac{1}{\sqrt{\Delta(2\pi)^n}}\,\exp\left\{-\frac{1}{2}(R^{-1}(\mathbf{x} - \mathbf{m}), (\mathbf{x} - \mathbf{m}))\right\}, \tag{1.18}$$

where R^{-1} is the inverse of the matrix R, $\Delta = \det(R)$ is the determinant of R.

Let us give a series of statements which are well known from the probability theory, e. g., see [71].

1. In the expression (1.17), $\mathbf{m} = (m_1, \ldots, m_n)^T$ is the vector of expectations, and R is the correlation function: $\mathbf{m} = \mathbf{E}\xi$, $r_{jk} = \mathbf{E}[(\xi_j - m_j)(\xi_k - m_k)]$.
2. If the correlation function of a Gaussian vector ξ is not singular, then there exists an n-dimensional probability density $f(\mathbf{x})$ which is defined by (1.18).
3. The mutual distribution of any group of components of a Gaussian vector is Gaussian.
4. If $\xi = (\xi_1, \ldots, \xi_n)^T$ is a Gaussian vector, and random vectors $\xi' = (\xi_1, \ldots, \xi_r)^T$, $\xi'' = (\xi_{r+1}, \ldots, \xi_n)^T$ are noncorrelated, then ξ' and ξ'' are independent.
5. Gaussian distributions remain Gaussian under linear transformation.

Let us give some other properties of the Gaussian distributions which are useful in practice. Assume we are given two vectors with Gaussian distributions: $\xi = (\xi_1, \ldots, \xi_n)^T$ and $\eta = (\eta_1, \ldots, \eta_m)^T$. We are interested in the conditional distribution of the vector ξ, assuming that η is fixed. Without loss of generality we suppose that the correlation matrix R_{22} of the vector η is nonsingular. Indeed, if R_{22} is singular, it means some components of η are linearly dependent on the other components. Then, we exclude these components, and the dimension of η is decreased. So let $\mathbf{m}_1 = \mathbf{E}\xi$, $\mathbf{m}_2 = \mathbf{E}\eta$, and let R_{11} be the correlation matrix of the vector ξ, and R_{12} be the cross-correlation matrix of the vectors ξ and η: $R_{12} = \mathbf{E}(\xi - \mathbf{m}_1)\eta - \mathbf{m}_2)^*$. Let us introduce a vector $\tilde{\xi} = \mathbf{m}_1 + R_{12}\, R_{22}^{-1}(\eta - \mathbf{m}_2)$. Then the conditional distribution of the vector ξ, under the condition that η is fixed, is Gaussian with the conditional expectation $\mathbf{E}(\xi|\eta) = \tilde{\xi}$ and the conditional correlation matrix

$$\mathbf{E}\left\{\left(\xi - \tilde{\xi}\right)\left(\xi - \tilde{\xi}\right)^* \middle| \eta\right\} = R_{11} - R_{12}R_{22}^{-1}R_{21}.$$

Notice the following important property: the matrix of the conditional correlations of the vector ξ, η fixed, is not random, and in particular, it does not depend on the value of η.

1.3.3 Gaussian random functions

A vector n-dimensional random function $\xi(\theta) = \{\xi_1(\theta), \ldots \xi_n(\theta)\}$ is called a Gaussian random function if the mutual distribution function of all components of the random vectors $\xi(\theta_1), \ldots, \xi(\theta_n)$ is Gaussian. The correlation matrix R of the mutual distribution of the vectors $\xi(\theta_1), \ldots, \xi(\theta_n)$ has a dimension $sn \times sn$ and can be di-

vided in square blocks of size $s \times s$ as follows:

$$R = \begin{pmatrix} R(\theta_1, \theta_1) & R(\theta_1, \theta_2) & \dots & R(\theta_1, \theta_n) \\ R(\theta_2, \theta_1) & R(\theta_2, \theta_2) & \dots & R(\theta_2, \theta_n) \\ \dots & \dots & \dots & \dots \\ R(\theta_n, \theta_1) & R(\theta_n, \theta_2) & \dots & R(\theta_n, \theta_n) \end{pmatrix},$$

where $R(\theta_1, \theta_2)$ is the correlation matrix of the function $\xi(\theta)$.

The reverse statement is true: for any real-valued vector function $m(\theta)$ and a non-negative definite symmetric matrix function $R(\theta_1, \theta_2)$ there exists an r-dimensional Gaussian random function for which $m(\theta)$ is the expectation, and $R(\theta_1, \theta_2)$ is the correlation matrix.

The Gaussian random functions play an extremely important role in many practical problems. This can be explained generally as follows. The real processes are usually affected by many random independent factors and the resulting superposition of these factors tends to a Gaussian distribution. This can be rigorously formulated as a limit theorem of normal correlations, which is a generalization of the well-known central limit theorem. Let us present this statement.

A sequence of random functions $\xi_n(\theta)$, $\theta \in \Theta$, $n = 1, \dots$ is said to be weakly convergent to a random function $\xi(\theta)$, $\theta \in \Theta$ if for any s the mutual distribution of the series of random variables $\{\theta_n(\theta_1), \dots, \xi(\theta_s)\}$ is weakly convergent, as $n \to \infty$, to the distribution of $\{\xi(\theta_1), \dots, \xi(\theta_s)\}$.

