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Nikolai Saveliev

LECTURES ON THE TOPOLOGY OF 3-MANIFOLDS

AN INTRODUCTION TO THE CASSON INVARIANT

2ND EDITION

De Gruyter Textbook

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Preface

This short book grew out of lectures the author gave at the University of Michigan in the Fall of 1997. The purpose of the course was to introduce second year graduate students to the theory of 3-dimensional manifolds and its role in the modern 4-dimensional topology and gauge theory. The course assumed only familiarity with the basic concepts of topology including: the fundamental group, the (co)homology theory of manifolds, and the Poincaré duality.

Progress in low-dimensional topology has been very fast over the last two decades, leading to the solution of many difficult problems. One of the consequences of this "acceleration of history" is that many results have only appeared in professional journals and monographs. Among these are Casson's results on the Rohlin invariant of homotopy 3-spheres, as well as his λ-invariant. The monograph "Casson's invariant for oriented homology 3-spheres: an exposition" by S. Akbulut and J. McCarthy, though beautifully written, is hardly accessible to students who have completed only a basic course in algebraic topology. The purpose of this book is to provide a much-needed bridge to these topics.

Casson's construction of his λ-invariant is rather elementary compared to further developments related to gauge theory. This book is in no way intended to explore this subject, as it requires an extensive knowledge of Riemannian geometry and partial differential equations.

The book begins with topics that may be considered standard for a book in 3-manifolds: existence of Heegaard splittings, Singer's theorem about the uniqueness of a Heegaard splitting up to stable equivalence, and the mapping class group of a closed surface. Then we introduce Dehn surgery on framed links, give a detailed description of the Kirby calculus of framed links in S^3, and use this calculus to prove that any oriented closed 3-manifold bounds a smooth simply-connected parallelizable 4-manifold.

The second part of the book is devoted to Rohlin's invariant and its properties. We first review some facts about 4-manifolds and their intersection forms, then we do some knot theory. The latter includes Seifert surfaces and matrices, the Alexander polynomial and Conway's formula, and the Arf-invariant and its relation to the Alexander polynomial. Our approach differs from the common one in that we work in a homology sphere rather than in S^3, though the difference here is more technical than conceptual. This part concludes with a geometric proof of the Rohlin Theorem (after M. Freedman and R. Kirby), and with the surgery formula for the Rohlin invariant.

The last part of the book deals with Casson's invariant and its applications, mostly along the lines of Akbulut and McCarthy's book. We employ a more intuitive approach here to emphasize the ideas behind the construction, and refer the reader to the aforementioned book for technical details.

The book is full of examples. Seifert fibered manifolds appear consistently among these examples. We discuss their Heegaard splittings, Dehn surgery description, classification, Rohlin invariant, SU(2)-representation spaces, twisted cohomology, Casson invariant, etc.

Throughout the book, we mention the latest developments whenever it seems appropriate. For example, in the section on 4-manifold topology, we give a review of recent results relating 4-manifolds and unimodular forms, including the "10/8-conjecture" and Donaldson polynomials. The Rohlin invariant gives restrictions on the genus of surfaces embedded in a smooth 4-manifold. When describing this old result, we also survey the results that follow from the Thom conjecture, proved a few years ago by Kronheimer and Mrowka with the help of Seiberg–Witten theory.

The topology of 3-manifolds includes a variety of topics not discussed in this book, among which are hyperbolic manifolds, Thurston's geometrization conjecture, incompressible surfaces, prime decompositions of 3-manifolds, and many others.

The book has brief notes on further developments, and a list of exercises at the end of each lecture.

The book is closely related, in several instances, both in content and method, to the books Akbulut–McCarthy [2] and Fomenko–Matveev [49], from which I have borrowed quite shamelessly. However, it is hoped that the present treatment will serve its purpose of providing an accessible introduction to certain topics in the topology of 3-manifolds. Other major sources I relied upon while writing this book include Browder [24], Fintushel–Stern [45], Freedman–Kirby [52], Guillou–Marin [64], Kirby [84], Livingston [105], Matsumoto [107], McCullough [110], Neumann–Raymond [122], Rolfsen [137] and Taubes [152].

