Progress in Commutative Algebra 2

Progress in Commutative Algebra 2

Closures, Finiteness and Factorization

edited by
Christopher Francisco
Lee Klingler
Sean Sather-Wagstaff
Janet C. Vassilev

De Gruyter

Mathematics Subject Classification 2010
13D02, 13D40, 05E40, 13D45, 13D22, 13H10, 13A35, 13A15, 13A05, 13B22, 13F15

ISBN 978-3-11-027859-0
e-ISBN 978-3-11-027860-6

Library of Congress Cataloging-in-Publication Data

A CIP catalog record for this book has been applied for at the Library of Congress.

Bibliographic information published by the Deutsche Nationalbibliothek

The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at http://dnb.dnb.de.

Typesetting: Da-TeX Gerd Blumenstein, Leipzig, www.da-tex.de
Printing: Hubert & Co. GmbH & Co. KG, Göttingen
∞ Printed on acid-free paper

Printed in Germany

www.degruyter.com

Preface

This collection of papers in commutative algebra stemmed out of the 2009 Fall Southeastern American Mathematical Society Meeting which contained three special sessions in the field:

- Special Session on Commutative Ring Theory, a Tribute to the Memory of James Brewer, organized by Alan Loper and Lee Klingler;
- Special Session on Homological Aspects of Module Theory, organized by Andy Kustin, Sean Sather-Wagstaff, and Janet Vassilev; and
- Special Session on Graded Resolutions, organized by Chris Francisco and Irena Peeva.

Much of the commutative algebra community has split into two camps, for lack of a better word: the Noetherian camp and the non-Noetherian camp. Most researchers in commutative algebra identify with one camp or the other, though there are some notable exceptions to this. We had originally intended this to be a Proceedings Volume for the conference as the sessions had a nice combination of both Noetherian and non-Noetherian talks. However, the project grew into two Volumes with invited papers that are blends of survey material and new research. We hope that members from the two camps will read each others' papers and that this will lead to increased mathematical interaction between the camps.

As the title suggests, this volume, Progress in Commutative Algebra II, contains surveys on aspects of closure operations, finiteness conditions and factorization. Contributions to this volume have come mainly from speakers in the first and second sessions and from invited articles on closure operations, test ideals, Noetherian rings without finite normaliztion and non-unique factorization. The collection documents some current trends in two of the most active areas of commutative algebra.

Closure operations on ideals and modules are a bridge between Noetherian and non-Noetherian commutative algebra. The Noetherian camp typically study structures related to a particular closure operation such as the core or the test ideal or how particular closure operations yield nice proofs of hard theorems. The non-Noetherian camp approach closure operations from the view of multiplicative ideal theory and the relationship to Kronecker function rings. This volume contains a nice guide to closure operations by Epstein, but also contains an article on test ideals by Schwede and Tucker and one by Enescu which discusses the action of the Frobenius on finite dimensional vector spaces both of which are related to tight closure.

Finiteness properties of rings and modules or the lack of them come up in all aspects of commutative algebra. For instance, the division between the Noetherian and the non-Noetherian crowd comes down to the property that all ideals in a Noetherian ring are finitely generated, by definition. However, in the study of non-Noetherian rings it is much easier to find a ring having a finite number of prime ideals. We have included papers by Boynton and Sather-Wagstaff and by Watkins that discuss the relationship of rings with finite Krull dimension and their finite extensions. Finiteness properties in commutative group rings are discussed in Glaz and Schwarz's paper. And Olberding's selection presents us with constructions that produce rings whose integral closure in their field of fractions is not finitely generated.

The final three papers in this volume investigate factorization in a broad sense. The first paper by Celikbas and Eubanks-Turner discusses the partially ordered set of prime ideals of the projective line over the integers. We have also included a paper on zero divisor graphs by Coykendall, Sather-Wagstaff, Sheppardson and Spiroff. The final paper, by Chapman and Krause, concerns non-unique factorization.

The first session was a Tribute to the Memory of James Brewer. As many of the authors participated in this session, we dedicate this volume to Brewer's memory. Enjoy!

March 2012

Sean Sather-Wagstaff
Chris Francisco
Lee Klingler
Janet C. Vassilev

Contents

Preface v

Neil Epstein
A Guide to Closure Operations in Commutative Algebra
1 Introduction 1
2 What Is a Closure Operation? 2
2.1 The Basics 2
2.2 Not-quite-closure Operations 6
3 Constructing Closure Operations 7
3.1 Standard Constructions 7
3.2 Common Closures as Iterations of Standard Constructions 9
4 Properties of Closures 10
4.1 Star-, Semi-prime, and Prime Operations 10
4.2 Closures Defined by Properties of (Generic) Forcing Algebras 16
4.3 Persistence 17
4.4 Axioms Related to the Homological Conjectures 18
4.5 Tight Closure and Its Imitators 20
4.6 (Homogeneous) Equational Closures and Localization 21
5 Reductions, Special Parts of Closures, Spreads, and Cores 22
5.1 Nakayama Closures and Reductions 22
5.2 Special Parts of Closures 23
6 Classes of Rings Defined by Closed Ideals 25
6.1 When Is the Zero Ideal Closed? 26
6.2 When Are 0 and Principal Ideals Generated by Non-zerodivisors Closed? 26
6.3 When Are Parameter Ideals Closed (Where R Is Local)? 27
6.4 When Is Every Ideal Closed? 28
7 Closure Operations on (Sub)modules 29
7.1 Torsion Theories 31

Karl Schwede and Kevin Tucker
A Survey of Test Ideals
1 Introduction 39
2 Characteristic p Preliminaries 41
2.1 The Frobenius Endomorphism 41
2.2 F-purity 42

3 The Test Ideal . . . 44
3.1 Test Ideals of Map-pairs . . . 44
3.2 Test Ideals of Rings . . . 47
3.3 Test Ideals in Gorenstein Local Rings . . . 48
4 Connections with Algebraic Geometry . . . 50
4.1 Characteristic 0 Preliminaries . . . 50
4.2 Reduction to Characteristic $p > 0$ and Multiplier Ideals . . . 52
4.3 Multiplier Ideals of Pairs . . . 54
4.4 Multiplier Ideals vs. Test Ideals of Divisor Pairs . . . 56
5 Tight Closure and Applications of Test Ideals . . . 57
5.1 The Briançon–Skoda Theorem . . . 61
5.2 Tight Closure for Modules and Test Elements . . . 61
6 Test Ideals for Pairs $(R, \mathfrak{a}^t)$ and Applications . . . 63
6.1 Initial Definitions of $\mathfrak{a}^t$-test Ideals . . . 63
6.2 $\mathfrak{a}^t$-tight Closure . . . 65
6.3 Applications . . . 66
7 Generalizations of Pairs: Algebras of Maps . . . 68
8 Other Measures of Singularities in Characteristic p . . . 71
8.1 F-rationality . . . 71
8.2 F-injectivity . . . 72
8.3 F-signature and F-splitting Ratio . . . 73
8.4 Hilbert–Kunz(–Monsky) Multiplicity . . . 75
8.5 F-ideals, F-stable Submodules, and F-pure Centers . . . 78
A Canonical Modules and Duality . . . 80
A.1 Canonical Modules, Cohen–Macaulay and Gorenstein Rings . . . 80
A.2 Duality . . . 81
B Divisors . . . 83
C Glossary and Diagrams on Types of Singularities . . . 85
C.1 Glossary of Terms . . . 86

Florian Enescu
Finite-dimensional Vector Spaces with Frobenius Action
1 Introduction . . . 101
2 A Noncommutative Principal Ideal Domain . . . 102
3 Ideal Theory and Divisibility in Noncommutative PIDs . . . 104
3.1 Examples in $K\{F\}$ . . . 107
4 Matrix Transformations over Noncommutative PIDs . . . 109
5 Module Theory over Noncommutative PIDs . . . 111
6 Computing the Invariant Factors . . . 114
6.1 Injective Frobenius Actions on Finite Dimensional Vector Spaces over a Perfect Field . . . 118
7 The Antinilpotent Case . . . 121

Sarah Glaz and Ryan Schwarz
Finiteness and Homological Conditions in Commutative Group Rings
1 Introduction 129
2 Finiteness Conditions 130
3 Homological Dimensions and Regularity 133
4 Zero Divisor Controlling Conditions 136

Jason G. Boynton and Sean Sather-Wagstaff
Regular Pullbacks
1 Introduction 145
2 Some Background 147
3 Pullbacks of Noetherian Rings 151
4 Pullbacks of Prüfer Rings 153
5 Pullbacks of Coherent Rings 156
6 The n-generator Property in Pullbacks 159
7 Factorization in Pullbacks 165

Bruce Olberding
Noetherian Rings without Finite Normalization
1 Introduction 171
2 Normalization and Completion 174
3 Examples between DVRs 176
4 Examples Birationally Dominating a Local Ring 185
5 A Geometric Example 189
6 Strongly Twisted Subrings of Local Noetherian Domains 190

John J. Watkins
Krull Dimension of Polynomial and Power Series Rings
1 Introduction 205
2 A Key Property of $R[x]$ 207
3 The Main Theorem 208
4 Additional Applications 212
5 The Dimension of Power Series Rings 217

Ela Celikbas and Christina Eubanks-Turner
The Projective Line over the Integers
1 Introduction 221
2 Definitions and Background 222
3 The Coefficient Subset and Radical Elements of Proj$(\mathbb{Z}[h,k])$ 226
4 The Conjecture for Proj$(\mathbb{Z}[h,k])$ and Previous Partial Results 229
5 New Results Supporting the Conjecture 232
6 Summary and Questions 238

Jim Coykendall, Sean Sather-Wagstaff, Laura Sheppardson, and Sandy Spiroff
On Zero Divisor Graphs
1 Introduction . . . 241
2 Survey of Past Research on Zero Divisor Graphs . . . 243
2.1 Beck's Zero Divisor Graph . . . 243
2.2 Anderson and Livingston's Zero Divisor Graph . . . 244
2.3 Mulay's Zero Divisor Graph . . . 246
2.4 Other Zero Divisor Graphs . . . 247
3 Star Graphs . . . 250
4 Graph Homomorphisms and Graphs Associated to Modules . . . 270
5 Cliques . . . 273
6 Girth and Cut Vertices . . . 282
6.1 Girth . . . 282
6.2 Cut Vertices . . . 286
7 Chromatic Numbers and Clique Numbers . . . 288
7.1 Chromatic/Clique Number 1 . . . 289
7.2 Chromatic/Clique Number 2 . . . 290
7.3 Chromatic/Clique Number 3 . . . 290
A Tables for Example 3.14 . . . 293
B Graph Theory . . . 295

Scott T. Chapman and Ulrich Krause
A Closer Look at Non-unique Factorization via Atomic Decay and Strong Atoms
1 Introduction . . . 301
2 Strong Atoms and Prime Ideals . . . 303
3 Atomic Decay in the Ring of Integers of an Algebraic Number Field . . . 305
4 The Fundamental Example of the Failure of Unique Factorization: $\mathbb{Z}[\sqrt{-5}]$ 309
5 A More Striking Example . . . 311
6 Concluding Remarks and Questions . . . 314

Progress in Commutative Algebra 2, 1–37

A Guide to Closure Operations in Commutative Algebra

Neil Epstein

Abstract. This article is a survey of closure operations on ideals in commutative rings, with an emphasis on structural properties and on using tools from one part of the field to analyze structures in another part. The survey is broad enough to encompass the radical, tight closure, integral closure, basically full closure, saturation with respect to a fixed ideal, and the v-operation, among others.

