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De Gruyter

Mathematics Subject Classification 2010: Primary: 28-01, 28-02, 60-01; Secondary: 28A33, 28C15, 46E30, 46N10, 49J45, 60B05, 60B10.

ISBN 978-3-11-027640-4
e-ISBN 978-3-11-028051-7

Library of Congress Cataloging-in-Publication Data

A CIP catalog record for this book has been applied for at the Library of Congress.

Bibliographic information published by the Deutsche Nationalbibliothek

The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at http://dnb.dnb.de.

Typesetting: Da-TeX Gerd Blumenstein, Leipzig, www.da-tex.de
Printing and binding: Hubert & Co. GmbH & Co. KG, Göttingen
∞ Printed on acid-free paper

Printed in Germany

www.degruyter.com

This book is dedicated
to our wonderful spouses,

Cristina and Roger

Preface

In recent years, technological progress created a great need for complex mathematical models. Many practical problems can be formulated using optimization theory and they hope to obtain an optimal solution. In most cases, such optimal solution can not be found.

So, non-convex optimization problems (arising, e.g., in variational calculus, optimal control, nonlinear evolutions equations) may not possess a classical minimizer because the minimizing sequences have typically rapid oscillations. This behavior requires a relaxation of notion of solution for such problems; often we can obtain a such relaxation by means of Young measures.

The Young measures generalize measurable functions. Thus, a Young measure is herself a measurable application that, to every point t of Ω, associates a probability τ_t on a topological space X; for all Borel set $A \subseteq X$, $\tau_t(A)$ may be interpreted as the probability that the value in t of the "function" $\tau_.$ belongs to A. In the particular case, a measurable application $u : \Omega \to X$ is a Young measure, where, for all $t \in \Omega$, $\tau_t = \delta_{u(t)}$ ($\delta_{u(t)}$ indicates the mass of Dirac in $u(t)$).

Young measures' theory has a long history; it begins with the work of L. C. Young which, in 1937, introduces the so-called "generalized curves" in order to provide extended solutions for some non-convex problems in variational calculus. A milestone in this history is the appearance of the monograph of J. Warga, "Optimal Control of Differential and Functional Equations" (Academic Press, 1972); here is systematically developed a theory of relaxed control in compact metric spaces. The extension of theory on locally compact metric spaces was made by H. Berliocchi and J. M. Lasry in 1973.

The study of Young measures was extended to Polish and Suslin spaces by the works of E. J. Balder (since 1984) and M. Valadier (1990).

Lately, Young measures were the object of an intense research due to their applications in obtaining relaxed solutions; here are some of the areas in which these relaxed solutions find applications: non-convex variational problems and differential inclusions, non-linear homogenization problems, micro-magnetic phenomena in ferro-magnetic materials, Nash equilibrium in games theory, Gamma-convergence, different phenomena in continuum mechanics (as elasticity, micro-structures' theory), optimal design and shape optimization problems.

On this subject, recent monographs appeared:

(i) Roubiček, T.—*Relaxation in optimization theory and variational calculus*, Walter de Gruyter, Berlin. New York, 1997.

(ii) Pedregal, P.—*Parametrized Measures and Variational Principles*, Birkhäuser Verlag, Basel. Boston. Berlin, 1997.

(iii) Castaing, Ch., Raynaud de Fitte, P. and Valadier, M.—*Young measures on topological spaces. With applications in control theory and probability theory*, Kluwer Academic Publ. Dordrecht. Boston. London, 2004.

The focus of the first two books is mainly on the applications; therefore, Young measures are used as generalized solutions to non-convex problems of variational calculus, optimization theory, or game theory.

The last monograph considers theoretical aspects of the theory of Young measures as well as the applications in control theory and probability theory. Many of the results presented here make reference to a wide bibliography; thus, the work is difficult to use for beginners.

The literature on the applications of Young measures in various areas (lower semicontinuity, optimal relaxed control, Gamma-convergence and homogenization, differential games, elasticity, hysteresis, etc.) is extremely rich and the existing monographs main focus on applications rather than on theoretical aspects. We found difficult for a young researcher who wants to clarify the theoretical aspects, to go through the extensive bibliography which is usually referred. Thus, our goal was to write a book where to be gathered all the theoretical aspects related to defining of Young measures (measurability, disintegration, stable convergence, compactness), book which to be a useful tool for those interested in theoretical foundations of the theory: the postgraduate students, the students in the doctoral study, but also to all those interested in measure theory and relaxed control.

The developing of Young measures' theory involves some compactness results for measures on abstract spaces and topological spaces. Hence, to achieve our goal, we considered useful to provide a complete set of classical and recent compactness results in measure and function spaces.

The book is organized in three chapters (Weak compactness in measure spaces, Bounded measures on topological spaces, Young measures). For a good comprehension of the subject, we developed in the first two chapters the results used in the third (biting lemma in the abstract measure theory and Prohorov's theorem in the measure theory on topological spaces).

The first chapter covers background material on measure theory in abstract frame. Therefore, we present some results of duality and weakly compactness in $ca(A)$ and $L^1(\lambda)$. However, here we prove some extensions of Dunford–Pettis theorem like biting lemma of Brooks–Chacon or subsequence splitting lemma of H. P. Rosenthal.

In Chapter two, we treat the measure theory on topological spaces. The framework is offered by Suslin spaces; on the one hand, these spaces are Radon and on the other hand, they cover the particular case of a separable Banach space provided with his weak topology. We introduce the narrow topology and then we prove the Prohorov's compactness theorem. In the particular case of Polish spaces, the narrow topology is metrizable; we present the compatible metrics of Dudley and of Lévy–Prohorov. As an application of Prohorov's theorem, we prove in the last paragraph the existence of Wiener's measure on $C[0, 1]$.

With some exceptions, in Chapters 1 and 2 are presented classical compactness results for measures on abstract spaces, or on topological spaces. The originality consists in the selection and ordering of these results and the accompanying remarks and examples. However, we note some approaches and new results, such as: the modulus of λ-continuity (1.79) and theorems 1.80 and 1.81, a-convergence of nets in L^1 and the extension of Dunford–Pettis theorem (1.93), a new proof for Rosenthal's Subsequence Splitting Lemma using Biting Lemma, the modulus of narrow compactness (2.65), a-convergence of nets in $ca(\mathcal{B}(T))$, theorem 2.69 and obtaining, as corollary of this theorem, a new proof of Prohorov's compactness theorem. Finally, in the last section of 2 we give a simple and self-contained presentation of Wiener measure (2.6).

Compactness results from the first two chapters are used to study Young measures in Chapter three. We prove the disintegration theorem for product measures and we use it to present Young measures as parametrized measures; the frame is that of a regular Suslin space. We remark that the space of Young measures contains the space of measurable mappings as dense subspace and that the narrow topology is an extension of the topology of convergence in measure. Prohorov's theorem in the case of Young measures highlights the role played by tightness in compactness results. We present a vector version for biting lemma and an extension of this result to some special non-bounded sets of measurable mappings: finite-tight sets. In the seventh paragraph, we will study the two types of products for the Young measures and will give the fiber product lemma.

In the last three sections of the book are presented some applications; thus, Prokhorov's theorem for Young measures was used in the ninth paragraph in the study of strong compactness in $L^p(\mu, E)$. We obtain, as corollaries, the theorems of Visintin–Balder, Rossi–Savaré, Lions–Aubin and Gutman; in the scalar case, the compactness criterion of Riesz–Fréchet–Kolmogorov is obtained.

In the tenth paragraph, we consider some applications of quasiconvexity to the study of gradient Young measures and to the lower semicontinuity. Are studied the Young measures generated by sequences and particularly the gradient Young measures. We pay special attention to quasiconvexity and its various equivalent definitions. The quasiconvexity is essentially used in the Kinderlehrer–Pedregal's characterization of gradient Young measures, but also in the study of lower semicontinuity of energy functional that appears in variational calculus. Finally, in paragraph eleven, we present some results of existence of solutions in relaxed variational calculus.

There are also, in this chapter, some new concepts and results among which: new proofs for theorems 3.30, 3.32 and 3.33, the density result 3.49 and the proof of theorem 3.50, theorems 3.51, 3.66, 3.67 and propositions 3.54 and 3.56, introduction of finite-tight sets (3.75) and use them to obtain extensions of biting lemma (3.84) and Saadoune–Valadier's theorem (3.85), Jordan finite-tight sets (3.91) and their utility in obtain of a compactness result in Sobolev spaces (3.102) and an alternative to Rellich–Kondrachov theorem (3.105).

All results are accompanied by full demonstrations; for many of these results, are given different proofs from those referred in the literature.

The bibliography gives the main references relevant to the content of the book; it is no exhaustive.

Understanding the text requires basic knowledge of general topology, functional analysis and Lebesgue integration that may be found in any textbook on the subject. In rest, all the statements are fully justified and proved.

To conclude, this text is intended as a postgraduate textbook as well as a reference for more experienced researchers.

The book was written over several years of collaboration between authors, with the occasion of stages that the first author has made, as a visiting professor, at the University of Brest.

An important role in setting the ideas and in the organization of book's material was played by discussions with various mathematicians met under these occasions.

First, we mention the authors of monograph "Young measures on topological spaces", C. Castaing, M. Valadier and P. Raynaud de Fitte, that supported and inspired us in writing the last chapter. Also, we have had useful discussions with E. Balder and T. Roubiček at the international conference "*Mesures de Young et Contrôle Stochastique*" (Brest, 2002); in this way, we thank them all.

Iaşi/Brest, December 2011

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Contents

Chapter 1

Weak Compactness in Measure Spaces

We will present in this first chapter the main properties of the measure spaces and of the space of integrable functions. We recall the classic results of weak compactness (like Vitali–Hahn–Saks, Radon–Nikodym and Dunford–Pettis theorems) but we will also mention more recent results such as Brooks–Chacon biting lemma or Rosenthal's lemma.

1.1 Measure Spaces

In this introductory section, we recall the definitions and classic properties of the additive and σ-additive measures. We finish this section by the Saks' theorem, the Vitali–Hahn–Saks and Nikodym theorems that we will use in the following sections for a study of weak compactness on $ca(\mathcal{A})$.

For beginning, we will specify the definitions and the notations to be employed henceforward. We consider as known the theory of integration relating to a positive, σ-additive and σ-finite measure.

We designate by X an arbitrary set and by $\mathcal{A}$ a σ–algebra of subsets of X; an $\mathcal{A}$-partition of $A \in \mathcal{A}$ is a partition of A with the elements in $\mathcal{A}$.

According to the usual notations, if μ is a positive, σ-additive and σ-finite measure, we shall denote by $\mathcal{L}^1(\mu) = \mathcal{L}^1(X, \mathcal{A}, \mu)$ the set of all real mappings f defined on X with the property that f is $\mathcal{A}$-measurable and μ-integrable and by $L^1(\mu) = L^1(X, \mathcal{A}, \mu)$ the quotient space $\mathcal{L}^1(\mu)/_{\doteq}$, where $\doteq$ is equality μ – almost everywhere.

In the following, we recall the definition of the signed measures.

Definition 1.1. A set function $\lambda : \mathcal{A} \to \bar{\mathbb{R}} = [-\infty, +\infty]$ is a *finitely additive measure*, or shortly an *additive measure*, if

(i) $\lambda(\emptyset) = 0$,
(ii) $\lambda(A \cup B) = \lambda(A) + \lambda(B)$, for every $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$,
(iii) λ assumes at most one of the values $+\infty$ and $-\infty$.

An additive measure λ on $\mathcal{A}$ is a *σ-additive measure* or a *countably additive measure* if, for every sequence of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ (i.e.