I am indebted to Boris Apanasov, Olivier Collin, John Dean, Max Forester, Slawomir Kwasik, Walter Neumann, Liviu Nicolaescu, Frank Raymond, Thang T. Q. Le, and Vladimir Turaev for sharing their expertise and advice, and for their help and support during my work on this book. I would also like to thank the graduate students who took my course at the University of Michigan. I wish to express my gratitude to John Dean who read the manuscript to polish the English usage. I was partially supported by NSF Grant DMS-97-04204 and by Max-Planck-Institut für Mathematik in Bonn, Germany, during my work on this book.

Comments on this edition

In the twelve years since the publication of this book, the face of low-dimensional topology has been profoundly changed by the proof of the three-dimensional Poincaré conjecture. The effect this had on the Casson invariant was that its original application to proving that the Rohlin invariant of a homotopy 3-sphere must vanish was rendered moot. Despite this, Casson's contribution remains as relevant as ever: in fact, a lot of the modern day low-dimensional topology, including a number of Floer homology theories, can be traced back to his λ-invariant. These Floer homology theories have been also linked to contact topology and Khovanov homology, and together they constitute a very active area of research.

I did not attempt to cover any of these new topics in the second edition. However, I added a couple of brief sections, where it seemed appropriate, to indicate how the material in this book is relevant to Heegaard Floer homology and open book decompositions. Other than that, I added a few updates and exercises, and corrected a number of typos.

I am thankful to everyone who has commented on the book, and especially to Ken Baker, Ivan Dynnikov, Jochen Kroll, and Marina Prokhorova.

Miami, August 2011 Nikolai Saveliev

Contents

Introduction

A topological space M is called a (topological) n-dimensional *manifold*, or n-manifold, if each point of M has an open neighborhood homeomorphic to $\mathbb{R}^n$. In other words, a manifold is a locally Euclidean space. To avoid pathological examples, it is standard to assume that all manifolds are Hausdorff and have a countable base of topology, and we will follow this convention. Most manifolds we consider will also be compact and connected.

Let U and V be two open sets in an n-manifold M each homeomorphic to $\mathbb{R}^n$ via homeomorphisms $\phi\colon U \to \mathbb{R}^n$ and $\psi\colon V \to \mathbb{R}^n$. Then

$$\psi \circ \phi^{-1}\colon \phi(U \cap V) \to \psi(U \cap V) \tag{1}$$

is a homeomorphism of open sets in Euclidean space $\mathbb{R}^n$. A manifold M is *smooth* if there is an open covering $\mathcal{U}$ of M such that for any open sets $U, V \in \mathcal{U}$ the map (1) is a diffeomorphism. A manifold M is called *piecewise linear* or simply *PL* if there is an open covering $\mathcal{U}$ of M such that for any open sets $U, V \in \mathcal{U}$ the map (1) is a piecewise linear homeomorphism. Another way to describe PL manifolds is as follows.

A triangulation of a polyhedron is called *combinatorial* if the link of each its vertex is PL-homeomorphic to a PL-sphere. Every PL-manifold admits a combinatorial triangulation. Any polyhedron which admits a combinatorial triangulation is a PL-manifold.

A Hausdorff topological space M whose topology has a countable base is called an *n-manifold with boundary* if each point of M has an open neighborhood homeomorphic to either Euclidean space $\mathbb{R}^n$ or closed upper half-space $\mathbb{R}^n_+$. The union of points of the second type is either empty or an $(n-1)$-dimensional manifold, which is denoted by ∂M and called the *boundary* of M. Note that the boundary of ∂M is empty. A manifold M is called *closed* if it is compact and its boundary is empty. Analogous definitions hold for smooth and PL manifolds.