Keywords. Closure Operation, Tight Closure, Integral Closure, Star-operation.

2010 Mathematics Subject Classification. Primary: 13-02; Secondary: 13A15, 13A35, 13B22, 13A99.

1 Introduction

There have been quite a few books and survey articles on tight closure (e.g. [40, 49, 50, 71, 72]), on integral closure (e.g. [51, 73, 74]), and on star-operations on integral domains (e.g. [23; 25, Chapters 32 and 34] but as far as this author knows, no such article on closure operations in general. However, several authors (e.g. [9, 19, 75]) have recently found it useful to consider closure operations as a subject in itself, so I write this article as an attempt to provide an overall framework. This article is intended both for the expert in one closure operation or another who wants to see how it relates to the rest, and for the lay commutative algebraist who wants a first look at what closure operations are. For the most part, this article will not go into the reasons why any given closure operation is important. Instead, I will concentrate on the structural aspects of closure operations, how closure operations arise, how to think about them, and how to analyze them.

The reader may ask: “If the only closure operation I am interested in is c, why should I care about other closure operations?” Among other reasons: the power of analogical thinking is central to what mathematicians do. If the d-theorists have discovered or used a property of their closure operation d, the c-theorist may use this to investigate the analogous property for c, *and may not have thought to do so without knowledge of* d-*closure*. Morover, what holds trivially for one closure operation can be a deep theorem (or only hold in special cases) for another – and vice versa. A good example is persistence (see Section 4.3).

The author was partially supported by a grant from the DFG (German Research Foundation).

In most survey articles, one finds a relatively well-defined subject and a more-or-less linear progression of ideas. The subject exists as such in the minds of those who practice it before the article is written, and the function of the article is to introduce new people to the already extant system of ideas. The current article serves a somewhat different function, as the ideas in this paper are not linked sociologically so much as axiomatically. Indeed, there are at least three socially distinct groups of people studying these things, some of whom seem barely to be aware of each other's existence. In this article, one of my goals is to bridge that gap.

The structure of the article is as follows: In Section 2, I introduce the notion of closures, eleven typical examples, and some non-examples. In Section 3, I exhibit six simple constructions and show how all the closure operations from Section 2 arise from these. The next section, Section 4, concerns properties that closures may have; it comprises more than $1/4$ of the paper! In it, we spend a good bit of time on star- and semi-prime operations, after which we devote a subsection each to forcing algebras, persistence, homological conjectures, tight closure-like properties, and (homogeneous) equational closures. The short Section 5 explores a tightly related set of notions involving what happens when one looks at the collection of *sub*ideals that have the same closure as a given ideal. In Section 6, we explore ring properties that arise from certain ideals being closed. Finally in Section 7, we talk about various ways to extend to closures on (sub)*modules*. Beyond the material in Sections 2 and 3, the reader may read the remaining sections in almost any order.

Throughout this paper, R is a commutative ring with unity. At this point, one would normally either say that R will be assumed Noetherian, or that R will not necessarily be assumed to be Noetherian. However, one of the reasons for the gap mentioned above is that people are often scared off by such statements. It is true that many of the examples I present here seem to work best (and are most studied) in the Noetherian context. On the other hand, I have also included some of the main examples and constructions that are most interesting in the non-Noetherian case. As my own training is among those who work mainly with Noetherian rings, it probably is inevitable that I will sometimes unknowingly assume a ring is Noetherian. In any case, the article should remain accessible and interesting to all readers.

2 What Is a Closure Operation?

2.1 The Basics

Definition 2.1.1. Let R be a ring. A *closure operation* cl on a set of ideals $\mathcal{I}$ of R is a set map $\mathrm{cl} : \mathcal{I} \to \mathcal{I}$ ($I \mapsto I^{\mathrm{cl}}$) satisfying the following conditions:

(i) (Extension) $I \subseteq I^{\mathrm{cl}}$ for all $I \in \mathcal{I}$.

(ii) (Idempotence) $I^{\mathrm{cl}} = (I^{\mathrm{cl}})^{\mathrm{cl}}$ for all $I \in \mathcal{I}$.

(iii) (Order-preservation) If $J \subseteq I$ are ideals of $\mathcal{I}$, then $J^{\mathrm{cl}} \subseteq I^{\mathrm{cl}}$.

If $\mathcal{I}$ is the set of *all* ideals of R, then we say that cl is a closure operation *on* R. We say that an ideal $I \in \mathcal{I}$ is cl-*closed* if $I = I^{\mathrm{cl}}$.

As far as I know, this concept is due to E. H. Moore [60], who defined it (over a century ago!) more generally for subsets of a set, rather than ideals of a ring. Moore's context was mathematical analysis. His has been the accepted definition of "closure operation" in lattice theory and universal algebra ever since (e.g. [5, V.1] or [12, 7.1]). Oddly, this general definition of closure operation does not seem to have gained currency in commutative algebra until the late 1980s [62, 64], although more special structures already had standard terminologies associated to them (see 4.1).

Example 2.1.2 (Examples of closures). The reader is invited to find his/her favorite closure(s) on the following list. Alternately, the list may be skipped and referred back to when an unfamiliar closure is encountered in the text.

(i) The *identity closure*, sending each ideal to itself, is a closure operation on R. (In multiplicative ideal theory, this is usually called the d-*operation*.)

(ii) The *indiscrete closure*, sending each ideal to the unit ideal R, is also a closure operation on R.

(iii) The *radical* is the first nontrivial example of a closure operation on an arbitrary ring R. It may be defined in one of two equivalent ways. Either

$$\sqrt{I} := \{f \in R \mid \exists \text{ a positive integer } n \text{ such that } f^n \in I\}$$

or

$$\sqrt{I} := \bigcap \{\mathfrak{p} \in \operatorname{Spec} R \mid I \subseteq \mathfrak{p}\}.$$

The importance of the radical is basic in the field of algebraic geometry, due to Hilbert's Nullstellensatz (cf. any introductory textbook on algebraic geometry).

(iv) Let $\mathfrak{a}$ be a fixed ideal of R. Then $\mathfrak{a}$-*saturation* is a closure operation on R. Using the usual notation of $(- : \mathfrak{a}^\infty)$, we may define it as follows:

$$(I : \mathfrak{a}^\infty) := \bigcup_{n \in \mathbb{N}} (I : \mathfrak{a}^n) = \{r \in R \mid \exists n \in \mathbb{N} \text{ such that } \mathfrak{a}^n r \subseteq I\}$$

This operation is important in the study of *local cohomology*. Indeed,

$$H^0_{\mathfrak{a}}(R/I) = \frac{(I : \mathfrak{a}^\infty)}{I}.$$

(v) The *integral closure* is a closure operation as well. One of the many equivalent definitions is as follows: For an element $r \in R$ and an ideal I of R,[1] $r \in I^-$ if

[1] Some may find my choice of notation surprising. Popular notations for integral closure include I_a and $\overline{I}$. I avoid the first of these because it looks like a variable subscript, as the letter a does not seem to stand for anything. The problem with the second notation is that it is overly ambiguous. Such notation can mean integral closure of rings, integral closure of ideals, a quotient module, and so forth. So in my articles, I choose to use the I^- notation to make it more consistent with the notation of other closure operations (such as tight, Frobenius, and plus closures).

there exist $n \in \mathbb{N}$ and $a_i \in I^i$ for $1 \le i \le n$ such that

$$r^n + \sum_{i=1}^{n} a_i r^{n-i} = 0.$$

Integral closure is a big topic. See for instance the books [51, 74].

(vi) Let R be an integral domain. Then *plus closure* is a closure operation. It is traditionally linked with tight closure (see below), and defined as follows: For an ideal I and an element $x \in R$, we say that $x \in I^+$ if there is some injective map $R \to S$ of integral domains, which makes S a finite R-module, such that $x \in IS$. (See Section 3.2 (vi) for general R.)

(vii) Let R be a ring of prime characteristic $p > 0$. Then *Frobenius closure* is a closure operation on R. To define this, we need the concept of bracket powers. For an ideal I, $I^{[p^n]}$ is defined to be the ideal generated by all the p^nth powers of elements of I. For an ideal I and an element $x \in R$, we say that $x \in I^F$ if there is some $n \in \mathbb{N}$ such that $x^{p^n} \in I^{[p^n]}$.

(viii) Let R be a ring of prime characteristic $p > 0$. Then *tight closure* is a closure operation on R. For an ideal I and an element $x \in R$, we say that $x \in I^*$ if there is some power $e_0 \in \mathbb{N}$ such that the ideal $\bigcap_{e \ge e_0}(I^{[p^e]} : x^{p^e})$ is not contained in any minimal prime of R.

(ix) Let R be a complete local domain. For an R-algebra S, we say that S is *solid* if $\operatorname{Hom}_R(S, R) \neq 0$. We define *solid closure* on R by saying that $x \in I^\star$ if $x \in IS$ for some solid R-algebra S. (See 3.2(ix) for general R.)

(x) Let Δ be a multiplicatively closed set of ideals. The Δ-*closure* [64] of an ideal I is $I^\Delta := \bigcup_{K \in \Delta}(IK : K)$. Ratliff [64] shows close connections between Δ-closure and integral closure for appropriate choices of Δ.

(xi) If $(R, \mathfrak{m})$ is local, and I is $\mathfrak{m}$-primary, then the *basically full closure* [31] of I is $I^{\mathrm{bf}} := (I\mathfrak{m} : \mathfrak{m})$. (Note: This is a closure operation even for non-$\mathfrak{m}$-primary ideals I. However, only for $\mathfrak{m}$-primary I does it produce the smallest so-called "basically full" ideal that contains I.)

Additional examples of closures include the v-, t-, and w-operations (4.1), various tight closure imitators (4.5), continuous and axes closures [8], natural closure [20], and weak subintegral closure [77]. (See the references for more details on these last four.)

Some properties follow from the axiomatic definition of a closure operation:

Proposition 2.1.3. *Let R be a ring and* cl *a closure operation. Let I be an ideal and $\{I_\alpha\}_{\alpha \in \mathcal{A}}$ a set of ideals.*

(i) *If every I_α is* cl*-closed, so is $\bigcap_\alpha I_\alpha$.*

(ii) $\bigcap_\alpha {I_\alpha}^{\mathrm{cl}}$ *is* cl*-closed.*

(iii) I^{cl} *is the intersection of all* cl*-closed ideals that contain* I.

