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Hans Föllmer, Alexander Schied

STOCHASTIC FINANCE

AN INTRODUCTION IN DISCRETE TIME

3RD EDITION

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Stochastic Finance

An Introduction in Discrete Time

Third revised and extended edition

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Preface to the third edition

This third edition of our book appears in the de Gruyter graduate textbook series. We have therefore included more than one hundred exercises. Typically, we have used the book as an introductory text for two major areas, either combined into one course or in two separate courses. The first area comprises static and dynamic arbitrage theory in discrete time. The corresponding core material is provided in Chapters 1, 5, and 6. The second area deals with mathematical aspects of financial risk as developed in Chapters 2, 4, and 11. Most of the exercises we have included in this edition are therefore contained in these core chapters. The other chapters of this book can be used both as complementary material for the introductory courses and as basis for special-topics courses.

In recent years, there has been an increasing awareness, both among practitioners and in academia, of the problem of model uncertainty in finance and economics, often called *Knightian uncertainty*; see, e.g., [259]. In this third edition we have put more emphasis on this issue. The theory of risk measures can be seen as a case study how to deal with model uncertainty in mathematical terms. We have therefore updated Chapter 4 on static risk measures and added the new Chapter 11 on dynamic risk measures. Moreover, in Section 2.5 we have extended the characterization of robust preferences in terms of risk measures from the coherent to the convex case. We have also included the new Sections 3.5 and 8.3 on robust variants of the classical problems of optimal portfolio choice and efficient hedging.

It is a pleasure to express our thanks to all students and colleagues whose comments have helped us to prepare this third edition, in particular to Aurélien Alfonsi, Günter Baigger, Francesca Biagini, Julia Brettschneider, Patrick Cheridito, Samuel Drapeau, Maren Eckhoff, Karl-Theodor Eisele, Damir Filipovic, Zicheng Hong, Kostas Kardaras, Thomas Knispel, Gesine Koch, Heinz König, Volker Krätschmer, Christoph Kühn, Michael Kupper, Mourad Lazgham, Sven Lickfeld, Mareike Massow, Irina Penner, Ernst Presman, Michael Scheutzow, Melvin Sim, Alla Slynko, Stephan Sturm, Gregor Svindland, Long Teng, Florian Werner, Wiebke Wittmüß, and Lei Wu. Special thanks are due to Yuliya Mishura and Georgiy Shevchenko, our translators for the Russian edition.

Berlin and Mannheim, November 2010 *Hans Föllmer*
Alexander Schied

Preface to the second edition

Since the publication of the first edition we have used it as the basis for several courses. These include courses for a whole semester on Mathematical Finance in Berlin and also short courses on special topics such as risk measures given at the Institut Henri Poincaré in Paris, at the Department of Operations Research at Cornell University, at the Academia Sinica in Taipei, and at the 8th Symposium on Probability and Stochastic Processes in Puebla. In the process we have made a large number of minor corrections, we have discovered many opportunities for simplification and clarification, and we have also learned more about several topics. As a result, major parts of this book have been improved or even entirely rewritten. Among them are those on robust representations of risk measures, arbitrage-free pricing of contingent claims, exotic derivatives in the CRR model, convergence to the Black–Scholes model, and stability under pasting with its connections to dynamically consistent coherent risk measures. In addition, this second edition contains several new sections, including a systematic discussion of law-invariant risk measures, of concave distortions, and of the relations between risk measures and Choquet integration.

It is a pleasure to express our thanks to all students and colleagues whose comments have helped us to prepare this second edition, in particular to Dirk Becherer, Hans Bühler, Rose-Anne Dana, Ulrich Horst, Mesrop Janunts, Christoph Kühn, Maren Liese, Harald Luschgy, Holger Pint, Philip Protter, Lothar Rogge, Stephan Sturm, Stefan Weber, Wiebke Wittmüß, and Ching-Tang Wu. Special thanks are due to Peter Bank and to Yuliya Mishura and Georgiy Shevchenko, our translators for the Russian edition. Finally, we thank Irene Zimmermann and Manfred Karbe of de Gruyter Verlag for urging us to write a second edition and for their efficient support.

Berlin, September 2004

Hans Föllmer
Alexander Schied

Preface to the first edition

This book is an introduction to probabilistic methods in Finance. It is intended for graduate students in mathematics, and it may also be useful for mathematicians in academia and in the financial industry. Our focus is on stochastic models in discrete time. This limitation has two immediate benefits. First, the probabilistic machinery is simpler, and we can discuss right away some of the key problems in the theory of pricing and hedging of financial derivatives. Second, the paradigm of a complete financial market, where all derivatives admit a perfect hedge, becomes the exception rather than the rule. Thus, the discrete-time setting provides a shortcut to some of the more recent literature on incomplete financial market models.

As a textbook for mathematicians, it is an introduction at an intermediate level, with special emphasis on martingale methods. Since it does not use the continuous-time methods of Itô calculus, it needs less preparation than more advanced texts such as [99], [98], [107], [171], [252]. On the other hand, it is technically more demanding than textbooks such as [215]: We work on general probability spaces, and so the text captures the interplay between probability theory and functional analysis which has been crucial for some of the recent advances in mathematical finance.

The book is based on our notes for first courses in Mathematical Finance which both of us are teaching in Berlin at Humboldt University and at Technical University. These courses are designed for students in mathematics with some background in probability. Sometimes, they are given in parallel to a systematic course on stochastic processes. At other times, martingale methods in discrete time are developed in the course, as they are in this book. Usually the course is followed by a second course on Mathematical Finance in continuous time. There it turns out to be useful that students are already familiar with some of the key ideas of Mathematical Finance.

The core of this book is the dynamic arbitrage theory in the first chapters of Part II. When teaching a course, we found it useful to explain some of the main arguments in the more transparent one-period model before using them in the dynamical setting. So one approach would be to start immediately in the multi-period framework of Chapter 5, and to go back to selected sections of Part I as the need arises. As an alternative, one could first focus on the one-period model, and then move on to Part II.

We include in Chapter 2 a brief introduction to the mathematical theory of expected utility, even though this is a classical topic, and there is no shortage of excellent expositions; see, for instance, [187] which happens to be our favorite. We have three reasons for including this chapter. Our focus in this book is on incompleteness, and incompleteness involves, in one form or another, preferences in the face of risk and uncertainty. We feel that mathematicians working in this area should be aware, at least to some extent, of the long line of thought which leads from Daniel Bernoulli via von Neumann–Morgenstern and Savage to some more recent developments which are motivated by shortcomings of the classical paradigm. This is our first reason. Second, the analysis of risk measures has emerged as a major topic in mathematical finance, and this is closely related to a robust version of the Savage theory. Third, but not least, our experience is that this part of the course was found particularly enjoyable, both by the students and by ourselves.

We acknowledge our debt and express our thanks to all colleagues who have contributed, directly or indirectly, through their publications and through informal discussions, to our understanding of the topics discussed in this book. Ideas and methods developed by Freddy Delbaen, Darrell Duffie, Nicole El Karoui, David Heath, Yuri Kabanov, Ioannis Karatzas, Dimitri Kramkov, David Kreps, Stanley Pliska, Chris Rogers, Steve Ross, Walter Schachermayer, Martin Schweizer, Dieter Sondermann and Christophe Stricker play a key role in our exposition. We are obliged to many others; for instance the textbooks [73], [99], [98], [155], and [192] were a great help when we started to teach courses on the subject.

We are grateful to all those who read parts of the manuscript and made useful suggestions, in particular to Dirk Becherer, Ulrich Horst, Steffen Krüger, Irina Penner, and to Alexander Giese who designed some of the figures. Special thanks are due to Peter Bank for a large number of constructive comments. We also express our thanks to Erhan Çinlar, Adam Monahan, and Philip Protter for improving some of the language, and to the Department of Operations Research and Financial Engineering at Princeton University for its hospitality during the weeks when we finished the manuscript.

Berlin, June 2002

Hans Föllmer
Alexander Schied

Contents

Part I

Mathematical finance in one period

Chapter 1
Arbitrage theory

In this chapter, we study the mathematical structure of a simple one-period model of a financial market. We consider a finite number of assets. Their initial prices at time $t = 0$ are known, their future prices at time $t = 1$ are described as random variables on some probability space. Trading takes place at time $t = 0$. Already in this simple model, some basic principles of mathematical finance appear very clearly. In Section 1.2, we single out those models which satisfy a condition of *market efficiency*: There are no trading opportunities which yield a profit without any downside risk. The absence of such *arbitrage opportunities* is characterized by the existence of an equivalent *martingale measure*. Under such a measure, discounted prices have the martingale property, that is, trading in the assets is the same as playing a fair game. As explained in Section 1.3, any equivalent martingale measure can be identified with a pricing rule: It extends the given prices of the primary assets to a larger space of *contingent claims*, or financial *derivatives*, without creating new arbitrage opportunities. In general, there will be several such extensions. A given contingent claim has a unique price if and only if it admits a *perfect hedge*. In our one-period model, this will be the exception rather than the rule. Thus, we are facing *market incompleteness*, unless our model satisfies the very restrictive conditions discussed in Section 1.4. The geometric structure of an arbitrage-free model is described in Section 1.5.

The one-period market model will be used throughout the first part of this book. On the one hand, its structure is rich enough to illustrate some of the key ideas of the field. On the other hand, it will provide an introduction to some of the mathematical methods which will be used in the dynamic hedging theory of the second part. In fact, the multi-period situation considered in Chapter 5 can be regarded as a sequence of one-period models whose initial conditions are contingent on the outcomes of previous periods. The techniques for dealing with such contingent initial data are introduced in Section 1.6.

1.1 Assets, portfolios, and arbitrage opportunities

Consider a financial market with $d + 1$ assets. The assets can consist, for instance, of equities, bonds, commodities, or currencies. In a simple one-period model, these assets are priced at the initial time $t = 0$ and at the final time $t = 1$. We assume that the i^{th} asset is available at time 0 for a price $\pi^i \geq 0$. The collection

$$\overline{\pi} = (\pi^0, \pi^1, \dots, \pi^d) \in \mathbb{R}_+^{d+1}$$

is called a *price system*. Prices at time 1 are usually not known beforehand at time 0. In order to model this uncertainty, we fix a measurable space $(\Omega, \mathcal{F})$ and describe the asset prices at time 1 as non-negative measurable functions

$$S^0, S^1, \ldots, S^d$$

on $(\Omega, \mathcal{F})$ with values in $[0, \infty)$. Every $\omega \in \Omega$ corresponds to a particular scenario of market evolution, and $S^i(\omega)$ is the price of the i^{th} asset at time 1 if the scenario ω occurs.

However, not all asset prices in a market are necessarily uncertain. Usually there is a riskless bond which will pay a *sure* amount at time 1. In our simple model for one period, such a riskless investment opportunity will be included by assuming that

$$\pi^0 = 1 \quad \text{and} \quad S^0 \equiv 1 + r$$

for a constant r, the return of a unit investment into the riskless bond. In most situations it would be natural to assume $r \geq 0$, but for our purposes it is enough to require that $S^0 > 0$, or equivalently that

$$r > -1.$$

In order to distinguish S^0 from the risky assets $S^1, \ldots, S^d$, it will be convenient to use the notation

$$\overline{S} = (S^0, S^1, \ldots, S^d) = (S^0, S),$$

and in the same way we will write $\overline{\pi} = (1, \pi)$.

At time $t = 0$, an investor will choose a *portfolio*

$$\overline{\xi} = (\xi^0, \xi) = (\xi^0, \xi^1, \ldots, \xi^d) \in \mathbb{R}^{d+1},$$

where ξ^i represents the number of shares of the i^{th} asset. The price for buying the portfolio $\overline{\xi}$ equals

$$\overline{\pi} \cdot \overline{\xi} = \sum_{i=0}^{d} \pi^i \xi^i.$$

At time $t = 1$, the portfolio will have the value

$$\overline{\xi} \cdot \overline{S}(\omega) = \sum_{i=0}^{d} \xi^i S^i(\omega) = \xi^0(1 + r) + \xi \cdot S(\omega),$$

depending on the scenario $\omega \in \Omega$. Here we assume implicitly that buying and selling assets does not create extra costs, an assumption which may not be valid for a small investor but which becomes more realistic for a large financial institution. Note our convention of writing $x \cdot y$ for the inner product of two vectors x and y in Euclidean space.