The following fact is very important for us: if $n \leq 3$ then the concepts of topological, smooth, and PL manifolds coincide, see Bing [15] and Moise [116]. More precisely, any topological manifold M of dimension less than or equal to 3 admits a smooth and a PL-structure. These are unique in that there is a diffeomorphism or a PL-homeomorphism between any two smooth or PL-manifolds that are homeomorphic to M. Moreover, if a PL-manifold of dimension $n \leq 3$ is homeomorphic to a smooth manifold then there is a homeomorphism between them whose restriction to each simplex of a certain triangulation is a smooth embedding.

In dimension 4, every PL-manifold has a unique smooth structure, and vice versa, see Cairns [27] and Hirsch [75]. However, there exist topological manifolds in dimension 4 that admit no smooth structure, and there are topological 4-manifolds with more than one smooth structure. These questions will be discussed in more detail in Lecture 5. Furthermore, there exists a closed 4-dimensional topological manifold which is not homeomorphic to *any* simplicial complex, much less a combinatorial one. A key ingredient in the construction of such a manifold is the Casson invariant, which is defined later in these lectures.

The relationships between topological, smooth, and PL-manifolds are more complicated in dimensions 5 and higher. They will be briefly discussed in Lecture 18.

Glossary

We explain some standard geometric and topological background material used in the book. Shown in *italic* are terms whose meaning is explained somewhere in the glossary text.

CW-complexes. A topological space X is called a CW-complex if X can be represented as a union

$$X = \bigcup_{q=0}^{\infty} X^{(q)}$$

where the 0-skeleton $X^{(0)}$ is a countable (possibly finite) discrete set of points, and each $(q+1)$-skeleton $X^{(q+1)}$ is obtained from the q-skeleton $X^{(q)}$ by attaching $(q+1)$-cells. More explicitly, for each q there is a collection $\{e_j \mid j \in J_{q+1}\}$ where

(1) each e_j is a subset of $X^{(q+1)}$ such that if $e'_j = e_j \cap X^{(q)}$, then $e_j \setminus e'_j$ is disjoint from $e_k \setminus e'_k$ if $j, k \in J_{q+1}$ with $j \neq k$,

(2) for each $j \in J_{q+1}$, there is a characteristic map $g_j : (D^{q+1}, \partial D^{q+1}) \to (X^{(q+1)}, X^{(q)})$ such that g_j is a quotient map from D^{q+1} to e_j, which maps $D^{q+1} \setminus \partial D^{q+1}$ homeomorphically onto $e_j \setminus e'_j$,

(3) a subset of X is closed if and only if its intersection with each skeleton $X^{(q)}$ is closed.

Each $e_j \setminus e'_j$ is called a $(q+1)$-cell. When all characteristic maps are embeddings, the CW-complex is called regular.

Cellular homology. Let X be a *CW-complex*, and R a commutative ring with an identity element. For each q, let $C_q(X, R)$ be the free R-module with basis the q-cells. We will define the boundary homomorphism $\partial_{q+1}: C_{q+1}(X, R) \to C_q(X, R)$. To define $\partial_{q+1}(c)$, where c is a fixed $(q+1)$-cell, fix an orientation for D^{q+1}, thus determining an orientation for the q-sphere ∂D^{q+1}, and look at how the characteristic map g of c carries ∂D^{q+1} to $X^{(q)}$. For each e_k in $X^{(q)}$, fix a point z_k in $c_k = e_k \setminus e'_k$. One can show that g is homotopic to a map such that for each k, the preimage of z_k is a finite set of points $p_{k,1}, \dots, p_{k,n_k}$. Moreover g takes a neighborhood of each $p_{k,j}$ homeomorphically to a neighborhood of z_k (by compactness, the preimage of z_k is empty for all but finitely many k). For each j with $1 \leq j \leq n_k$, let $\varepsilon_{k,j} = \pm 1$

according to whether g restricted to the neighborhood of $p_{k,j}$ preserves or reverses orientation. Let

$$\varepsilon_k = \sum_{j=1}^{n_k} \varepsilon_{k,j} \quad \text{and} \quad \partial_{q+1}(c) = \sum_{k=1}^{\infty} \varepsilon_k c_k$$

where all but finitely many ε_k are equal to zero.