(iv) $\left(\sum_\alpha {I_\alpha}^{\mathrm{cl}}\right)^{\mathrm{cl}} = \left(\sum_\alpha I_\alpha\right)^{\mathrm{cl}}$.

Proof. Let I and $\{I_\alpha\}$ be as above.

(i) For any $\beta \in \mathcal{A}$, we have $\bigcap_\alpha I_\alpha \subseteq I_\beta$, so since cl is order-preserving, we have $(\bigcap_\alpha I_\alpha)^{\mathrm{cl}} \subseteq {I_\beta}^{\mathrm{cl}} = I_\beta$. Since this holds for any β, we have $(\bigcap_\alpha I_\alpha)^{\mathrm{cl}} \subseteq \bigcap_\beta I_\beta = \bigcap_\alpha I_\alpha$.

(ii) This follows directly from part (i).

(iii) Let J be an ideal such that $I \subseteq J = J^{\mathrm{cl}}$. Then by order-preservation, $I^{\mathrm{cl}} \subseteq J^{\mathrm{cl}} = J$, so I^{cl} is contained in the given intersection. But since $I^{\mathrm{cl}} = (I^{\mathrm{cl}})^{\mathrm{cl}}$ is one of the ideals being intersected, the conclusion follows.

(iv) '$\supseteq$': By the Extension property, ${I_\alpha}^{\mathrm{cl}} \supseteq I_\alpha$, so $\sum_\alpha {I_\alpha}^{\mathrm{cl}} \supseteq \sum_\alpha I_\alpha$. Then the conclusion follows from order-preservation.

'$\subseteq$': For any $\beta \in \mathcal{A}$, $I_\beta \subseteq \sum_\alpha I_\alpha \subseteq (\sum_\alpha I_\alpha)^{\mathrm{cl}}$, so by order-preservation and idempotence, ${I_\beta}^{\mathrm{cl}} \subseteq ((\sum_\alpha I_\alpha)^{\mathrm{cl}})^{\mathrm{cl}} = (\sum_\alpha I_\alpha)^{\mathrm{cl}}$. Since this holds for all $\beta \in \mathcal{A}$, we have $\sum_\alpha {I_\alpha}^{\mathrm{cl}} \subseteq (\sum_\alpha I_\alpha)^{\mathrm{cl}}$. □

We finish the subsection on "basics" by giving two alternate characterizations of closure operations on R:

Remark 2.1.4. Here is a "low-tech" way of looking at closure operations, due essentially to Moore [60]. Namely, giving a closure operation is *equivalent* to giving a collection $\mathcal{C}$ of ideals such that the intersection of any subcollection is also in $\mathcal{C}$.

For suppose cl is a closure operation on R. Let $\mathcal{C}$ be the class of cl-closed ideals. That is, $I \in \mathcal{C}$ iff $I = I^{\mathrm{cl}}$. By Proposition 2.1.3 (i), the intersection of any subcollection of ideals in $\mathcal{C}$ is also in $\mathcal{C}$.

Conversely, suppose $\mathcal{C}$ is a collection of ideals for which the intersection of any subcollection is in $\mathcal{C}$. For an ideal I, let $I^{\mathrm{cl}} := \bigcap\{J \mid I \subseteq J \in \mathcal{C}\}$. All three of the defining properties of closure operations follow easily. Hence, cl is a closure operation.

The applicability of this observation is obvious: Given any collection of ideals in a ring, one may obtain a closure operation from it by extending it to contain all intersections of the ideals in the collection, and letting these be the closed ideals. The resulting closure operation may then be used to analyze the property that defined the original class of ideals.

Remark 2.1.5. On the other hand, here is a "high-tech" way of looking at closure operations. Let R be a ring, and $\mathcal{C}$ be the *category* associated to the partially ordered set of ideals of R. Then a closure operation on R is the same thing as a *monad in the category* $\mathcal{C}$ (see [58, VI.1] for the definition of monad in a category). It is easy to see that any monad in a poset is idempotent, and the theory of idempotent monads is central in the study of so-called "localization functors" in algebraic topology (thanks to G. Biedermann and G. Raptis, who mention [1, Chapter 2] as a good source, for pointing me in this direction).

2.2 Not-quite-closure Operations

It should be noted that the three given axioms of closure operations are independent of each other; many operations on ideals satisfy two of the axioms without satisfying the third. For example, the operation on ideals that sends every ideal to the 0 ideal is idempotent (condition (ii)) and is order-preserving (condition (iii)), but of course is not extensive unless R is the zero ring.

For an operation that is extensive (i) and order-preserving (iii), but is not idempotent, let f be a fixed element (or ideal) of R, and consider the operation $I \mapsto (I : f)$. This is almost never idempotent. For example one always has $((f^2 : f) : f) = (f^2 : f^2) = R$ of course, but if f is any nonzero element of the Jacobson radical of R, then $(f^2 : f) \neq R$.

Another non-idempotent operation that is extensive and order-preserving is the so-called "$\mathfrak{a}$-tight closure", for a fixed ideal $\mathfrak{a}$ [28], denoted $(\cdot)^{*\mathfrak{a}}$. By definition, $x \in I^{*\mathfrak{a}}$ if there is some $e_0 \in \mathbb{N}$ such that the ideal $\bigcap_{e \geq e_0}(I^{[p^e]} : \mathfrak{a}^{p^e} x^{p^e})$ is not contained in any minimal prime. In their Remark 1.4, Hara and Yoshida note that if $\mathfrak{a} = (f)$ is a principal ideal, then $I^{*\mathfrak{a}} = (I^* : \mathfrak{a})$, an operation which we have already noted fails to be idempotent. (For a similarly-defined operation which actually *is* a closure operation, see [79].)

Consider the operation which sends each ideal I to its *unmixed part* I^{unm} [50]. This is defined by looking at the primary ideals (commonly called *components*) in an irredundant minimal primary decomposition of I, and then intersecting those components that have *maximum dimension*. Although the decomposition is not uniquely determined, the components of maximum dimension are, so this is a well-defined operation. Moreover, this operation is extensive (i) and idempotent (ii) (since all the components of I^{unm} already have the same dimension), but is *not order-preserving* in general. For an example, let $R = k[x, y]$ be a polynomial ring in two variables over a field k, and let $J := (x^2, xy)$ and $I := (x^2, xy, y^2)$. Then $J \subseteq I$, but $J^{\text{unm}} = (x) \nsubseteq I^{\text{unm}} = I$. Similar comments apply in the 3-variable case when $J = (xy, xz)$ and $I = (y, z)$.

For another extensive, idempotent operation which is not order-preserving, consider the "Ratliff–Rush closure" (or "Ratliff–Rush operation"), given in [65], defined on so-called *regular ideals* (where an ideal I of R is *regular* if it contains an R-regular element), defined by $\tilde{I} := \bigcup_{n=1}^{\infty}(I^{n+1} : I^n)$. In [30, 1.11] (resp. in [29, 1.1]), the domain $R := k[\![x^3, x^4]\!]$ (resp. $R := k[x, y]$) is given where k is any field and x, y indeterminates over k, along with nonzero ideals $J \subseteq I$ of R such that $\widetilde{J} \nsubseteq \widetilde{I}$.

Many of the topics and questions explored in this article could be applied to these not-quite-closure operations as well, but additional care is needed.

3 Constructing Closure Operations

There are, however, some actions one can take which always produce closure operations.

3.1 Standard Constructions

Construction 3.1.1. Let U be an R-module. Then the operation $I \mapsto I^{\mathrm{cl}} := \{f \in R \mid fU \subseteq IU\} = (IU :_R U)$ gives a closure operation on R. Extension and order-preservation are clear. As for idempotence, suppose $f \in (I^{\mathrm{cl}})^{\mathrm{cl}}$. Then $fU \subseteq I^{\mathrm{cl}}U$. But for any $g \in I^{\mathrm{cl}}$, $gU \subseteq IU$, whence $I^{\mathrm{cl}}U \subseteq IU$, so $fU \subseteq IU$ as required.

As we shall see, this is a very productive way to obtain closure operations, especially when U is an R-algebra. For example, letting $\mathfrak{a}$ be an ideal of R and $U := R/\mathfrak{a}$, we see that the assignment $I \mapsto I + \mathfrak{a}$ gives a closure operation. On the other hand, letting $U := \mathfrak{a}$, the resulting closure operation becomes $I \mapsto (I\mathfrak{a} : \mathfrak{a})$, which is the basis for the Δ-*closures* of [64] and for the *basically full closure* of [31].

Construction 3.1.2. We give here a variant on Construction 3.1.1.

Let $\phi : R \to S$ be a ring homomorphism and let d be a closure operation on S. For ideals I of R, define $I^{\mathrm{c}} := \phi^{-1}((\phi(I)S)^{\mathrm{d}})$. (One might loosely write $I^{\mathrm{c}} := (IS)^{\mathrm{d}} \cap R$.) Then c is a closure operation on R.

Extension and order-preservation are clear. As for idempotence, if $f \in (I^{\mathrm{c}})^{\mathrm{c}}$, then $\phi(f) \in ((I^{\mathrm{c}})S)^{\mathrm{d}} \subseteq ((IS)^{\mathrm{d}})^{\mathrm{d}} = (IS)^{\mathrm{d}}$, so that $f \in I^{\mathrm{c}}$.

Construction 3.1.3. Let $\{\mathrm{c}_\lambda\}_{\lambda\in\Lambda}$ be an *arbitrary collection* of closure operations on ideals of R. Then $I^{\mathrm{c}} := \bigcap_{\lambda\in\Lambda} I^{\mathrm{c}_\lambda}$ gives a closure operation as well.[2]

Again, extension and order-preservation are clear. As for idempotence, suppose $f \in (I^{\mathrm{c}})^{\mathrm{c}}$. Then for every $\lambda \in \Lambda$, we have $f \in (I^{\mathrm{c}})^{\mathrm{c}_\lambda}$. But since $I^{\mathrm{c}} \subseteq I^{\mathrm{c}_\lambda}$ and c_λ preserves order, we have

$$f \in (I^{\mathrm{c}})^{\mathrm{c}_\lambda} \subseteq (I^{\mathrm{c}_\lambda})^{\mathrm{c}_\lambda} = I^{\mathrm{c}_\lambda},$$

where the last property follows from the idempotence of c_λ. Since $\lambda \in \Lambda$ was chosen arbitrarily, $f \in I^{\mathrm{c}}$ as required.

For the next construction, we need to mention the natural partial order on closure operations on a ring R. Namely, if c and d are closure operations, we write $\mathrm{c} \leq \mathrm{d}$ if for every ideal I, $I^{\mathrm{c}} \subseteq I^{\mathrm{d}}$.