Our definition of a portfolio allows the components ξ^i to be negative. If $\xi^0 < 0$, this corresponds to taking out a loan such that we receive the amount $|\xi^0|$ at $t = 0$ and pay back the amount $(1+r)|\xi^0|$ at time $t = 1$. If $\xi^i < 0$ for $i \geq 1$, a quantity of $|\xi^i|$ shares of the i^{th} asset is sold without actually owning them. This corresponds to a *short sale* of the asset. In particular, an investor is allowed to take a short position $\xi^i < 0$, and to use up the received amount $\pi^i|\xi^i|$ for buying quantities $\xi^j \geq 0$, $j \neq i$, of the other assets. In this case, the price of the portfolio $\overline{\xi} = (\xi^0, \xi)$ is given by $\overline{\xi} \cdot \overline{\pi} = 0$.

Remark 1.1. So far we have not assumed that anything is known about probabilities that might govern the realization of the various scenarios $\omega \in \Omega$. Such a situation is often referred to as *Knightian uncertainty*, in honor of F. Knight [176], who introduced the distinction between "risk" which refers to an economic situation in which the probabilistic structure is assumed to be known, and "uncertainty" where no such assumption is made. $\diamond$

Let us now assume that a probability measure P is given on $(\Omega, \mathcal{F})$. The asset prices $S^1, \dots, S^d$ and the portfolio values $\overline{\xi} \cdot \overline{S}$ can thus be regarded as random variables on $(\Omega, \mathcal{F}, P)$.

Definition 1.2. A portfolio $\overline{\xi} \in \mathbb{R}^{d+1}$ is called an *arbitrage opportunity* if $\overline{\pi} \cdot \overline{\xi} \leq 0$ but $\overline{\xi} \cdot \overline{S} \geq 0$ P-a.s. and $P[\,\overline{\xi} \cdot \overline{S} > 0\,] > 0$.

Intuitively, an arbitrage opportunity is an investment strategy that yields with positive probability a positive profit and is not exposed to any downside risk. The existence of such an arbitrage opportunity may be regarded as a market inefficiency in the sense that certain assets are not priced in a reasonable way. In real-world markets, arbitrage opportunities are rather hard to find. If such an opportunity would show up, it would generate a large demand, prices would adjust, and the opportunity would disappear. Later on, the absence of such arbitrage opportunities will be our key assumption. Absence of arbitrage implies that S^i vanishes P-a.s. once $\pi^i = 0$. Hence, there is no loss of generality if we assume from now on that

$$\pi^i > 0 \quad \text{for } i = 1, \dots, d.$$

Remark 1.3. Note that the probability measure P enters the definition of an arbitrage opportunity only through the null sets of P. In particular, the definition can be formulated without any explicit use of probabilities if Ω is countable. In this case, we can simply apply Definition 1.2 with an arbitrary probability measure P such that $P[\{\omega\}] > 0$ for every $\omega \in \Omega$. Then an arbitrage opportunity is a portfolio $\overline{\xi}$ with $\overline{\pi} \cdot \overline{\xi} \leq 0$, with $\overline{\xi} \cdot \overline{S}(\omega) \geq 0$ for all $\omega \in \Omega$, and such that $\overline{\xi} \cdot \overline{S}(\omega_0) > 0$ for at least one $\omega_0 \in \Omega$. $\diamond$

The following lemma shows that absence of arbitrage is equivalent to the following property of the market: Any investment in risky assets which yields with positive probability a better result than investing the same amount in the risk-free asset must be exposed to some downside risk.

Lemma 1.4. *The following statements are equivalent.*

(a) *The market model admits an arbitrage opportunity.*

(b) *There is a vector $\xi \in \mathbb{R}^d$ such that*

$$\xi \cdot S \geq (1+r)\xi \cdot \pi \ \ P\text{-a.s.} \quad \text{and} \quad P[\,\xi \cdot S > (1+r)\xi \cdot \pi\,] > 0.$$

Proof. To see that (a) implies (b), let $\overline{\xi}$ be an arbitrage opportunity. Then $0 \geq \overline{\xi} \cdot \overline{\pi} = \xi^0 + \xi \cdot \pi$. Hence,

$$\xi \cdot S - (1+r)\xi \cdot \pi \geq \xi \cdot S + (1+r)\xi^0 = \overline{\xi} \cdot \overline{S}.$$

Since $\overline{\xi} \cdot \overline{S}$ is P-a.s. non-negative and strictly positive with non-vanishing probability, the same must be true of $\xi \cdot S - (1+r)\xi \cdot \pi$.

Next let ξ be as in (b). We claim that the portfolio (ξ^0, ξ) with $\xi^0 := -\xi \cdot \pi$ is an arbitrage opportunity. Indeed, $\overline{\xi} \cdot \overline{\pi} = \xi^0 + \xi \cdot \pi = 0$ by definition. Moreover, $\overline{\xi} \cdot \overline{S} = -(1+r)\xi \cdot \pi + \xi \cdot S$, which is P-a.s. non-negative and strictly positive with non-vanishing probability. □

Exercise 1.1.1. On $\Omega = \{\omega_1, \omega_2, \omega_3\}$ we fix a probability measure P with $P[\,\omega_i\,] > 0$ for $i = 1, 2, 3$. Suppose that we have three assets with prices

$$\overline{\pi} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

at time 0 and

$$\overline{S}(\omega_1) = \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}, \quad \overline{S}(\omega_2) = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}, \quad \overline{S}(\omega_3) = \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix}$$

at time 1. Show that this market model admits arbitrage. ◇

Exercise 1.1.2. We consider a market model with a single risky asset defined on a probability space with a finite sample space Ω and a probability measure P that assigns strictly positive probability to each $\omega \in \Omega$. We let

$$a := \min_{\omega \in \Omega} S(\omega) \quad \text{and} \quad b := \max_{\omega \in \Omega} S(\omega).$$

Show that the model does not admit arbitrage if and only if $a < \pi(1+r) < b$. ◇

Exercise 1.1.3. Show that the existence of an arbitrage opportunity implies the following seemingly stronger condition.

(a) There exists an arbitrage opportunity $\overline{\xi}$ such that $\overline{\pi} \cdot \overline{\xi} = 0$.

Show furthermore that the following condition implies the existence of an arbitrage opportunity.

(b) There exists $\overline{\eta} \in \mathbb{R}^{d+1}$ such that $\overline{\pi} \cdot \overline{\eta} < 0$ and $\overline{\eta} \cdot \overline{S} \geq 0$ P-a.s.

What can you say about the implication (a)⇒(b)? ◇

1.2 Absence of arbitrage and martingale measures

In this section, we are going to characterize those market models which do not admit any arbitrage opportunities. Such models will be called *arbitrage-free*.

Definition 1.5. A probability measure P^* is called a *risk-neutral measure*, or a *martingale measure*, if

$$\pi^i = E^*\left[\frac{S^i}{1+r}\right], \quad i = 0, 1, \ldots, d. \tag{1.1}$$

Remark 1.6. In (1.1), the price of the i^{th} asset is identified as the expectation of the discounted payoff under the measure P^*. Thus, the pricing formula (1.1) can be seen as a classical valuation formula which does not take into account any risk aversion, in contrast to valuations in terms of expected utility which will be discussed in Section 2.3. This is why a measure P^* satisfying (1.1) is called risk-neutral. The connection to martingales will be made explicit in Section 1.6. ◇

The following basic result is sometimes called the "fundamental theorem of asset pricing" or, in short, FTAP. It characterizes arbitrage-free market models in terms of the set

$$\mathcal{P} := \left\{ P^* \mid P^* \text{ is a risk-neutral measure with } P^* \approx P \right\}$$

of risk-neutral measures which are *equivalent* to P. Recall that two probability measures P^* and P are said to be equivalent ($P^* \approx P$) if, for $A \in \mathcal{F}$, $P^*[A] = 0$ if and only if $P[A] = 0$. This holds if and only if P^* has a strictly positive density dP^*/dP with respect to P; see Appendix A.2. An equivalent risk-neutral measure is also called a *pricing measure* or an *equivalent martingale measure*.

Theorem 1.7. *A market model is arbitrage-free if and only if $\mathcal{P} \neq \emptyset$. In this case, there exists a $P^* \in \mathcal{P}$ which has a bounded density dP^*/dP.*

We show first that the existence of a risk-neutral measure implies the absence of arbitrage.

Proof of the implication "⇐" of Theorem 1.7. Suppose that there exists a risk-neutral measure $P^* \in \mathcal{P}$. Take a portfolio $\overline{\xi} \in \mathbb{R}^{d+1}$ such that $\overline{\xi} \cdot \overline{S} \geq 0$ P-a.s. and $E[\,\overline{\xi} \cdot \overline{S}\,] > 0$. Both properties remain valid if we replace P by the equivalent measure P^*. Hence,

$$\overline{\pi} \cdot \overline{\xi} = \sum_{i=0}^{d} \pi^i \xi^i = \sum_{i=0}^{d} E^*\Big[\frac{\xi^i S^i}{1+r}\Big] = E^*\Big[\frac{\overline{\xi} \cdot \overline{S}}{1+r}\Big] > 0.$$

Thus, $\overline{\xi}$ cannot be an arbitrage opportunity. □

For the proof of the implication ⇒ of Theorem 1.7, it will be convenient to introduce the random vector $Y = (Y^1, \dots, Y^d)$ of *discounted net gains*:

$$Y^i := \frac{S^i}{1+r} - \pi^i, \quad i = 1, \dots, d. \tag{1.2}$$

With this notation, Lemma 1.4 implies that the absence of arbitrage is equivalent to the following condition:

$$\text{For } \xi \in \mathbb{R}^d: \quad \xi \cdot Y \geq 0 \ P\text{-a.s.} \implies \xi \cdot Y = 0 \ P\text{-a.s.} \tag{1.3}$$

Since Y^i is bounded from below by $-\pi^i$, the expectation $E^*[\,Y^i\,]$ of Y^i under any measure P^* is well-defined, and so P^* is a risk-neutral measure if and only if

$$E^*[\,Y\,] = 0. \tag{1.4}$$

Here, $E^*[\,Y\,]$ is a shorthand notation for the d-dimensional vector with components $E^*[\,Y^i\,]$, $i = 1, \dots, d$. The assertion of Theorem 1.7 can now be read as follows: Condition (1.3) holds if and only if there exists some $P^* \approx P$ such that $E^*[\,Y\,] = 0$, and in this case, P^* can be chosen such that the density dP^*/dP is bounded.

Proof of the implication "⇒" of Theorem 1.7. We have to show that (1.3) implies the existence of some $P^* \approx P$ such that (1.4) holds and such that the density dP^*/dP is bounded. We will do this first in the case in which

$$E[\,|Y|\,] < \infty.$$

Let $\mathcal{Q}$ denote the convex set of all probability measures $Q \approx P$ with bounded densities dQ/dP, and denote by $E_Q[\,Y\,]$ the d-dimensional vector with components $E_Q[\,Y^i\,]$, $i = 1, \dots, d$. Due to our assumption $|Y| \in \mathcal{L}^1(P)$, all these expectations are finite. Let

$$\mathcal{C} := \{E_Q[\,Y\,] \mid Q \in \mathcal{Q}\},$$

and note that $\mathcal{C}$ is a convex set in $\mathbb{R}^d$: If $Q_1, Q_0 \in \mathcal{Q}$ and $0 \le \alpha \le 1$, then $Q_\alpha := \alpha Q_1 + (1-\alpha)Q_0 \in \mathcal{Q}$ and

$$\alpha E_{Q_1}[\,Y\,] + (1-\alpha)E_{Q_0}[\,Y\,] = E_{Q_\alpha}[\,Y\,],$$

which lies in $\mathcal{C}$.

Our aim is to show that $\mathcal{C}$ contains the origin. To this end, we suppose by way of contradiction that $0 \notin \mathcal{C}$. Using the "separating hyperplane theorem" in the elementary form of Proposition A.1, we obtain a vector $\xi \in \mathbb{R}^d$ such that $\xi \cdot x \ge 0$ for all $x \in \mathcal{C}$, and such that $\xi \cdot x_0 > 0$ for some $x_0 \in \mathcal{C}$. Thus, ξ satisfies $E_Q[\,\xi \cdot Y\,] \ge 0$ for all $Q \in \mathcal{Q}$ and $E_{Q_0}[\,\xi \cdot Y\,] > 0$ for some $Q_0 \in \mathcal{Q}$. Clearly, the latter condition yields that $P[\,\xi \cdot Y > 0\,] > 0$. We claim that the first condition implies that $\xi \cdot Y$ is P-a.s. non-negative. This fact will be a contradiction to our assumption (1.3) and thus will prove that $0 \in \mathcal{C}$.