$A_n \cap A_m = \emptyset$, for every $n \neq m$),

$$\lambda\left(\bigcup_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{+\infty} \lambda(A_n).$$

A σ-additive measure λ is *finite* or *real valued* if its range is contained in $\mathbb{R}$. λ is *σ-finite* if, for every $A \in \mathcal{A}$, there exists a sequence $(A_n)_{n\in\mathbb{N}}$ such that $A = \bigcup_{n\in\mathbb{N}} A_n$ and $\lambda(A_n) \in \mathbb{R}$, for every $n \in \mathbb{N}$.

We will designate by

$ba(\mathcal{A})$—the set of all *real valued bounded additive measures* on $\mathcal{A}$,

$ca(\mathcal{A})$—the set of all *real valued σ-additive measures* on $\mathcal{A}$.

$ba(\mathcal{A})$ and $ca(\mathcal{A})$ are vector spaces under the usual addition and scalar multiplication operations.

$ca^+(\mathcal{A})$ ($ba^+(\mathcal{A})$)—the subsets of all *positive measures* of $ca(\mathcal{A})$ ($ba(\mathcal{A})$).

The following properties are easy to demonstrate.

Proposition 1.2. *Let $\lambda : \mathcal{A} \to \bar{\mathbb{R}}$ be an additive measure and let $A, B \in \mathcal{A}$ with $B \subseteq A$.*

(i) *If $|\lambda(A)| < +\infty$, then $|\lambda(B)| < +\infty$.*

(ii) *If $|\lambda(B)| < +\infty$, then $\lambda(A \setminus B) = \lambda(A) - \lambda(B)$.*

Proof. $A = B \cup (A \setminus B)$ and so $\lambda(A) = \lambda(B) + \lambda(A \setminus B)$.

(i) If $\lambda(B) = +\infty(-\infty)$, then $\lambda(A) = +\infty(-\infty)$, what contradicts hypothesis. Therefore, $\lambda(B)$ is finite.

(ii) If $|\lambda(B)| < +\infty$, then $\lambda(A) - \lambda(B) = \lambda(A \setminus B)$. □

Proposition 1.3. *Let λ be a σ-additive measure and let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$.*

(i) *If $(A_n)_n$ is an increasing sequence, then $\lambda(\cup_{n=0}^{\infty} A_n) = \lim_{n\to\infty} \lambda(A_n)$.*

(ii) *If $(A_n)_n$ is a decreasing sequence and $|\lambda(A_0)| < +\infty$, then $\lambda(\cap_{n=0}^{\infty} A_n) = \lim_{n\to\infty} \lambda(A_n)$.*

Proof. (i) Let $A = \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$. Firstly, we suppose that there is $n_0 \in \mathbb{N}$ such that $|\lambda(A_{n_0})| = +\infty$. According to (i) of Proposition 1.2, $|\lambda(A)| = +\infty = |\lambda(A_n)|$, for every $n \geq n_0$; since λ assumes at most one of the values $+\infty$ and $-\infty$, $\lambda(A) = \lim_n \lambda(A_n)$.

If, for every $n \in \mathbb{N}, |\lambda(A_n)| < +\infty$, then we define the pairwise disjoint sequence $(B_n)_n \subseteq \mathcal{A}$ letting: $B_0 = A_0, B_n = A_n \setminus A_{n-1}, \quad \forall n \geq 1$; then $A =$

$\cup_{n=0}^{\infty} B_n$ and, using (ii) of Proposition 1.2, we obtain $\lambda(A) = \sum_{n=0}^{\infty} \lambda(B_n) = \lim_n \sum_{k=0}^{n} \lambda(B_k) = \lim_n \lambda(A_n)$.

(ii) If $(A_n)_{n\in\mathbb{N}}$ is decreasing and $|\lambda(A_0)| < +\infty$, then the sequence $(B_n)_n$, where $B_n = A_0 \setminus A_n$, is increasing and the result follows from the first part of the proof. □

Definition 1.4. Let $\lambda : \mathcal{A} \to \bar{\mathbb{R}}$ be a σ-additive measure and let $A \in \mathcal{A}$;

A is called λ-*positive* if, for every $B \in \mathcal{A}$, $\lambda(A \cap B) \geq 0$.

A is called λ-*negative* if, for every $B \in \mathcal{A}$, $\lambda(A \cap B) \leq 0$.

A is called λ-*null* if it is λ-positive and λ-negative. A is λ-null set if and only if, for every measurable set $B \subseteq A$, $\lambda(B) = 0$.

For the following two results, see Theorem A, p. 121 in [93].

Proposition 1.5. *Let $\lambda : \mathcal{A} \to (-\infty, +\infty]$ be a σ-additive measure and let $A \in \mathcal{A}$ with $\lambda(A) < 0$. There exists a λ-negative set $B \subseteq A$ such that $\lambda(B) < 0$.*

Proof. If A is λ-negative, then $B = A$.

Otherwise, there exists $C \in \mathcal{A}$ such that $\lambda(C \cap A) > 0$. Let n_1 be the smallest positive integer for which there exists $A_1 \in \mathcal{A}$, $A_1 \subseteq A$ with $\lambda(A_1) > \frac{1}{n_1}$.

If $A \setminus A_1$ is λ-negative, then $B = A \setminus A_1$; since $\lambda(A) < 0$, $\lambda(A) \in \mathbb{R}$ and then, by Theorem 1.2, $\lambda(A_1) \in \mathbb{R}$ and $\lambda(B) = \lambda(A) - \lambda(A_1) < \lambda(A) < 0$.

If $A \setminus A_1$ is not λ-negative, let n_2 be the smallest positive integer for which there exists $A_2 \in \mathcal{A}$, $A_2 \subseteq A \setminus A_1$ with $\lambda(A_2) > \frac{1}{n_2}$; obviously, $n_2 > n_1$.

If the above construct does not produce a solution of problem after a finite number of steps, then we obtain a sequence of pairwise disjoint sets $(A_k)_{k\geq 1} \subseteq \mathcal{A}$, $A_k \subseteq A \setminus \cup_{i=1}^{k-1} A_i$, with $\lambda(A_k) > \frac{1}{n_k}$, for every $k \geq 1$ and $n_k \uparrow +\infty$.

Let $B = A \setminus \bigcup_{k=1}^{\infty} A_k$; then $\lambda(B) = \lambda(A) - \sum_{k=1}^{\infty} \lambda(A_k) < 0$.

For every $C \in \mathcal{A}, C \subseteq B$ and for every $k \in \mathbb{N}^*$, $C \subseteq A \setminus \cup_{i=1}^{k-1} A_i$ so that $\lambda(C) \leq \frac{1}{n_k - 1}$ (n_k is the smallest positive integer for which there is $A_k \subseteq A \setminus \cup_{i=1}^{k-1} A_i$ with $\lambda(A_k) > \frac{1}{n_k}$). Then $\lambda(C) \leq 0$ and therefore B is λ-negative. □

Theorem 1.6 (Hahn decomposition theorem). *Let $\lambda : \mathcal{A} \to \bar{\mathbb{R}}$ be a σ-additive measure; there exists a λ-positive set $H \in \mathcal{A}$ such that $H^c = X \setminus H$ is λ-negative.*

For any other pair $\{H_1, H_1^c\} \subseteq \mathcal{A}$ with H_1 λ-positive and H_1^c λ-negative, $H \Delta H_1$ is a λ-null set.

Proof. First, let us suppose that $\lambda(\mathcal{A}) \subseteq (-\infty, +\infty]$.

Let $a = \inf\{\lambda(A) : A \in \mathcal{A}, A = \lambda\text{-negative}\}$, let $(A_n)_n$ be a sequence of λ-negative sets such that $\lambda(A_n) \to a$ and let $H = X \setminus \bigcup_{n=1}^{\infty} A_n$; then $H^c = \bigcup_{n=1}^{\infty} A_n$.

If we define $(B_n)_{n\in\mathbb{N}^*}$ letting $B_1 = A_1$ and, for every $n \geq 2$, $B_n = A_n \setminus \cup_{i=1}^{n-1} A_i$, then, for every $C \in \mathcal{A}$, $\lambda(C \cap H^c) = \lambda(C \cap \bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \lambda(C \cap B_n) \leq 0$. Therefore H^c is λ-negative.

Moreover, for every $n \in \mathbb{N}^*$,

$$\lambda(H^c) = \lambda(A_n) + \lambda(H^c \setminus A_n) \leq \lambda(A_n)$$

so that $\lambda(H^c) = a$.

If we suppose that H is not λ-positive, then there exists $C \in \mathcal{A}, C \subseteq H$ such that $\lambda(C) < 0$. The previous proposition assures us on the existence of a λ-negative set $B \subseteq C$ with $\lambda(B) < 0$. Then $H^c \cup B$ is λ-negative and

$$\lambda(H^c \cup B) = \lambda(H^c) + \lambda(B) = a + \lambda(B) < a$$

and this contradicts the definition of a.

In the case where $\lambda(\mathcal{A}) \subseteq [-\infty, +\infty), -\lambda : \mathcal{A} \to (-\infty, +\infty]$ is a σ-additive measure. Let H be a $(-\lambda)$-positive set and H^c be a $(-\lambda)$-negative set; then H^c is λ-positive set and H is λ-negative set.

Let now H_1 λ-positive and H_1^c λ-negative an other pair. For every $B \in \mathcal{A}$ with $B \subseteq H \setminus H_1 = H \cap H_1^c, \lambda(B) \leq 0$ and $\lambda(B) \geq 0$, hence $\lambda(B) = 0$. Then $H \setminus H_1$ is a λ-null set. Similarly, $H_1 \setminus H$ is λ-null and then $H \Delta H_1 = (H \setminus H_1) \cup (H_1 \setminus H)$ is a λ-null set. □

Definition 1.7. Every pair of sets $\{H, H^c\} \subseteq \mathcal{A}$, with the property that H is λ-positive and H^c is λ-negative, is called a *Hahn decomposition* of X relatively to the measure λ.

Remark 1.8.

(i) Hahn's decomposition of X relatively to a measure λ is not unique (we can replace H by $H \cup N$ where N is a λ-null set).

(ii) The Hahn decomposition theorem says that, for every σ-additive measure λ, there exists a Hahn decomposition of X relatively to λ.

Proposition 1.9. *Let $\lambda : \mathcal{A} \to \bar{\mathbb{R}}$ be a σ-additive measure, let $\{H, H^c\}$ be a Hahn decomposition of X relatively to λ and let $\lambda^+,\ \lambda^- : \mathcal{A} \to \bar{\mathbb{R}}_+$ defined by $\lambda^+(A) = \lambda(A \cap H),\ \lambda^-(A) = -\lambda(A \setminus H)$, for every $A \in \mathcal{A}$.*

Then λ^+, λ^- are two σ-additive positive measures (one of them finite), $\lambda = \lambda^+ - \lambda^-$ and $\lambda^+(H^c) = \lambda^-(H) = 0$.

If $\lambda_1^+,\ \lambda_1^-$ are two other σ-additive positive measures (one of them finite) such that $\lambda = \lambda_1^+ - \lambda_1^-$ and if $\lambda_1^+(H_1^c) = \lambda_1^-(H_1) = 0$ for a set $H_1 \in \mathcal{A}$, then $\lambda^+ = \lambda_1^+$ and $\lambda^- = \lambda_1^-$.

Proof. The first part of the proposition is obvious. We shall prove only the uniqueness of decomposition of λ as difference of positive measures.