The numbers ε_k can also be described as follows. The quotient space $X^{(q)}/X^{(q-1)}$ is homeomorphic to a one-point union of q-dimensional spheres, one for each q-cell $c_k = e_k \setminus e'_k$. Given a $(q+1)$-cell c, its characteristic map $g : (D^{q+1}, \partial D^{q+1}) \to (X^{(q+1)}, X^{(q)})$ induces the map

$$\varphi_k : \partial D^{q+1} \to X^{(q)} \to X^{(q)}/X^{(q-1)} \to S^q,$$

where the last arrow maps the sphere S^q corresponding to the cell c_k identically to itself, while contracting all other spheres to a point. The *degree* of φ_k is ε_k. This description of ε_k ensures that $\partial_{q+1}(c)$ is well-defined.

This defines the homomorphism ∂_{q+1} on the generators, and the definition extends by linearity to the entire free R-module $C_{q+1}(X, R)$. One can prove that $\partial_q \partial_{q+1} = 0$. The reason is that algebraically, the q-sphere ∂D^{q+1} acts as though it were a regular CW-complex with one q-cell corresponding to each preimage point of a z_k. Since ∂D^{q+1} is a manifold, the boundaries of these q-cells form a collection of $(q-1)$-cells, each appearing as part of the boundary of two q-cells, but with opposite orientations. Consequently, the algebraic sum of the boundaries of these q-cells is 0. Applying ∂_q to $\partial_{q+1}(c)$ simply adds up the images of the boundaries of those q-cells in $C_{q-1}(X, R)$, and the pairs with opposite signs cancel out, giving 0.

An element of $C_q(X, R)$ is a formal finite sum $\sum r_k c_k$, where each c_k is a q-cell; such a sum is called a q-chain. Now form a sequence of R-modules and homomorphisms

$$\cdots \to C_{q+1}(X, R) \xrightarrow{\partial_{q+1}} C_q(X, R) \xrightarrow{\partial_q} C_{q-1}(X, R) \to \cdots \to C_0(X, R) \to 0. \quad (2)$$

This is called a chain complex, since $\partial_q \partial_{q+1} = 0$ for all q. This implies that the image of ∂_{q+1} is contained in the kernel of ∂_q for each q. If the image of ∂_{q+1} equals the kernel of ∂_q for each q, the sequence is called exact. If not, we measure its deviation from exactness by defining cellular homology groups

$$H_q(X; R) = \ker(\partial_q)/\operatorname{im}(\partial_{q+1}).$$

Elements of $\ker(\partial_q)$ are called cycles, and elements of $\operatorname{im}(\partial_{q+1})$ are called boundaries. Explicitly, an element of $H_q(X; R)$ is a coset $a_q + \partial_{q+1}(C_{q+1}(X, R))$, where $\partial_q a_q = 0$, but it is usually written as $[a_q]$. Note that $[a_q] = [a'_q]$ if and only if $a_q = a'_q + \partial_{q+1}(b_{q+1})$ for some $(q+1)$-chain b_{q+1}.

To complete the definition of H_* as a *homology theory*, we need to define f_* for all continuous maps $f: X \to Y$. We first define $C_q(f): C_q(X, R) \to C_q(Y, R)$. By the Cellular Approximation Theorem, f may be changed within its homotopy class so that $f(X^{(q)}) \subset Y^{(q)}$ for all q. Then, define $C_q(f)(c)$ similarly to the way that $\partial_q(c)$ was defined above. Then $f_*([c]) = [C_q(f)(c)]$.