To prove the claim that $\xi \cdot Y \ge 0$ P-a.s., let $A := \{\xi \cdot Y < 0\}$, and define functions

$$\varphi_n := \left(1 - \frac{1}{n}\right) \cdot \mathrm{I}_A + \frac{1}{n} \cdot \mathrm{I}_{A^c}.$$

We take φ_n as densities for new probability measures Q_n:

$$\frac{dQ_n}{dP} := \frac{1}{E[\,\varphi_n\,]} \cdot \varphi_n, \quad n = 2, 3, \dots.$$

Since $0 < \varphi_n \le 1$, it follows that $Q_n \in \mathcal{Q}$, and thus that

$$0 \le \xi \cdot E_{Q_n}[\,Y\,] = \frac{1}{E[\,\varphi_n\,]}\, E[\,\xi \cdot Y\, \varphi_n\,].$$

Hence, Lebesgue's dominated convergence theorem yields that

$$E[\,\xi \cdot Y\, \mathrm{I}_{\{\xi \cdot Y < 0\}}\,] = \lim_{n \uparrow \infty} E[\,\xi \cdot Y\, \varphi_n\,] \ge 0.$$

This proves the claim that $\xi \cdot Y \ge 0$ P-a.s. and completes the proof of Theorem 1.7 in case $E[\,|Y|\,] < \infty$.

If Y is not P-integrable, then we simply replace the probability measure P by a suitable equivalent measure $\tilde{P}$ whose density $d\tilde{P}/dP$ is bounded and for which $\tilde{E}[\,|Y|\,] < \infty$. For instance, one can define $\tilde{P}$ by

$$\frac{d\tilde{P}}{dP} = \frac{c}{1 + |Y|} \quad \text{for } c := \left(E\left[\,\frac{1}{1 + |Y|}\,\right]\right)^{-1}.$$

Recall from Remark 1.3 that replacing P with an equivalent probability measure does not affect the absence of arbitrage opportunities in our market model. Thus, the first part of this proof yields a risk-neutral measure P^* which is equivalent to $\tilde{P}$ and whose density $dP^*/d\tilde{P}$ is bounded. Then $P^* \in \mathcal{P}$, and

$$\frac{dP^*}{dP} = \frac{dP^*}{d\tilde{P}} \cdot \frac{d\tilde{P}}{dP}$$

is bounded. Hence, P^* is as desired, and the theorem is proved. □

Remark 1.8. Note that neither the absence of arbitrage nor the definition of the class $\mathcal{P}$ involve the full structure of the probability measure P, they only depend on the class of nullsets of P. In particular, the preceding theorem can be formulated in a situation of Knightian uncertainty, i.e., without fixing any initial probability measure P, whenever the underlying set Ω is countable. ◇

Remark 1.9. Our assumption that asset prices S are non-negative implies that the components of Y are bounded from below. Note however that this assumption was not needed in our proof. Thus, Theorem 1.7 also holds if we only assume that S is finite-valued and $\pi \in \mathbb{R}^d$. In this case, the definition of a risk-neutral measure P^* via (1.1) is meant to include the assumption that S^i is integrable with respect to P^* for $i = 1, \dots, d$. ◇

Example 1.10. Let P be any probability measure on the finite set $\Omega := \{\omega_1, \dots, \omega_N\}$ that assigns strictly positive probability p_i to each singleton $\{\omega_i\}$. Suppose that there is a single risky asset defined by its price $\pi = \pi^1$ at time 0 and by the random variable $S = S^1$. We may assume without loss of generality that the values $s_i := S(\omega_i)$ are distinct and arranged in increasing order: $s_1 < \dots < s_N$. According to Theorem 1.7, this model does not admit arbitrage opportunities if and only if

$$\pi(1+r) \in \{\tilde{E}[\,S\,] \mid \tilde{P} \approx P\} = \Big\{\sum_{i=1}^N s_i\,\tilde{p}_i \;\Big|\; \tilde{p}_i > 0,\ \sum_{i=1}^N \tilde{p}_i = 1\Big\} = (s_1, s_N),$$

and P^* is a risk-neutral measure if and only if the probabilities $p_i^* := P^*[\,\{\omega_i\}\,] \geq 0$ solve the linear equations

$$\begin{aligned} s_1 p_1^* + \dots + s_N p_N^* &= \pi(1+r), \\ p_1^* + \dots + p_N^* &= 1. \end{aligned}$$

If a solution exists, it will be unique if and only if $N = 2$, and there will be infinitely many solutions for $N > 2$. ◇

Exercise 1.2.1. On $\Omega = \{\omega_1, \omega_2, \omega_3\}$ we fix a probability measure P with $P[\,\omega_i\,] > 0$ for $i = 1, 2, 3$. Suppose that we have three assets with prices

$$\overline{\pi} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

at time 0 and

$$\overline{S}(\omega_1) = \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}, \quad \overline{S}(\omega_2) = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}, \quad \overline{S}(\omega_3) = \begin{pmatrix} 1 \\ 5 \\ 13 \end{pmatrix}$$

at time 1. Show that this market model does not admit arbitrage and find all risk-neutral measures. Note that this model differs from the one in Exercise 1.1.1 only in the value of $S^2(\omega_3)$. $\diamond$

Exercise 1.2.2. Consider a market model with one risky asset that is such that $\pi^1 > 0$ and the distribution of S^1 has a strictly positive density function $f : (0, \infty) \to (0, \infty)$. That is, $P[\, S^1 \le x \,] = \int_0^x f(y)\, dy$ for $x > 0$. Find an equivalent risk-neutral measure P^*. $\diamond$

Remark 1.11. The economic reason for working with the *discounted* asset prices

$$X^i := \frac{S^i}{1+r}, \quad i = 0, \dots, d, \tag{1.5}$$

is that one should distinguish between one unit of a currency (e.g. €) at time $t = 0$ and one unit at time $t = 1$. Usually people tend to prefer a certain amount today over the same amount which is promised to be paid at a later time. Such a preference is reflected in an interest $r > 0$ paid by the riskless bond: Only the amount $1/(1+r)$ € must be invested at time 0 to obtain 1 € at time 1. This effect is sometimes referred to as the *time value of money*. Similarly, the price S^i of the i^{th} asset is quoted in terms of € at time 1, while π^i corresponds to time-zero euros. Thus, in order to compare the two prices π^i and S^i, one should first convert them to a common standard. This is achieved by taking the riskless bond as a *numéraire* and by considering the *discounted* prices in (1.5). $\diamond$

Remark 1.12. One can choose as numéraire any asset which is strictly positive. For instance, suppose that $\pi^1 > 0$ and $P[\, S^1 > 0 \,] = 1$. Then all asset prices can be expressed in units of the first asset by considering

$$\widetilde{\pi}^i := \frac{\pi^i}{\pi^1} \quad \text{and} \quad \frac{S^i}{S^1}, \quad i = 0, \dots, d.$$

Clearly, the definition of an arbitrage opportunity is independent of the choice of a particular numéraire. Thus, an arbitrage-free market model should admit a risk-neutral measure with respect to the new numéraire, i.e., a probability measure $\tilde{P}^* \approx P$ such that

$$\tilde{\pi}^i = \tilde{E}^* \left[\frac{S^i}{S^1} \right], \quad i = 0, \dots, d.$$

Let us denote by $\tilde{\mathcal{P}}$ the set of all such measures $\tilde{P}^*$. Then

$$\tilde{\mathcal{P}} = \left\{ \tilde{P}^* \,\Big|\, \frac{d\tilde{P}^*}{dP^*} = \frac{S^1}{E^*[\,S^1\,]} \text{ for some } P^* \in \mathcal{P} \right\}.$$

Indeed, if $\tilde{P}^*$ lies in the set on the right, then

$$\tilde{E}^*\Big[\frac{S^i}{S^1} \Big] = \frac{E^*[\,S^i\,]}{E^*[\,S^1\,]} = \frac{\pi^i}{\pi^1} = \tilde{\pi}^i,$$

and so $\tilde{P}^* \in \tilde{\mathcal{P}}$. Reversing the roles of $\tilde{\mathcal{P}}$ and $\mathcal{P}$ then yields the identity of the two sets. Note that

$$\mathcal{P} \cap \tilde{\mathcal{P}} = \emptyset$$

as soon as S^1 is not P-a.s. constant, because Jensen's inequality then implies that

$$\frac{1}{\pi^1} = \tilde{\pi}^0 = \tilde{E}^*\Big[\frac{1+r}{S^1} \Big] > \frac{1+r}{\tilde{E}^*[\,S^1\,]}$$

and hence $\tilde{E}^*[\,S^1\,] > E^*[\,S^1\,]$ for all $\tilde{P}^* \in \tilde{\mathcal{P}}$ and $P^* \in \mathcal{P}$. ◇

Let

$$\mathcal{V} := \left\{ \overline{\xi} \cdot \overline{S} \mid \overline{\xi} \in \mathbb{R}^{d+1} \right\}$$

denote the linear space of all payoffs which can be generated by some portfolio. An element of $\mathcal{V}$ will be called an *attainable payoff*. The portfolio that generates $V \in \mathcal{V}$ is in general not unique, but we have the following *law of one price*.

Lemma 1.13. *Suppose that the market model is arbitrage-free and that $V \in \mathcal{V}$ can be written as $V = \overline{\xi} \cdot \overline{S} = \overline{\zeta} \cdot \overline{S}$ P-a.s. for two different portfolios $\overline{\xi}$ and $\overline{\zeta}$. Then $\overline{\pi} \cdot \overline{\xi} = \overline{\pi} \cdot \overline{\zeta}$.*

Proof. We have $(\overline{\xi} - \overline{\zeta}) \cdot \overline{S} = 0$ P^*-a.s. for any $P^* \in \mathcal{P}$. Hence,

$$\overline{\pi} \cdot \overline{\xi} - \overline{\pi} \cdot \overline{\zeta} = E^*\Big[\frac{(\overline{\xi} - \overline{\zeta}) \cdot \overline{S}}{1+r} \Big] = 0,$$

due to (1.1). □

By the preceding lemma, it makes sense to define the *price* of $V \in \mathcal{V}$ as

$$\pi(V) := \overline{\pi} \cdot \overline{\xi} \quad \text{if } V = \overline{\xi} \cdot \overline{S}, \tag{1.6}$$

whenever the market model is arbitrage-free.

Remark 1.14. Via (1.6), the price system π can be regarded as a linear form on the finite-dimensional vector space $\mathcal{V}$. For any $P^* \in \mathcal{P}$ we have

$$\pi(V) = E^*\Big[\frac{V}{1+r}\Big], \qquad V \in \mathcal{V}.$$

Thus, an equivalent risk-neutral measure P^* defines a linear extension of π onto the larger space $\mathcal{L}^1(P^*)$ of P^*-integrable random variables. Since this space is usually infinite-dimensional, one cannot expect that such a pricing measure is in general unique; see however Section 1.4. ◇

We have seen above that, in an arbitrage-free market model, the condition $\overline{\xi} \cdot \overline{S} = 0$ P-a.s. implies that $\overline{\pi} \cdot \overline{\xi} = 0$. In fact, one may assume without loss of generality that

$$\overline{\xi} \cdot \overline{S} = 0 \ P\text{-a.s.} \quad \Longrightarrow \quad \overline{\xi} = 0, \tag{1.7}$$

for otherwise we can find $i \in \{0, \dots, d\}$ such that $\xi^i \neq 0$ and represent the i^{th} asset as a linear combination of the remaining ones

$$\pi^i = -\frac{1}{\xi^i} \sum_{j \neq i} \xi^j \pi^j \quad \text{and} \quad S^i = -\frac{1}{\xi^i} \sum_{j \neq i} \xi^j S^j.$$

In this sense, the i^{th} asset is redundant and can be omitted.

Definition 1.15. The market model is called *non-redundant* if (1.7) holds.