The pair $\{H_1, H_1^c\}$ is again a Hahn decomposition of X relatively to λ. Indeed, for every $A \in \mathcal{A}$, $\lambda(A \cap H_1) = \lambda_1^+(A \cap H_1) - \lambda_1^-(A \cap H_1) = \lambda_1^+(A \cap H_1) \geq 0$ and $\lambda(A \setminus H_1) = \lambda_1^+(A \setminus H_1) - \lambda_1^-(A \setminus H_1) = -\lambda_1^-(A \setminus H_1) \leq 0$. According to Hahn decomposition theorem, $H \Delta H_1$ is a λ-null set. Therefore, for every $A \in \mathcal{A}$, $\lambda^+(A) = \lambda(A \cap H) = \lambda(A \cap H_1) = \lambda_1^+(A)$ so that $\lambda^+ = \lambda_1^+$. Similarly, $\lambda^- = \lambda_1^-$. □

Definition 1.10. We say that the unique pair $\{\lambda^+, \lambda^-\}$ of σ-additive positive measures (one of them finite) with $\lambda = \lambda^+ - \lambda^-$ and $\lambda^+(H^c) = \lambda^-(H) = 0$ for a set $H \in \mathcal{A}$, is the *Jordan decomposition of* λ.

Definition 1.11. A positive measure λ on $\mathcal{A}$ is *concentrated on the set* $D \in \mathcal{A}$ if $\lambda(D) = \lambda(X)$.

Remark 1.12. If λ is a σ-additive measure on $\mathcal{A}$, if $\{H, H^c\}$ is a Hahn decomposition of X relatively to λ and if $\{\lambda^+, \lambda^-\}$ is the Jordan decomposition of λ, then λ^+ is concentrated on the set H and λ^- is concentrated on H^c.

Theorem 1.13. *Let λ be a σ-additive measure on $\mathcal{A}$; there exist $A_m, A_M \in \mathcal{A}$ such that*

$$\lambda(A_m) = \inf\{\lambda(A) : A \in \mathcal{A}\} \leq 0 \leq \sup\{\lambda(A) : A \in \mathcal{A}\} = \lambda(A_M).$$

Every σ-additive measure is bounded either from below or from above.

Proof. Let $\{H, H^c\}$ a Hahn decomposition of X relatively to λ and let $\{\lambda^+, \lambda^-\}$ the Jordan decomposition of λ; for every $A \in \mathcal{A}$, $\lambda^+(A) = \lambda(A \cap H) \leq \lambda(H)$ and $\lambda^-(A) = -\lambda(A \setminus H) \leq \lambda^-(H^c) = \lambda(H^c)$. Then

$$-\lambda(H^c) \leq -\lambda^-(A) \leq \lambda^+(A) - \lambda^-(A) = \lambda(A) \leq \lambda^+(A) \leq \lambda(H).$$

Therefore we can take $A_m = H^c$ and $A_M = H$. □

The following result is a corollary of Theorem 1.13.

Corollary 1.14. *Every measure $\lambda \in ca(\mathcal{A})$ is bounded; therefore*

$$ca(\mathcal{A}) \subseteq ba(\mathcal{A}).$$

Remark 1.15. If $\lambda : \mathcal{A} \to \mathbb{R}$ is only an additive measure, it is not compulsory that λ should be bounded, as it is shown in the following example.

Example 1.16. Let $\sum_{n=0}^{\infty} a_n$ be a conditionally convergent series (a convergent series for which $\sum_{n=0}^{\infty} |a_n| = +\infty$), let $\mathcal{A}$ be the algebra of all sets $A \subseteq \mathbb{N}$ such that A or $\mathbb{N} \setminus A$ is a finite and let $\mu : \mathcal{A} \to \mathbb{R}$ defined by

$$\mu(A) = \begin{cases} \sum\limits_{n \in A} a_n, & A \neq \emptyset, \\ 0, & A = \emptyset. \end{cases}$$

Then μ is an additive measure, but it is not bounded on the algebra $\mathcal{A}$. We notice that $\mathcal{A}$ is not a σ-algebra, but, since all additive function on an algebra can be extended to an additive function on the generated σ-algebra – in our case $\mathcal{P}(\mathbb{N})$ – (see [29], p.185 and [175],1.8.), it is evident that the extension itself is not bounded.

The total variation defined below is introduced in order to define a complete norm on $ba(\mathcal{A})$ or on its subspace $ca(\mathcal{A})$ of σ-additive measures (see Definition III.1.4 and Lemma III.1.6 of [62]).

Theorem 1.17. *For every additive measure λ, let $|\lambda| : \mathcal{A} \to \overline{\mathbb{R}}_+$ defined by*

$$|\lambda|(A) = \sup \left\{ \sum_{i=1}^{n} |\lambda(A_i)| : n \in \mathbb{N}^*, \quad \{A_1, \ldots, A_n\} = \mathcal{A} - \textit{partition of } A \right\}.$$

Then:

(i) $\sup_{B \in \mathcal{A}, B \subseteq A} |\lambda(B)| \leq |\lambda|(A) \leq 2 \sup_{B \in \mathcal{A}, B \subseteq A} |\lambda(B)|$.

(ii) $|\lambda|$ *is additive.*

(iii) *If $\lambda \in ba(\mathcal{A})$, then $|\lambda| \in ba^+(\mathcal{A})$; moreover $|\lambda|$ is the smallest element of the set $\mathfrak{M} = \{\nu \in ba^+(\mathcal{A}) : |\lambda(A)| \leq \nu(A), \quad \forall A \in \mathcal{A}\}$.*

(iv) *If λ is σ-additive, then $|\lambda|$ is σ-additive and $|\lambda|(H) = \lambda(H)$ and $|\lambda|(H^c) = -\lambda(H^c)$, where $\{H, H^c\}$ is a Hahn decomposition of X.*

If $\lambda \in ca(\mathcal{A})$, then $|\lambda| \in ca^+(\mathcal{A})$.

Proof. (i) For every $B \in \mathcal{A}$ with $B \subseteq A$, $\{B, \ A \setminus B\}$ is an $\mathcal{A}$-partition of A and so $|\lambda|(A) \geq |\lambda(B)| + |\lambda(A \setminus B)| \geq |\lambda(B)|$, from where

$$\sup\{|\lambda(B)| : B \in \mathcal{A}, B \subseteq A\} \leq |\lambda|(A).$$

Now, let $n \in \mathbb{N}^*$ and let $\{A_1, \ldots, A_n\}$ be an $\mathcal{A}$-partition of the set A. We can suppose that $\lambda(A_1), \ldots, \lambda(A_p) \geq 0$ and $\lambda(A_{p+1}), \ldots, \lambda(A_n) < 0$. Then

$$\begin{aligned}\sum_{i=1}^{n} |\lambda(A_i)| &= \sum_{i=1}^{p} \lambda(A_i) - \sum_{j=p+1}^{n} \lambda(A_j) = \lambda\left(\bigcup_{i=1}^{p} A_i\right) - \lambda\left(\bigcup_{j=p+1}^{n} A_j\right) \\ &= \left|\lambda\left(\bigcup_{i=1}^{p} A_i\right)\right| + \left|\lambda\left(\bigcup_{j=p+1}^{n} A_j\right)\right| \\ &\leq 2 \sup\{|\lambda(B)| : B \in \mathcal{A}, B \subseteq A\}.\end{aligned}$$

(ii) Let $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$ and let $C = A \cup B$. If $|\lambda|(A) = +\infty$, then $+\infty = \sup\{|\lambda(D)| : D \in \mathcal{A}, D \subseteq A\} \leq \sup\{|\lambda(D)| : D \in \mathcal{A}, D \subseteq C\} \leq |\lambda|(C)$ and so $|\lambda|(C) = +\infty = |\lambda|(A) + |\lambda|(B)$.

In the same way, if $|\lambda|(B) = +\infty$, we have $|\lambda|(C) = +\infty = |\lambda|(A) + |\lambda|(B)$.

Now suppose that $|\lambda|(A) < +\infty$ and $|\lambda|(B) < +\infty$. Then, for every $\varepsilon > 0$, there exists an $\mathcal{A}$-partition $\{A_1, \ldots, A_n\}$ of A and an $\mathcal{A}$-partition of B, $\{B_1, \ldots, B_m\}$, such that

$$|\lambda|(A) - \frac{\varepsilon}{2} < \sum_{i=1}^{n} |\lambda(A_i)| \quad \text{and} \quad |\lambda|(B) - \frac{\varepsilon}{2} < \sum_{j=1}^{m} |\lambda(B_j)|.$$

Then $\{A_1, \ldots, A_n, B_1, \ldots, B_m\}$ is an $\mathcal{A}$-partition of C and therefore

$$|\lambda|(A) + |\lambda|(B) - \varepsilon < \sum_{i=1}^{n} |\lambda(A_i)| + \sum_{j=1}^{m} |\lambda(B_j)| \leq |\lambda|(C),$$

from where, ε being arbitrary,

$$|\lambda|(A) + |\lambda|(B) \leq |\lambda|(C). \tag{1}$$

For every $\mathcal{A}$-partition $\{C_1, \ldots, C_p\}$ of C, let's note $A_i = C_i \cap A$ and $B_i = C_i \cap B$, for all $i = 1, \ldots, p$. Then $A_i, B_i \in \mathcal{A}$ and $C_i = A_i \cup B_i$. Therefore $\{A_1, \ldots, A_p\}$ is an $\mathcal{A}$-partition of A and $\{B_1, \ldots, B_p\}$ is an $\mathcal{A}$-partition of B.

$$\begin{aligned}\sum_{i=1}^{p} |\lambda(C_i)| &= \sum_{i=1}^{p} |\lambda(A_i) + \lambda(B_i)| \\ &\leq \sum_{i=1}^{p} |\lambda(A_i)| + \sum_{i=1}^{p} |\lambda(B_i)| \leq |\lambda|(A) + |\lambda|(B).\end{aligned}$$

As $\{C_1, \ldots, C_p\}$ is an arbitrary partition of C, we have:

$$|\lambda|(C) \leq |\lambda|(A) + |\lambda|(B). \tag{2}$$

From (1) and (2) we have that $|\lambda|$ is additive.

(iii) If $\lambda \in ba(\mathcal{A})$, then λ is bounded; from *(i)*, we obtain that,

$$0 \leq |\lambda|(A) \leq 2\sup\{|\lambda(B)| : B \in \mathcal{A}\} = M < +\infty, \quad \text{for every} \quad A \in \mathcal{A}$$

and so $\sup\{|\lambda|(A) : A \in \mathcal{A}\} \leq M < +\infty$.

Therefore $|\lambda| : \mathcal{A} \to \mathbb{R}_+$ is a positive bounded additive measure on $\mathcal{A}$, which means that $|\lambda| \in ba^+(\mathcal{A})$.

Since $|\lambda(A)| \leq \sup\{|\lambda(B)| : B \in \mathcal{A}, B \subseteq A\} \leq |\lambda|(A)$, for all $A \in \mathcal{A}$, it is clear that $|\lambda| \in \mathfrak{M} = \{\nu \in ba^+(\mathcal{A}) : |\lambda(A)| \leq \nu(A)$, for all $A \in \mathcal{A}\}$.

It remains to show that $|\lambda|$ is the smallest element of $\mathfrak{M}$.

Let $\nu \in \mathfrak{M}$, $A \in \mathcal{A}$ and let $\{A_1, \ldots, A_n\}$ be an $\mathcal{A}$-partition of A; we have

$$\sum_{i=1}^{n} |\lambda(A_i)| \leq \sum_{i=1}^{n} \nu(A_i) = \nu\left(\bigcup_{i=1}^{n} A_i\right) = \nu(A),$$

from where, $|\lambda|(A) \leq \nu(A)$, for every $A \in \mathcal{A}$ and so $|\lambda| \leq \nu$.