It is not easy to prove that this is well-defined and satisfies the *Eilenberg–Steenrod axioms*, but it can be done. In particular, $H_*(X; R)$ does not depend on the *CW-complex* structure chosen for X since the identity map induces an isomorphism on the homologies defined using two different *CW-complex* structures on X, and f_* depends only on the homotopy class of f.

When A is a subcomplex of X define the relative homology groups $H_q(X, A; R)$ by setting $C_q(X, A, R) = C_q(X, R)/C_q(A, R)$ and noting that ∂_q induces $\partial_q : C_q(X, A, R) \to C_{q-1}(X, A, R)$. Then, $H_q(X, A; R)$ is defined as the homology of the chain complex $C_*(X, A, R)$. The long exact sequence of the second *Eilenberg–Steenrod axiom* is then a purely algebraic consequence of the existence of short exact sequences

$$0 \to C_q(A, R) \to C_q(X, R) \to C_q(X, A, R) \to 0.$$

Note that every element of $H_q(X, A; R)$ can be represented by a q-chain whose boundary lies in A.

Cohomology of spaces. Once *cellular*, *simplicial*, or *singular* homology is defined, cohomology can be defined algebraically. This is based on the following fact. If A and B are R-modules, and $\varphi: A \to B$ is an R-module homomorphism, then there is an R-module homomorphism $\varphi^*: \mathrm{Hom}(B, R) \to \mathrm{Hom}(A, R)$ defined by $\varphi^*(\alpha) = \alpha \circ \varphi$. Clearly $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$, so if we define the coboundary homomorphism by $\delta_q = \partial_q^*$, then $\delta_{q+1}\delta_q = \partial_{q+1}^*\partial_q^* = (\partial_q\partial_{q+1})^* = 0^* = 0$. Therefore, abbreviating $\mathrm{Hom}(C_q(X), R)$ to $C^q(X, R)$, we have a cochain complex

$$0 \to C^0(X, R) \to \cdots \to C^{q-1}(X, R) \xrightarrow{\delta_q} C^q(X, R) \xrightarrow{\delta_{q+1}} C^{q+1}(X, R) \to \cdots \tag{3}$$

whose deviation from exactness is measured by the cohomology groups

$$H^q(X, R) = \ker \delta_{q+1} / \operatorname{im} \delta_q.$$

A continuous map $f: X \to Y$ induces homomorphisms $f^*: H^q(Y, R) \to H^q(X, R)$ with $(f \circ g)^* = g^* \circ f^*$, and there are corresponding versions of the *Eilenberg–Steenrod axioms* and *Mayer–Vietoris exact sequence* for cohomology.

An important case is when $R = F$ is a field. Then it can be proved that $H^q(X; F) \cong \mathrm{Hom}(H_q(X; F), F)$, the dual vector space of $H_q(X, F)$. Hence $H^q(X; F)$ and $H_q(X; F)$ are vector spaces of the same rank, although there is no natural isomorphism between them.

Connected sums. Let M_1 and M_2 be closed oriented manifolds of dimension n, and $D^n \subset M_k$, $k = 1, 2$, a pair of n-discs embedded in M_1 and M_2. A connected sum of M_1 and M_2 is defined as the manifold $M_1 \# M_2 = (M_1 \setminus \operatorname{int} D^n) \cup (M_2 \setminus \operatorname{int} D^n)$ obtained by *gluing* the manifolds $M_k \setminus \operatorname{int}(D^n)$ along their common boundary S^{n-1} via an orientation reversing homeomorphism $r\colon S^{n-1} \to S^{n-1}$. The manifold $M_1 \# M_2$ inherits an orientation from those on M_1 and M_2. The manifolds $M_1 \# M_2$ and $M_1 \#(-M_2)$, where $-M_2$ stands for the manifold M_2 with reversed orientation, need not be homeomorphic. Note also that if the manifolds M_1 and M_2 are smooth, a choice of smoothly embedded discs in M_1 and M_2 and a smooth identification map provides us with a smooth manifold $M_1 \# M_2$.