Exercise 1.2.3. Show that in a non-redundant market model the components of the vector Y of discounted net gains are linearly independent in the sense that

$$\xi \cdot Y = 0 \ P\text{-a.s.} \quad \Longrightarrow \quad \xi = 0. \tag{1.8}$$

Show then that condition (1.8) implies non-redundance if the market model is arbitrage-free. ◇

Exercise 1.2.4. Show that in a non-redundant and arbitrage-free market model the set

$$\big\{\overline{\xi} \in \mathbb{R}^{d+1} \mid \overline{\pi} \cdot \overline{\xi} = w \text{ and } \overline{\xi} \cdot \overline{S} \geq 0 \ P\text{-a.s.}\big\}$$

is compact for any $w > 0$. ◇

Definition 1.16. Suppose that the market model is arbitrage-free and that $V \in \mathcal{V}$ is an attainable payoff such that $\pi(V) \neq 0$. Then the *return* of V is defined by

$$R(V) := \frac{V - \pi(V)}{\pi(V)}.$$

Note that we have already seen the special case of the risk-free return

$$r = \frac{S^0 - \pi^0}{\pi^0} = R(S^0).$$

If an attainable payoff V is a linear combination $V = \sum_{k=1}^{n} \alpha_k V_k$ of non-zero attainable payoffs V_k, then

$$R(V) = \sum_{k=1}^{n} \beta_k R(V_k) \quad \text{for } \beta_k = \frac{\alpha_k \pi(V_k)}{\sum_{i=1}^{n} \alpha_i \pi(V_i)}.$$

The coefficient β_k can be interpreted as the proportion of the investment allocated to V_k. As a particular case of the formula above, we have that

$$R(V) = \sum_{i=0}^{d} \frac{\pi^i \, \xi^i}{\overline{\pi} \cdot \overline{\xi}} \cdot R(S^i)$$

for all non-zero attainable payoffs $V = \overline{\xi} \cdot \overline{S}$ (recall that we have assumed that all π^i are strictly positive).

Proposition 1.17. *Suppose that the market model is arbitrage-free, and let $V \in \mathcal{V}$ be an attainable payoff such that $\pi(V) \neq 0$.*

(a) *Under any risk-neutral measure P^*, the expected return of V is equal to the risk-free return r*

$$E^*[\, R(V)\,] = r.$$

(b) *Under any measure $Q \approx P$ such that $E_Q[\,|\overline{S}|\,] < \infty$, the expected return of V is given by*

$$E_Q[\, R(V)\,] = r - \operatorname*{cov}_Q\left(\frac{dP^*}{dQ}, R(V)\right),$$

where P^* is an arbitrary risk-neutral measure in $\mathcal{P}$ and cov_Q *denotes the covariance with respect to Q.*

Proof. (a): Since $E^*[\, V\,] = \pi(V)(1 + r)$, we have

$$E^*[\, R(V)\,] = \frac{E^*[\, V\,] - \pi(V)}{\pi(V)} = r.$$

(b): Let $P^* \in \mathcal{P}$ and $\varphi^* := dP^*/dQ$. Then

$$\begin{aligned}\operatorname*{cov}_Q(\varphi^*, R(V)) &= E_Q[\,\varphi^* R(V)\,] - E_Q[\,\varphi^*\,] \cdot E_Q[\, R(V)\,] \\ &= E^*[\, R(V)\,] - E_Q[\, R(V)\,].\end{aligned}$$

Using part (a) yields the assertion. □

Remark 1.18. Let us comment on the extension of the fundamental equivalence in Theorem 1.7 to market models with an infinity of tradable assets $S^0, S^1, S^2, \dots$. We assume that $S^0 \equiv 1 + r$ for some $r > -1$ and that the random vector

$$S(\omega) = (S^1(\omega), S^2(\omega), \dots)$$

takes values in the space ℓ^∞ of bounded real sequences. This space is a Banach space with respect to the norm

$$\|x\|_\infty := \sup_{i \geq 1} |x^i| \quad \text{for } x = (x^1, x^2, \dots) \in \ell^\infty.$$

A portfolio $\overline{\xi} = (\xi^0, \xi)$ is chosen in such a way that $\xi = (\xi^1, \xi^2, \dots)$ is a sequence in the space ℓ^1, i.e., $\sum_{i=1}^\infty |\xi^i| < \infty$. We assume that the corresponding price system $\overline{\pi} = (\pi^0, \pi)$ satisfies $\pi \in \ell^\infty$ and $\pi^0 = 1$. Clearly, this model class includes our model with $d + 1$ traded assets as a special case.

Our first observation is that the implication $\Leftarrow$ of Theorem 1.7 remains valid, i.e., the existence of a measure $P^* \approx P$ with the properties

$$E^*[\, \|S\|_\infty \,] < \infty \quad \text{and} \quad E^*\left[\frac{S^i}{1+r} \right] = \pi^i$$

implies the absence of arbitrage opportunities. To this end, suppose that $\overline{\xi}$ is a portfolio strategy such that

$$\overline{\xi} \cdot \overline{S} \geq 0 \;\; P\text{-a.s.} \quad \text{and} \quad E[\, \overline{\xi} \cdot \overline{S} \,] > 0. \tag{1.9}$$

Then we can replace P in (1.9) by the equivalent measure P^*. Hence, $\overline{\xi}$ cannot be an arbitrage opportunity since

$$\overline{\xi} \cdot \overline{\pi} = \sum_{i=0}^\infty \xi^i E^*\left[\frac{S^i}{1+r} \right] = E^*\left[\frac{\overline{\xi} \cdot \overline{S}}{1+r} \right] > 0.$$

Note that interchanging summation and integration is justified by dominated convergence, because

$$|\xi^0| + \|S\|_\infty \sum_{i=0}^\infty |\xi^i| \in \mathcal{L}^1(P^*).$$

The following example shows that the implication $\Rightarrow$ of Theorem 1.7, namely that absence of arbitrage opportunities implies the existence of a risk-neutral measure, may no longer be true for an infinite number of assets. ◇

Example 1.19. Let $\Omega = \{1, 2, \dots\}$, and choose any probability measure P which assigns strictly positive probability to all singletons $\{\omega\}$. We take $r = 0$ and define a price system $\pi^i = 1$, for $i = 0, 1, \dots$. Prices at time 1 are given by $S^0 \equiv 1$ and, for $i = 1, 2, \dots$, by

$$S^i(\omega) = \begin{cases} 0 & \text{if } \omega = i, \\ 2 & \text{if } \omega = i+1, \\ 1 & \text{otherwise.} \end{cases}$$

Let us show that this market model is arbitrage-free. To this end, suppose that $\overline{\xi} = (\xi^0, \xi)$ is a portfolio such that $\xi \in \ell^1$ and such that $\overline{\xi} \cdot \overline{S}(\omega) \geq 0$ for each $\omega \in \Omega$, but such that $\overline{\pi} \cdot \overline{\xi} \leq 0$. Considering the case $\omega = 1$ yields

$$0 \leq \overline{\xi} \cdot \overline{S}(1) = \xi^0 + \sum_{k=2}^{\infty} \xi^k = \overline{\pi} \cdot \overline{\xi} - \xi^1 \leq -\xi^1.$$

Similarly, for $\omega = i > 1$,

$$0 \leq \overline{\xi} \cdot \overline{S}(\omega) = \xi^0 + 2\xi^{i-1} + \sum_{\substack{k=1 \\ k \neq i, i-1}}^{\infty} \xi^k = \overline{\pi} \cdot \overline{\xi} + \xi^{i-1} - \xi^i \leq \xi^{i-1} - \xi^i.$$

It follows that $0 \geq \xi^1 \geq \xi^2 \geq \cdots$. But this can only be true if all ξ^i vanish, since we have assumed that $\xi \in \ell^1$. Hence, there are no arbitrage opportunities.

However, there exists no probability measure $P^* \approx P$ such that $E^*[\,S^i\,] = \pi^i$ for all i. Such a measure P^* would have to satisfy

$$\begin{aligned} 1 = E^*[S^i] &= 2P^*[\{i+1\}] + \sum_{\substack{k=1 \\ k \neq i, i+1}}^{\infty} P^*[\{k\}] \\ &= 1 + P^*[\{i+1\}] - P^*[\{i\}] \end{aligned}$$

for $i > 1$. This relation implies that $P^*[\,\{i\}\,] = P^*[\,\{i+1\}\,]$ for all $i > 1$, contradicting the assumption that P^* is a probability measure and equivalent to P. $\diamondsuit$

1.3 Derivative securities

In real financial markets, not only the primary assets are traded. There is also a large variety of securities whose payoff depends in a non-linear way on the primary assets $S^0, S^1, \dots, S^d$, and sometimes also on other factors. Such financial instruments are usually called *options*, *contingent claims*, *derivative securities*, or just *derivatives*.

Example 1.20. Under a *forward contract*, one agent agrees to sell to another agent an asset at time 1 for a price K which is specified at time 0. Thus, the owner of a forward contract on the i^{th} asset gains the difference between the actual market price S^i and the *delivery price* K if S^i is larger than K at time 1. If $S^i < K$, the owner loses the amount $K - S^i$ to the issuer of the forward contract. Hence, a forward contract corresponds to the random payoff

$$C^{\text{fw}} = S^i - K. \qquad \diamond$$

Example 1.21. The owner of a *call option* on the i^{th} asset has the right, but not the obligation, to buy the i^{th} asset at time 1 for a fixed price K, called the *strike price*. This corresponds to a payoff of the form

$$C^{\text{call}} = (S^i - K)^+ = \begin{cases} S^i - K & \text{if } S^i > K, \\ 0 & \text{otherwise.} \end{cases}$$

Conversely, a *put option* gives the right, but not the obligation, to sell the asset at time 1 for a strike price K. The corresponding random payoff is given by

$$C^{\text{put}} = (K - S^i)^+ = \begin{cases} K - S^i & \text{if } S^i < K, \\ 0 & \text{otherwise.} \end{cases}$$

Call and put options with the same strike K are related through the formula

$$C^{\text{call}} - C^{\text{put}} = S^i - K.$$

Hence, if the price $\pi(C^{\text{call}})$ of a call option has already been fixed, then the price $\pi(C^{\text{put}})$ of the corresponding put option is determined by linearity through the *put-call parity*

$$\pi(C^{\text{call}}) = \pi(C^{\text{put}}) + \pi^i - \frac{K}{1+r}. \tag{1.10}$$

$\diamond$

Example 1.22. An option on the value $V = \overline{\xi} \cdot \overline{S}$ of a portfolio of several risky assets is sometimes called a *basket* or *index option*. For instance, a basket call would be of the form $(V - K)^+$. The asset on which the option is written is called the *underlying asset* or just the *underlying*. $\diamond$

Put and call options can be used as building blocks for a large class of derivatives.

Example 1.23. A *straddle* is a combination of “at-the-money” put and call options on a portfolio $V = \overline{\xi} \cdot \overline{S}$, i.e., on put and call options with strike $K = \pi(V)$:

$$C = (\pi(V) - V)^+ + (V - \pi(V))^+ = |V - \pi(V)|.$$

Thus, the payoff of the straddle increases proportionally to the change of the price of $\overline{\xi}$ between time 0 and time 1. In this sense, a straddle is a bet that the portfolio price will move, no matter in which direction. $\diamond$

Example 1.24. The payoff of a *butterfly spread* is of the form

$$C = (K - |V - \pi(V)|)^+,$$

where $K > 0$ and where $V = \overline{\xi} \cdot \overline{S}$ is the price of a given portfolio or the value of a stock index. Clearly, the payoff of the butterfly spread is maximal if $V = \pi(V)$ and decreases if the price at time 1 of the portfolio $\overline{\xi}$ deviates from its price at time 0. Thus, the butterfly spread is a bet that the portfolio price will stay close to its present value. ◇

Exercise 1.3.1. Draw the payoffs of put and call options, a straddle, and a butterfly spread as functions of its underlying. ◇

Exercise 1.3.2. Consider a butterfly spread as in Example 1.24 and write its payoff as a combination of

(a) call options,

(b) put options

on the underlying. As in the put-call parity (1.10), such a decomposition determines the price of a butterfly spread once the prices of the corresponding put or call options have been fixed. ◇

Example 1.25. The idea of *portfolio insurance* is to increase exposure to rising asset prices, and to reduce exposure to falling prices. This suggests to replace the payoff $V = \overline{\xi} \cdot \overline{S}$ of a given portfolio by a modified profile $h(V)$, where h is convex and increasing. Let us first consider the case where $V \geq 0$. Then the corresponding payoff $h(V)$ can be expressed as a combination of investments in bonds, in V itself, and in basket call options on V. To see this, recall that convexity implies that

$$h(x) = h(0) + \int_0^x h'(y)\,dy$$

for the increasing right-hand derivative $h' := h'_+$ of h; see Appendix A.1. By the arguments in Lemma A.19, the increasing rightcontinuous function h' can be represented as the distribution function of a positive Radon measure γ on $[0, \infty)$: $h'(x) = \gamma([0, x])$ for $x \geq 0$. Recall that a positive Radon measure is a σ-additive measure that assigns to each Borel set $A \subset [0, \infty)$ a value in $\gamma(A) \in [0, \infty]$, which is finite when A is compact. An example is the Lebesgue measure on $[0, \infty)$. Using the representation $h'(x) = \gamma([0, x])$, Fubini's theorem implies that

$$\begin{aligned} h(x) &= h(0) + \int_0^x \int_{[0,y]} \gamma(dz)\,dy \\ &= h(0) + \gamma(\{0\})\,x + \int_{(0,\infty)} \int_{\{y \mid z \leq y \leq x\}} dy\,\gamma(dz). \end{aligned}$$

Since the inner integral equals $(x-z)^+$, we obtain

$$h(V) = h(0) + h'(0)\, V + \int_{(0,\infty)} (V-K)^+\, \gamma(dK). \tag{1.11}$$

$\diamond$

The formula (1.11) yields a representation of $h(V)$ in terms of investments in bonds, in $V = \overline{\xi}\cdot\overline{S}$ itself, and in call options on V. It requires, however, that V is nonnegative and that both $h(0)$ and $h'(0)$ are finite. Also, it is sometimes more convenient to have a development around the initial value $\pi^V := \overline{\pi}\cdot\overline{\xi}$ of the portfolio $\overline{\xi}$ than to have a development around zero. Corresponding extensions of formula (1.11) are explored in the following exercise.