(iv) According to (ii), $|\lambda|$ is finite additive. Let $(E_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be a sequence of pairwise disjoint sets and let $E = \bigcup_{n\in\mathbb{N}} E_n$. Then, for every $m \in \mathbb{N}$,

$$|\lambda|(E) \geq |\lambda|(\cup_{n\leq m} E_n) = \sum_{n=0}^{m} |\lambda|(E_n) \quad \text{and so} \quad |\lambda|(E) \geq \sum_{n=0}^{\infty} |\lambda|(E_n).$$

To demonstrate the inverse inequality, let $(F_i)_{i\leq k}$ be an $\mathcal{A}$-partition of E. Then,

$$\begin{aligned}\sum_{i=1}^{k} |\lambda(F_i)| &= \textstyle\sum_{i=1}^{k} |\lambda(F_i \cap (\cup_{n\in\mathbb{N}} E_n)| = \sum_{i=1}^{k} \left|\sum_{n=0}^{\infty} \lambda(F_i \cap E_n)\right| \\ &\leq \textstyle\sum_{n=0}^{\infty} \sum_{i=1}^{k} |\lambda(F_i \cap E_n)| \leq \sum_{n=0}^{\infty} |\lambda|(E_n)\end{aligned}$$

from where $|\lambda|(E) \leq \sum_{n=0}^{\infty} |\lambda|(E_n)$. Therefore $|\lambda|$ is σ-additive.

Let now $\{H, H^c\}$ be a Hahn decomposition of X relatively to λ. Then H is λ-positive and then, for every $\mathcal{A}$ - partition of H, $\{A_1, \ldots, A_n\}$, $\sum_{i=1}^{n} |\lambda(A_i)| = \sum_{i=1}^{n} \lambda(A_i) = \lambda(H)$; so that $|\lambda|(H) = \lambda(H)$. Similarly, $|\lambda|(H^c) = -\lambda(H^c)$.

If $\lambda \in ca(\mathcal{A})$, then λ is σ-additive and bounded (see Corollary 1.14). Therefore $|\lambda|$ is σ-additive and by *(i)*, $|\lambda|$ belongs to $ca(\mathcal{A})$. □

Remark 1.18. Let λ be a σ-additive measure, let $\{H, H^c\}$ be a Hahn decomposition of X relatively to λ and let $\{\lambda^+, \lambda^-\}$ be the Jordan decomposition of λ; then

$$\lambda^+ = \frac{1}{2}(|\lambda| + \lambda), \quad \lambda^- = \frac{1}{2}(|\lambda| - \lambda) \quad \text{and} \quad |\lambda| = \lambda^+ + \lambda^-.$$

Indeed, if we note $\lambda_1^+ = \frac{1}{2}(|\lambda|+\lambda)$ and $\lambda_1^- = \frac{1}{2}(|\lambda|-\lambda)$, then λ_1^+ and λ_1^- are σ-additive positive measures, $\lambda = \lambda_1^+ - \lambda_1^-$ and $\lambda_1^+(H^c) = \frac{1}{2}(-\lambda(H^c)+\lambda(H^c)) = 0 = \lambda^-(H)$. Therefore $\{\lambda_1^+, \lambda_1^-\}$ is the Jordan decomposition of λ and then $\lambda^+ = \lambda_1^+$ and $\lambda^- = \lambda_1^-$.

Definition 1.19. Let $\lambda : \mathcal{A} \to \bar{\mathbb{R}}$ be an additive measure; $|\lambda|$ is called *the total variation* of λ.

Let $\lambda \in ba(\mathcal{A})$; according to previous remark, we say that $\lambda^+(\lambda^-)$ are *the positive variation* (*negative variation*) of λ, where $\lambda^+(\lambda^-) : \mathcal{A} \to \mathbb{R}_+$ is defined by

$$\lambda^+(A) = \frac{1}{2}(\,|\lambda|(A) + \lambda(A)) \quad \text{for every} \quad A \in \mathcal{A}$$
$$(\lambda^-(A) = \frac{1}{2}(\,|\lambda|(A) - \lambda(A)) \quad \text{for every} \quad A \in \mathcal{A}).$$

Obviously, $\lambda^+, \lambda^- \in ba^+(\mathcal{A})$, $\lambda = \lambda^+ - \lambda^-$ and $|\lambda| = \lambda^+ + \lambda^-$.

If $\lambda \in ca(\mathcal{A})$, then $\lambda^+, \lambda^- \in ca^+(\mathcal{A})$.

Remark 1.20. It results that every bounded additive measure is a difference between two bounded positive additive measures. This decomposition is not unique. Indeed, if $\lambda \in ba(\mathcal{A})$, for all $\mu \in ba^+(\mathcal{A})$, $\lambda = (\lambda^+ + \mu) - (\lambda^- + \mu)$ is another decomposition of λ.

In the following results, we will mention some direct consequences of Theorem 1.13.

Corollary 1.21. *If $\lambda \in ca(\mathcal{A})$, then the sets A_m and A_M, introduced in Theorem* 1.13*, have the following properties:*

(i) $\lambda(A) \geq 0, \quad \forall A \in \mathcal{A}, A \subseteq A_M,$

$\lambda(A) \leq 0, \quad \forall A \in \mathcal{A}, A \subseteq A_m.$

(ii) $\lambda(A) = 0, \quad \forall A \in \mathcal{A}, A \subseteq A_m \cap A_M.$

(iii) $\lambda(A \setminus (A_m \cup A_M)) = 0, \quad \forall A \in \mathcal{A}$.

(iv) $\lambda(A) = \lambda(A \cap A_M) + \lambda(A \cap A_m), \quad \forall A \in \mathcal{A}$.

Proof. (i) This point is demonstrated in the previous proposition.

(ii) is a consequence of (i).

(iii) Suppose that there exists a set $A_0 \in \mathcal{A}$ such that $\lambda(A_0 \setminus (A_m \cup A_M)) \neq 0$. Let $B_0 = A_0 \setminus (A_m \cup A_M)$.

If $\lambda(B_0) > 0$, let $B_1 = B_0 \cup A_M$; then $\lambda(B_1) = \lambda(B_0) + \lambda(A_M) > \lambda(A_M)$ which contradicts the maximality of A_M.

If $\lambda(B_0) < 0$, let $B_1 = B_0 \cup A_m$; then $\lambda(B_1) = \lambda(B_0) + \lambda(A_m) < \lambda(A_m)$ which contradicts the minimality of A_m. Therefore (iii) is satisfied.

(iv) For every $A \in \mathcal{A}, \lambda(A) = \lambda(A \cap A_M) + \lambda(A \setminus A_M) = \lambda(A \cap A_M) + \lambda((A \setminus A_M) \cap A_m) + \lambda((A \setminus A_M) \setminus A_m) = \lambda(A \cap A_M) + \lambda(A \cap A_m) - \lambda(A_m \cap A_M) + \lambda[A \setminus (A_m \cup A_M)]$.

According to (ii) and (iii), the last two terms are null. □

Corollary 1.22. *Let $\lambda \in ca(\mathcal{A})$ and A_m, A_M the already defined sets. Then, for every $A \in \mathcal{A}$,*

(i) $\lambda(A \cap A_M) = \sup\{\lambda(E) : E \in \mathcal{A}, E \subseteq A\}$,

$\lambda(A \cap A_m) = \inf\{\lambda(E) : E \in \mathcal{A}, E \subseteq A\}$.

(ii) $|\lambda|(A) = \lambda(A \cap A_M) - \lambda(A \cap A_m)$.

(iii) $\lambda^+(A) = \lambda(A \cap A_M), \lambda^-(A) = -\lambda(A \cap A_m)$.

Proof. (i) Obviously, $\lambda(A \cap A_M) \leq \sup\{\lambda(E) : E \in \mathcal{A}, E \subseteq A\}$.

We suppose that $\lambda(A \cap A_M) < \sup\{\lambda(E) : E \in \mathcal{A}, E \subseteq A\}$. Then there exists $E_0 \in \mathcal{A}, E_0 \subseteq A$ such that $\lambda(A \cap A_M) < \lambda(E_0)$. $\lambda(A_M) = \lambda(A \cap A_M) + \lambda(A_M \setminus A) < \lambda(E_0) + \lambda(A_M \setminus A) = \lambda(E_0 \cup (A_M \setminus A))$, which contradicts the maximality of the set A_M. The second equality is proved in a similar manner.

(ii) For every $A \in \mathcal{A}, \{A \cap A_M, A \setminus A_M\}$ is an $\mathcal{A}$-partition of A; therefore $|\lambda|(A) \geq |\lambda(A \cap A_M)| + |\lambda(A \setminus A_M)|$. According to (iv) of Corollary 1.21 $\lambda(A \setminus A_M) = \lambda((A \setminus A_M) \cap A_M) + \lambda((A \setminus A_M) \cap A_m) = \lambda(A \cap A_m \setminus A_M)$; by (ii) of Theorem 1.21, $\lambda(A \cap A_m \setminus A_M) = \lambda(A \cap A_m \cap A_M) + \lambda(A \cap A_m \setminus A_M) = \lambda(A \cap A_m)$. Finally, using (i) of Theorem 1.21,

$$|\lambda|(A) \geq \lambda(A \cap A_M) - \lambda(A \cap A_m). \tag{1}$$

If we note $\lambda_1(A) = \lambda(A \cap A_M) - \lambda(A \cap A_m)$, then λ_1 is a positive measure and, according to (iv) and (i) of Theorem 1.21, for every $A \in \mathcal{A}$,

$$|\lambda(A)| = |\lambda(A \cap A_M) + \lambda(A \cap A_m)| \leq \lambda(A \cap A_M) - \lambda(A \cap A_m) = \lambda_1(A).$$

According to (iii) of Theorem 1.17,

$$|\lambda|(A) \leq \lambda_1(A) = \lambda(A \cap A_M) - \lambda(A \cap A_m). \tag{2}$$

By (1) and (2), $|\lambda|(A) = \lambda(A \cap A_M) - \lambda(A \cap A_m)$.

(iii) According to (ii) and to (iv) of Corollary 1.21, $\lambda^+(A) = \frac{1}{2}(|\lambda|(A) + \lambda(A)) = \frac{1}{2}(\lambda(A \cap A_M) - \lambda(A \cap A_m) + \lambda(A \cap A_M) + \lambda(A \cap A_m)) = \lambda(A \cap A_M)$ and $\lambda^-(A) = \frac{1}{2}(|\lambda|(A) - \lambda(A)) = \frac{1}{2}(\lambda(A \cap A_M) - \lambda(A \cap A_m) - \lambda(A \cap A_M) - \lambda(A \cap A_m)) = -\lambda(A \cap A_m)$. □

Theorem 1.17 allows us to introduce a norm on $ba(\mathcal{A})$ equivalent to the norm $\|\cdot\|_\infty$ of the uniform convergence.

Theorem 1.23. *The applications $\|\cdot\|, \|\cdot\|_\infty : ba(\mathcal{A}) \to \mathbb{R}_+$ defined as $\|\lambda\| = |\lambda|(X) = \lambda^+(X) + \lambda^-(X)$ and $\|\lambda\|_\infty = \sup_{A\in\mathcal{A}}(|\lambda(A)|)$ are two equivalent norms on $ba(\mathcal{A})$.*

The spaces $(ba(\mathcal{A}), \|\cdot\|)$ and $(ba(\mathcal{A}), \|\cdot\|_\infty)$ are Banach spaces; $ca(\mathcal{A})$ is a Banach subspace of $ba(\mathcal{A})$.

Proof. Firstly, we show that $\|\cdot\|, \|\cdot\|_\infty$ are norms. According to Theorem 1.17(i), for every $\lambda \in ba(\mathcal{A})$

$$\|\lambda\|_\infty \leq \|\lambda\| \leq 2\|\lambda\|_\infty.$$

Therefore we have $\|\lambda\| = 0 \Leftrightarrow \|\lambda\|_\infty = 0 \Leftrightarrow \sup_{A\in\mathcal{A}} |\lambda(A)| = 0$ what comes back to $\lambda = 0$. For every $a \in \mathbb{R}$, $\|a\lambda\| = |a\lambda|(X) = \sup\{\sum_{i=1}^n |a\lambda(A_i)| : \{A_1, \ldots, A_n\} = \mathcal{A}\text{-partition of } X\} = |a| \cdot \|\lambda\|$ and $\|a\lambda\|_\infty = \sup_{A\in\mathcal{A}} |a\lambda(A)| = |a| \cdot \|\lambda\|_\infty$.