Construction 3.1.4. Let $\{\mathrm{c}_\lambda\}_{\lambda\in\Lambda}$ be a *directed set* of closure operations. That is, for any $\lambda_1, \lambda_2 \in \Lambda$, there exists some $\mu \in \Lambda$ such that $\mathrm{c}_{\lambda_i} \leq \mathrm{c}_\mu$ for $i = 1, 2$. Moreover, assume that R is Noetherian. Then $I^{\mathrm{c}} := \bigcup_{\lambda\in\Lambda} I^{\mathrm{c}_\lambda}$ gives a closure operation.

[2] Similar considerations in the context of star-operations on integral domains are used in [2] to give lattice structures on certain classes of closure operations.

First note that I^{c} is indeed an ideal. It is the sum $\sum_{\lambda\in\Lambda} I^{\mathrm{c}_\lambda}$. After this, extension and order-preservation are clear. Next, we note that for any ideal I, there is some $\mu \in \Lambda$ such that $I^{\mathrm{c}} = I^{\mathrm{c}_\mu}$. To see this, we use the fact that I^{c} is finitely generated along with the directedness of the set $\{\mathrm{c}_\lambda \mid \lambda \in \Lambda\}$. Namely, $I^{\mathrm{c}} = (f_1, \ldots, f_n)$; each $f_i \in I^{\mathrm{c}_{\lambda_i}}$; then let c_μ be such that $\mathrm{c}_{\lambda_i} \leq \mathrm{c}_\mu$ for $i = 1, \ldots, n$.

To show idempotence, take any ideal I. By what we just showed, there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $(I^{\mathrm{c}})^{\mathrm{c}} = (I^{\mathrm{c}})^{\mathrm{c}_{\lambda_1}}$ and $I^{\mathrm{c}} = I^{\mathrm{c}_{\lambda_2}}$. Choose $\mu \in \Lambda$ such that $\mathrm{c}_{\lambda_i} \leq \mathrm{c}_\mu$ for $i = 1, 2$. Then

$$(I^{\mathrm{c}})^{\mathrm{c}} = (I^{\mathrm{c}_{\lambda_2}})^{\mathrm{c}_{\lambda_1}} \subseteq (I^{\mathrm{c}_\mu})^{\mathrm{c}_\mu} = I^{\mathrm{c}_\mu} \subseteq I^{\mathrm{c}}.$$

Construction 3.1.5. Let d be an operation on (ideals of) R that satisfies properties (i) and (iii) of Definition 2.1.1, but is not idempotent. Let $\mathcal{S}$ be the set of all closure operations on R defined by the property that $\mathrm{c} \in \mathcal{S}$ if and only if $I^{\mathrm{d}} \subseteq I^{\mathrm{c}}$ for all ideals I of R. Then by Construction 3.1.3, the assignment $I \mapsto I^{\mathrm{d}^\infty} := \bigcap_{\mathrm{c}\in\mathcal{S}} I^{\mathrm{c}}$ is itself a closure operation, called the *idempotent hull* of d [15, Section 4.6]. It is obviously the smallest closure operation lying above d.

If R is Noetherian, it is equivalent to do the following: Let $\mathrm{d}^1 := \mathrm{d}$, and for each integer $n \geq 2$, we inductively define d^n by setting $I^{\mathrm{d}^n} := (I^{\mathrm{d}^{n-1}})^{\mathrm{d}}$. Let $I^{\mathrm{d}'} := \bigcup_n I^{\mathrm{d}^n}$ for all I. One may routinely check that d' is an extensive, order-preserving operation on ideals of R, and idempotence follows from the ascending chain condition on the ideals $\{I^{\mathrm{d}^n}\}_{n\in\mathbb{N}}$. Clearly, $I^{\mathrm{d}'} = I^{\mathrm{d}^\infty}$.

Construction 3.1.6. This construction is only relevant when R is not necessarily Noetherian.

Let c be a closure operation. Then we define c_f by setting

$$I^{\mathrm{c}_f} := \bigcup \{J^{\mathrm{c}} \mid J \text{ a finitely generated ideal such that } J \subseteq I\}.$$

This is a closure operation: Extension follows from looking at the principal ideals (x) for all $x \in I$. Order-preservation is obvious. As for idempotence, suppose $z \in (I^{\mathrm{c}_f})^{\mathrm{c}_f}$. Then there is some finitely generated ideal $J \subseteq I^{\mathrm{c}_f}$ such that $z \in J^{\mathrm{c}}$. Let $\{z_1, \ldots, z_n\}$ be a finite generating set for J. Since each $z_i \in I^{\mathrm{c}_f}$, there exist finitely generated ideals $K_i \subseteq I$ such that $z_i \in K_i^{\mathrm{c}}$. Now let $K := \sum_{i=1}^n K_i$. Then $J \subseteq K^{\mathrm{c}}$, so that

$$z \in J^{\mathrm{c}} \subseteq (K^{\mathrm{c}})^{\mathrm{c}} = K^{\mathrm{c}},$$

and since K is a finitely generated sub-ideal of I, it follows that $z \in I^{\mathrm{c}_f}$.

If $\mathrm{c} = \mathrm{c}_f$, we say that c is of *finite type*. Clearly c_f is of finite type for any closure operation c, and it is the largest finite-type closure operation d such that $\mathrm{d} \leq \mathrm{c}$. Connected with this, we have the following:

Proposition 3.1.7. *Let* c *be a closure operation of finite type on* R*. Then every* c*-closed ideal is contained in a* c*-closed ideal that is maximal among* c*-closed ideals.*

The proof is a standard Zorn's lemma argument. The point is that the union of a chain of c-closed ideals is c-closed because c is of finite type.

3.2 Common Closures as Iterations of Standard Constructions

Here we will show that essentially all the closures we gave in Example 2.1.2 result as iterations of the constructions just given:

(i) The *identity* needs no particular construction.

(ii) The *indiscrete closure* is an example of Construction 3.1.1, by letting $U = 0$.

(iii) As for the *radical*, we use the characterization of it being the intersection of the prime ideals that contain the ideal. Consider the maps $\pi_{\mathfrak{p}} : R \to R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} =: \kappa(\mathfrak{p})$ for prime ideals $\mathfrak{p}$. Note that

$$\pi_{\mathfrak{p}}^{-1}(I\kappa(\mathfrak{p})) = \begin{cases} \mathfrak{p}, & \text{if } I \subseteq \mathfrak{p}, \\ R, & \text{otherwise.} \end{cases}$$

Let $I^{\mathfrak{p}} := \pi_{\mathfrak{p}}^{-1}(I\kappa(\mathfrak{p}))$. This is an instance of Construction 3.1.1 with $U = \kappa(\mathfrak{p})$, and the intersection of all such closures (Construction 3.1.3) is the radical. That is, $\sqrt{I} = \bigcap_{\mathfrak{p}\in\operatorname{Spec} R} I^{\mathfrak{p}}$.

(iv) The $\mathfrak{a}$-*saturation* may be obtained in one of two ways. Assuming one already has the extensive, order-preserving operation $(- : \mathfrak{a})$, then applying Construction 3.1.5 yields the $\mathfrak{a}$-saturation.

Alternately, let $\{a_\lambda \mid \lambda \in \Lambda\}$ be a generating set for $\mathfrak{a}$, with each $a_\lambda \neq 0$. Let $\ell_\lambda : R \to R_{a_\lambda}$ be the localization map. Then each $(I : a_\lambda^\infty) = \ell_\lambda^{-1}(IR_{a_\lambda}) =: I^\lambda$ is an instance of Construction 3.1.1 (or 3.1.2, if you like), and we may apply Construction 3.1.3 to get $(I : \mathfrak{a}^\infty) = \bigcap_{\lambda\in\Lambda}(I : a_\lambda^\infty) = \bigcap_{\lambda\in\Lambda} I^\lambda$.

(v) For *integral closure*, let $\mathfrak{p}$ be a minimal prime of R, let V be a valuation ring (or if $R/\mathfrak{p}$ is Noetherian, it's enough to let V be a rank 1 discrete valuation ring) between $R/\mathfrak{p}$ and its fraction field, let $j_V : R \to V$ be the natural map, and let $I^V := j_V^{-1}(IV)$ (which gives a closure operation via Construction 3.1.1 with $U = V$). Then it is a theorem (e.g. [51, Theorem 6.8.3]) that $I^- = \bigcap_{\text{all such } V} I^V$, which is an application of Construction 3.1.3.

(vi) For *plus closure*, *when R is an integral domain*, let Q be its fraction field, $\overline{Q}$ an algebraic closure of Q, and let R^+ be the integral closure of R in $\overline{Q}$. That is, R^+ consists of all elements of $\overline{Q}$ that satisfy a *monic* polynomial over R. Then we let $I^+ := IR^+ \cap R$, by way of Construction 3.1.1 with $U = R^+$.

In the general case, where R is not necessarily a domain, for each minimal prime $\mathfrak{p}$ of R we let $\pi_{\mathfrak{p}} : R \to R/\mathfrak{p}$ be the natural surjection. Then $I^+ := \bigcap_{\text{all such } \mathfrak{p}} \pi_{\mathfrak{p}}^{-1}((IR/\mathfrak{p})^+)$, via Construction 3.1.2.

(vii) For *Frobenius closure* (when R has positive prime characteristic p), we introduce the left R-modules ${}^eR$ for all $e \in \mathbb{N}$. ${}^eR$ has the same additive group structure as R (with elements being denoted ${}^er$ for each $r \in R$), and R-module structure given as follows: For $a \in R$ and ${}^er \in {}^eR$, $a \cdot {}^er = {}^e(a^{p^e}r)$. Let $f_e : R \to {}^eR$ be the R-module map given by $a \mapsto a \cdot {}^e1 = {}^e(a^{p^e})$. Let F_e be the closure operation given by $I^{F_e} := f_e^{-1}(I \cdot {}^eR)$, via Construction 3.1.1. Note that this is a *totally ordered set* (and hence a directed set) of closure operations, in that $F_e \leq F_{e+1}$ for all e, due to the R-module maps ${}^eR \to {}^{e+1}R$ given by ${}^er \mapsto {}^{e+1}(r^p)$. Thus, we may use Construction 3.1.4 to get $I^F := \bigcup_{e \in \mathbb{N}} I^{F_e}$.

(viii) For *tight closure* (when R has positive prime characteristic p), we cannot use these constructions directly. However, recall the theorem [37, Theorem 8.6] that under quite mild assumptions on R (namely the same ones that guarantee persistence of tight closure, see 4.3), $I^* = I^\star$, and use the constructions for solid closure below.

(ix) For *solid closure* (when R is a complete local domain), letting $i_S : R \to S$ for solid R-algebras S and $I^S := i_S^{-1}(IS)$ by way of Construction 3.1.1 with $U = S$, we note that this is a directed set of closure operations, since [37, Proposition 2.1a] if S and T are solid R-algebras, so is $S \otimes_R T$. Thus, we have $I^\star = \bigcup_{\text{all such } i_S} I^S$ via Construction 3.1.4.