Exercise 1.3.3. In this exercise, we consider the situation of Example 1.25 without insisting that the payoff $V = \overline{\xi}\cdot\overline{S}$ takes only nonnegative values. In particular, the portfolio $\overline{\xi}$ may also contain short positions. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function.

(a) Show that for convex h there exists a nonnegative Radon measure γ on $\mathbb{R}$ such that the payoff $h(V)$ can be realized by holding bonds, forward contracts, and a mixture of call and put options on V:

$$\begin{aligned} h(V) &= h(\pi^V) + h'(\pi^V)(V-\pi^V) \\ &\quad + \int_{(-\infty,\pi^V]} (V-K)^+\, \gamma(dK) + \int_{(\pi^V,\infty)} (K-V)^+\, \gamma(dK). \end{aligned}$$

Note that the put and call options occurring in this formula are "out of the money" in the sense that their "intrinsic value", i.e., their value when V is replaced by its present value π^V, is zero.

(b) Now let h be any twice continuously differentiable function on $\mathbb{R}$. Deduce from part (a) that

$$\begin{aligned} h(V) &= h(\pi^V) + h'(\pi^V)(V-\pi^V) \\ &\quad + \int_{-\infty}^{\pi^V} (V-K)^+ h''(K)\, dK + \int_{\pi^V}^{\infty} (K-V)^+ h''(K)\, dK. \end{aligned}$$

This formula is sometimes called the *Breeden–Litzenberger formula*. $\diamond$

Example 1.26. A *reverse convertible bond* pays interest which is higher than that earned by an investment into the riskless bond. But at maturity $t = 1$, the issuer may convert the bond into a predetermined number of shares of a given asset S^i instead of paying the nominal value in cash. The purchase of this contract is equivalent to the purchase of a standard bond and the sale of a certain put option. More precisely, suppose that 1 is the price of the reverse convertible bond at $t = 0$, that its nominal value at maturity is $1 + \tilde{r}$, and that it can be converted into x shares of the i^{th} asset. This conversion will happen if the asset price S^i is below $K := (1 + \tilde{r})/x$. Thus, the payoff of the reverse convertible bond is equal to

$$1 + \tilde{r} - x(K - S^i)^+,$$

i.e., the purchase of this contract is equivalent to a risk-free investment of the amount $(1+\tilde{r})/(1+r)$ with interest r and the sale of the put option $x(K - S^i)^+$ for the price $(\tilde{r} - r)/(1 + r)$. ◇

Example 1.27. A *discount certificate* on $V = \overline{\xi} \cdot \overline{S}$ pays off the amount

$$C = V \wedge K,$$

where the number $K > 0$ is often called the *cap*. Since

$$C = V - (V - K)^+,$$

buying the discount certificate is the same as purchasing $\overline{\xi}$ and selling the basket call option $C^{\text{call}} := (V - K)^+$. If the price $\pi(C^{\text{call}})$ has already been fixed, then the price of C is given by $\pi(C) = \pi(V) - \pi(C^{\text{call}})$. Hence, the discount certificate is less expensive than the portfolio $\overline{\xi}$ itself, and this explains the name. On the other hand, it participates in gains of $\overline{\xi}$ only up to the cap K. ◇

Example 1.28. For an insurance company, it may be desirable to shift some of its insurance risk to the financial market. As an example of such an *alternative risk transfer*, consider a *catastrophe bond* issued by an insurance company. The interest paid by this security depends on the occurrence of certain special events. For instance, the contract may specify that no interest will be paid if more than a given number of insured cars are damaged by hail on a single day during the lifetime of the contract; as a compensation for taking this risk, the buyer will be paid an interest above the usual market rate if this event does not occur. ◇

Mathematically, it will be convenient to focus on contingent claims whose payoff is non-negative. Such a contingent claim will be interpreted as a contract which is sold at time 0 and which pays a random amount $C(\omega) \geq 0$ at time 1. A derivative security whose terminal value may also become negative can usually be reduced to a combination of a non-negative contingent claim and a short position in some of the primary assets $S^0, S^1, \dots, S^d$. For instance, the terminal value of a reverse convertible bond is bounded from below so that it can be decomposed into a short position in cash and into a contract with positive value. From now on, we will work with the following formal definition of the term "contingent claim".

Definition 1.29. A *contingent claim* is a random variable C on the underlying probability space $(\Omega, \mathcal{F}, P)$ such that

$$0 \leq C < \infty \quad P\text{-a.s.}$$

A contingent claim C is called a *derivative* of the primary assets $S^0, \dots, S^d$ if it is measurable with respect to the σ-field $\sigma(S^0, \dots, S^d)$ generated by the assets, i.e., if

$$C = f(S^0, \dots, S^d)$$

for a measurable function f on $\mathbb{R}^{d+1}$.

So far, we have only fixed the prices π^i of our primary assets S^i. Thus, it is not clear what the correct price should be for a general contingent claim C. Our main goal in this section is to identify those possible prices which are compatible with the given prices in the sense that they do not generate arbitrage. Our approach is based on the observation that trading C at time 0 for a price π^C corresponds to introducing a new asset with the prices

$$\pi^{d+1} := \pi^C \quad \text{and} \quad S^{d+1} := C. \tag{1.12}$$

Definition 1.30. A real number $\pi^C \geq 0$ is called an *arbitrage-free price* of a contingent claim C if the market model extended according to (1.12) is arbitrage-free. The set of all arbitrage-free prices for C is denoted $\Pi(C)$.

In the previous definition, we made the implicit assumption that the introduction of a contingent claim C as a new asset does not affect the prices of primary assets. This assumption is reasonable as long as the trading volume of C is small compared to that of the primary assets. In Section 3.6 we will discuss the equilibrium approach to asset pricing, where an extension of the market will typically change the prices of *all* traded assets.

The following result shows in particular that we can always find an arbitrage-free price for a given contingent claim C if the initial model is arbitrage-free.

Theorem 1.31. *Suppose that the set $\mathcal{P}$ of equivalent risk-neutral measures for the original market model is non-empty. Then the set of arbitrage-free prices of a contingent claim C is non-empty and given by*

$$\Pi(C) = \left\{ E^*\left[\frac{C}{1+r} \right] \,\middle|\, P^* \in \mathcal{P} \text{ such that } E^*[\,C\,] < \infty \right\}. \tag{1.13}$$

Proof. By Theorem 1.7, π^C is an arbitrage-free price for C if and only if there exists an equivalent risk-neutral measure $\hat{P}$ for the market model extended via (1.12), i.e.,

$$\pi^i = \hat{E}\Big[\frac{S^i}{1+r}\Big] \quad \text{for } i = 1, \dots, d+1.$$

In particular, $\hat{P}$ is necessarily contained in $\mathcal{P}$, and we obtain the inclusion $\subseteq$ in (1.13). Conversely, if $\pi^C = E^*[C/(1+r)]$ for some $P^* \in \mathcal{P}$, then this P^* is also an equivalent risk-neutral measure for the extended market model, and so the two sets in (1.13) are equal.

To show that $\Pi(C)$ is non-empty, we first fix some measure $\tilde{P} \approx P$ such that $\tilde{E}[C] < \infty$. For instance, we can take $d\tilde{P} = c(1+C)^{-1}\,dP$, where c is the normalizing constant. Under $\tilde{P}$, the market model is arbitrage-free. Hence, Theorem 1.7 yields $P^* \in \mathcal{P}$ such that $dP^*/d\tilde{P}$ is bounded. In particular, $E^*[C] < \infty$ and $\pi^C = E^*[C/(1+r)] \in \Pi(C)$. □

Exercise 1.3.4. Show that the set $\Pi(C)$ of arbitrage-free prices of a contingent claim is convex and hence an interval. ◇

The following theorem provides a dual characterization of the lower and upper bounds

$$\pi_{\inf}(C) := \inf \Pi(C) \quad \text{and} \quad \pi_{\sup}(C) := \sup \Pi(C),$$

which are often called *arbitrage bounds* for C.

Theorem 1.32. *In an arbitrage-free market model, the arbitrage bounds of a contingent claim C are given by*

$$\begin{aligned}\pi_{\inf}(C) &= \inf_{P^*\in\mathcal{P}} E^*\Big[\frac{C}{1+r}\Big]\\ &= \max\Big\{m \in [0,\infty) \;\Big|\; \exists\, \xi \in \mathbb{R}^d \textit{ with } m + \xi\cdot Y \le \frac{C}{1+r}\ P\textit{-a.s.}\Big\}\end{aligned} \tag{1.14}$$

and

$$\begin{aligned}\pi_{\sup}(C) &= \sup_{P^*\in\mathcal{P}} E^*\Big[\frac{C}{1+r}\Big]\\ &= \min\Big\{m \in [0,\infty] \;\Big|\; \exists\, \xi \in \mathbb{R}^d \textit{ with } m + \xi\cdot Y \ge \frac{C}{1+r}\ P\textit{-a.s.}\Big\}.\end{aligned}$$

Proof. We only prove the identities for the upper arbitrage bound. The ones for the lower bound are obtained in a similar manner; see Exercise 1.3.5. We take $m \in [0,\infty]$ and $\xi \in \mathbb{R}^d$ such that $m + \xi\cdot Y \ge C/(1+r)$ P-a.s., and we denote by M the set of all such m. Taking the expectation with $P^* \in \mathcal{P}$ yields $m \geq E^*[\,C/(1+r)\,]$, and we get

$$\begin{aligned}\inf M &\geq \sup_{P^* \in \mathcal{P}} E^*\Big[\frac{C}{1+r}\Big] \\ &\geq \sup\Big\{E^*\Big[\frac{C}{1+r}\Big] \,\Big|\, P^* \in \mathcal{P},\, E^*[\,C\,] < \infty\Big\} = \pi_{\sup}(C),\end{aligned} \tag{1.15}$$

where we have used Theorem 1.31 in the last identity.

Next we show that all inequalities in (1.15) are in fact identities. This is trivial if $\pi_{\sup}(C) = \infty$. For $\pi_{\sup}(C) < \infty$, we will show that $m > \pi_{\sup}(C)$ implies $m \geq \inf M$. By definition, $\pi_{\sup}(C) < m < \infty$ requires the existence of an arbitrage opportunity in the market model extended by $\pi^{d+1} := m$ and $S^{d+1} := C$. That is, there is $(\xi, \xi^{d+1}) \in \mathbb{R}^{d+1}$ such that $\xi \cdot Y + \xi^{d+1}(C/(1+r) - m)$ is almost-surely non-negative and strictly positive with positive probability. Since the original market model is arbitrage-free, ξ^{d+1} must be non-zero. In fact, we have $\xi^{d+1} < 0$ as taking expectations with respect to $P^* \in \mathcal{P}$ for which $E^*[\,C\,] < \infty$ yields

$$\xi^{d+1}\Big(E^*\Big[\frac{C}{1+r}\Big] - m\Big) \geq 0,$$

and the term in parenthesis is negative since $m > \pi_{\sup}(C)$. Thus, we may define $\zeta := -\xi/\xi^{d+1} \in \mathbb{R}^d$ and obtain $m + \zeta \cdot Y \geq C/(1+r)$ P-a.s., hence $m \geq \inf M$.