If the manifolds M_1 and M_2 have non-empty boundaries, one can still form their connected sum by choosing the n-discs in their interiors. One can also form their boundary connected sum, $M_1 \natural M_2$, by identifying $(n-1)$-discs $D^{n-1} \subset \partial M_k$, $k = 1, 2$, via an orientation reversing homeomorphism. The boundary of $M_1 \natural M_2$ is $(\partial M_1) \# (\partial M_2)$.

Cutting open. This is an operation which is "inverse" to the *gluing* of spaces. Let Y be a closed subspace of a connected space X such that the closure of $X \setminus Y$ coincides with X. Suppose that $X \setminus Y$ consists of a finite number of connected components, $X_1, \dots, X_n$. Consider the space

$$X' = \bigcup X_i \times \{i\} \subset X \times \mathbb{R},$$

that is, move the components apart from each other. The closure of X' in the product topology on $X \times \mathbb{R}$ is the result of cutting X open along Y.

Degree of a map. Let $f\colon (M, \partial M) \to (N, \partial N)$ be a continuous map between oriented connected compact manifolds of identical dimension n. The degree of f is an integer $\deg f$ satisfying $f_*[M, \partial M] = \deg f \cdot [N, \partial N]$, where $[M, \partial M]$ and $[N, \partial N]$ are the *fundamental classes* of the manifolds M and N, and $f_*\colon H_n(M, \partial M) \to H_n(N, \partial N)$ the induced map. If $f\colon M \to N$ is a smooth map between smooth closed oriented manifolds, choose any point $y \in N$ such that f is *transversal* to y. Then the degree of f coincides with the integer

$$\deg f = \sum_{x \in f^{-1}(y)} \operatorname{sign}(\det d_x f),$$

where $d_x f\colon T_x M \to T_y N$ is the derivative of f at a point $x \in M$, and is independent of the choice of y.

Eilenberg–MacLane spaces. The Eilenberg–MacLane spaces $K(\pi, n)$ are the fundamental building blocks of homotopy theory. They are *CW-complexes* characterized uniquely up to homotopy equivalence as having a single non-trivial homotopy group:

$$\pi_i(K(\pi,n)) = \begin{cases} \pi, & \text{if } i = n, \\ 0, & \text{if } i \neq n. \end{cases}$$

Of course, the group π is required to be Abelian if $n \geq 2$. Standard examples of Eilenberg–MacLane spaces include $K(\mathbb{Z},1) = S^1$ and $K(\mathbb{Z},2) = \mathbb{C}P^\infty$, where $\mathbb{C}P^\infty$ is defined as the limiting space of the tower of complex projective spaces $\mathbb{C}P^1 \subset \mathbb{C}P^2 \subset \mathbb{C}P^3 \subset \cdots$ with respect to the natural inclusions. Eilenberg–MacLane spaces are classifying spaces for cohomology in that

$$H^n(X;\pi) = [X, K(\pi,n)] \tag{4}$$

for any space X and Abelian group π, where the brackets denote the set of all homotopy classes of continuous maps. The isomorphism (4) is obtained as follows. From the *Hurewicz theorem* and the *Universal Coefficient theorem*, it is easy to see that $H^n(K(\pi,n);\pi) \cong \mathrm{Hom}(\pi,\pi)$. Let $\iota\colon \pi \to \pi$ be the identity map. Associate to any $f\colon X \to K(\pi,n)$ the cohomology class $f^*\iota \in H^n(X;\pi)$; this is the correspondence (4).

Gluing construction. Let X and Y be topological spaces, and $f\colon Z \to Y$ a continuous map where $Z \subset X$ is a subspace of X. Consider the disjoint union $X \cup Y$ and introduce the equivalence relation generated by $z \sim f(z)$ whenever $z \in Z$. The space $X \cup_f Y = (X \cup Y)/\sim$ with the quotient topology is said to be obtained by gluing X and Y along f. In most cases we consider, the map f will be a homeomorphism of Z onto its image $f(Z) \subset Y$.