For general R: Let $\mathfrak{m}$ be a maximal ideal of R, $\widehat{R}^{\mathfrak{m}}$ the completion of $R_{\mathfrak{m}}$ at its maximal ideal, $\mathfrak{p}$ a minimal prime of $\widehat{R}^{\mathfrak{m}}$, and $u_{\mathfrak{m},\mathfrak{p}} : R \to \widehat{R}^{\mathfrak{m}}/\mathfrak{p}$ the natural map. Then we use Constructions 3.1.3 and 3.1.2 to get $I^\star := \bigcap_{\text{all such pairs } \mathfrak{m},\mathfrak{p}} u_{\mathfrak{m},\mathfrak{p}}^{-1}((I\widehat{R}^{\mathfrak{m}}/\mathfrak{p})^\star)$.

(x) For Δ-*closure*, first note that for any ideal $K \in \Delta$, $I^K := (IK : K)$ gives a closure operation via Construction 3.1.1 with $U = K$. Next, note that the closure operations $\{(-)^K \mid K \in \Delta\}$ form a *directed set*, since for any $H, K \in \Delta$, $I^H + I^K \subseteq I^{KH}$. Thus, Construction 3.1.4 applies to give $I^\Delta := \bigcup_{K \in \Delta} I^K$.

(xi) For *basically full closure*, we merely apply Construction 3.1.1 with $U = \mathfrak{m}$.

4 Properties of Closures

4.1 Star-, Semi-prime, and Prime Operations

Definition 4.1.1. Let cl be a closure operation for a ring R. We say that cl is

(i) *semi-prime* [63] if for all ideals I, J of R, we have $I \cdot J^{\mathrm{cl}} \subseteq (IJ)^{\mathrm{cl}}$. (Equivalently, $\left(I^{\mathrm{cl}}J^{\mathrm{cl}}\right)^{\mathrm{cl}} = (IJ)^{\mathrm{cl}}$ for all I, J.)

(ii) a *star-operation* [25, Chapter 32 and see below] if for every ideal J and every non-zerodivisor x of R, $(xJ)^{\mathrm{cl}} = x \cdot (J^{\mathrm{cl}})$.

(iii) *prime* [55, 56, 64] if it is a semi-prime star-operation.

Sociological Comment. In the literature of so-called "multiplicative ideal theory" (which is, roughly, that branch of commutative algebra that uses [25] as its basic textbook), the definition of *star-operation* is somewhat different from the above. Namely, one assumes first that R is an integral domain, one defines star-operations on *fractional ideals of* R. However, when R is a domain, it is equivalent to do as I have done above. Moreover, the terminology of star-operations is different from the terminology of this article. For instance, if c is a star-operation on a domain R, then I^{c} is not called the c-closure, but rather the c-*envelope* (or sometimes c-*image*) of I, and if $I = I^{\mathrm{c}}$, then I is said to be a c-*ideal*. For the sake of self-containedness, I have elected rather to use the terminology I was raised on.

The field of closure operations on Noetherian rings has remained nearly disjoint from the field of star- (and "semistar-") operations on integral domains. I think this is largely because the two groups of people have historically been interested in very different problems and baseline assumptions. Multiplicative ideal theorists do not like to assume their rings are Noetherian, for example. But I feel it would save a good deal of energy if the two fields would come together to some extent. After all, there are very reasonable assumptions under which tight, integral, plus, and Frobenius closures are prime- (and hence star-) operations (see below). This provides the star-operation theorists with a fresh infusion of star-operations to study, and it provides those who study said closures with a fresh arsenal of tools with which to study them.

I take the point of view natural to one of my training, in which one generalizes from integral domains to general commutative rings.

First note the following:

Lemma 4.1.2. *Let* cl *be a closure operation on a ring* R.

(i) cl *is a semi-prime operation if and only if for all* $x \in R$ *and ideals* $J \subseteq R$, *we have* $x \cdot J^{\mathrm{cl}} \subseteq (xJ)^{\mathrm{cl}}$.

(ii) *If* R *is an integral domain, then* cl *is a star-operation if and only if it is prime.*

Proof. If cl is a semi-prime operation, then for any $x \in R$ and ideal $J \subseteq R$, we have

$$x \cdot J^{\mathrm{cl}} = (x) \cdot J^{\mathrm{cl}} \subseteq ((x)J)^{\mathrm{cl}} = (xJ)^{\mathrm{cl}}.$$

Conversely, suppose $xJ^{\mathrm{cl}} \subseteq (xJ)^{\mathrm{cl}}$ for all x and J. Let I be an ideal of R, and let $\{a_\lambda\}_{\lambda\in\Lambda}$ be a generating set for I. Then

$$I \cdot J^{\mathrm{cl}} = \sum_{\lambda\in\Lambda} a_\lambda \cdot J^{\mathrm{cl}} \subseteq \sum_\lambda (a_\lambda J)^{\mathrm{cl}} \subseteq \Big(\sum_\lambda a_\lambda J\Big)^{\mathrm{cl}} = (IJ)^{\mathrm{cl}},$$

so that cl is semi-prime.

Now suppose R is an integral domain. By definition any prime operation must be a star-operation. So let cl be a star-operation on R. To see that it is semi-prime, we use part (i). For any $x \in R$, either $x = 0$ or x is a non-zerodivisor. Clearly $0 \cdot J^{\mathrm{cl}} = 0 \subseteq 0^{\mathrm{cl}} = (0J)^{\mathrm{cl}}$. And if x is a non-zerodivisor, then $x \cdot J^{\mathrm{cl}} = (xJ)^{\mathrm{cl}}$ by definition of star-operation. □

One reason why the star-operation property is useful is as follows: any star-operation admits a unique extension to the set of fractional ideals of R (where a *fractional ideal* is defined to be a submodule M of Q, the total quotient ring of R, such that for some non-zerodivisor f of R, $fM \subseteq R$). Namely, if cl is a star-operation and M is a fractional ideal, an element $x \in Q$ is in M^{cl} if $fx \in (fM)^{\mathrm{cl}}$, where f is a non-zerodivisor of R such that $fM \subseteq R$. After this observation, another important property of star-operations is that if two fractional ideals M, N are isomorphic, their closures are isomorphic as well.

Star-operations are important in the study of so-called *Kronecker function rings* (for a historical and topical overview of this connection, see [23]). However, the star-operation property is somewhat limiting. For instance, I leave it as an exercise for the reader to show that if R is a local Noetherian ring, then the radical operation on R is a star-operation if and only if depth $R = 0$. The only star-operation on a rank 1 discrete valuation ring is the identity. On the other hand, it is well known that integral closure is a star-operation on R if and only if R is normal. This is true of tight closure as well:

Proposition 4.1.3. *Consider the following property for a closure operation* cl*:*

$$\textit{For any non-zerodivisor } x \in R \textit{ and any ideal } I,\ I^{\mathrm{cl}} = ((xI)^{\mathrm{cl}} : x). \qquad (\#)$$

(i) *A closure operation* cl *is a star-operation if and only if it satisfies* (#) *and* $(x)^{\mathrm{cl}} = (x)$ *for all non-zerodivisors* $x \in R$.

(ii) *Closure operations that satisfy* (#) *include plus-closure (when* R *is a domain), integral closure, tight closure (in characteristic* $p > 0$*), and Frobenius closure.*

Proof. (i) Suppose cl is a star-operation, x is a non-zerodivisor, and I an ideal. Then $(x)^{\mathrm{cl}} = ((x)R)^{\mathrm{cl}} = x \cdot R^{\mathrm{cl}} = (x)$, and $((xI)^{\mathrm{cl}} : x) = (x(I^{\mathrm{cl}}) : x)$ (since cl is a star-operation) $= I^{\mathrm{cl}}$ (since x is a non-zerodivisor).

Conversely, suppose cl satisfies (#) and that all principal ideals generated by non-zerodivisors are cl-closed. For a non-zerodivisor x and ideal I, we have $x \cdot I^{\mathrm{cl}} = x \cdot ((xI)^{\mathrm{cl}} : x) \subseteq (xI)^{\mathrm{cl}}$, so we need only show that $(xI)^{\mathrm{cl}} \subseteq x \cdot I^{\mathrm{cl}}$. So suppose $g \in (xI)^{\mathrm{cl}}$. Since $xI \subseteq (x)$, it follows that $g \in (x)^{\mathrm{cl}} = (x)$, so $g = xf$ for some $f \in R$. Thus, $xf \in (xI)^{\mathrm{cl}}$, so $f \in ((xI)^{\mathrm{cl}} : x) = I^{\mathrm{cl}}$, whence $g = xf \in x \cdot I^{\mathrm{cl}}$ as required.

(ii) (Frobenius closure): Let $g \in ((xI)^F : x)$. Then $xg \in (xI)^F$, so there is some $q = p^n$ such that

$$x^q g^q = (xg)^q \in (xI)^{[q]} = x^q I^{[q]}.$$

Since x^q is a non-zerodivisor, $g^q \in I^{[q]}$, whence $g \in I^F$.

(Tight closure): The proof is similar to the Frobenius closure case.

(Plus closure): If $xg \in (xI)^+$, then there is some module-finite domain extension $R \subseteq S$ such that $xg \in xIS$. But since x is a non-zero element of the domain S (hence a non-zerodivisor on S), it follows that $g \in IS$, whence $g \in I^+$.

(Integral closure): Suppose $xg \in (xI)^-$. Then there is some $n \in \mathbb{N}$ and elements $a_i \in (xI)^i$ $(1 \leq i \leq n)$ such that

$$(xg)^n + \sum_{i=1}^{n} a_i (xg)^{n-i} = 0.$$

But each $a_i \in (xI)^i = x^i I^i$, so for some $b_i \in I^i$ (for each i), we have $a_i = x^i b_i$. Then the displayed equation yields:

$$x^n \Big(g^n + \sum_{i=1}^{n} b_i g^{n-i} \Big) = 0,$$

and since x^n is a non-zerodivisor, it follows that $g \in I^-$. □

Semi-prime operations, however, are ubiquitous. (In fact, some authors [54] even include the property in their basic definition of what a closure operation is!) One can, of course, cook up a non-semi-prime closure operation, even on a rank 1 discrete valuation ring [75, Example 2.3]. However, *essentially all the examples and constructions explored so far yield semi-prime operations*, in the following sense (noting that all of the following statements have easy proofs):

- Any closure arising from Construction 3.1.1 is semi-prime.
- In Construction 3.1.2, if d is a semi-prime operation on S, then c is a semi-prime operation on R.
- In Construction 3.1.3, if every c_λ is semi-prime, then so is c.
- In Construction 3.1.4, if every c_λ is semi-prime, then so is c.
- In Construction 3.1.5, if $I \cdot J^{\mathrm{d}} \subseteq (IJ)^{\mathrm{d}}$ for all ideals I, J of R, then d^∞ is semi-prime.
- In Construction 3.1.6, if c is semi-prime, then so is c_f.
- Hence by 3.2, all of the closures from Example 2.1.2 are semi-prime.[3]

Here are some nice properties of semi-prime closure operations:

Proposition 4.1.4. *Let* cl *be a semi-prime closure operation on* R*. Let* I*,* J *be ideals of* R*, and* W *a multiplicatively closed subset of* R*.*

(i) $(I : J)^{\mathrm{cl}} \subseteq (I^{\mathrm{cl}} : J)$*. Hence if* I *is* cl*-closed, then so is* $(I : J)$*.*

(ii) $(I^{\mathrm{cl}} : J)$ *is* cl*-closed.*

[3] One need not go through solid closure to show that tight closure must always be semi-prime.