We now prove that $\inf M$ belongs to M. To this end, we may assume without loss of generality that $\inf M < \infty$ and that the market model is non-redundant in the sense of Definition 1.15. For a sequence $m_n \in M$ that decreases towards $\inf M = \pi_{\sup}(C)$, we fix $\xi_n \in \mathbb{R}^d$ such that $m_n + \xi_n \cdot Y \geq C/(1+r)$ P-almost surely. If $\liminf_n |\xi_n| < \infty$, there exists a subsequence of (ξ_n) that converges to some $\xi \in \mathbb{R}^d$. Passing to the limit yields $\pi_{\sup}(C) + \xi \cdot Y \geq C/(1+r)$ P-a.s., which gives $\pi_{\sup}(C) \in M$. But this is already the desired result, since the following argument will show that the case $\liminf_n |\xi_n| = \infty$ cannot occur. Indeed, after passing to some subsequence if necessary, $\eta_n := \xi_n/|\xi_n|$ converges to some $\eta \in \mathbb{R}^d$ with $|\eta| = 1$. Under the assumption that $|\xi_n| \to \infty$, passing to the limit in

$$\frac{m_n}{|\xi_n|} + \eta_n \cdot Y \geq \frac{C}{|\xi_n|(1+r)} \quad P\text{-a.s.}$$

yields $\eta \cdot Y \geq 0$. The absence of arbitrage opportunities thus implies $\eta \cdot Y = 0$ P-a.s., whence $\eta = 0$ by non-redundance of the model. But this contradicts the fact that $|\eta| = 1$. □

Exercise 1.3.5. Prove the identity (1.14). ◇

Remark 1.33. Theorem 1.32 shows that $\pi_{\sup}(C)$ is the lowest possible price of a portfolio $\overline{\xi}$ with

$$\overline{\xi} \cdot \overline{S} \geq C \quad P\text{-a.s.}$$

Such a portfolio is often called a "superhedging strategy" or "superreplication" of C, and the identities for $\pi_{\inf}(C)$ and $\pi_{\sup}(C)$ obtained in Theorem 1.32 are often called *superhedging duality relations*. When using $\overline{\xi}$, the *seller* of C would be protected against any possible future claims of the buyer of C. Thus, a natural goal for the seller would be to finance such a superhedging strategy from the proceeds of C. Conversely, the objective of the *buyer* would be to cover the price of C from the sale of a portfolio $\overline{\eta}$ with

$$\overline{\eta} \cdot \overline{S} \leq C \quad P\text{-a.s.},$$

which is possible if and only if $\overline{\pi} \cdot \overline{\eta} \leq \pi_{\inf}(C)$. Unless C is an attainable payoff, however, neither objective can be fulfilled by trading C at an arbitrage-free price, as shown in Corollary 1.35 below. Thus, any arbitrage-free price involves a trade-off between these two objectives. ◇

For a portfolio $\overline{\xi}$ the resulting payoff $V = \overline{\xi} \cdot \overline{S}$, if positive, may be viewed as a contingent claim, and in particular as a derivative. Those claims which can be replicated by a suitable portfolio will play a special role in the sequel.

Definition 1.34. A contingent claim C is called *attainable* (*replicable, redundant*) if $C = \overline{\xi} \cdot \overline{S}$ P-a.s. for some $\overline{\xi} \in \mathbb{R}^{d+1}$. Such a portfolio strategy $\overline{\xi}$ is then called a *replicating portfolio* for C.

If one can show that a given contingent claim C can be replicated by some portfolio $\overline{\xi}$, then the problem of determining a price for C has a straightforward solution: The price of C is unique and equal to the cost $\overline{\xi} \cdot \overline{\pi}$ of its replication, due to the law of one price. The following corollary also shows that the attainable contingent claims are in fact the only ones which admit a unique arbitrage-free price.

Corollary 1.35. *Suppose the market model is arbitrage-free and C is a contingent claim.*

(a) *C is attainable if and only if it admits a unique arbitrage-free price.*

(b) *If C is not attainable, then $\pi_{\inf}(C) < \pi_{\sup}(C)$ and*

$$\Pi(C) = (\pi_{\inf}(C), \pi_{\sup}(C)).$$

Proof. To prove part (a), note first that $|\Pi(C)| = 1$ if C is attainable. The converse implication will follow from (b).

In order to prove part (b), note first that $\Pi(C)$ is an interval due to Exercise 1.3.4. To show that this interval is open, it suffices to exclude the possibility that it contains one of its boundary points $\pi_{\inf}(C)$ and $\pi_{\sup}(C)$. To this end, we use Theorem 1.32 to get $\xi \in \mathbb{R}^d$ such that

$$\pi_{\inf}(C) + \xi \cdot Y \leq \frac{C}{1+r} \quad P\text{-a.s.}$$

Since C is not attainable, this inequality cannot be an almost-sure identity. Hence, with $\xi^0 := \pi \cdot \xi - \pi_{\inf}(C)$, the strategy $(\xi^0, -\xi, 1) \in \mathbb{R}^{d+2}$ is an arbitrage opportunity in the market model extended by $\pi^{d+1} := \pi_{\inf}(C)$ and $S^{d+1} := C$. Therefore $\pi_{\inf}(C)$ is not an arbitrage-free price for C. The possibility $\pi_{\sup}(C) \in \Pi(C)$ is excluded by a similar argument. □

Remark 1.36. In Theorem 1.32, the set $\mathcal{P}$ of *equivalent* risk-neutral measures can be replaced by the set $\tilde{\mathcal{P}}$ of risk-neutral measures that are merely *absolutely continuous* with respect to P. That is,

$$\pi_{\inf}(C) = \inf_{\tilde{P} \in \tilde{\mathcal{P}}} \tilde{E}\Big[\frac{C}{1+r}\Big] \quad \text{and} \quad \pi_{\sup}(C) = \sup_{\tilde{P} \in \tilde{\mathcal{P}}} \tilde{E}\Big[\frac{C}{1+r}\Big], \tag{1.16}$$

for any contingent claim C. To prove this, note first that $\mathcal{P} \subset \tilde{\mathcal{P}}$, so that we get the two inequalities "$\geq$" and "$\leq$" in (1.16). On the other hand, for $\tilde{P} \in \tilde{\mathcal{P}}$, $P^* \in \mathcal{P}$ with $E^*[\,C\,] < \infty$, and $\varepsilon \in (0, 1]$, the measure $P^*_\varepsilon := \varepsilon P^* + (1-\varepsilon)\tilde{P}$ belongs to $\mathcal{P}$ and satisfies $E^*_\varepsilon[\,C\,] = \varepsilon E^*[\,C\,] + (1-\varepsilon)\tilde{E}[\,C\,]$. Sending $\varepsilon \downarrow 0$ yields the converse inequalities. ◇

Remark 1.37. Consider any arbitrage-free market model, and let $C^{\text{call}} = (S^i - K)^+$ be a call option on the i^{th} asset with strike $K > 0$. Clearly, $C^{\text{call}} \leq S^i$ so that

$$E^*\Big[\frac{C^{\text{call}}}{1+r}\Big] \leq \pi^i$$

for any $P^* \in \mathcal{P}$. From Jensen's inequality, we obtain the following lower bound:

$$E^*\Big[\frac{C^{\text{call}}}{1+r}\Big] \geq \Big(E^*\Big[\frac{S^i}{1+r}\Big] - \frac{K}{1+r}\Big)^+ = \Big(\pi^i - \frac{K}{1+r}\Big)^+.$$

Thus, the following universal bounds hold for any arbitrage-free market model:

$$\Big(\pi^i - \frac{K}{1+r}\Big)^+ \leq \pi_{\inf}(C^{\text{call}}) \leq \pi_{\sup}(C^{\text{call}}) \leq \pi^i. \tag{1.17}$$

For a put option $C^{\text{put}} = (K - S^i)^+$, one obtains the universal bounds

$$\Big(\frac{K}{1+r} - \pi^i\Big)^+ \leq \pi_{\inf}(C^{\text{put}}) \leq \pi_{\sup}(C^{\text{put}}) \leq \frac{K}{1+r}. \tag{1.18}$$

If $r \geq 0$, then the lower bound in (1.17) can be further reduced to $\pi_{\inf}(C^{\text{call}}) \geq (\pi^i - K)^+$. Informally, this inequality states that the value of the right to buy the i^{th} asset at $t = 0$ for a price K is strictly less than any arbitrage-free price for C^{call}. This fact is sometimes expressed by saying that the *time value* of a call option is non-negative. The quantity $(\pi^i - K)^+$ is called the *intrinsic value* of the call option. Observe that an analogue of this relation usually fails for put options: The left-hand side of (1.18) can only be bounded by its intrinsic value $(K - \pi^i)^+$ if $r \leq 0$. If the intrinsic value of a put or call option is positive, then one says that the option is "in the money". For $\pi^i = K$ one speaks of an "at-the-money" option. Otherwise, the option is "out of the money". $\diamond$

In many situations, the universal arbitrage bounds (1.17) and (1.18) are in fact attained, as illustrated by the following example.

Example 1.38. Take any market model with a single risky asset $S = S^1$ such that the distribution of S under P is concentrated on $\{0, 1, \dots\}$ with positive weights. Without loss of generality, we may assume that S has under P a Poisson distribution with parameter 1, i.e., S is P-a.s. integer-valued and

$$P[\, S = k\,] = \frac{e^{-1}}{k!} \quad \text{for } k = 0, 1, \dots.$$

If we take $r = 0$ and $\pi = 1$, then P is a risk-neutral measure and the market model is arbitrage-free. We are going to show that the upper and lower bounds in (1.17) are attained for this model by using Remark 1.36. To this end, consider the measure $\tilde{P} \in \tilde{\mathcal{P}}$ which is defined by its density

$$\frac{d\tilde{P}}{dP} = e \cdot \mathrm{I}_{\{S=1\}}\,.$$

We get

$$\tilde{E}[\,(S-K)^+\,] = (1-K)^+ = (\pi - K)^+,$$

so that the lower bound in (1.17) is attained, i.e., we have

$$\pi_{\inf}((S-K)^+) = (\pi - K)^+.$$

To see that also the upper bound is sharp, we define

$$g_n(k) := \left(e - \frac{e}{n}\right) \cdot \mathrm{I}_{\{0\}}(k) + (n-1)! \cdot e \cdot \mathrm{I}_{\{n\}}(k), \quad k = 0, 1, \dots$$

It is straightforward to check that

$$d\tilde{P}_n := g_n(S)\, dP$$

defines a measure $\tilde{P}_n \in \tilde{\mathcal{P}}$ such that

$$\tilde{E}_n[\,(S-K)^+\,] = \left(1 - \frac{K}{n}\right)^+.$$

By sending $n \uparrow \infty$, we see that also the upper bound in (1.17) is attained

$$\pi_{\sup}((S-K)^+) = \pi.$$

Furthermore, the put-call parity (1.10) shows that the universal bounds (1.18) for put options are attained as well. ◇

Exercise 1.3.6. We consider the market model from Exercise 1.1.2 and suppose that $a < \pi(1+r) < b$ so that the model is arbitrage-free. Let C be a derivative that is given by $C = h(S)$, where $h \geq 0$ is a convex function. Show that

$$\pi_{\sup}(C) = \frac{h(b)}{1+r} \cdot \frac{(1+r)\pi - a}{b-a} + \frac{h(a)}{1+r} \cdot \frac{b-(1+r)\pi}{b-a}. \qquad ◇$$

Exercise 1.3.7. In an arbitrage-free market model, we consider a derivative C that is given by $C = h(S^1)$, where $h \geq 0$ is a convex function. Derive the following arbitrage bounds for C:

$$\pi_{\inf}(C) \geq \frac{h(\pi^1(1+r))}{1+r} \quad \text{and} \quad \pi_{\sup}(C) \leq \frac{h(0)}{1+r} + \lim_{x\uparrow\infty} \frac{h(x)}{x}\pi^1. \qquad ◇$$

1.4 Complete market models

Our goal in this section is to characterize the particularly transparent situation in which all contingent claims are attainable.

Definition 1.39. An arbitrage-free market model is called *complete* if every contingent claim is attainable.

The following theorem characterizes the class of all complete market models. It is sometimes called the "second fundamental theorem of asset pricing".

Theorem 1.40. *An arbitrage-free market model is complete if and only if there exists exactly one risk-neutral probability measure, i.e., if* $|\mathcal{P}| = 1$.

Proof. If the model is complete, then the indicator I_A of each set $A \in \mathcal{F}$ is an attainable contingent claim. Hence, Corollary 1.35 implies that $P^*[\,A\,] = E^*[\,I_A\,]$ is independent of $P^* \in \mathcal{P}$. Consequently, there is just one risk-neutral probability measure.