Now we prove the triangular inequality; according to Theorem 1.17, for every $\varepsilon > 0$, there exists an $\mathcal{A}$-partition of X, $\{A_1, \ldots, A_n\}$, such that

$$\begin{aligned}\|\lambda + \mu\| - \varepsilon = |\lambda + \mu|(X) - \varepsilon &< \sum_{i=1}^n |\lambda(A_i) + \mu(A_i)| \\ &\leq \sum_{i=1}^n |\lambda(A_i)| + \sum_{i=1}^n |\mu(A_i)| \leq \|\lambda\| + \|\mu\|,\end{aligned}$$

from where $\|\lambda + \mu\| \leq \|\lambda\| + \|\mu\|$.

In the same way for $\|\cdot\|_\infty$, for every $\varepsilon > 0$, there exists $A_\varepsilon \in \mathcal{A}$ such that

$$\|\lambda + \mu\|_\infty - \varepsilon < |(\lambda + \mu)(A_\varepsilon)| \leq |\lambda(A_\varepsilon)| + |\mu(A_\varepsilon)| \leq \|\lambda\|_\infty + \|\mu\|_\infty$$

from where $\|\lambda + \mu\|_\infty \leq \|\lambda\|_\infty + \|\mu\|_\infty$.

Therefore, in the light of the inequalities mentioned at the beginning of the demonstration, $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent norms on $ba(\mathcal{A})$.

It is evident that

$$\lambda_n \xrightarrow{\|\cdot\|} \lambda \Leftrightarrow \lambda_n \xrightarrow{\|\cdot\|_\infty} \lambda \Leftrightarrow \lambda_n \underset{\mathcal{A}}{\xrightarrow{u}} \lambda.$$

Similarly, if (λ_n) is a sequence $\|\cdot\|$-Cauchy (and so $\|\cdot\|_\infty$-Cauchy), then $(\lambda_n(A))_{n\in\mathbb{N}}$ is a Cauchy sequence, uniformly in $A \in \mathcal{A}$; then there exists $\lambda : \mathcal{A} \to \mathbb{R}$ such that $\lambda_n \underset{\mathcal{A}}{\xrightarrow{u}} \lambda$. Therefore λ is additive and bounded on $\mathcal{A}$; that is to say $\lambda \in ba(\mathcal{A})$. (λ_n) converges to λ in $(ba(\mathcal{A}), \|\cdot\|)$ and also in $(ba(\mathcal{A}), \|\cdot\|_\infty)$ so that $(ba(\mathcal{A}), \|\cdot\|)$ and $(ba(\mathcal{A}), \|\cdot\|_\infty)$ are Banach spaces.

Finally, in order to establish that $ca(\mathcal{A})$ is a closed subspace of $(ba(\mathcal{A}), \|\cdot\|)$, let $\lambda \in ba(\mathcal{A})$ and $(\lambda_n) \subseteq ca(\mathcal{A})$ such that $\lambda_n \xrightarrow{\|\cdot\|} \lambda$; we show that $\lambda \in ca(\mathcal{A})$.

Let $\{A_p : p \in \mathbb{N}^*\} \subseteq \mathcal{A}$ be a pairwise disjoint family of sets and let $A = \cup_{p=1}^{\infty} A_p \in \mathcal{A}$. Since $\lambda_n \underset{\mathcal{A}}{\xrightarrow{u}} \lambda$, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$|\lambda_n(B) - \lambda(B)| < \varepsilon, \quad \forall n \geq n_0, \quad \forall B \in \mathcal{A}. \tag{1}$$

Because $\lambda_{n_0} \in ca(\mathcal{A})$, there exists $n_1 > n_0$ such that, for every $n \geq n_1$

$$\left|\lambda_{n_0}(A) - \sum_{k=1}^{n} \lambda_{n_0}(A_k)\right| < \varepsilon. \tag{2}$$

Then, for all $n \geq n_1$,

$$\begin{aligned}\left|\lambda(A) - \sum_{k=1}^{n} \lambda(A_k)\right| &= \left|\lambda\left(A \setminus \bigcup_{k=1}^{n} A_k\right)\right| \leq \left|\lambda\left(A \setminus \bigcup_{k=1}^{n} A_k\right) - \lambda_{n_0}\left(A \setminus \bigcup_{k=1}^{n} A_k\right)\right| \\ &\quad + \left|\lambda_{n_0}\left(A \setminus \bigcup_{k=1}^{n} A_k\right)\right| = \left|\lambda\left(A \setminus \bigcup_{k=1}^{n} A_k\right) - \right. \\ &\quad \left. \lambda_{n_0}\left(A \setminus \bigcup_{k=1}^{n} A_k\right)\right| + \left|\lambda_{n_0}(A) - \sum_{k=1}^{n} \lambda_{n_0}(A_k)\right|.\end{aligned}$$

From (1) and (2), we obtain

$$\left|\lambda(A) - \sum_{k=1}^{n} \lambda(A_k)\right| < 2\varepsilon, \quad \forall n \geq n_1, \tag{3}$$

so that $\lambda \in ca(\mathcal{A})$. □

Before mentioning the definition of the integral in relation to a signed measure, we need to clarify a number of notations and properties of the integral relatively to a positive measure.

Let λ be a positive σ-additive measure on $\mathcal{A}$ and let $f : X \to \mathbb{R}$ be an $\mathcal{A}$-measurable mapping. We recall that f is λ-integrable if $f^+ = \sup\{f, 0\}$ and $f^- = \sup\{-f, 0\}$ are λ-integrable. Then, $\int_X f d\lambda = \int_X f^+ d\lambda - \int_X f^- d\lambda$; let $\mathcal{L}^1(\lambda)$ be the set of all λ-integrable mappings and let $L^1(\lambda) = L^1(X, \mathcal{A}, \lambda)$ be the quotient space $\mathcal{L}^1(\lambda)/\doteq$, where $\doteq$ is equality λ-almost everywhere.

If at least one of the two functions f^+ and f^- is λ-integrable, the difference of the integrals is always defined and will be marked by $\int_X f d\lambda$.

Definition 1.24. Let $\lambda \in ca(\mathcal{A})$ and let $f : X \to \mathbb{R}$ be an $\mathcal{A}$-measurable mapping; we say that *f is λ-integrable* if $f \in \mathcal{L}^1(\lambda^+) \cap \mathcal{L}^1(\lambda^-)$. Let us denote

$$\mathcal{L}^1(\lambda) = \mathcal{L}^1(\lambda^+) \cap \mathcal{L}^1(\lambda^-), L^1(\lambda) = L^1(\lambda^+) \cap L^1(\lambda^-) \quad \text{and}$$
$$\int_A f d\lambda = \int_A f d\lambda^+ - \int_A f d\lambda^-, \quad \text{for every} \quad A \in \mathcal{A},$$

where f marks, according to the context, the function f or the equivalence class of a function f. It is clear that $L^1(\lambda)$ is a vector space and that $\int_A$ is a linear operator on $L^1(\lambda)$.

Proposition 1.25. *Let $\lambda \in ca^+(\mathcal{A})$ and $f : X \to \mathbb{R}$ be an $\mathcal{A}$-measurable mapping such that at least one of mappings f^+ and f^- is λ-integrable. Let $\mu(A) = \int_A f^+ d\lambda - \int_A f^- d\lambda$. Then, μ is a σ-additive measure on $\mathcal{A}$ and $\mu^+(A) = \int_A f^+ d\lambda$, $\mu^-(A) = \int_A f^- d\lambda$.*

If $f \in \mathcal{L}^1(\lambda)$, then $\mu \in ca(\mathcal{A})$ and $\|\mu\| = |\mu|(X) = \int_X |f| d\lambda$.

Proof. Let $H = \{x \in X : f(x) \geq 0\}$; $H \in \mathcal{A}$, $f^+\chi_H = f^+$ and $f^-\chi_H = 0$. $\int_A f^+ d\lambda = \int_A f^+\chi_H d\lambda = \int_{A\cap H} f^+ d\lambda$ and $\int_{A\cap H} f^- d\lambda = \int_A f^-\chi_H d\lambda = 0$.

Therefore $\mu(A \cap H) = \int_A f^+ d\lambda$. If $B \subset A$, then $\mu(B) \leq \int_B f^+ d\lambda \leq \int_A f^+ d\lambda$. Then, according to Corollary 1.22, $\int_A f^+ d\lambda = \sup\{\mu(B) : B \in \mathcal{A}, B \subset A\} = \mu^+(A) = \mu(A \cap H)$, which leads to $\int_A f^- d\lambda = \mu^-(A) = -\mu(A \cap H^c)$.

If $f \in \mathcal{L}^1(\lambda)$, then $\mu(A) = \int_A f d\lambda$, for every $A \in \mathcal{A}$, hence $\mu \in ca(\mathcal{A})$. According to Definition 1.19, $\|\mu\| = |\mu|(X) = \mu^+(X) + \mu^-(X) = \int_X f^+ d\lambda + \int_X f^- d\lambda = \int_X |f| d\lambda$. □

Proposition 1.26.

(i) $L^1(\lambda) = L^1(|\lambda|)$ *and* $|\int_A f d\lambda| \le \int_A |f| d|\lambda|, \quad \forall A \in \mathcal{A}, \quad \forall f \in L^1(\lambda).$

(ii) $|\lambda|(A) = \sup\{|\int_A f d\lambda| : f \in L^1(\lambda), |f| \le 1\}.$

(iii) *The mapping* $\|\cdot\|_1 : L^1(\lambda) \to \mathbb{R}_+, \|f\|_1 = \int_X |f| d|\lambda|$ *is a norm on* $L^1(\lambda)$ *and* $(L^1(\lambda), \|\cdot\|_1)$ *is a Banach space.*

Proof. (i) Let $f \in L^1(\lambda)$; then f is $\mathcal{A}$-measurable and $f \in L^1(\lambda^+) \cap L^1(\lambda^-)$. $\int_X |f| d|\lambda| = \int_X |f| d\lambda^+ + \int_X |f| d\lambda^- < +\infty$ and then $f \in L^1(|\lambda|)$.

Reciprocally, if $f \in L^1(|\lambda|)$, $\int_X |f| d\lambda^+ \le \int_X |f| d|\lambda| < +\infty$ and so $f \in L^1(\lambda^+)$; similarly, $f \in L^1(\lambda^-)$ and, therefore, $f \in L^1(\lambda)$.

The inequality follows immediately:

$$\begin{aligned}\left|\int_A f d\lambda\right| &= \left|\int_A f d\lambda^+ - \int_A f d\lambda^-\right| \\ &\le \left|\int_A f d\lambda^+\right| + \left|\int_A f d\lambda^-\right| \le \int_A |f| d\lambda^+ + \int_A |f| d\lambda^- \\ &= \int_A |f| d|\lambda|.\end{aligned}$$

(ii)

$$|\lambda|(A) = \sup\left\{\sum_{i=1}^n |\lambda(A_i)| : n \in \mathbb{N}^* : \{A_1, \dots, A_n\} = \mathcal{A} - \text{partition of } A\right\}.$$

Let $\{A_1, \dots, A_n\}$ be an arbitrary partition of A. We can assume that $\lambda(A_i) \ge 0$, for every $i = 1, \dots, p$ and $\lambda(A_j) < 0$, for every $j = p+1, \dots, n$. Let $f = \chi_{\cup_1^p A_i} - \chi_{\cup_{p+1}^n A_j}$; then $|f| = 1$, f is $\mathcal{A}$-measurable and

$$\begin{aligned}\sum_1^n |\lambda(A_i)| = \lambda(\cup_1^p A_i) - \lambda(\cup_{p+1}^n A_j) &= \int_A f d\lambda \\ &\le \sup\left\{\left|\int_A f d\lambda\right| : f \in L^1(\lambda), |f| \le 1\right\}.\end{aligned}$$

We have therefore $|\lambda|(A) \le \sup\left\{|\int_A f d\lambda| : f \in L^1(\lambda), |f| \le 1\right\}$.