Theorem 1.1 (see [71]). *Assume we are given a family of sums of random functions*

$$\eta_n(\theta) = \sum_{k=1}^{m_n} \alpha_{nk}(\theta), \quad \theta \in \Theta, \quad n = 1, 2 \dots$$

and the following conditions are satisfied:
1. *For fixed n, the random variables $\alpha_{n1}(\theta_1), \alpha_{n2}(\theta_2), \dots, \alpha_{nm}(\theta_m)$ are all mutually independent for each $\theta_1, \theta_2, \dots \theta_m$, and have finite second moments such that*

$$\mathbf{E}\alpha_{nk}(\theta) = 0, \quad \mathbf{E}\alpha_{nk}^2(\theta) = b_{nk}^2(\theta).$$

2. *The correlation function $R_n(\theta_1, \theta_2) = \mathbf{E}[\eta_n(\theta_1)\eta_n(\theta_2)]$, converges to a limit:*

$$\lim_{n \to \infty} R_n(\theta_1, \theta_2) = R(\theta_1, \theta_2).$$

3. *The sums $\eta_n(\theta) = \sum_{k=1}^{m_n} \alpha_{nk}(\theta)$ satisfy, for each θ, the Lindeberg condition: for each $\tau > 0$*

$$\frac{1}{B_n^2} \sum_{k=1}^{m_n} \int_{|x| > \tau B_n} x^2 \, d\Pi_{nk}(\theta, x) \to 0,$$

where $\Pi_{nk}(\theta, x)$ *is the distribution function of the random variable* $\alpha_{nk}(\theta)$*, and*

$$B_n^2 = \sum_{k=1}^{m_n} b_{nk}^2(\theta) = R_n(\theta, \theta).$$

Then the random function $\eta_n(\theta)$ *is weakly convergent, as* $n \to \infty$*, to a Gaussian random function with zero expectation and correlation function* $R(\theta_1, \theta_2)$*.*

1.3.4 Random fields

Assume that our parameter θ is a point $\mathbf{x} \in I\!R^n$. Let $\zeta(\mathbf{x}) = (\zeta^1(\mathbf{x}), \ldots, \zeta^d(\mathbf{x}))^T$ be a vector random function possibly with complex values, defined for all $\mathbf{x} \in I\!R^n$. It is called a sl random field. Note that if $n = 1$, one uses the notion of *random process*. The random field $\zeta(\mathbf{x})$ is called homogeneous in a broad sense if

$$\mathbf{E}\zeta(\mathbf{x}) = \mathbf{m} = \text{ const}, \quad \mathbf{E}(\zeta(\mathbf{x}) - \mathbf{m})(\zeta(\mathbf{x}) - \mathbf{m})^* = R(\mathbf{x} - \mathbf{y}),$$

where $R(\mathbf{x})$ is a continuous matrix function, the correlation of the homogeneous random field. The matrix function $R(\mathbf{x})$ is nonnegative definite. This means that for any d-dimensional complex vectors z_k, points $\mathbf{x}_k \in I\!R^m$, $k = 1, \ldots, n$, and for any integer n

$$\sum_{k,j=1}^{n} \mathbf{z}_k^* R(\mathbf{x}_k - \mathbf{x}_j)\mathbf{z}_j \geq 0.$$

A random field $\zeta(\mathbf{x})$, $\mathbf{x} \in I\!R^m$ is called meansquare continuous (m.s.c.) if from $\mathbf{x}^{(n)} \to \mathbf{x}$ it follows that $\mathbf{E}|\xi(\mathbf{x}^{(n)}) - \xi(\mathbf{x})|^2 \to 0$ as $n \to \infty$.

The correlation functions and nonnegative definite functions are related by the following theorem.

Theorem 1.2 (Bochner–Khinchin theorem). *A matrix function* $R(\boldsymbol{x}), \boldsymbol{x} \in I\!R^d$ *is a correlation function of a homogeneous, m.s.c. random field if and only if it can be represented in the form*

$$R(\boldsymbol{x}) = \int_{\boldsymbol{R}^d} e^{\mathrm{i}(\boldsymbol{x}, \boldsymbol{f}\boldsymbol{u})} F(d\,\mathbf{u}), \tag{1.19}$$

where $F(A)$ *is a matrix-valued complex countable-additive function defined on Borel sets in* $I\!R^m$ *such that* $\mathbf{z}^* F(A)\mathbf{z} \geq 0$ *for any complex vector* $\mathbf{z}$ *and any Borel set* $A \subset I\!R^d$*, and the trace* $Sp\, F(I\!R^m) < \infty$*.*

A random field $\zeta(\mathbf{x})$ is called *isotropic* if it is homogeneous and it correlation function $R(\mathbf{x})$ depends on the length $|\mathbf{x}|$. Thus for an isotropic random field $R(\mathbf{x}) = R(\rho)$

1.3.5 Stochastic measures and integrals

In this section we introduce integrals with respect to stochastic measures, known as stochastic integrals.