Conversely, suppose that $\mathcal{P} = \{P^*\}$. If C is a contingent claim, then Theorem 1.31 states that the set $\Pi(C)$ of arbitrage-free prices is non-empty and given by

$$\Pi(C) = \left\{ E^*\left[\frac{C}{1+r} \right] \;\Big|\; P^* \in \mathcal{P} \text{ such that } E^*[\,C\,] < \infty \right\}.$$

Since $\mathcal{P}$ has just one element, the same must hold for $\Pi(C)$. Hence, Corollary 1.35 implies that C is attainable. □

We will now show that every complete market model has a finite structure and can be reduced to a *finite* probability space. To this end, observe first that in every market model the following inclusion holds for each $P^* \in \mathcal{P}$:

$$\begin{aligned} \mathcal{V} = \{\overline{\xi} \cdot \overline{S} \mid \overline{\xi} \in \mathbb{R}^{d+1}\} &\subseteq L^1(\Omega, \sigma(S^1, \dots, S^d), P^*) \\ &\subseteq L^0(\Omega, \mathcal{F}, P^*) = L^0(\Omega, \mathcal{F}, P); \end{aligned} \tag{1.19}$$

see Appendix A.7 for the definition of L^p-spaces. If the market is complete then all of these inclusions are in fact equalities. In particular, $\mathcal{F}$ coincides with $\sigma(S^1, \dots, S^d)$ modulo P-null sets, and every contingent claim coincides P-a.s. with a derivative of the traded assets. Since the linear space $\mathcal{V}$ is finite-dimensional, it follows that the same must be true of $L^0(\Omega, \mathcal{F}, P)$. But this means that the model can be reduced to a finite number of relevant scenarios. This observation can be made precise by using the notion of an *atom* of the probability space $(\Omega, \mathcal{F}, P)$. Recall that a set $A \in \mathcal{F}$ is called an atom of $(\Omega, \mathcal{F}, P)$, if $P[\,A\,] > 0$ and if each $B \in \mathcal{F}$ with $B \subseteq A$ satisfies either $P[\,B\,] = 0$ or $P[\,B\,] = P[\,A\,]$.

Proposition 1.41. *For $p \in [0, \infty]$, the dimension of the linear space $L^p(\Omega, \mathcal{F}, P)$ is given by*

$$\begin{aligned} &\dim L^p(\Omega, \mathcal{F}, P) \\ &= \sup\{n \in \mathbb{N} \mid \exists \textit{partition } A^1, \dots, A^n \textit{ of } \Omega \textit{ with } A^i \in \mathcal{F} \textit{ and } P[\,A^i\,] > 0\}. \end{aligned} \tag{1.20}$$

Moreover, $n := \dim L^p(\Omega, \mathcal{F}, P) < \infty$ if and only if there exists a partition of Ω into n atoms of $(\Omega, \mathcal{F}, P)$.

Proof. Suppose that there is a partition $A^1, \dots, A^n$ of Ω such that $A^i \in \mathcal{F}$ and $P[\,A^i\,] > 0$. The corresponding indicator functions $\mathrm{I}_{A^1}, \dots, \mathrm{I}_{A^n}$ can be regarded as linearly independent vectors in $L^p := L^p(\Omega, \mathcal{F}, P)$. Thus $\dim L^p \geq n$. Consequently, it suffices to consider only the case in which the right-hand side of (1.20) is a finite number, n_0. If $A^1, \dots, A^{n_0}$ is a corresponding partition, then each A^i is an atom because otherwise n_0 would not be maximal. Thus, any $Z \in L^p$ is P-a.s. constant on each A^i. If we denote the value of Z on A^i by z^i, then

$$Z = \sum_{i=1}^{n_0} z_i \, \mathrm{I}_{A^i} \qquad P\text{-a.s.}$$

Hence, the indicator functions $\mathrm{I}_{A^1}, \dots, \mathrm{I}_{A^{n_0}}$ form a basis of L^p, and this implies $\dim L^p = n_0$. $\square$

Since for a complete market model the inclusions in (1.19) are in fact equalities, we have

$$\dim L^0(\Omega, \mathcal{F}, P) = \dim \mathcal{V} \le d + 1,$$

with equality when the model is non-redundant. Together with Proposition 1.41, this implies the following result on the structure of complete market models.

Corollary 1.42. *For every complete market model there exists a partition of* Ω *into at most* $d+1$ *atoms of* $(\Omega, \mathcal{F}, P)$.

Example 1.43. Consider the simple situation where the sample space Ω consists of two elements ω^+ and ω^-, and where the measure P is such that

$$p := P[\{\omega^+\}] \in (0, 1).$$

We assume that there is one single risky asset, which takes at time $t = 1$ the two values b and a with the respective probabilities p and $1-p$, where a and b are such that $0 \le a < b$:

$$\begin{tikzcd}[column sep=large] & S(\omega^+) = b \\ \pi \arrow[ur, "p"] \arrow[dr, "1-p"'] & \\ & S(\omega^-) = a \end{tikzcd}$$

This model does not admit arbitrage if and only if

$$\pi(1+r) \in \big\{\tilde{E}[\,S\,] \mid \tilde{P} \approx P\big\} = \big\{\tilde{p}\,b + (1-\tilde{p})a \mid \tilde{p} \in (0,1)\big\} = (a, b); \tag{1.21}$$

see also Example 1.10. In this case, the model is also complete: Any risk-neutral measure P^* must satisfy

$$\pi(1+r) = E^*[\,S\,] = p^* b + (1-p^*)a,$$

and this condition uniquely determines the parameter $p^* = P^*[\{\omega^+\}]$ as

$$p^* = \frac{\pi(1+r) - a}{b - a} \in (0, 1).$$

Hence $|\mathcal{P}| = 1$, and completeness follows from Theorem 1.40. Alternatively, we can directly verify completeness by showing that a given contingent claim C is attainable if (1.21) holds. Observe that the condition

$$C(\omega) = \xi^0\, S^0(\omega) + \xi\, S(\omega) = \xi^0(1+r) + \xi\, S(\omega) \quad \text{for all } \omega \in \Omega$$

is a system of two linear equations for the two real variables ξ^0 and ξ. The solution is given by

$$\xi = \frac{C(\omega^+) - C(\omega^-)}{b-a} \quad \text{and} \quad \xi^0 = \frac{C(\omega^-)b - C(\omega^+)a}{(b-a)(1+r)}.$$

Therefore, the unique arbitrage-free price of C is

$$\pi(C) = \overline{\pi} \cdot \overline{\xi} = \frac{C(\omega^+)}{1+r} \cdot \frac{\pi(1+r)-a}{b-a} + \frac{C(\omega^-)}{1+r} \cdot \frac{b-\pi(1+r)}{b-a}.$$

For a call option $C = (S-K)^+$ with strike $K \in [a,b]$, we have

$$\pi((S-K)^+) = \frac{b-K}{b-a} \cdot \pi - \frac{(b-K)a}{b-a} \cdot \frac{1}{1+r}. \tag{1.22}$$

Note that this price is *independent* of p and *increasing* in r, while the classical discounted expectation with respect to the "objective" measure P,

$$E\left[\frac{C}{1+r}\right] = \frac{p(b-K)}{1+r},$$

is *decreasing* in r and *increasing* in p.

In this example, one can illustrate how options can be used to modify the risk of a position. Consider the particular case in which the risky asset can be bought at time $t = 0$ for the price $\pi = 100$. At time $t = 1$, the price is either $S(\omega^+) = b = 120$ or $S(\omega^-) = a = 90$, both with positive probability. If we invest in the risky asset, the corresponding returns are given by

$$R(S)(\omega^+) = +20\,\% \quad \text{or} \quad R(S)(\omega^-) = -10\,\%.$$

Now consider a call option $C := (S-K)^+$ with strike $K = 100$. Choosing $r = 0$, the price of the call option is

$$\pi(C) = \frac{20}{3} \approx 6.67$$

from formula (1.22). Hence the return

$$R(C) = \frac{(S-K)^+ - \pi(C)}{\pi(C)}$$

on the initial investment $\pi(C)$ equals

$$R(C)(\omega^+) = \frac{20 - \pi(C)}{\pi(C)} = +200\,\%$$

or

$$R(C)(\omega^-) = \frac{0 - \pi(C)}{\pi(C)} = -100\,\%,$$

according to the outcome of the market at time $t = 1$. Here we see a dramatic increase of both profit opportunity and risk; this is sometimes referred to as the *leverage effect* of options.

On the other hand, we could reduce the risk of holding the asset by holding a combination

$$\tilde{C} := (K - S)^+ + S$$

of a put option and the asset itself. This “portfolio insurance” will of course involve an additional cost. If we choose our parameters as above, then the put-call parity (1.10) yields that the price of the put option $(K - S)^+$ is equal to $20/3$. Thus, in order to hold both S and a put, we must invest the capital $100 + 20/3$ at time $t = 0$. At time $t = 1$, we have an outcome of either 120 or of 100 so that the return of $\tilde{C}$ is given by

$$R(\tilde{C})(\omega^+) = +12.5\,\% \quad \text{and} \quad R(\tilde{C})(\omega^-) = -6.25\,\%. \qquad \diamond$$

Exercise 1.4.1. We consider the following three market models.

(A) $\Omega = \{\omega_1, \omega_2\}$ with $r = \frac{1}{9}$ and one risky asset with prices

$$\pi^1 = 5 \quad S^1(\omega_1) = \frac{20}{3}, \quad S^1(\omega_2) = \frac{49}{9}.$$

(B) $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with $r = \frac{1}{9}$ and one risky asset with prices

$$\pi^1 = 5 \quad S^1(\omega_1) = \frac{20}{3}, \quad S^1(\omega_2) = \frac{49}{9}, \quad S^1(\omega_3) = \frac{10}{3}.$$

(C) $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with $r = \frac{1}{9}$ and two risky assets with prices

$$\pi = \begin{pmatrix} 5 \\ 10 \end{pmatrix}, \quad S(\omega_1) = \begin{pmatrix} \frac{20}{3} \\ \frac{40}{3} \end{pmatrix}, \quad S(\omega_2) = \begin{pmatrix} \frac{20}{3} \\ \frac{80}{9} \end{pmatrix}, \quad S(\omega_3) = \begin{pmatrix} \frac{40}{9} \\ \frac{80}{9} \end{pmatrix}.$$

Each of these models is endowed with a probability measure that assigns strictly positive probability to each element of the corresponding sample space Ω.

(a) Which of these models are arbitrage-free? For those that are, describe the set $\mathcal{P}$ of equivalent risk-neutral measures. For those that are not, find an arbitrage opportunity.

(b) Discuss the completeness of those models that are arbitrage-free. For those that are not complete find non-attainable contingent claims. $\diamond$

Exercise 1.4.2. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ be endowed with a probability measure P such that $P[\{\omega_i\}] > 0$ for $i = 1, 2, 3$ and consider the market model with $r = 0$ and one risky asset with prices $\pi^1 = 1$ and $0 < S^1(\omega_1) < S^2(\omega_2) < S^3(\omega_3)$. We suppose that the model is arbitrage-free.

(a) Describe the following objects as subsets of three-dimensional Euclidean space $\mathbb{R}^3$:

  (i) the set $\mathcal{P}$ of equivalent risk neutral measures;

  (ii) the set $\tilde{\mathcal{P}}$ of absolutely continuous risk neutral measures;

  (iii) the set of attainable contingent claims.

(b) Find an example for a non-attainable contingent claim.

(c) Show that the supremum

$$\sup_{\tilde{P} \in \tilde{\mathcal{P}}} \tilde{E}[\,C\,] \tag{1.23}$$

is attained for every contingent claim C.

(d) Let C be a contingent claim. Give a direct and elementary proof of the fact that the map that assigns to each $\tilde{P} \in \tilde{\mathcal{P}}$ the expectation $\tilde{E}[\,C\,]$ is constant if and only if the supremum (1.23) is attained in some element of $\mathcal{P}$. ◇

Exercise 1.4.3. Let $\Omega = \{\omega_1, \dots, \omega_N\}$ be endowed with a probability measure P such that $P[\{\omega_i\}] > 0$ for $i = 1, \dots, N$. On this probability space we consider a market model with interest rate $r = 0$ and with one risky asset whose prices satisfy $\pi^1 = 1$ and

$$0 < S^1(\omega_1) < S^1(\omega_2) < \dots < S^1(\omega_N).$$

Show that there are strikes $K_1, \dots, K_{N-2} > 0$ and prices $\pi^C(K_i)$ such that the corresponding call options $(S^1 - K_i)^+$ complete the market in the following sense: the market model extended by the risky assets with prices

$$\pi^i := \pi^C(K_{i-1}) \quad \text{and} \quad S^i := (S^1 - K_{i-1})^+, \quad i = 2, \dots, N-1,$$

is arbitrage-free and complete. ◇

Exercise 1.4.4. Let $\Omega = \{\omega_1, \dots, \omega_{N+1}\}$ be endowed with a probability measure P such that $P[\{\omega_i\}] > 0$ for $i = 1, \dots, N+1$.