On the other hand, for every $f \in L^1(\lambda)$ with $|f| \le 1$, by (i), we have

$$\left|\int_A f d\lambda\right| \le \int_A |f| d|\lambda| \le \int_A 1 d|\lambda| = |\lambda|(A),$$

from where $\sup\{|\int_A f d\lambda| : f \in L^1(\lambda), |f| \le 1\} \le |\lambda|(A)$.

(iii) Obviously, $\|\cdot\|_1$ is a norm on $L^1(\lambda) = L^1(|\lambda|)$. It is easy to see that $(L^1(|\lambda|), \|\cdot\|_1)$ is a Banach space. □

In the following, we will present Saks' theorem which will be useful in the study of the weak compactness on $(ca(\mathcal{A}), \|\cdot\|)$ (see [62], Lemma III.7.1 and [57], Theorem 8, p. 86).

Theorem 1.27 (Saks). *Let $\lambda : \mathcal{A} \to \overline{\mathbb{R}}_+$ be a positive and σ-additive measure. The mapping $d_\lambda : \mathcal{A} \times \mathcal{A} \to \mathbb{R}_+$, defined by $d_\lambda(A, B) = \arctan(\lambda(A \triangle B))$, for every $(A, B) \in \mathcal{A} \times \mathcal{A}$, is a pseudo-metric on $\mathcal{A}$. The pseudo-metric space $(\mathcal{A}, d_\lambda)$ is complete and the binary operations $(A, B) \mapsto A \cup B$, $(A, B) \mapsto A \cap B$ and $(A, B) \mapsto A \setminus B$ are continuous maps on $(\mathcal{A} \times \mathcal{A}, d_\lambda \times d_\lambda)$.*

Proof. Let's recall that if $\lambda(A \triangle B)$ is finite, $d_\lambda(A, B) = \arctan(\lambda(A \triangle B)) \in [0, \frac{\pi}{2}[$ and if $\lambda(A \triangle B) = +\infty$, then $d_\lambda(A, B) = \frac{\pi}{2}$.

We can put $D_\lambda(A, B) = \lambda(A \triangle B)$; obviously $d_\lambda = \arctan D_\lambda$ is a pseudo-metric on $\mathcal{A}$. In the following we say that d_λ is *the pseudo-metric associated* to λ.

It remains to demonstrate that $(\mathcal{A}, d_\lambda)$ is complete. Let $(A_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathcal{A}, d_\lambda)$; for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $m, n \geq n_0$, $D_\lambda(A_m, A_n) = \lambda(A_m \triangle A_n) < \varepsilon$.

Step by step, we define a strictly increasing sequence of integers $(k_n)_{n\in\mathbb{N}}$ such that, for every $n \in \mathbb{N}$, $\lambda(A_{k_n} \triangle A_{k_{n+1}}) < \frac{1}{2^n}$.

Let $N = \limsup_n (A_{k_n} \triangle A_{k_{n+1}}) = \bigcap_{n=0}^{\infty} \bigcup_{i=n}^{\infty} (A_{k_i} \triangle A_{k_{i+1}}) \in \mathcal{A}$; for every $n \in \mathbb{N}$, $N \subseteq \bigcup_{i=n}^{\infty} (A_{k_i} \triangle A_{k_{i+1}})$ and then $\lambda(N) \leq \sum_{i=n}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n-1}}$. Therefore $\lambda(N) = 0$.

Let $A = \liminf_n A_{k_n} = \bigcup_{n=0}^{\infty} \bigcap_{i=n}^{\infty} A_{k_i} \in \mathcal{A}$; then

$$X \setminus N \subseteq A \cup \left[\liminf_n (X \setminus A_{k_n})\right]. \tag{1}$$

Indeed, for every $x \in X \setminus N = \bigcup_{n=0}^{\infty} \bigcap_{i=n}^{\infty} [X \setminus (A_{k_i} \triangle A_{k_{i+1}})]$, there exists $n_0 \in \mathbb{N}$ such that, for all $i \geq n_0$, $x \notin A_{k_i} \triangle A_{k_{i+1}}$, or $\chi_{A_{k_i}}(x) = \chi_{A_{k_{i+1}}}(x)$; therefore $\chi_{A_{k_i}}(x) = 1$, for every $i \geq n_0$, or $\chi_{A_{k_i}}(x) = 0$, for every $i \geq n_0$, from where $x \in (\liminf_n A_{k_n}) \cup [\liminf_n (X \setminus A_{k_n})]$.

From (1), we have

$$A_{k_p} \triangle A \subseteq N \cup \left[\liminf_n (A_{k_p} \triangle A_{k_n})\right], \quad \text{for every} \quad p \in \mathbb{N}. \tag{2}$$

Indeed, from (1), one gets that, for every $p \in \mathbb{N}$,

$$(A_{k_p} \setminus A) \setminus N \subseteq A_{k_p} \cap \left[\liminf_n (X \setminus A_{k_n})\right] = \liminf_n (A_{k_p} \setminus A_{k_n}) \quad \text{and}$$
$$(A \setminus A_{k_p}) \setminus N \subseteq \left(\liminf_n A_{k_n}\right) \setminus A_{k_p} = \liminf_n (A_{k_n} \setminus A_{k_p}).$$

From (2), we have,

$$\begin{aligned}
\lambda(A_{k_p} \triangle A) &\le \lambda(\liminf_n (A_{k_p} \triangle A_{k_n})) \le \liminf_n \lambda(A_{k_p} \triangle A_{k_n}) \\
&\le \liminf_n \left[\lambda(A_{k_p} \triangle A_{k_{p+1}}) + \cdots + \lambda(A_{k_{n-1}} \triangle A_{k_n})\right] \\
&\le \liminf_n \left(\frac{1}{2^p} + \cdots + \frac{1}{2^{n-1}}\right) = \frac{1}{2^{p-1}}, \quad \text{for every} \quad p \in \mathbb{N}
\end{aligned}$$

hence we obtain

$$d_\lambda(A_{k_p}, A) \le \arctan\left(\frac{1}{2^{p-1}}\right), \quad \text{for every} \quad p \in \mathbb{N}. \tag{3}$$

From (3), it results that $A_{k_p} \xrightarrow{d_\lambda} A$ and consequently $A_n \xrightarrow{d_\lambda} A$.

Therefore $(\mathcal{A}, d_\lambda)$ is complete.

Suppose now that $A_n \xrightarrow{d_\lambda} A$ and $B_n \xrightarrow{d_\lambda} B$.

From the following inclusions:

$$\begin{aligned}
(A_n \cup B_n) \triangle (A \cup B) &\subseteq (A_n \triangle A) \cup (B_n \triangle B), \\
(A_n \cap B_n) \triangle (A \cap B) &\subseteq (A_n \triangle A) \cup (B_n \triangle B), \\
(A_n \setminus B_n) \triangle (A \setminus B) &\subseteq (A_n \triangle A) \cup (B_n \triangle B),
\end{aligned}$$

we obtain

$$\begin{aligned}
D_\lambda(A_n \cup B_n, A \cup B) &\le D_\lambda(A_n, A) + D_\lambda(B_n, B), \\
D_\lambda(A_n \cap B_n, A \cap B) &\le D_\lambda(A_n, A) + D_\lambda(B_n, B), \\
D_\lambda(A_n \setminus B_n, A \setminus B) &\le D_\lambda(A_n, A) + D_\lambda(B_n, B).
\end{aligned}$$

The inequality $\arctan(x + y) \le \arctan x + \arctan y$ implies that

$$\begin{aligned}
d_\lambda(A_n \cup B_n, A \cup B) &\le d_\lambda(A_n, A) + d_\lambda(B_n, B), \\
d_\lambda(A_n \cap B_n, A \cap B) &\le d_\lambda(A_n, A) + d_\lambda(B_n, B), \\
d_\lambda(A_n \setminus B_n, A \setminus B) &\le d_\lambda(A_n, A) + d_\lambda(B_n, B),
\end{aligned}$$

from where it results that

$$\begin{aligned}
(A_n \cup B_n) &\xrightarrow{d_\lambda} (A \cup B), \\
(A_n \cap B_n) &\xrightarrow{d_\lambda} (A \cap B), \\
(A_n \setminus B_n) &\xrightarrow{d_\lambda} (A \setminus B).
\end{aligned}$$

which demonstrates the continuity of the applications $\cup$, $\cap$ and $\setminus$. $\square$

Definition 1.28. Let $\lambda : \mathcal{A} \to \overline{\mathbb{R}}_+$ be an additive measure and let $\mu \in ba(\mathcal{A})$; we say that μ est *absolutely continuous* with respect to λ if, for every $A \in \mathcal{A}$,

$$\lambda(A) = 0 \Longrightarrow \mu(A) = 0.$$

We note this by $\mu \ll \lambda$.

Remark 1.29. Let $\lambda \in ba^+(\mathcal{A})$ and $\mu \in ba(\mathcal{A})$; then

$$\mu \ll \lambda \Leftrightarrow |\mu| \ll \lambda \Leftrightarrow \mu^+ \ll \lambda \quad \text{and} \quad \mu^- \ll \lambda.$$

Indeed, if $\mu \ll \lambda$, then, for every $A \in \mathcal{A}$ with $\lambda(A) = 0$ and for every $B \in \mathcal{A}$, $B \subseteq A$, we have $\lambda(B) = 0$ and so $\mu(B) = 0$. Then, by (i) of Theorem 1.17, $|\mu|(A) = 0$, from where $|\mu| \ll \lambda$. The implication $|\mu| \ll \lambda \Rightarrow \mu^+ \ll \lambda$ and $\mu^- \ll \lambda$ is obvious and, from $\mu^+ \ll \lambda, \mu^- \ll \lambda$ and $\mu = \mu^+ - \mu^-$, it results immediately that $\mu \ll \lambda$.

The following proposition shows that, for the real σ-additive measures, the property of a measure to be absolutely continuous with respect to another one is a property of continuity (see Definition III.4.12 and Lemma III.4.13 of [62]).

Proposition 1.30. *Let $\lambda : \mathcal{A} \to \overline{\mathbb{R}}_+$ be a σ-additive measure and let $\mu \in ca(\mathcal{A})$; then the following properties are equivalent:*

(i) *μ is absolutely continuous with respect to λ ($\mu \ll \lambda$),*

(ii) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $A \in \mathcal{A}$ satisfying $\lambda(A) < \delta$, we have $|\mu|(A) < \varepsilon$,*

(iii) *$\mu : (\mathcal{A}, d_\lambda) \to \mathbb{R}$ is a d_λ-continuous function.*

Proof. (i) $\Longrightarrow$ (ii). Let us suppose that (ii) is not satisfied. There exist $\varepsilon > 0$ and $(A_n)_n \subseteq \mathcal{A}$ such that $\lambda(A_n) < \frac{1}{2^n}$ and $|\mu|(A_n) \geq \varepsilon$, for every $n \in \mathbb{N}$. Let $A = \limsup_n A_n = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k \in \mathcal{A}$. Then, for every $n \in \mathbb{N}$, $|\mu|(\bigcup_{k=n}^\infty A_k) \geq \varepsilon$ and $\lambda(\bigcup_{k=n}^\infty A_k) \leq \sum_{k=n}^\infty \lambda(A_k) \leq \sum_{k=n}^\infty \frac{1}{2^n} = \frac{1}{2^{n-1}}$. We have then $\lambda(A) = \lim_n \lambda(\bigcup_{k=n}^\infty A_k) = 0$ and $|\mu|(A) = \lim_{n\to\infty} |\mu|(\bigcup_{k=n}^\infty A_k) \geq \varepsilon$. Therefore, (i) is not satisfied.