(iii) *If R is Noetherian and I is* cl*-closed, then* $(IW^{-1}R) \cap R$ *is* cl*-closed.*

(iv) *If R is Noetherian and I is* cl*-closed, then all the* minimal primary components *of I are* cl*-closed. Hence, if $I = I^{\mathrm{cl}}$ has no embedded components, it has a* primary decomposition by cl-closed ideals.

(v) *The maximal elements of the set* $\{I \mid I^{\mathrm{cl}} = I \neq R\}$ *are prime ideals.*

Proof. (i) Let $f \in (I : J)^{\mathrm{cl}}$. Then $Jf \subseteq J(I : J)^{\mathrm{cl}} \subseteq (J \cdot (I : J))^{\mathrm{cl}} \subseteq I^{\mathrm{cl}}$.

(ii) follows directly from part (i).

(iii) Let $J := (IW^{-1}R) \cap R$. Then $J = \{f \in R \mid \exists w \in W$ such that $wf \in I\}$. But J is finitely generated (since R is Noetherian); say $J = (f_1, \dots, f_n)$. Then for each $1 \leq i \leq n$, there exists $w_i \in W$ such that $w_i f_i \in I$. Let $w := \prod_{i=1}^n w_i$. Then $wJ \subseteq I$, so $J \subseteq (I : w)$. But it is obvious that $(I : w) \subseteq J$, so $J = (I : w)$. Then the conclusion follows from part (i).

(iv) The minimal primary components of I look like $IR_P \cap R$, for each minimal prime P over I. Then the conclusion follows from part (iii).

(v) Let I be such a maximal element. Let $x, y \in R$ such that $xy \in I$ and $y \notin I$. Then $(I : x)$ is a cl-closed ideal (by part (i)) that properly contains I (since $y \in (I : x) \setminus I$), so since I is maximal among proper cl-closed ideals, it follows that $(I : x) = R$, which means that $x \in I$. □

Finally, here is a construction on semi-prime operations:

Construction 4.1.5. Let c be a semi-prime closure operation on R. Let c_f-Max R (see Construction 3.1.6 for the definition of c_f) denote the set of c_f-closed ideals which are maximal among the set of all c_f-closed ideals. By Proposition 4.1.4 (v), c_f-Max R consists of prime ideals, and by Proposition 3.1.7, every c_f-closed ideal is contained in a member of c_f-Max R. Then we define c_w as follows:

$$I^{\mathrm{c}_w} := \{x \in R \mid \forall \mathfrak{p} \in \mathrm{c}_f\text{-}\operatorname{Max} R,\ \exists d \in R \setminus \mathfrak{p} \text{ such that } dx \in I\}.$$

In other words, I^{c_w} consists of all the elements of R that land in the extension of I to all localizations $R \to R_{\mathfrak{p}}$ for $\mathfrak{p} \in \mathrm{c}_f$-Max R. As this arises from Constructions 3.1.1 and 3.1.3, c_w is a semi-prime closure operation.

Moreover, $\mathrm{c}_w \leq \mathrm{c}_f$. To see this, let $x \in I^{\mathrm{c}_w}$. Then for all $\mathfrak{p} \in \mathrm{c}_f$-Max R, there exists $d_{\mathfrak{p}} \in R \setminus \mathfrak{p}$ with $d_{\mathfrak{p}} x \in I$. Let J be the ideal generated by the set $\{d_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{c}_f$-Max $R\}$. Then $Jx \subseteq I$ and $J^{\mathrm{c}_f} = R$, so since c_f is semi-prime, we have

$$x = 1 \cdot x \in R(x) = J^{\mathrm{c}_f}(x) \subseteq (Jx)^{\mathrm{c}_f} \subseteq I^{\mathrm{c}_f}.$$

If R is a domain, and c is a star-operation (i.e. prime), then this construction is essentially due to [3], who show that in this context c_w distributes over finite intersections, is of finite type, and is the *largest* star-operation d of finite type that distributes over finite intersection such that $\mathrm{d} \leq \mathrm{c}$.

The v-operation

Arguably the most important star-operation (at least in the theory of star-operations *per se*) is the so-called v-operation. Classically it was only defined when R is a domain [25, Chapters 16, 32, 34], but it works in general. Most star-operations in the literature (among those that are identified as star-operations) are based in one way or another on the v-operation:

Definition 4.1.6. Let R be a ring and Q its total quotient ring. For an ideal I, the set I_{v} is defined to be the intersection of all cyclic R-submodules M of Q such that $I \subseteq M$.

Proposition 4.1.7. (i) v *is a star-operation.*

(ii) *For any star-operation* cl *on R, $I^{\mathrm{cl}} \subseteq I_{\mathrm{v}}$ for all ideals I of R. (That is,* v *is the* largest *star-operation on R.)*

(iii) *There exists a ring R for which* v *is* not *semi-prime (and hence not prime).*

Proof. It is easy to see that v is a closure operation. For the star-operation property, let x be a non-zerodivisor and I an ideal of R. Let $a \in I_{\mathrm{v}}$. Let M be a cyclic submodule of Q that contains xI. Then $M = R \cdot \frac{r}{s}$ for some $r, s \in R$ with s a non-zerodivisor. Moreover, $xI \subseteq M$ implies that $I \subseteq R \cdot \frac{r}{sx}$. Since $a \in I_{\mathrm{v}}$, we have $a \in R \cdot \frac{r}{sx}$ as well, so that $xa \in R \cdot \frac{r}{s} = M$. Thus, $xa \in (xI)_{\mathrm{v}}$ as required.

For the opposite inclusion, we first note that principal ideals are v-closed because they are cyclic R-submodules of Q. Now let $a \in (xI)_{\mathrm{v}}$. Since $xI \subseteq (x)$, we have $a \in (xI)_{\mathrm{v}} \subseteq (x)_{\mathrm{v}} = (x)$, so $a = xb$ for some $b \in R$. Let $M = R \cdot \frac{r}{s}$ be a cyclic submodule of Q that contains I. Then $xI \subseteq xM = R \cdot \frac{xr}{s}$, so since $a \in (xI)_{\mathrm{v}}$, it follows that $xb = a \in xM = R \cdot \frac{xr}{s} = x \cdot R \cdot \frac{r}{s}$. Since x is a non-zerodivisor on Q, we can cancel it to get $b \in R \cdot \frac{r}{s} = M$, whence $b \in I_{\mathrm{v}}$. Thus, $a = xb \in x \cdot I_{\mathrm{v}}$, as required.

Now let cl be an arbitrary star-operation on R, and I an ideal. Let $M = R \cdot \frac{r}{s}$ be a cyclic submodule of R that contains I (so that $r, s \in R$ and s is not a zerodivisor). Then $sI \subseteq rR = (r)$, so that

$$s \cdot I^{\mathrm{cl}} = (sI)^{\mathrm{cl}} \subseteq (r)^{\mathrm{cl}} = (r).$$

That is, $I^{\mathrm{cl}} \subseteq R \cdot \frac{r}{s} = M$. Thus, $I^{\mathrm{cl}} \subseteq I_{\mathrm{v}}$.

For the counterexample, let

$$R := k[X, Y]/(X^2, XY, Y^2) = k[x, y],$$

where k is a field and X, Y are indeterminates over k (with the images in R denoted x, y respectively). Let $\mathfrak{m} := (x, y)$ be the unique maximal ideal of R. Since R is an Artinian local ring, it is equal to its own total ring of quotients, and so the cyclic R-submodules of said ring of quotients are just the principal ideals of R. Since $\mathfrak{m}$ is not contained in any proper principal ideal of R, we have $\mathfrak{m}_{\mathrm{v}} = R$. Thus, $\mathfrak{m} \cdot \mathfrak{m}_{\mathrm{v}} =$

$\mathfrak{m} \cdot R = \mathfrak{m}$. On the other hand, $(\mathfrak{m}\mathfrak{m})_{\mathrm{v}} = (\mathfrak{m}^2)_{\mathrm{v}} = 0_{\mathrm{v}} = 0$, which shows that $\mathfrak{m} \cdot \mathfrak{m}_{\mathrm{v}} \not\subseteq (\mathfrak{m}\mathfrak{m})_{\mathrm{v}}$, whence v is not semi-prime. □

Remark 4.1.8. When R is an integral domain, parts (i) and (ii) of the above Proposition are well known (and since by Lemma 4.1.2, any star-operation on a domain is prime, the analogue of (iii) is false). Two other well-known properties of the v-operation in the domain case are as follows:

- $\mathrm{Hom}_R(\mathrm{Hom}_R(I, R), R) \cong I_{\mathrm{v}}$ as R-modules. For this reason, the v-operation is sometimes also called the *reflexive hull* operation on ideals.
- $I_{\mathrm{v}} = (I^{-1})^{-1}$. (Recall that for a fractional ideal J of R, $J^{-1} := \{x \in Q \mid xJ \subseteq R\}$, where Q is the quotient field of R.) For this reason, the v-operation is sometimes also called the *divisorial closure*.

These ideas have obvious connections to Picard groups and divisor class groups.

The t- and w-operations (see e.g. [80]) should be mentioned here as well. By definition, $\mathrm{t} := \mathrm{v}_f$ (via Construction 3.1.6). When R is a domain, the w-operation is defined by $\mathrm{w} := \mathrm{v}_w$ (by Construction 4.1.5). So if R is a domain, then t is semi-prime, but otherwise it need not be (as the counterexample in Proposition 4.1.7 shows), and w may not even be well-defined in the non-domain case.

4.2 Closures Defined by Properties of (Generic) Forcing Algebras

Let R be a ring, I a (finitely generated) ideal and $f \in R$. Then a *forcing algebra* [37] for $[I; f]$ consists of an R-algebra A such that (the image of) $f \in IA$. In particular, given a generating set $I = (f_1, \dots, f_n)$, one may construct the *generic forcing algebra* A for the data $[f_1, \dots, f_n; f]$, given by

$$A := R[T_1, \dots, T_n] \Big/ \Big(f + \sum_{i=1}^{n} f_i T_i\Big).$$

Clearly A is a forcing algebra for $[I; f]$. Moreover, if B is any other forcing algebra for $[I; f]$, there is an R-algebra map $A \to B$. To see this, if $f \in IB$, then there exist $b_1, \dots, b_n \in B$ such that $f + \sum_{i=1}^{n} f_i b_i = 0$. Then we define the map $A \to B$ by sending each $T_i \mapsto b_i$.