Let (Ω, F, P) be a probability space, $\mathbf{E}f = \int dP$ is the expectation, $L_2(\Omega) = L_2(\Omega, F, P)$ is a class of random variables ζ with finite second moment. Let X be a set, and $\mathcal{K}$ be a semi-ring of the subsets of X. Assume that each $\Delta \in \mathcal{K}$ is related to a complex-valued random variable $\zeta(\Delta)$ which satisfies the following conditions:

1. $\zeta(\Delta) \in L_2(\Omega), \quad \zeta(\emptyset) = 0,$
2. $\zeta(\Delta_1 \bigcup)\Delta_2) = \zeta(\Delta_1) + \zeta(\Delta_2)(\mathrm{mod}(P), \text{ if } \Delta_1 \bigcap \Delta_2 = \emptyset,$
3. $\mathbf{E}\zeta(\Delta_1)\overline{\zeta(\Delta_2)} = m(\Delta_1 \bigcap \Delta_2),$

where $m(\Delta)$ is a function defined on the sets of $\mathcal{K}$.

A family of random variables $\{\zeta(\Delta), \Delta \in \mathcal{K}\}$ satisfying the conditions 1–3 is called an elementary orthogonal stochastic measure, and $m(\Delta)$ is its structure measure. The orthogonality property of the stochastic measure is expressed by condition 3: if $\Delta_1 \bigcap \Delta_2) = \emptyset$, then $\mathbf{E}\zeta(\Delta_1)\overline{\zeta(\Delta_2)} = 0$. From the definition of $m(\Delta)$ follows that it is nonnegative:

$$m(\Delta) = \mathbf{E}|\zeta(\Delta)|^2 \geq 0, \quad m(\emptyset) = 0$$

and additive, i. e., if $\Delta_1 \bigcap \Delta_2) = 0$, then

$$\begin{aligned} m\Big(\Delta_1 \bigcup \Delta_2\Big) &= \mathbf{E}|\zeta(\Delta_1) + \zeta(\Delta_2)|^2 \\ &= m(\Delta_1) + m(\Delta_2) + 2m\Big(\Delta_1 \bigcap \Delta_2\Big) \\ &= m(\Delta_1) + m(\Delta_2). \end{aligned}$$

Now, a class $L_0\{\mathcal{K}\|$ of *simple functions* is introduced:

$$f(x) = \sum_{r=1}^{n} c_r \chi_{\Delta_r}(x), \quad \Delta_r \in \mathcal{K}, \quad r = 1, 2, \ldots n, \tag{1.20}$$

where n is arbitrary, and $\chi_A(x)$ is the indicator of the set A, $c_1, \ldots c_r$ are complex numbers.

The *stochastic integral* of a simple function $f(x) \in L_0(\mathcal{K})$ with respect to a stochastic measure $\zeta(\Delta)$ is defined by the formula

$$\eta = \int f(x)\zeta(dx) = \sum_{r=1}^{n} c_r \zeta(\Delta_r). \tag{1.21}$$

For any two functions $f(x), g(x) \in L_0(\mathcal{K})$ the following equality holds:

$$\mathbf{E}\Big(\int f(x)\zeta(dx) \int \overline{g(x)\zeta(dx)} \Big) = \int f(x)\bar{g}(x)\, m(dx). \tag{1.22}$$

Assume that m satisfies the semi-additive condition and hence can be prolonged to a complete measure $\{X, \mathcal{B}, m\}$. Then $L_0\{\mathcal{K}\}$ is a linear subset of the Hilbert space $L_2(m) = L_2\{\{X, \mathcal{B}, m\}$. Denote by $L_2\{\mathcal{K}\}$ the close of $L_0(\mathcal{K})$ in $L_2(m)$. Now we introduce a linear span $L_0\{\zeta\}$ of the family of random variables $\zeta(\Delta), \Delta \in \mathcal{K}\}$ i. e., a set of random variables which can be represented in the form (1.21); the space $L_2(\zeta)$ is defined as a closure of $L_0(\zeta)$ in the Hilbert space of random variables $L_2(\Omega, F, P)$. Notice that the relation (1.21) establishes an isometrical mapping $\eta = \psi(f)$ between $L_0(\mathcal{K})$ and $L_0(\zeta)$. This mapping can be prolonged to an isometry between $L_2\{\mathcal{K}\}$ and $L_2\{\zeta\}$. If $\eta = \psi(f)$, $f \in L_2\{\mathcal{K}\}$, then we define $\eta = \psi(f) = \int f(x)\zeta(dx)$. The random variable η is then called a stochastic integral of f with respect to the measure ζ. From this follows the following.

The following statement is true (see [71].

Theorem 1.3.

(a) *For a simple function* (1.20) *the stochastic integral is defined by the formula* (1.21).