(ii) $\Longrightarrow$ (iii). According to (ii), for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $B \in \mathcal{A}$ such that $\lambda(B) < \delta, |\mu|(B) < \varepsilon$ and, by continuity of the mapping $\tan$ in 0, there exists $\eta > 0$ such that $\arctan \lambda(B) < \eta$ implies $\lambda(B) < \delta$ and so $|\mu|(B) < \varepsilon$.

Let now $A \in \mathcal{A}$ and let $B \in \mathcal{A}$ with $d_\lambda(A, B) < \eta$; then $\lambda(A \triangle B) < \delta$ and so $|\mu|(A \triangle B) < \varepsilon$. Therefore

$$\begin{aligned}|\mu(A) - \mu(B)| &= |\mu(A) - \mu(A \cap B) + \mu(A \cap B) - \mu(B)| \\ &= |\mu(A \setminus B) - \mu(B \setminus A)| \le |\mu(A \setminus B)| + |\mu(B \setminus A)| \\ &\le |\mu|(A \triangle B) < \varepsilon\end{aligned}$$

and so μ is d_λ-continuous in A.

(iii) $\Longrightarrow$ (i). Since μ is continuous on $\mathcal{A}$, it is continuous at $\emptyset \in \mathcal{A}$. Then, for every $\varepsilon > 0$, there exists $\delta \in]0, 1[$ such that, for every $A \in \mathcal{A}$ satisfying $d_\lambda(A, \emptyset) = \lambda(A) < \delta, |\mu(A)| < \varepsilon$. Let $A \in \mathcal{A}$ with $\lambda(A) = 0$; then $d_\lambda(A, \emptyset) < \delta$ and hence $|\mu(A)| < \varepsilon$ and, as ε is arbitrary, $\mu(A) = 0$. Therefore $\mu \ll \lambda$. □

Proposition 1.31. *Let $\lambda : \mathcal{A} \to \overline{\mathbb{R}}_+$ be a σ-additive measure and let $\mu : \mathcal{A} \to \mathbb{R}$ be an additive measure. If μ is d_λ-continuous, then $\mu \in ca(\mathcal{A})$.*

Proof. Let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be a sequence of pairwise disjoint sets and let $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

Since μ is d_λ-continuous, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every B and C of $\mathcal{A}$ satisfying $\lambda(B \triangle C) < \delta, |\mu(B) - \mu(C)| < \varepsilon$. Since λ is σ-additive, there exists $n_0 \in \mathbb{N}$ such that, for every $n \ge n_0$,

$$\left|\lambda(A) - \sum_{k=1}^{n} \lambda(A_k)\right| = \lambda\left(A \setminus \bigcup_{k=1}^{n} A_k\right) = \lambda\left(A \triangle \bigcup_{k=1}^{n} A_k\right) < \delta.$$

Then $|\mu(A) - \sum_1^n \mu(A_k)| = |\mu(A) - \mu(\cup_1^n A_k)| < \varepsilon$ and so $\mu \in ca(\mathcal{A})$. □

Remark 1.32. The result of Proposition 1.30 asserts that, if μ is σ- additive, then the absolute continuity of μ with respect to λ is equivalent to the d_λ-continuity of μ; this result is no longer valid if μ is only additive.

In fact, let μ be as in the example of Remark 1.16; μ is additive, it is not bounded and, according to Corollary 1.14, its extension to $\mathcal{P}(\mathbb{N})$, still noted μ, is not σ - additive.

Let $\lambda : \mathcal{P}(\mathbb{N}) \to \mathbb{R}_+, \lambda(A) = \sum_{n\in A} \delta_n(A)$, where δ_n is the Dirac measure that gives to singleton set $\{n\}$ the measure 1. Then λ is a σ-additive measure and, as μ is not σ-additive, according to Proposition 1.31, μ is not d_λ- continuous. However μ is absolutely continuous with respect to λ. Indeed, let $A \in \mathcal{P}(\mathbb{N})$ with $\lambda(A) = 0$; then $A = \emptyset$ and therefore $\mu(A) = 0$.

In the case where $\mu \in ba(\mathcal{A}) \setminus ca(\mathcal{A})$, we have the following implications among the conditions of Proposition 1.30: (ii)$\Longleftrightarrow$(iii) $\Longrightarrow$(i).

In relation to Proposition 1.30, if λ is σ-additive, then in order to avoid the use of the difficult formulation of "d_λ-continuity" or the longer one "absolutely continuous with respect to λ", we will give the following definition:

Definition 1.33. A measure $\mu \in ca(\mathcal{A})$, continuous on the space $(\mathcal{A}, d_\lambda)$, is called *λ-continuous*.

We will note by $ca_\lambda(\mathcal{A})$ the subset of all λ-continuous measure of $ca(\mathcal{A})$.

We can find the following theorem in [57] (see Theorem 9, p. 87).

Theorem 1.34. *Let $\lambda : \mathcal{A} \to \overline{\mathbb{R}}_+$ be a σ-additive measure, let d_λ be the pseudo-metric associated and let $\mathcal{K}$ be a family of measures of $ca(\mathcal{A})$; then the following properties are equivalent:*

(i) *the family $\mathcal{K}$ is d_λ-equicontinuous at some $E \in \mathcal{A}$.*

(ii) *the family $\mathcal{K}$ is d_λ-equicontinuous at the point $\emptyset \in \mathcal{A}$.*

(iii) *the family $\mathcal{K}$ is uniformly d_λ-equicontinuous on $\mathcal{A}$.*

Each of these conditions entails the following:

(iv) *the family $\mathcal{K}$ is uniformly σ-additive.*

Proof. (i)$\Longrightarrow$(ii). Let $\mathcal{K}$ be d_λ-equicontinuous at $E \in \mathcal{A}$; then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $A \in \mathcal{A}$ with $d_\lambda(A, E) < \delta$ and for every $\mu \in \mathcal{K}$, $|\mu(A) - \mu(E)| < \varepsilon$. Let $A \in \mathcal{A}$ such that $d_\lambda(A, \emptyset) < \delta$, that is $\lambda(A) < \eta = \tan(\delta)$; then

$$D_\lambda(A \cup E, E) = \lambda\left[(A \cup E) \triangle E\right] = \lambda(A \setminus E) \le \lambda(A) < \eta \quad \text{and}$$
$$D_\lambda(E \setminus A, E) = \lambda\left[(E \setminus A) \triangle E\right] = \lambda(A \cap E) \le \lambda(A) < \eta.$$

Therefore, for every $\mu \in \mathcal{K}$, $|\mu(A \cup E) - \mu(E)| < \varepsilon$ and $|\mu(E) - \mu(E \setminus A)| < \varepsilon$. We have then:

$$\begin{aligned}|\mu(A)| &= |\mu(A \cup E) - \mu(E \setminus A)| \\ &\le |\mu(A \cup E) - \mu(E)| + |\mu(E) - \mu(E \setminus A)| < 2\varepsilon.\end{aligned}$$

$\mathcal{K}$ is therefore d_λ-equicontinuous at $\emptyset$.

(ii) $\Longrightarrow$(iii). $\mathcal{K}$ being d_λ-equicontinuous at $\emptyset$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $E \in \mathcal{A}$ satisfying $d_\lambda(E, \emptyset) = \arctan \lambda(E) < \delta$, we have $|\mu(E)| < \varepsilon$, for every $\mu \in \mathcal{K}$.

If $d_\lambda(C, D) = \arctan \lambda(C \triangle D) < \delta$ then $d_\lambda(C \setminus D, \emptyset) < \delta$ and $d_\lambda(D \setminus C, \emptyset) < \delta$.

Then, for every $\mu \in \mathcal{K}$,

$$\begin{aligned}|\mu(C) - \mu(D)| &= |\mu(C \setminus D) + \mu(C \cap D) - \mu(C \cap D) - \mu(D \setminus C)| \\ &\leq |\mu(C \setminus D)| + |\mu(D \setminus C)| < 2\varepsilon.\end{aligned}$$

Therefore $\mathcal{K}$ is uniformly d_λ-equicontinuous on $\mathcal{A}$.

Obviously, (iii) $\Longrightarrow$(i).

(ii) $\Longrightarrow$(iv). Let $\mathcal{K} \subseteq ca(\mathcal{A})$ be a family satisfying (ii), let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be a sequence of pairwise disjoint sets and let $A = \cup_1^\infty A_n$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $E \in \mathcal{A}$ with $\lambda(E) < \delta$, we have $|\mu(E)| < \varepsilon$, for every $\mu \in \mathcal{K}$. λ being σ-additive and positive, there exists $n_0 \in \mathbb{N}$ such that $|\lambda(A) - \sum_{k=1}^n \lambda(A_k)| = \lambda(A \setminus \cup_{k=1}^n A_k) < \delta$, for every $n \geq n_0$.

Then, for every $\mu \in \mathcal{K}$, $|\mu(A) - \sum_{k=1}^n \mu(A_k)| = |\mu(A \setminus \cup_{k=1}^n A_k)| < \varepsilon$ from where it results that $\mathcal{K}$ is uniformly σ-additive. □

In a consistent manner with Theorem 1.33, we give the following definition:

Definition 1.35. A family of measures $\mathcal{K} \subseteq ca(\mathcal{A})$, d_λ-equicontinuous at $\emptyset$ (and therefore on $\mathcal{A}$) is called *λ-equicontinuous*.

We need to emphasize that the definitions of λ-continuity and λ-equicontinuity refer only to the real σ-additive measures, meaning that they do not refer to the σ-additive measures taking at most one of the values $+\infty$ or $-\infty$. However, the previous results can be extended by replacing the σ-additive and positive measure λ by a σ-additive measure λ of finite total variation $|\lambda|$ and the measure $\mu \in ca(A)$ by a σ-additive measure with values in a Banach space.

We will now give a very important result of equicontinuity (Vitali–Hahn–Saks theorem) which allows us to establish the analogue of the uniform boundedness principle from Functional Analysis for the Measure Theory (see Theorem III.7.2 and Corollary III.7.3 in [62] or Theorem 2.53 of [85]).

Theorem 1.36 (Vitali–Hahn–Saks). *Let $\lambda : \mathcal{A} \to \overline{\mathbb{R}}_+$ be a σ-additive measure and let $(\mu_n)_{n\in\mathbb{N}} \subseteq ca(\mathcal{A})$ be a sequence of λ-continuous measures.*

Assume that $\lim_n \mu_n(A) = \mu(A) \in \mathbb{R}$ *exists, for every $A \in \mathcal{A}$; then:*

(i) *$\{\mu_n : n \in \mathbb{N}\}$ is λ-equicontinuous,*

(ii) *$\mu \in ca(\mathcal{A})$,*

(iii) *μ is λ-continuous.*

Proof. According to Proposition 1.30, $\mu_n : (\mathcal{A}, d_\lambda) \to \mathbb{R}$ is a continuous function, for every $n \in \mathbb{N}$.

For every $\varepsilon > 0$ and for all couple $(n, m) \in \mathbb{N} \times \mathbb{N}$, let's note

$$\mathcal{A}_{n,m}(\varepsilon) = \{A \in \mathcal{A} : |\mu_n(A) - \mu_m(A)| \leq \varepsilon\}.$$

$\mathcal{A}_{n,m}(\varepsilon)$ are closed sets in the complete space $(\mathcal{A}, d_\lambda)$; then, for every $p \in \mathbb{N}$,

$$\mathcal{A}_p(\varepsilon) = \bigcap_{m,n \geq p} \mathcal{A}_{n,m}(\varepsilon)$$

is a closed set in $(\mathcal{A}, d_\lambda)$.