Handles. Let X be a smooth n-manifold with boundary, and $0 \leq k \leq n$. An n-dimensional k-handle is a copy of $D^k \times D^{n-k}$, attached to the boundary of X along $(\partial D^k) \times D^{n-k}$ using an embedding $f : (\partial D^k) \times D^{n-k} \to \partial X$. The corners that arise can be smoothed out, see for instance Chapter 1 of Conner–Floyd [31], hence $X \cup_f (D^k \times D^{n-k})$ is again a smooth manifold. For example, a 1-handle is a product $D^1 \times D^{n-1}$ attached along a pair of $(n-1)$-balls, $S^0 \times D^{n-1}$. A 2-handle is a product $D^2 \times D^{n-2}$ attached along $S^1 \times D^{n-2}$. For more details see Gompf–Stipsicz [61] or Rourke–Sanderson [138].

Homology theory. Let R be a commutative ring with an identity element. Sometimes R will be required to be a principal ideal domain. By a homology theory we mean a functor from the category of pairs of spaces and continuous maps to the category of graded R-modules and graded homomorphisms. That is, for each pair (X, A), where A is a subspace of X, there is an R-module

$$H_*(X, A; R) = \bigoplus_{q=0}^{\infty} H_q(X, A; R),$$

and for each continuous map of pairs $f\colon (X, A) \to (Y, B)$ there are homomorphisms $f_*\colon H_q(X, A; R) \to H_q(Y, B; R)$ for every q, so that $(f \circ g)_* = f_* \circ g_*$. We abbreviate $H_q(X, A; R)$ to $H_q(X, A)$ and $H_q(X, \emptyset)$ to $H_q(X)$. It will be clear from the context what the ring R is. The following *Eilenberg–Steenrod axioms* must hold:

(1) (homotopy invariance) If $f, g\colon (X, A) \to (Y, B)$ are homotopic then $f_* = g_*$.

(2) (long exact sequence) For every pair (X, A) and every q there are homomorphisms $\partial\colon H_q(X, A) \to H_{q-1}(A)$ fitting into a long exact sequence

$$\cdots \to H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A) \to \cdots \to H_0(X, A) \to 0,$$

where $i\colon A \to X$ and $j\colon (X, \emptyset) \to (X, A)$ are inclusion maps.

(3) (excision axiom) If U is an open subspace of X whose closure is contained in the interior of A, then the inclusion map $j\colon (X \setminus U, A \setminus U) \to (X, A)$ induces isomorphisms $j_*\colon H_q(X \setminus U, A \setminus U) \to H_q(X, A)$ for all q.

(4) (coefficient module) If P is a one point space, then $H_0(P) = R$ and $H_q(P) = 0$ for $q \geq 1$.

The module in axiom (4) is called the coefficient ring for the homology theory. We often refer to $H_q(X, A)$ as homology groups. Strictly speaking, one should say homology modules, but for the common cases $R = \mathbb{Z}$ and $R = \mathbb{Z}/n$, the homology modules are Abelian groups.

There are many ways to define homology groups. For a fixed ring R, all the standard ways produce the same results when X is a *simplicial* or a *CW-complex* and A is a subcomplex. The most widely used theories are *simplicial, singular*, and *cellular homology*. We will be working with the latter most of the time.

The Eilenberg–Steenrod axioms imply the *Mayer–Vietoris exact sequence*, which is very powerful for computation of homology. It applies in quite general situations, but we will only state it for *CW-complexes*. Suppose that A and B are subcomplexes of a CW-complex X, with $X = A \cup B$. Then there are homomorphisms $\partial\colon H_q(X) \to H_{q-1}(A \cap B)$ fitting into a long exact sequence

$$\cdots \to H_q(A \cap B) \xrightarrow{(i_*, -j_*)} H_q(A) \oplus H_q(B) \xrightarrow{I_* + J_*} H_q(X) \xrightarrow{\partial} H_{q-1}(A \cap B) \to \cdots$$

where $i\colon A \cap B \to A$, $j\colon A \cap B \to B$, $I\colon A \to X$, and $J\colon B \to X$ are inclusion maps.