(b) *For any f and g from $L_2\{m\} = L_2\{X, \mathcal{B}, m\}$, the equality* (1.22) *holds.*

(c) $\int \left[\alpha\, f(x) + \beta\, g(x)\right] \zeta(dx) = \alpha \int f(x)\zeta(dx) + \beta \int g(x)\zeta(dx).$

(d) *For an arbitrary sequence of functions $f^{(n)}(x) \in L_2\{X, \mathcal{B}, m\}$ such that*

$$\int |f(x) - f^{(n)}(x)|^2\, m(dx) \to 0,$$

the following relation is true:

$$\int f(x)\, \zeta(dx) = \underset{n\to\infty}{\text{l.i.m.}} \int f^{(n)}(x)\, \zeta(dx).$$

Here l.i.m. *means a limit in mean square sense, i. e.,* $\underset{n\to\infty}{\text{l.i.m.}}\, \xi_n = \xi$ *implies* $\lim\limits_{n\to\infty} \boldsymbol{E}|\xi_n - \xi|^2 = 0$.

The existence of a sequence of simple functions approximating an arbitrary function $f(x0 \in L_2\{X, \mathcal{B}, m\}$ follows from the general theorems of the measure theory. Thus the stochastic integral can be considered as a mean square limit of the relevant integral sums.

Let us denote by $\mathcal{B}_0$ the class of all subsets $A \in \mathcal{B}$ with $m(A) < \infty$, and define a random function of sets $\tilde{\zeta}(A)$ by

$$\tilde{\zeta}(A) = \int \chi_A(x)\zeta(dx) = \int_A \zeta(dx). \tag{1.23}$$

We list the following properties of this function:

1. $\tilde{\zeta}(A)$ is defined on the class of sets $\mathcal{B}_0$;
2. if $A_n \in \mathcal{B}_0, n = 0, 1, \ldots, A_0 = \bigcup_{n=1}^{\infty} A_n, q \quad A_k \bigcap A_r = \emptyset$ for $k \neq r, k > 0$, then $\tilde{\zeta}(A_0) = \sum_{n=1}^{\infty} \tilde{\zeta}(A_n)$ in mean square sense;
3. $\mathbf{E}\tilde{\zeta}(A)\tilde{\zeta}(B) = m(A \bigcap B), \quad A, B \in \mathcal{B}_0$;
4. $\tilde{\zeta}(\Delta) = \zeta(\Delta) \quad$ for $\Delta \in \mathcal{K}$.

An *orthogonal stochastic measure* is defined as a random set function $\tilde{\zeta}$ satisfying the above conditions 1–4.

Note that property 4 means that $\tilde{\zeta}$ is a prolongation of the elementary stochastic measure $\zeta(\Delta)$. Thus the following statement is true [71].

Theorem 1.4. *If the structure measure of the elementary stochastic measure $\zeta(\Delta)$. is semi-additive, then $\zeta(\Delta)$ can be prolonged to a stochastic measure $\tilde{\zeta}(\Delta)$.*

Note that

$$\int f(x)\zeta(dx) = \int f(x)\tilde{\zeta}(dx)$$

since $L_2\{\zeta\} = L_2\{\tilde{\zeta}\}$.

1.3.6 Integral representation of random functions

Using the results of the previous section we can represent random functions via stochastic integrals. Let Θ be an arbitrary parameter set, and (Ω, F, P) a probability space. Assume first that a p-dimensional random vector function $\xi(\theta), \theta \in \Theta)$ can be written in the form

$$\xi(\theta) = \int g(\theta, x)\zeta(dx), \tag{1.24}$$

where $\zeta()$ is a stochastic measure on a measurable space $(X, \mathcal{B})$ with its values in C^p and a structure measure $m()$, and $g(\theta, x)$ is a scalar function such that for any $\theta \in \Theta$

$$g(\theta, x) \in L_2(m_0) = L_2(X, \mathcal{B}, m_0), \quad m_0(\Delta) = \operatorname{Sp} m(\Delta) = \sum_{k=1}^{p} m_{kk}(\Delta).$$

The correlation function reads

$$B(\theta_1, \theta_2) = \mathbf{E}\xi(\theta_1)\xi^*(\theta_2) = \int g(\theta_1, x)\overline{g(\theta_2, x)}\, m(dx). \tag{1.25}$$

We recall that $(X, \mathcal{B}, m_0)$ is a space with a complete measure, $L_2(m_0) = L_2(X, \mathcal{B}, m_0)$ is a Hilbert space of $\mathcal{b}$-measurable complex-valued functions which

are square m_0-integrable:

$$L_2(m_0) = \left\{ f : X \to C : \int |f(x)|^2 m_0(dx) < \infty \right\}.$$

By $L_2\{g\}$ we denote the closure in $L_2(m_0)$ of a linear span generated by the family of functions $\{g(\theta, x),\ \theta \in \Theta\}$. Then, $L_2\{g\}$ is a linear closed subspace of $L_2(m_0)$.

If $L_2\{g\} = L_2(m_0)$, then the family of functions $\{g(\theta, x),\ \theta \in \Theta\}$ is called complete in $L_2(m_0) = L_2(X, \mathcal{B}, m_0)$.

Let $\{g(\theta),\ \theta \in \Theta\}$ be a Hilbert random function with its values in C^p, and $L_0\{\xi\}$ is a set of all random vectors

$$\eta = \sum_{k=1}^{n} c_k \xi(\theta_k), \quad n = 1, 2, \ldots, \quad \theta_k \in \Theta,$$

where c_k are arbitrary complex numbers, and $L_2\{\xi\}$ is a closure of $L_0\{\xi\}$ in the sense of mean square convergence of random vectors.