Since $\lim_n \mu_n(A) \in \mathbb{R}$, for every $A \in \mathcal{A}$,

$$\mathcal{A} = \bigcup_{p=1}^{\infty} \mathcal{A}_p(\varepsilon).$$

According to Baire theorem, there exists $p_0 \in \mathbb{N}$ such that $\mathcal{A}_{p_0}(\varepsilon)$ has nonempty interior in $(\mathcal{A}, d_\lambda)$. Therefore, there exists $A_0 \in \mathcal{A}$, there exists $r > 0$ such that the ball $S(A_0, \arctan r) \subseteq \mathcal{A}_{p_0}(\varepsilon)$, i.e.,

$$|\mu_n(A) - \mu_m(A)| < \varepsilon, \quad \forall A \in \mathcal{A} \quad \text{with} \quad \lambda(A \triangle A_0) < r, \quad \forall m, n \geq p_0. \tag{1}$$

Since the set $\{\mu_1, \ldots, \mu_{p_0}\}$ is λ-equicontinuous, there exists $\delta \in]0, r[$ such that

$$|\mu_n(B)| < \varepsilon, \quad \text{for all} \quad B \in \mathcal{A} \quad \text{with} \quad \lambda(B) < \delta, \quad \forall n = 1, \ldots, p_0. \tag{2}$$

Let $A \in \mathcal{A}$ with $\lambda(A) < \delta$; then

$$\lambda((A \cup A_0) \triangle A_0) = \lambda(A \setminus A_0) \leq \lambda(A) < \delta < r \quad \text{and}$$
$$\lambda((A_0 \setminus A) \triangle A_0) = \lambda(A_0 \cap A) \leq \lambda(A) < \delta < r.$$

By (1), we have

$$|\mu_n(A \cup A_0) - \mu_{p_0}(A \cup A_0)| < \varepsilon, \quad \forall n \geq p_0 \tag{3}$$

and

$$|\mu_n(A_0 \setminus A) - \mu_{p_0}(A_0 \setminus A)| < \varepsilon, \quad \forall n \geq p_0. \tag{4}$$

By (2), (3) and (4), we deduct that, for every $n \geq p_0$,

$$\begin{aligned}
|\mu_n(A)| &= |\mu_{p_0}(A) + [\mu_n(A) - \mu_{p_0}(A)]| \\
&\leq |\mu_{p_0}(A)| + |\mu_n(A \cup A_0) - \mu_{p_0}(A \cup A_0) + \mu_{p_0}(A_0 \setminus A) - \mu_n(A_0 \setminus A)| \\
&\leq |\mu_{p_0}(A)| + |\mu_n(A \cup A_0) - \mu_{p_0}(A \cup A_0)| + |\mu_{p_0}(A_0 \setminus A) - \mu_n(A_0 \setminus A)| \\
&< 3\varepsilon.
\end{aligned}$$

Therefore, according to (2), $|\mu_n(A)| < 3\varepsilon$, for every $n \in \mathbb{N}$. $\{\mu_n : n \in \mathbb{N}\}$ is therefore d_λ-equicontinuous at $\emptyset$ and so it is λ-equicontinuous.

Since (μ_n) converges punctually to μ, μ is additive on $\mathcal{A}$.

We still need to show that μ is λ-continuous.

According to Theorem 1.34, since $\{\mu_n\}$ is λ-equicontinuous, it is uniformly d_λ-equicontinuous on $\mathcal{A}$. Therefore, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all A and B of $\mathcal{A}$ with $\lambda(A \triangle B) < \delta$, for any $n \in \mathbb{N}$, $|\mu_n(A) - \mu_n(B)| < \varepsilon$. Now let n tend to ∞; so we obtain $|\mu(A) - \mu(B)| \leq \varepsilon$, from where it results that μ is uniformly - d_λ-continuous and so it is d_λ-continuous.

From Proposition 1.31, it results that $\mu \in ca(\mathcal{A})$ and, according to Proposition 1.30, μ is λ-continuous. □

The previous theorem accepts as corollary the following result (see Corollary III.7.4 and Theorem IV.9.8 of [62] and [57], p. 90):

Theorem 1.37 (Nikodym). *Let $(\mu_n) \subseteq ca(\mathcal{A})$ be a sequence of measures such that, for every $E \in \mathcal{A}$, there exists* $\lim_n \mu_n(E) = \mu(E) \in \mathbb{R}$. *Then:*

(i) $\mu \in ca(\mathcal{A})$,

(ii) $\{\mu_n : n \in \mathbb{N}\}$ *is uniformly σ-additive and*

(iii) $\{\mu_n : n \in \mathbb{N}\}$ *is bounded in the space $(ca(\mathcal{A}), \|\cdot\|)$.*

Proof. (i) + (ii) Let $\lambda : \mathcal{A} \to \mathbb{R}_+$ be defined by

$$\lambda(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|\mu_n|(A)}{1 + \|\mu_n\|}, \quad \forall A \in \mathcal{A}.$$

It is clear that, for all $A \in \mathcal{A}$, $\lambda(A) < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

Let $\{E_p : p \in \mathbb{N}\} \subseteq \mathcal{A}$ be a family of disjoint sets and let $E = \bigcup_{p=1}^{\infty} E_p$. For all $\varepsilon > 0$ let $n_0 \in \mathbb{N}$ such that $\frac{1}{2^{n_0-2}} < \varepsilon$. Since $\mu_n \in ca(\mathcal{A})$, $|\mu_n| \in ca(\mathcal{A})$ and then there exists $k_0 \in \mathbb{N}$ such that, for every $k \geq k_0$ and every $n = 1, \dots, n_0$,

$$\left| |\mu_n|(E) - \sum_{p=1}^{k} |\mu_n|(E_p) \right| < \frac{\varepsilon}{2}(1 + \|\mu_n\|).$$

Then

$$\begin{aligned}\left|\lambda(E)-\sum_{p=1}^{k}\lambda(E_p)\right| &= \left|\sum_{n=1}^{\infty}\left(\frac{1}{2^n}\cdot\frac{|\mu_n|(E)}{1+\|\mu_n\|}-\sum_{p=1}^{k}\frac{1}{2^n}\cdot\frac{|\mu_n|(E_p)}{1+\|\mu_n\|}\right)\right| \\ &\le \sum_{n=1}^{n_0}\frac{1}{2^n}\left|\frac{|\mu_n|(E)}{1+\|\mu_n\|}-\sum_{p=1}^{k}\frac{|\mu_n|(E_p)}{1+\|\mu_n\|}\right|+\sum_{n=n_0+1}^{\infty}\frac{1}{2^n}\cdot 2 \\ &< \sum_{n=1}^{n_0}\frac{1}{2^n}\cdot\frac{\varepsilon}{2}+\frac{1}{2^{n_0-1}}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Therefore $\lambda \in ca^+(\mathcal{A})$.

Moreover, it is evident that, for every $n \in \mathbb{N}$, $\mu_n \ll \lambda$. We are, therefore, in the conditions to apply the Vitali–Hahn–Saks theorem. Therefore $\mu \in ca(\mathcal{A})$, $\{\mu_n : n \in \mathbb{N}\}$ is λ-equicontinuous and, according to (iv) of Theorem 1.34, $\{\mu_n : n \in \mathbb{N}\}$ is uniformly σ-additive.

(iii) As, for every $A \in \mathcal{A}$, $\mu_n(A) \to \mu(A) \in \mathbb{R}$, we have:

$$\sup_n |\mu_n(A)| < +\infty \quad \text{for every} \quad A \in \mathcal{A}. \tag{1}$$

Suppose that the family $\{\mu_n : n \in \mathbb{N}\}$ is not bounded in the space $(ca(\mathcal{A}), \|\cdot\|)$. Then $\sup_{n\in\mathbb{N}} \|\mu_n\| = +\infty$.

According to Theorem 1.23, for every $n \in \mathbb{N}$, $\|\mu_n\| \le 2\|\mu_n\|_\infty$.

We have therefore

$$\sup_n \|\mu_n\|_\infty = \sup_{n\in\mathbb{N}}(\sup_{A\in\mathcal{A}} |\mu_n(A)|) = +\infty. \tag{2}$$

By (1), $\sup_k |\mu_k(X)| < +\infty$. Let $\Sigma_1 = \sup_k |\mu_k(X)| + 1$; according to (2), there exists $n_1 \in \mathbb{N}$ such that

$$\sup_{A\in\mathcal{A}} |\mu_{n_1}(A)| > \Sigma_1$$

and therefore there exists $A_1 \in \mathcal{A}$ such that $|\mu_{n_1}(A_1)| > \Sigma_1$.

$$\begin{aligned}|\mu_{n_1}(X \setminus A_1)| &= |\mu_{n_1}(X) - \mu_{n_1}(A_1)| \\ &\ge |\mu_{n_1}(A_1)| - |\mu_{n_1}(X)| \ge |\mu_{n_1}(A_1)| - \sup_k |\mu_k(X)| > 1.\end{aligned}$$

Let's note $B_1 = X \setminus A_1$. We have obtained an $\mathcal{A}$- partition (A_1, B_1) of X such that

$$|\mu_{n_1}(A_1)| \ge 1, \quad |\mu_{n_1}(B_1)| \ge 1. \tag{3}$$

By (2), we have that

$$\sup_{n\in\mathbb{N}}(\sup_{A\in\mathcal{A}} |\mu_n(A\cap A_1)|) = +\infty. \tag{4}$$

or

$$\sup_{n\in\mathbb{N}}(\sup_{A\in\mathcal{A}} |\mu_n(A\cap B_1)|) = +\infty. \tag{5}$$

If (4) is satisfied, then we note $C_1 = B_1$ (otherwise $C_1 = A_1$). Because all finite subset of $ca(\mathcal{A})$ is uniformly bounded on $\mathcal{A}$ (this is an immediate consequence of Corollary 1.14), we have

$$\sup_{n>n_1}(\sup_{A\in\mathcal{A}} |\mu_n(A\cap A_1)|) = +\infty.$$

Can one restart this procedure by applying (1) to A_1.

Let $\Sigma_2 = \sup_k |\mu_k(A_1)|+2$; there exist $n_2 > n_1$ and an $\mathcal{A}$ - partition (A_2, B_2) of A_1 such that

$$|\mu_{n_2}(A_2)| \geq 2, \quad |\mu_{n_2}(B_2)| \geq 2$$

and

$$\sup_{n>n_2}(\sup_{A\in\mathcal{A}} |\mu_n(A\cap A_2)|) = +\infty. \tag{6}$$

or

$$\sup_{n>n_2}(\sup_{A\in\mathcal{A}} |\mu_n(A\cap B_2)|) = +\infty. \tag{7}$$

If (6) is satisfied, we note $C_2 = B_2$ (otherwise $C_2 = A_2$). $C_2 = B_2 \subseteq X \setminus C_1$.

Continuing in this fashion, we define a strictly increasing sequence of integers $(n_p)_{p\in\mathbb{N}}$ tending to infinity and a sequence of pairwise disjoint sets $(C_p)_{p\in\mathbb{N}} \subseteq \mathcal{A}$ such that,

$$|\mu_{n_p}(C_p)| \geq p, \quad \text{for every} \quad p \in \mathbb{N}. \tag{8}$$

Let $C = \bigcup_1^\infty C_p \in \mathcal{A}$; by (ii), (μ_n) are uniformly σ-additive. Therefore, for $\varepsilon = 1$, there exists $k_0 \in \mathbb{N}$ such that

$$|\mu_n(C) - \sum_{i=1}^{k} \mu_n(C_i)| < 1, \quad \text{for every} \quad k \geq k_0 \quad \text{and for all} \quad n \in \mathbb{N}.$$