Many closure operations may be characterized by properties of generic forcing algebras. This viewpoint is explored in some detail in [9], where connections are also made with so-called *Grothendieck topologies*. We list a few (taken from [9]), letting $f, I := (f_1, \dots, f_n)$, and A be as above:

- $f \in I$ (the identity closure of I) if and only if R is a forcing algebra for $[f; I]$. That is, $f \in I$ if and only if there is an R-algebra map $A \to R$. In geometric terms, one says that the structure map Spec $A \to$ Spec R has a *section*.

- $f \in \sqrt{I}$ if and only if $f \in IK$ for all fields K that are R-algebras, if and only if the ring map $R \to A$ has the *lying-over* property, if and only if the map Spec $A \to$ Spec R is *surjective* (as a set map).
- $f \in I^-$ if and only if $f \in IV$ for all rank-1 discrete valuations of R (appropriately defined), if and only if for all such V, there is an R-algebra map $A \to V$. It is not immediately obvious, but this is equivalent to the topological property that the map Spec $A \to$ Spec R is *universally submersive* (also called a *universal topological epimorphism*).
- If R has prime characteristic $p > 0$, $f \in I^F$ if and only if $f \in IR_\infty$, if and only if there is an R-algebra map $A \to R_\infty$.

The final closure to note in this context is solid closure (the connection of which to tight closure has already been noted). For simplicity, let $(R, \mathfrak{m})$ be a complete local domain of dimension d. We have $f \in I^\star$ if and only if there is some solid R-algebra S such that $f \in IS$, i.e. iff there is an R-algebra map $A \to S$. Hochster [37, Corollary 2.4] showed in turn that this is equivalent to the condition that $H^d_\mathfrak{m}(A) \neq 0$. (!) This viewpoint brings in all sorts of cohomological tools into the study of solid closure, and hence tight closure in characteristic p. Such tools were crucial in the proof that tight closure does not always commute with localization [10].

4.3 Persistence

Although it is possible to do so, usually one does not define a closure operation one ring at a time. The more common thing to do is define the closure operation for a whole class of rings. In such cases, the most important closure operations are *persistent*:

Definition 4.3.1. Let $\mathcal{R}$ be a subcategory of the category of commutative rings; let c be a closure operation defined on the rings of $\mathcal{R}$. We say that c is *persistent* if for any ring homomorphism $\phi : R \to S$ in $\mathcal{R}$ and any ideal I of R, one has $\phi(I^{\mathrm{c}})S \subseteq (\phi(I)S)^{\mathrm{c}}$.

Common choices for $\mathcal{R}$ are

- All rings and ring homomorphisms.
- Any full subcategory of the category of rings.
- Graded rings and graded homomorphisms.
- Local rings and local homomorphisms.

For instance, it is easy to show that radical and integral closure are persistent on the category of all rings (as are the identity and indiscrete closures), and that Frobenius closure is persistent on characteristic p rings. Tight closure is persistent along maps $R \to S$ of characteristic p rings, as long as either $R/\sqrt{0}$ is F-finite or R is essentially of finite type over an excellent local ring, although this is truly a deep theorem [44]. On the other hand, tight closure is persistent on the category of equal characteristic 0 rings because of the way it is defined (see the discussion after Theorem 4.5.1). Saturation (with respect to the maximal ideal) is persistent on the category of local rings and local homomorphisms, as well as on the category of graded rings and graded homomorphisms over a fixed base field.

Plus closure is also persistent on the category of integral domains, as is evident from the fact that the operation of taking absolute integral closure of the domains involved is *weakly functorial*, in the sense that any such map $R \to S$ extends (not necessarily uniquely) to a map $R^+ \to S^+$ [42, p. 139]. This argument may be extended to show that plus closure is persistent on the category of *all* rings as well, by considering minimal primes.

Basically full closure, however, is not persistent, even if we restrict to complete local rings of dimension one, $\mathfrak{m}$-primary ideals, and local homomorphisms $R \to S$ such that S becomes a finite R-module. For a counterexample, let k be any field, let x, y, z be analytic indeterminates over k, let $R := k[\![x, y]\!]/(x^2, xy)$, $I := yR$, $S := k[\![x, y, z]\!]/(x^2, xy, z^2)$, and let the map $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be the obvious inclusion. Then $I^{\mathrm{bf}} = (\mathfrak{m}y :_R \mathfrak{m}) = (x, y)$ because x is killed by all of $\mathfrak{m}$. But $x \notin (IS)^{\mathrm{bf}} = (\mathfrak{n}y :_S \mathfrak{n})$ because $zx \in \mathfrak{n}x \setminus \mathfrak{n}y$. (Indeed, IS is a basically full ideal of S.)

4.4 Axioms Related to the Homological Conjectures

A treatment of closure operations would not be complete without mentioning the so-called "homological conjectures" (for which we assume all rings are Noetherian). So named by Mel Hochster, these comprise a complex list of reasonable-sounding statements that have been central to research in commutative algebra since the 1970s. For the original treatment, see [35]. For a more modern treatment, see [39]. Rather than trying to cover the topic comprehensively, consider the following two conjectures:

Conjecture 4.4.1 (Direct Summand Conjecture). *Let $R \to S$ be an injective ring homomorphism, where R is a regular local ring, such that S is module-finite over R. Then R is a direct summand of S, considered as R-modules.*

Conjecture 4.4.2 (Cohen–Macaulayness of Direct Summands conjecture). *Let $A \to R$ be a ring homomorphism which makes A a direct summand of R, and suppose R is regular. Then A is Cohen–Macaulay.*

Assuming a "sufficiently good" closure operation in mixed characteristic, these conjectures would be theorems. Indeed, they *are* theorems in equal characteristic, a fact which can be seen as a consequence of the existence of tight closure, as discussed in Section 4.5 (although the proof in equal characteristic of Conjecture 4.4.1 predated tight closure by some 15 years [34], as did a proof of some special cases of Conjecture 4.4.2 that come from invariant theory [46]).

Consider the following axioms for a closure operation cl on a category $\mathcal{R}$ of Noetherian rings:

(i) (Persistence) For ring maps $R \to S$ in $\mathcal{R}$, we have $I^{\mathrm{cl}}S \subseteq (IS)^{\mathrm{cl}}$.

(ii) (Tightness) If R is a regular ring in $\mathcal{R}$, then $I^{\mathrm{cl}} = I$ for all ideals I of R.

(iii) (Plus-capturing) If $R \to S$ is a module-finite extensions of integral domains in $\mathcal{R}$ and I is any ideal of R, then $IS \cap R \subseteq I^{\mathrm{cl}}$. (i.e. $I^+ \subseteq I^{\mathrm{cl}}$)

(iv) (Colon-capturing) Let R be a *local* ring in $\mathcal{R}$ and $x_1, \dots, x_d$ a system of parameters. Then for all $0 \le i \le d-1$, $(x_1, \dots, x_i) : x_{i+1} \subseteq (x_1, \dots, x_i)^{\mathrm{cl}}$.

Notation. Let $d \in \mathbb{N}$, and let a, b be numbers such that either $a = b$ is a rational prime number, $a = b = 0$, or $a = 0$ and b is a rational prime number. For any such triple (d, a, b), let $\mathcal{R}_{d,a,b}$ be the category of complete local domains $(R, \mathfrak{m})$ such that $\dim R = d$, $\operatorname{char} R = a$, and $\operatorname{char}(R/\mathfrak{m}) = b$.

Proposition 4.4.3. *Let (d, a, b) be a triple as above.*

Suppose a closure operation cl *exists on the category $\mathcal{R}_{d,a,b}$ that satisfies conditions* (i)*,* (ii)*, and* (iii) *above. Then the Direct Summand Conjecture holds in the category $\mathcal{R}_{d,a,b}$.*

Proof. Let $R \to S$ be a module-finite injective ring homomorphism in $\mathcal{R}_{d,a,b}$, where R is a regular local ring. Let I be any ideal of R. Then $I \subseteq IS \cap R \subseteq I^{\mathrm{cl}} = I$ by properties (iii) and (ii). That is, $R \to S$ is *cyclically pure*. But then by [36], since R is a Noetherian domain it follows that $R \to S$ is *pure*. Since S is module-finite over R, it follows that R is a direct summand of S [47, Corollary 5.3]. □

Proposition 4.4.4. *Let (d, a, b) be as above.*

Suppose a closure operation cl *exists on the category $\mathcal{R}_{d,a,b}$ that satisfies conditions* (i)*,* (ii)*, and* (iv) *above. Then Conjecture* 4.4.2 *holds in the category $\mathcal{R}_{d,a,b}$.*

Proof. Let $A \to R$ be a local ring homomorphism of Noetherian local rings such that A is a direct summand of R, and suppose R is regular. Let $x_1, \dots, x_d$ be a system of parameters for A, and pick some $0 \le i < d$. Then

$$\begin{aligned}(x_1, \dots, x_i) : x_{i+1} &\subseteq (x_1, \dots, x_i)^{\mathrm{cl}} \subseteq (x_1, \dots, x_i)^{\mathrm{cl}} R \cap A \\ &\subseteq ((x_1, \dots, x_i)R)^{\mathrm{cl}} \cap A = (x_1, \dots, x_i)R \cap A = (x_1, \dots, x_i).\end{aligned}$$

Thus, A is Cohen–Macaulay. □

Remark 4.4.5. These ideas are closely related to Hochster's *big Cohen–Macaulay modules* conjecture. Indeed, Conjecture 4.4.1 holds when R has a so-called 'big Cohen–Macaulay module' (see e.g. [39, the final remark of §4]). When R is a complete local domain, Dietz [14] has given axioms for a persistent, residual closure operation on *R-modules* (see Section 7 for the basics on module closures) that are equivalent to the existence of a big Cohen–Macaulay module over R.

4.5 Tight Closure and Its Imitators

Tight closure has been used, among other things, to carry out the program laid out in Section 4.4 in cases where the ring contains a field. Indeed, we have the following:

Theorem 4.5.1 ([41, 45]). *Consider the category $\mathcal{R} := \mathcal{R}_{d,p,p}$, where d is any nonnegative integer and $p \geq 0$ is either a prime number or zero. Then we have*

(i) *(Persistence) For ring maps $R \to S$ in $\mathcal{R}$, we have $I^*S \subseteq (IS)^*$.*

(ii) *(Tightness) If R is a regular ring in $\mathcal{R}$, then $I^* = I$ for all ideals I of R.*

(iii) *(Plus-capturing) If $R \to S$ is a module-finite extensions of integral domains in $\mathcal{R}$ and I is any ideal of R, then $IS \cap R \subseteq I^*$. (i.e. $I^+ \subseteq I^*$)*

(iv) *(Colon-capturing) Let R be a* local *ring in $\mathcal{R}$ and $x_1, \dots, x_d$ a system of parameters. Then for all $0 \leq i \leq d-1$, $(x_1, \dots, x_i) : x_{i+1} \subseteq (x_1, \dots, x_i)^*$.*

(v) *(Briançon–Skoda property) For any $R \in \mathcal{R}$ and any ideal I of R,*

$$(I^d)^- \subseteq I^* \subseteq I^-.$$

Hence, by Propositions 4.4.3 and 4.4.4, both the Direct Summand Conjecture and Conjecture 4.4.2 hold in equal characteristic.