A family of random vectors $\{\eta_\alpha, \alpha \in A\}$, $\eta_\alpha \in L_2(\Omega)$ is called subordinate to a random function $\xi(\theta),\ \theta \in \Theta\}$ if $\eta_\alpha \in L_2(\xi)$, $\alpha \in A$.

For random functions whose correlation function can be represented in the form (1.25) the following theorem can be formulated, see [71].

Theorem 1.5. *Assume that the correlation matrix of a random function* $\{\xi(\theta),\ \theta \in \Theta\}$ *can be written in the form* (1.25) *where* m *is a positive definite matrix measure on* $(X, \mathcal{B})$, $g(\theta, x) \in L_2(m_0)$, $\theta \in \Theta$ ($m_0 = Spm$), *and the family* $\{g(\theta, x), \theta \in \Theta\}$ *is complete in* $L_2(X, \mathcal{B}, m_0)$. *Then* $\xi(\theta)$ *can be represented in the form* (1.24) *where* $\{\zeta(B), B \in \mathcal{B}\}$ *is a stochastic orthogonal measure which is subordinate to the random function* $\{\xi(\theta),\ \theta \in \Theta\}$ *with a structure measure* $m()$, *and the equality* (1.24) *holds with probability one for each* θ.

Let $\xi(\mathbf{x}) = (\xi^1(\mathbf{x}), \ldots, \xi^d(\mathbf{x}))$, $\mathbf{x} \in I\!R^m$ is centered mean square continuous homogeneous (in a broad sense) random field with real- or complex values. Let $R(\mathbf{x})$ be the correlation function of this field. By the Bochner–Khinchin theorem, this correlation function obeys the representation (1.19). This representation is a particular case of (1.25) in which $X = \Theta = I\!R^m$, $g(\theta, \mathbf{x}) = \exp\{\mathrm{i}(\mathbf{x}, \theta)\}$. Since the family $\{\exp\{\mathrm{i}(\mathbf{x}, \theta)\},\ \theta \in I\!R^m\}$ is complete in $L_2(m_0) = L_2(X, \mathcal{B}, m_0)$ where m_0 is an arbitrary bounded measure on the sigma-algebra $\mathcal{B}^m$ of $I\!R^m$, we come to the following corollary of the previous theorem which is known as the *spectral theorem* (see [71]).

Theorem 1.6. *Any centered mean square continuous homogeneous vector random field* $\xi(\theta), \theta \in I\!R^m$ *can be represented in the form*

$$\xi(\theta) = \int e^{\mathrm{i}(x,\theta)}\, \zeta(d\boldsymbol{x}), \quad \theta \in I\!R^m,$$

where $\zeta(A)$, $A \in \mathcal{B}^m$ *is a vector orthogonal measure on* $\mathcal{B}^m$ *which is subordinate to* $\xi(0)$. *Between* $L_2\{\xi\}$ *and* $L_2\{F\}$, $F_0 = SpF$ *there is an isometry providing*

(a) $\xi(\theta) \in L_2\{\xi\} \leftrightarrow e^{i(x,\theta)} \in L_2\{F_0\}$;

(b) *if* $\eta_i \leftrightarrow g_i(\boldsymbol{x}), \eta_i \in L_2\{\xi\}, g_i \in L_2\{F_0\}$, $i = 1, 2$, *then*

$$\eta_i = \int g_i(\boldsymbol{x})\, \zeta(d\boldsymbol{x}), \quad \boldsymbol{E}\eta_1\eta_2^* = \int g_1(\boldsymbol{x})\overline{g_2(\boldsymbol{x})}\, F(d\boldsymbol{x}).$$

1.3.7 Random trajectories

A random trajectory can be defined as follows. A random trajectory is defined as a solution to an ordinary differential equation with a random field in the right hand side:

$$\frac{dX_i(t,\omega)}{dt} = f_i(X(t,\omega), t, \omega), \quad t \geq t_0,\ \omega \in \Omega,\ i = 1, \ldots, n, \tag{1.26}$$

where $f(x,t) = f(x,t,\omega) = (f_1(x,t,\omega), \ldots, f_n(x,t,\omega))$, $x \in I\!R^n$ is a vector random field, $X(t,\omega) = (X_1(t,\omega), \ldots, X_1(t,\omega))$. The samples of the random field $f(x,t)$ are assumed to be smooth enough in the sense that for each sample the classical solution of the deterministic equation (1.26) exists. An important difference to the classical deterministic ordinary differential equation is only in that in the considered case the right-hand side depends on a parameter $\omega \in \Omega$ where Ω is a probabilistic space. So in contrast to the stochastic differential equations of the Ito type, the solution is well defined in the classical sense, and there is no need to develop a special calculus for studying (1.26) . However it does not imply that all the question about the solutions to (1.26) can be answered by adapting the relevant results of the classical theory of differential equations. For example, the existence of the solution of an initial value problem for (1.26) is not equivalent to the existence of the solution for all (or almost all) samples (i. e., all values of ω), one needs the existence of the solutions as random processes on one common interval (not depending on ω). Let us give an illustrating example. Assume we solve the following equation:

$$\frac{dX}{dt} = \xi\, X^2, \quad X \in I\!R^1,\ t \in [0, \infty),$$

where $\xi = \xi(\omega)$ is a standard Gaussian random variable. The solution of this equation with the prescribed initial value $X(0) = x_0$ can be written explicitly as

$$X(t,\omega) = \begin{cases} 0, & \text{if } x_0 = 0, 0 \leq t < \infty \\ \left(\frac{1}{x_0} - \xi(\omega)t\right)^{-1}, & \text{if } |x_0| \neq 0, 0 \leq t < \delta. \end{cases}$$