Here are some consequences of the axioms and the Mayer–Vietoris sequence. Assume that K is a CW-complex and L is a subcomplex, possibly empty. Then, if K is n-dimensional, or more generally if every cell of $K \setminus L$ has dimension less than or equal to n, then $H_q(K, L) = 0$ for all $q > n$. Moreover, $H_0(K) = \oplus R$ with one summand for each path component of K.

A cohomology theory is defined similarly, together with cohomological versions of the Eilenberg–Steenrod axioms and the Mayer–Vietoris exact sequence.

Homotopy lifting property. A map $p\colon E \to B$ has the homotopy lifting property with respect to a space X if, for every two maps $f\colon X \to E$ and $G\colon X \times I \to B$ for which $pf = Gi$ (where $I = [0,1]$ and $i\colon X \to X \times I$ is the map $x \mapsto (x,0)$), there exists a continuous map $\tilde{G}\colon X \times I \to E$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "G"] \arrow[ur, "\tilde{G}"] & B \end{tikzcd} \tag{5}$$

A map $p\colon E \to B$ is called a fibration if it has the homotopy lifting property with respect to every space X. If $b \in B$, then $p^{-1}(b) = F$ is called a fiber. Different fibers of a fibration need not be homeomorphic, however, they all are homotopy equivalent. A map $p\colon E \to B$ is called a Serre fibration if it has the homotopy lifting property with respect to all CW-complexes X. *Locally trivial bundles* are Serre fibrations, and in fact fibrations if the base B is paracompact.

Let $p : E \to B$ be a fibration, and $\tilde{G}_0$, $\tilde{G}_1$ two maps making the above diagram commute. Then $\tilde{G}_0$ and $\tilde{G}_1$ are fiberwise homotopic rel $X \times \{0\}$, see for instance Spanier [150], Corollary 2.8.11.

Homotopy theory. We refer the reader to Hatcher [71] or Spanier [150] for the basics of the homotopy theory, including homotopy, homotopy equivalences, the fundamental group $\pi_1(X, x_0)$, van Kampen's theorem, covering spaces, higher homotopy groups $\pi_n(X, x_0)$ etc.

Hurewicz Theorem. Suppose $\sigma\colon (S^n, s_0) \to (X, x_0)$ is a map representing an element of $\pi_n(X, x_0)$. Let γ_n be a fixed generator of $H_n(S^n; \mathbb{Z}) = \mathbb{Z}$. The Hurewicz homomorphism $\rho\colon \pi_n(X, x_0) \to H_n(X; \mathbb{Z})$ is defined by $\rho([\sigma]) = \sigma_*(\gamma_n)$. One can show that this homomorphism is natural, that is, if $f\colon X \to Y$ is a continuous map, the diagram

$$\begin{tikzcd} \pi_n X \arrow[r, "\rho"] \arrow[d, "f_*"] & H_n X \arrow[d, "f_*"] \\ \pi_n Y \arrow[r, "\rho"] & H_n Y \end{tikzcd}$$

commutes. The basic relationship between homotopy groups and homology groups is given by the Hurewicz theorem, which in its simplest form asserts the following. Let X be a topological space such that $\pi_0 X = \pi_1 X = \ldots = \pi_{n-1} X = 0$ for some $n \geq 1$.

(1) If $n = 1$ then $\rho\colon \pi_1 X \to H_1 X$ is given by Abelianization and is surjective.

(2) If $n \geq 2$ then $\tilde{H}_0 X = H_1 X = \ldots = H_{n-1} X = 0$ and $\rho\colon \pi_n X \to H_n X$ is an isomorphism.