Tight closure is defined (in [45]) for finitely generated $\mathbb{Q}$-algebras by a process of "reduction to characteristic p", a time-honored technique that we will not get into here (but see Definition 6.3.1 for a baby version of it). Next, tight closure was defined on arbitrary (excellent) $\mathbb{Q}$-algebras in an "equational" way (see Section 4.6), and then Artin approximation must be employed to demonstrate that it has the properties given in Theorem 4.5.1. This is a long process, so various attempts have been made to give a closure operation in equal characteristic 0 that circumvents it. More importantly, though, people have been trying to obtain a closure operation in *mixed* characteristic that has the right properties.

The first such attempt was probably solid closure [37], already discussed. Unfortunately, it fails tightness for regular rings of dimension 3 [67]. Parasolid closure [7] (a variant of solid closure) agrees with tight closure in characteristic p, and it has all the right properties in equal characteristic 0, but is not necessarily easier to work with than tight closure, and it may or may not have the right properties in mixed characteristic. Other at least partially successful attempts include parameter tight closure [38], diamond closure [48] and dagger closure (defined in [43], but shown to satisfy tightness just recently in [11]).

The most successful progress on the homological conjectures since the advent of tight closure theory is probably represented by Heitmann's proof [33] of the direct summand conjecture (Conjecture 4.4.1) for $\mathcal{R}_{3,0,p}$ (i.e. in mixed characteristic in dimension 3), which he does according to the program laid out above. Indeed, he shows the analogue of Theorem 4.5.1 when $\mathcal{R} = \mathcal{R}_{3,0,p}$ and tight closure is replaced everywhere with *extended plus closure*, denoted epf, first defined in his earlier paper [32].

(So in fact, his proof works to show the dimension 3 version of Conjecture 4.4.2 as well.)

4.6 (Homogeneous) Equational Closures and Localization

Let cl be a closure operation on some category $\mathcal{R}$ of rings. We say that cl *commutes with localization* in $\mathcal{R}$ if for any ring R and any multiplicative set $W \subseteq R$ such that the localization map $R \to W^{-1}R$ is in $\mathcal{R}$, and for any ideal I of R, we have $I^{\mathrm{cl}}(W^{-1}R) = (IW^{-1}R)^{\mathrm{cl}}$.

Tight closure does not commute with localization [10], unlike Frobenius closure, plus closure, integral closure, and radical. In joint work with Mel Hochster [21], we investigated what it is that makes a persistent closure operation commute with localization, and in so doing we construct a tight-closure-like operation that *does* commute with localization. Our closure is in general smaller than tight closure, though in many cases (see below), it does in fact coincide with tight closure.

To do this, we introduce the concepts of *equational* and *homogeneous(ly equational)* closure operations. The former notion is given implicitly in [45], where Hochster and Huneke define the characteristic 0 version of tight closure in an equational way.

In the following, we let Λ be a fixed base ring. Often $\Lambda = \mathbb{F}_p$, $\mathbb{Z}$, or $\mathbb{Q}$.

Definition 4.6.1. Let $\mathcal{F}$ be the category of *finitely generated* Λ-algebras, and let c be a persistent closure operation on $\mathcal{F}$. Then the *equational* version of c, denoted ceq, is defined on the category of *all* Λ-algebras as follows: Let R be a Λ-algebra, $f \in R$, and I an ideal of R. Then $f \in I^{\mathrm{ceq}}$ if there exists $A \in \mathcal{F}$, an ideal J of A, $g \in A$, and a Λ-algebra map $\phi : A \to R$ such that $g \in J^{\mathrm{c}}$, $\phi(g) = f$, and $\phi(J) \subseteq I$.

Let $\mathcal{G}$ be the category of finitely generated $\mathbb{N}$-*graded* Λ-algebras A which have the property that $[A]_0 = \Lambda$. Let c be a persistent closure operation on $\mathcal{G}$. Then the *homogeneous(ly equational)* version of c, denoted ch, is defined on the category of *all* Λ-algebras as follows: Let R be a Λ-algebra, $f \in R$, and I an ideal of R. Then $f \in I^{\mathrm{ch}}$ if there exists $A \in \mathcal{G}$, a *homogeneous* ideal J of A, a *homogeneous* element $g \in A$, and a Λ-algebra map $\phi : A \to R$ such that $g \in J^{\mathrm{c}}$, $\phi(g) = f$, and $\phi(J) \subseteq I$.

If c is a closure operation on Λ-algebras, we say it is *equational* if one always has $I^{\mathrm{c}} = I^{\mathrm{ceq}}$, or *homogeneous* if one always has $I^{\mathrm{c}} = I^{\mathrm{ch}}$.

In [21], we prove the following theorem:

Theorem 4.6.2. *Suppose* c *is a homogeneous closure operation. Then it commutes with arbitrary localization. That is, if R is a Λ-algebra, I an ideal, and W a multiplicative subset of R, then* $(W^{-1}I)^{\mathrm{c}} = W^{-1}(I^{\mathrm{c}})$.

In particular, *homogeneous tight closure* ($I^{*\mathrm{h}}$) commutes with localization. The following theorem gives circumstances under which $I^* = I^{*\mathrm{h}}$. Parts (i) and (iii) involve cases where tight closure was already known to commute with localization, thus providing a "reason" that it commutes in these cases. Part (ii) gives a reason why the coefficient field in Brenner and Monsky's counterexample is transcendental over $\mathbb{F}_2$.

Theorem 4.6.3. *Let R be an excellent Noetherian ring which is either of prime characteristic $p > 0$ or of equal characteristic* 0*.*

(i) *If I is a parameter ideal (or more generally, if R/I has* finite phantom projective dimension *as an R-module), then $I^* = I^{*\mathrm{h}}$.*

(ii) *If R is a finitely generated and positively-graded k-algebra, where k is an algebraic extension of $\mathbb{F}_p$ or of $\mathbb{Q}$, and I is an ideal generated by forms of positive degree, then $I^* = I^{*\mathrm{h}}$.*

(iii) *If R is a* binomial ring *over any field k (that is, $R = k[X_1, \ldots, X_n]/J$, where the X_j are indeterminates and J is generated by polynomials with at most two terms each), then for any ideal I of R, $I^* = I^{*\mathrm{h}}$.*

5 Reductions, Special Parts of Closures, Spreads, and Cores

5.1 Nakayama Closures and Reductions

Definition 5.1.1. [18] Let $(R, \mathfrak{m})$ be a Noetherian local ring. A closure operation c on ideals of R is *Nakayama* if whenever J, I are ideals such that $J \subseteq I \subseteq (J + \mathfrak{m}I)^{\mathrm{c}}$, it follows that $J^{\mathrm{c}} = I^{\mathrm{c}}$.

It turns out that many closure operations are Nakayama. For example, integral closure [61], tight closure [18], plus closure [17], and Frobenius closure [19] are all Nakayama closures under extremely mild conditions. The fact that the identity closure is Nakayama is a special case of the classical Nakayama lemma (which is where the name of the condition comes from). However, radical is not Nakayama. For example, if $R = k[\![x]\!]$ (the ring of power series in one variable over a field k), $J = 0$, and $I = (x) = \mathfrak{m}$, then clearly $J \subseteq I \subseteq \sqrt{J + \mathfrak{m}I}$, but the radical ideals J and I are distinct.

Definition 5.1.2. Let R be a ring, c a closure operation on R, and $J \subseteq I$ ideals. We say that J is a c-*reduction* of I if $J^{\mathrm{c}} = I^{\mathrm{c}}$. A c-reduction $J \subseteq I$ is *minimal* if for all ideals $K \subsetneq J$, K is not a c-reduction of I.

For any Nakayama closure c, one can make a very strong statement about the existence of minimal c-reductions:

Lemma 5.1.3 ([18, Lemma 2.2]). *If* cl *is a Nakayama closure on R and I an ideal, then for any* cl*-reduction J of I, there is a minimal* cl*-reduction K of I contained in J. Moreover, in this situation any minimal generating set of K extends to a minimal generating set of J.*

This is another reason to see that radical is not Nakayama, as, for instance, the ideal (x) in the example above has no minimal $\sqrt{\ }$-reductions at all.

5.2 Special Parts of Closures

Consider the following notion from [19]:

Definition 5.2.1. Let $(R, \mathfrak{m})$ be a Noetherian local ring and c a closure operation on R. Let $\mathcal{I}$ be the set of all ideals of R. Then a set map csp : $\mathcal{I} \to \mathcal{I}$ is a *special part of* c if it satisfies the following properties for all $I, J \in \mathcal{I}$:

(i) (Trapped) $\mathfrak{m}I \subseteq I^{\mathrm{csp}} \subseteq I^{\mathrm{c}}$,

(ii) (Depends only on the closure) $(I^{\mathrm{c}})^{\mathrm{csp}} = I^{\mathrm{csp}}$,

(iii) (Order-preserving) If $J \subseteq I$ then $J^{\mathrm{csp}} \subseteq I^{\mathrm{csp}}$,

(iv) (Special Nakayama property) If $J \subseteq I \subseteq (J + I^{\mathrm{csp}})^{\mathrm{c}}$, then $I \subseteq J^{\mathrm{c}}$.

Of course, any closure operation c that admits a special part csp must be a Nakayama closure. Examples of special parts of closures include:

- "Special tight closure" (first defined by Adela Vraciu in [78]), when R is excellent and of prime characteristic p:
$$I^{*\mathrm{sp}} := \{f \in R \mid x^{q_0} \in (\mathfrak{m}I^{[q_0]})^* \text{ for some power } q_0 \text{ of } p\}.$$
- The special part of Frobenius closure (see [19]), when R has prime characteristic p:
$$I^{F\,\mathrm{sp}} := \{f \in R \mid x^{q_0} \in \mathfrak{m}I^{[q_0]} \text{ for some power } q_0 \text{ of } p\}.$$
- The special part of integral closure (see [19]):
$$I^{-\mathrm{sp}} := \{f \in R \mid x^n \in (\mathfrak{m}I^n)^- \text{ for some } n \in \mathbb{N}\}.$$
- The special part of plus closure (when R is a domain) (see [17]):
$$I^{+\mathrm{sp}} := \{f \in R \mid f \in \mathrm{Jac}(S)IS \text{ for some module-finite domain extension } R \to S\},$$
where $\mathrm{Jac}(S)$ is the Jacobson radical of S

All of these are in fact special parts of the corresponding closures. Moreover, we have the following:

Proposition 5.2.2.

- *If $(R, \mathfrak{m})$ has a perfect residue field, then $I^F = I + I^{F\,\mathrm{sp}}$ for all ideals I* [19].
- *If $(R, \mathfrak{m})$ is a Henselian domain with algebraically closed residue field, then $I^+ = I + I^{+\mathrm{sp}}$ for all ideals I* [17].