Here

$$\delta = \delta(x_0, \omega) = \begin{cases} \frac{1}{\xi\, x_0}, & \text{if } x_0\xi > 0 \\ \infty, & \text{if } x_0\xi < 0 \end{cases}$$

is the explosion time instant, i. e., $\lim_{t\to\delta} |X(t) = \infty$. This means, that for any $x_0 \neq 0$, the solution may explode arbitrarily quickly, with positive probability, i. e., for any $\varepsilon > 0$, we get $P(\delta < \varepsilon) > 0$. This implies that the problem has no sample solutions.

Another issue is the existence of statistical characteristics of the solution to (1.26). For example in the framework of the conventional theory of differential equations it is not possible to answer the question of whether or not the first moment $\mathbf{E}|X(t)|$ is finite, i. e., when $\int_\Omega |X(t,\omega)\, P(d\omega) < \infty$. Nevertheless, many results in stochastic differential equations of the type (1.26) were obtained (e. g., see [9, 37, 228]) via an extension of the relevant results of the classical differential equation theory. For example, for the existence of a sample solution to (1.26) on a finite interval, say, $I = [0,1]$, it is necessary to prove the existence of measurable random processes $\xi_1(t,\omega), \xi_2(t,\omega)$, $(t,\omega) \in I \times \omega$, such that $P\{\xi_i(t)\, dt < \infty\} = 1$, $i = 1,2$, and almost all samples of the right-hand side $f(x,t) = f(x,t,\omega)$ should satisfy the linear growth condition

$$|f(x,t)| \leq \xi_1(t) + \xi_2(t)\,|x| \quad \text{for all } x \in I\!R^n.$$

This result follows from the classical theory of differential equations (e. g., see [75]).

1.3.8 Stochastic differential, Ito integrals

As already mentioned above, the motion of fluid elements and aerosol particles are often described by Langevin stochastic differential equations, known in the theory of stochastic processes as Ito stochastic differential equations. We have given the definition of stochastic integrals with respect to general stochastic measure, and so we could just refer to a particular case of Wiener measure to define the Ito integrals. Let us however begin with considerations which are closer to physics.

1.3.9 Brownian motion

Let us again consider the ordinary differential equation (1.26), but now the randomness is specifically entering this equation as an additive noise:

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot \text{“random noise”}, \tag{1.27}$$

where b and σ are some given deterministic functions. So intuitively, the solution X_t is a random process with some distribution and correlations caused by the input noise $W_t =$ “random noise”. Again from physical intuition, we might assume that W_t has the following properties: (i) W_{t_1} and W_{t_2} are independent if $t_1 \neq t_2$; (ii) W_t is a stationary process, i. e., the joint distributions of $\{W_{t_1+t}, \ldots, W_{t_k+t}\}$ do not depend on t; (iii) $\mathbf{E}W_t = 0$ for all t.

However, it turns out there does not exist any “reasonable” stochastic process satisfying (i) and (ii): such a process cannot have continuous trajectories (e. g., see [70]). If

we assume $\mathbf{E}|W_t^2| = 1$, then the function $(t, \omega) \to W_t(\omega)$ cannot even be measurable, with respect to the σ-algebra $\mathcal{B} \times \mathcal{F}$ where $\mathcal{B}$ is the Borel σ-algebra on $[0, \infty)$ (e. g., see [158]). So the functions W_t belong to other class we considered above, generalized stochastic functions, constructed as a probability measure in the space of tempered distributions on $[0, \infty)$, and not as a probability measure on the much smaller space $\mathbb{R}^{0,\infty}$, like an ordinary stochastic process.

We will avoid this construction and turn to a description of stochastic differential equations which uses random processes with independent increments. Let us start with the physical process of Brownian motion.

Let us consider a particle which is moving on a line: the particle starts from the origin $\xi(0) = 0$, and during a small time increment Δt it makes a jump to the left with probability $1/2$, and to the right with probability $1/2$. So during time t the particle makes $n = t/\delta t$ jumps. We put

$$x_i = \begin{cases} \Delta x & \text{with probability 1/2,} \\ -\Delta x & \text{with probability 1/2.} \end{cases}$$

The total displacement of the particle reads as $\xi(t) = \sum_{i=1}^{n} x_i$. Assuming that x_i are mutually independent and equally distributed, the variance (dispersion) can be found as

$$D\xi(t) = D \sum_{i=1}^{n} x_i = n\, D\, x_i = n(\Delta x)^2 = \frac{t}{\Delta t}(\Delta x)^2.$$

Let us consider two arbitrary times s and t, where $s < t$. Then, the first and second summands in $\xi(t) = [\xi(t) - \xi(s)] + \xi(s)$ are independent, and in addition, the displacement during the time interval (s, t) depends obviously on $t - s$, since physically the character of the motion is not changing in the time, hence, $\xi(t) - \xi(s)$ and $\xi(t - s)$ are equally distributed. Therefore,

$$D\xi(t) = D[\xi(t) - \xi(s)] + D\xi(s) = D\xi(t - s) + D\xi(s),$$

which implies that $D\xi(t)$ is a linear function of t: $D\xi(t) = t\sigma^2$, where σ^2 is a constant called a diffusion coefficient. Thus we have

$$D\xi(t) = t\sigma^2 = \frac{t}{\Delta t}(\Delta x)^2 = n\, Dx_i,$$

and due to the Gaussian distribution of x_i

$$P\left(\frac{\xi(t)}{\sigma\sqrt{t}} < x\right) = P\left(\frac{1}{\sqrt{nDx_i}} \sum_{i=1}^{n} < x\right).$$

Taking a limit as $n \to \infty$, we conclude by the central limit theorem that

$$P\left(\frac{1}{\sqrt{nDx_i}} \sum_{i=1}^{n} < x\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}\, dz,$$

and hence

$$P\left(\frac{\xi(t)}{\sigma\sqrt{t}} < x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}\, dz,$$

which implies that $\xi(t)$ is a Gaussian random variable with zero mean and variance $\sigma^2 t$.

Since the increment $\xi(t-s) - \xi(s)$ is distributed as $\xi(s)$ for all $s \geq 0$, we conclude that $\xi(t-s) - \xi(s)$ has a Gaussian distribution with zero mean and variance $\sigma^2 t$, $s \geq 0$. In this context, we call a stochastic process $W(t)$ *Brownian motion* if $W(0)$ and $W(t-s) - W(s)$ are independent and have a Gaussian distribution with zero mean and variance $\sigma^2 t$ for all $s \geq 0$.

We are now in a position to give an interpretation of the stochastic differential equation of the Ito type we started with in (1.27). So let us take the time subdivision $0 = t_0 < t_1 < \cdots < t_m = t$ and write a discrete version of (1.27):

$$X_{k+1} - X_k = b(t_k, X_k)\, \Delta t_k + \sigma(t_k, X_k)\, W_k \Delta_k, \tag{1.28}$$

where $X_j = X(t_j)$, $W_k = W_{t_k}$, $\Delta_k = t_{k+1} - t_k$. Now we look at the value $W_k \Delta t_k$ as an increment of a random process $\{V_t\}_{t \geq 0}$, $\Delta V_k = V_{k+1} - V_k$, and assume, in accordance with the assumptions (i)–(iii) made above for W_t, that V_t have stationary independent increments with zero mean. It can be shown that the only such process with continuous samples is the Brownian motion B_t. Thus we put $V_t = B_t$ and obtain from (1.28)

$$X_k = X_0 + \sum_{j=0}^{k-1} b(t_j, X_j)\Delta t_j + \sum_{j=0}^{k-1} \sigma(t_j, X_j)\Delta B_j. \tag{1.29}$$

Now we can take a limit, as $\Delta t_j \to 0$, in the mean squared sense we introduced above in the general considerations, and we arrive at

$$X(t) = X(0) + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s. \tag{1.30}$$

Thus the solution is expressed through the stochastic integral $\int_0^t \sigma(s, X_s)\, dB_s$ where the stochastic measure dB_s is generated by the Brownian motion. The construction of this integral can be carried out following the general construction of stochastic integrals given above. Alternatively, it can be done following a direct scheme using the Wiener process begun at the origin. So suppose $0 \leq S < T$ and $f(t, \omega)$ are given. As in the general approach presented above, first the stochastic integral for a simple class of functions f is defined, and then it is extended using some approximating procedure. So first assume that the function f has the form

$$\varphi(t, \omega) = \sum_{j \geq 0} e_j(\omega)\, \chi_{[j \cdot 2^{-n}, (j+1)2^{-n}]}(t), \tag{1.31}$$

where ξ_I is the indicator of the interval I, and n is a natural number, and e_j are some functions to be defined properly. For functions φ the stochastic integral is defined by

$$\int_S^T \varphi(t,\omega)\, dB_t(\omega) = \sum_{j\geq 0} e_j(\omega)\big[B_{t_{j+1}} - B_{t_j}\big](\omega), \tag{1.32}$$

where

$$t_k = t_k^{(n)} = \begin{cases} k\cdot 2^{-n} & \text{if } S \leq k\cdot 2^{-n} \leq T, \\ S & \text{if } k\cdot 2^{-n} < S, \\ T & \text{if } k\cdot 2^{-n} > T. \end{cases}$$

Then the standard extension procedure is carried out similarly to what we presented above (for more on this see [158]).

1.3.10 Multidimensional diffusion and Fokker–Planck equation

In this book, we will often work with multidimensional diffusions governed by systems of stochastic differential equations. The relevant probability density function satisfies the multidimensional Fokker–Planck equation. So let us consider first a 1-dimensional stochastic differential equation

$$dX(t) = b(X_t,t)dt + \sigma(X_t,t)\, dB_t \tag{1.33}$$

for a random trajectory starting at $X(0) = x_0$, with drift $b(X_t,t)$ and diffusion coefficient $\sigma^2(X_t,t)$. The probability density $f(x,t) = \langle\delta(x - X(t,x_0)\rangle$ satisfies the Fokker–Planck equation

$$\frac{\partial f(x,t)}{\partial t} = -\frac{\partial}{\partial x}\big[b(x,t)\, f(x,t)\big] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\big[\sigma^2(X_t,t)\ f(x,t)\big]. \tag{1.34}$$

In many dimensions, we deal with $\mathbf{X}_t$, an n-dimensional random vector, and $\mathbf{W}_t$, an n-dimensional standard Wiener process. The multidimensional stochastic differential equation reads

$$d\mathbf{X}_t = \mathbf{b}(\mathbf{X}_t,t)\, dt + \sigma(\mathbf{X}_t)\, d\mathbf{W}_t \qquad \mathbf{X}_t(0) = \mathbf{x}_0, \tag{1.35}$$

where $\mathbf{b}(\mathbf{X}_t,t) = (b_1,\ldots,b_n)$ is the drift vector, and $\sigma(\mathbf{x},t)$ is a positive-definite matrix.

The probability density function $f(\mathbf{x},t)$ for the random vector $\mathbf{X}_t$ satisfies the Fokker–Planck equation which is the following parabolic equation

$$\frac{\partial f(\mathbf{x},t)}{\partial t} = -\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\big[b_j(\mathbf{x},t)\, f(\mathbf{x},t)\big] + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_i^2\,\partial x_j^2}\big[\sigma^2(\mathbf{x},t)\, f(\mathbf{x},t)\big], \tag{1.36}$$

where $\sigma^2(\mathbf{x},t)$ is the diffusion tensor, a matrix with the entries

$$\sigma_{ij}^2(\mathbf{x},t) = \frac{1}{2}\sum_{k=1}^{n}\sigma_{ij}(\mathbf{x},t)\sigma_{jk}(\mathbf{x},t).$$

1.3.11 Central limit theorem and convergence of a Poisson process to a Gaussian process

Let us start with the central limit theorem (CLT).

Assume that we have a sequence of zero mean, i.i.d. (*arbitrarily distributed*!) random variables $X_i,\ i = 1, \ldots, P_X(x)$ being the probability distribution of X_i, and σ^2, the variance of X_i. Let

$$Z = \frac{X_1 + \cdots X_n}{\sqrt{n}}.$$

The CLT says that if $\sigma^2 < \infty$, then Z converges (in distribution) to the Gaussian distribution $N(0, \sigma^2)$.

The proof is extremely simple. We show that the characteristic function $\varphi_Z(t) = \langle e^{\mathrm{i}tZ} \rangle$ converges to the characteristic function of the Gaussian variable $N(0, \sigma^2)$, i. e., to $\exp[-\sigma^2 t^2/2]$.

Indeed, for any distribution $P_X(x)$ we have for its characteristic function

$$\varphi_X(t) = \int e^{itx}\, dP_X(x) = 1 - \frac{1}{2}\sigma^2 t^2 + \cdots . \tag{1.37}$$

But, since X_i are all mutually independent, we can write, using the well-known property of characteristics functions $\varphi_X(t\sigma) = \varphi_{\sigma X}(t)$, that

$$\varphi_Z(t) = \left[\varphi_{X_1}\left(\frac{t}{\sqrt{n}}\right)\right]^n \approx \left(1 - \frac{1}{2}\,\frac{\sigma^2 t^2}{n}\right)^n \to \exp\,[-\sigma^2 t^2/2]. \tag{1.38}$$

Notice that the omitted terms in (1.37) give a $n^{-3/2}$ contribution to (1.38), hence it vanishes, as $n \to \infty$.

Remark 1.1. The assumption that the means of X_i are equal to zero is by no means necessary. Just notice that $\varphi_{X+m}(t) = e^{\mathrm{i}mt}\varphi_X(t)$ where $m = \langle X \rangle$.

Now let us prove that a Poisson distribution with a mean m (for simplicity we assume that m is an integer number), having the discrete probability distribution $P_i = \frac{m^i}{i!}\, e^{-m}$, and the characteristic function

$$\varphi(t) = \exp\,[m(e^{\mathrm{i}t} - 1)], \tag{1.39}$$

converges to a Gaussian distribution $N(m, m)$ if m is sufficiently large.

Clearly, from CLT we could immediately conclude this result if we could represent the Poissonian random variable as a sum of i.i.d variables. But this can indeed be done, since the random variable having a Poisson distribution with the mean m can be represented as a sum of m i.i.d. random numbers whose means are equal to 1. This follows immediately from the structure of the characteristic function (1.39). So, by the CLT this sum converges to $N(m, m)$; however, m should be large enough. In practice,