(a) On this probability space we consider a non-redundant and arbitrage-free market model with d risky assets and prices $\overline{\pi} \in \mathbb{R}^{d+1}$ and $\overline{S}$, where $d < N$. Show that this market model can be extended by additional assets with prices $\pi^{d+1}, \dots, \pi^N$ and $S^{d+1}, \dots, S^N$ in such a way that the extended market model is arbitrage-free and complete.

(b) Let specifically $N = 2$, $d = 1$, $\pi^1 = 2$, and

$$S^1(\omega_i) = \begin{cases} 1 & \text{for } i = 1, \\ 2 & \text{for } i = 2, \\ 3 & \text{for } i = 3. \end{cases}$$

We suppose furthermore that the risk-free interest rate r is chosen such that the model is arbitrage-free. Find a non-attainable contingent claim. Then find an extended model that is arbitrage-free and complete. Finally determine the unique equivalent risk-neutral measure P^* in the extended model. ◇

1.5 Geometric characterization of arbitrage-free models

The "fundamental theorem of asset pricing" in the form of Theorem 1.7 states that a market model is arbitrage-free if and only if the origin is contained in the set

$$M_b(Y, P) := \Big\{ E_Q[\,Y\,] \;\Big|\; Q \approx P,\ \frac{dQ}{dP} \text{ is bounded, } E_Q[\,|Y|\,] < \infty \Big\} \subset \mathbb{R}^d,$$

where $Y = (Y^1, \dots, Y^d)$ is the random vector of discounted net gains defined in (1.2). The aim of this section is to give a geometric description of the set $M_b(Y, P)$ as well as of the larger set

$$M(Y, P) := \big\{ E_Q[\,Y\,] \mid Q \approx P,\ E_Q[\,|Y|\,] < \infty \big\}.$$

To this end, it will be convenient to work with the distribution

$$\mu := P \circ Y^{-1}$$

of Y with respect to P. That is, μ is a Borel probability measure on $\mathbb{R}^d$ such that

$$\mu(A) = P[\,Y \in A\,] \quad \text{for each Borel set } A \subset \mathbb{R}^d.$$

If ν is a Borel probability measure on $\mathbb{R}^d$ such that $\int |y|\,\nu(dy) < \infty$, we will call $\int y\,\nu(dy)$ its *barycenter*.

Lemma 1.44. *We have*

$$M_b(Y, P) = M_b(\mu) := \Big\{ \int y\,\nu(dy) \;\Big|\; \nu \approx \mu,\ \frac{d\nu}{d\mu} \textit{ is bounded, } \int |y|\,\nu(dy) < \infty \Big\},$$

and

$$M(Y, P) = M(\mu) := \Big\{ \int y\,\nu(dy) \;\Big|\; \nu \approx \mu,\ \int |y|\,\nu(dy) < \infty \Big\}.$$

Proof. If $\nu \approx \mu$ is a Borel probability measure on $\mathbb{R}^d$, then the Radon–Nikodym derivative of ν with respect to μ evaluated at the random variable Y defines a probability measure $Q \approx P$ on $(\Omega, \mathcal{F})$:

$$\frac{dQ}{dP}(\omega) := \frac{d\nu}{d\mu}(Y(\omega)).$$

Clearly, $E_Q[Y] = \int y\,\nu(dy)$. This shows that $\mathcal{M}(\mu) \subseteq \mathcal{M}(Y, P)$ and $\mathcal{M}_b(\mu) \subseteq \mathcal{M}_b(Y, P)$.

Conversely, if $\tilde{Q}$ is a given probability measure on $(\Omega, \mathcal{F})$ which is equivalent to P, then the Radon–Nikodym theorem in Appendix A.2 shows that the distribution $\tilde{\nu} := \tilde{Q} \circ Y^{-1}$ must be equivalent to μ, whence $\mathcal{M}(Y, P) \subseteq \mathcal{M}(\mu)$. Moreover, it follows from Proposition A.11 that the density $d\tilde{\nu}/d\mu$ is bounded if $d\tilde{Q}/dP$ is bounded, and so $\mathcal{M}_b(Y, P) \subseteq \mathcal{M}_b(\mu)$ also follows. □

By the above lemma, the characterization of the two sets $\mathcal{M}_b(Y, P)$ and $\mathcal{M}(Y, P)$ is reduced to a problem for Borel probability measures on $\mathbb{R}^d$. Here and in the sequel, we do not need the fact that μ is the distribution of the lower bounded random vector Y of discounted net gains; our results are true for arbitrary μ such that $\int |y|\,\mu(dy) < \infty$; see also Remark 1.9.

Definition 1.45. The *support* of a Borel probability measure ν on $\mathbb{R}^d$ is the smallest closed set $A \subset \mathbb{R}^d$ such that $\nu(A^c) = 0$, and it will be denoted by $\operatorname{supp} \nu$.

The support of a measure ν can be obtained as the intersection of all closed sets A with $\nu(A^c) = 0$, i.e.,

$$\operatorname{supp} \nu = \bigcap_{\substack{A \text{ closed} \\ \nu(A^c)=0}} A.$$

We denote by

$$\begin{aligned}\Gamma(\mu) &:= \operatorname{conv}(\operatorname{supp} \mu) \\ &= \left\{ \sum_{k=1}^n \alpha_k y_k \;\middle|\; \alpha_k \geq 0,\ \sum_{k=1}^n \alpha_k = 1,\ y_k \in \operatorname{supp} \mu,\ n \in \mathbb{N} \right\}\end{aligned}$$

the *convex hull of the support of* μ. Thus, $\Gamma(\mu)$ is the smallest convex set which contains $\operatorname{supp} \mu$; see also Appendix A.1.

Example 1.46. Take $d = 1$, and consider the measure

$$\mu = \frac{1}{2}(\delta_{-1} + \delta_{+1}).$$

Clearly, the support of μ is equal to $\{-1, +1\}$ and so $\Gamma(\mu) = [-1, +1]$. A measure ν is equivalent to μ if and only if

$$\nu = \alpha\delta_{-1} + (1-\alpha)\delta_{+1}$$

for some $\alpha \in (-1, +1)$. Hence, $M_b(\mu) = M(\mu) = (-1, +1)$. ◇

The previous example gives the correct intuition, namely that one always has the inclusions

$$M_b(\mu) \subset M(\mu) \subset \Gamma(\mu).$$

But while the first inclusion will turn out to be an identity, the second inclusion is usually strict. Characterizing $M(\mu)$ in terms of $\Gamma(\mu)$ will involve the following concept.

Definition 1.47. The *relative interior* of a convex set $C \subset \mathbb{R}^d$ is the set of all points $x \in C$ such that for all $y \in C$ there exists some $\varepsilon > 0$ with

$$x - \varepsilon(y - x) \in C.$$

The relative interior of C is denoted $\operatorname{ri} C$.

If the convex set C has non-empty topological interior $\operatorname{int} C$, then $\operatorname{ri} C = \operatorname{int} C$, and the elementary properties of the relative interior collected in the following remarks become obvious. This applies in particular to the set $\Gamma(\mu)$ if the non-redundance condition (1.8) is satisfied. For the general case, proofs of these statements can be found, for instance, in § 6 of [221].

Remark 1.48. Let C be a non-empty convex subset of $\mathbb{R}^d$, and consider the *affine hull* $\operatorname{aff} C$ spanned by C, i.e., the smallest affine set which contains C. If we identify $\operatorname{aff} C$ with some $\mathbb{R}^n$, then the relative interior of C is equal to the topological interior of C, considered as a subset of $\operatorname{aff} C \cong \mathbb{R}^n$. In particular, each non-empty convex set has non-empty relative interior. ◇

Exercise 1.5.1. Let C be a non-empty convex subset of $\mathbb{R}^d$ and denote by $\overline{C}$ its closure. Show that for $x \in \operatorname{ri} C$,

$$\alpha x + (1-\alpha)y \in \operatorname{ri} C \quad \text{for all } y \in \overline{C} \text{ and all } \alpha \in (0, 1]. \tag{1.24}$$

In particular, $\operatorname{ri} C$ is convex. Moreover, show that the operations of taking the closure or the relative interior of a convex set C are consistent with each other

$$\operatorname{ri} \overline{C} = \operatorname{ri} C \quad \text{and} \quad \overline{\operatorname{ri} C} = \overline{C}. \tag{1.25}$$

◇

After these preparations, we can now state the announced geometric characterization of the set $M_b(\mu)$. Note that the proof of this characterization relies on the "fundamental theorem of asset pricing" in the form of Theorem 1.7.

Theorem 1.49. *The set of all barycenters of probability measures $\nu \approx \mu$ coincides with the relative interior of the convex hull of the support of μ. More precisely,*

$$M_b(\mu) = M(\mu) = \operatorname{ri}\Gamma(\mu).$$

Proof. In a first step, we show the inclusion $\operatorname{ri}\Gamma(\mu) \subseteq M_b(\mu)$. Suppose we are given $m \in \operatorname{ri}\Gamma(\mu)$. Let $\tilde{\mu}$ denote the translated measure

$$\tilde{\mu}(A) := \mu(A+m) \quad \text{for Borel sets } A \subset \mathbb{R}^d$$

where $A + m := \{x + m \mid x \in A\}$. Then $M_b(\tilde{\mu}) = M_b(\mu) - m$, and analogous identities hold for $M(\tilde{\mu})$ and $\Gamma(\tilde{\mu})$. It follows that there is no loss of generality in assuming that $m = 0$, i.e., we must show that $0 \in M_b(\mu)$ if $0 \in \operatorname{ri}\Gamma(\mu)$.

We claim that $0 \in \operatorname{ri}\Gamma(\mu)$ implies the following "no-arbitrage" condition:

$$\text{If } \xi \in \mathbb{R}^d \text{ is such that } \xi \cdot y \geq 0 \text{ for } \mu\text{-a.e. } y, \text{ then } \xi \cdot y = 0 \text{ for } \mu\text{-a.e. } y. \tag{1.26}$$

If (1.26) is false, then we can find some $\xi \in \mathbb{R}^d$ such that $\xi \cdot y \geq 0$ for μ-a.e. y but $\mu(\{y \mid \xi \cdot y > \delta\}) > 0$ for some $\delta > 0$. In this case, the support of μ is contained in the closed set $\{y \mid \xi \cdot y \geq 0\}$ but not in the hyperplane $\{y \mid \xi \cdot y = 0\}$. We conclude that $\xi \cdot y \geq 0$ for all $y \in \operatorname{supp}\mu$ and that there exists at least one $y^* \in \operatorname{supp}\mu$ such that $\xi \cdot y^* > 0$. In particular, $y^* \in \Gamma(\mu)$ so that our assumption $m = 0 \in \operatorname{ri}\Gamma(\mu)$ implies the existence of some $\varepsilon > 0$ such that $-\varepsilon y^* \in \Gamma(\mu)$. Consequently, $-\varepsilon y^*$ can be represented as a convex combination

$$-\varepsilon y^* = \alpha_1 y_1 + \cdots + \alpha_n y_n$$

of certain $y_1, \ldots, y_n \in \operatorname{supp}\mu$. It follows that

$$0 > -\varepsilon\xi \cdot y^* = \alpha_1 \xi \cdot y_1 + \cdots + \alpha_n \xi \cdot y_n,$$

in contradiction to our assumption that $\xi \cdot y \geq 0$ for all $y \in \operatorname{supp}\mu$. Hence, (1.26) must be true.

Applying the "fundamental theorem of asset pricing" in the form of Theorem 1.7 to $\Omega := \mathbb{R}^d$, $P := \mu$, and to the random variable $Y(y) := y$, yields a probability measure $\mu^* \approx \mu$ whose density $d\mu^*/d\mu$ is bounded and which satisfies $\int |y|\, \mu^*(dy) < \infty$ and $\int y\, \mu^*(dy) = 0$. This proves the inclusion $\operatorname{ri}\Gamma(\mu) \subseteq M_b(\mu)$.

Clearly, $M_b(\mu) \subset M(\mu)$. So the theorem will be proved if we can show the inclusion $M(\mu) \subset \operatorname{ri}\Gamma(\mu)$. To this end, suppose by way of contradiction that $\nu \approx \mu$ is such that

$$\int |y|\, \nu(dy) < \infty \quad \text{and} \quad m := \int y\, \nu(dy) \notin \operatorname{ri}\Gamma(\mu).$$

Again, we may assume without loss of generality that $m = 0$. Applying the separating hyperplane theorem in the form of Proposition A.1 with $\mathcal{C} := \operatorname{ri}\Gamma(\mu)$ yields some $\xi \in \mathbb{R}^d$ such that $\xi \cdot y \geq 0$ for all $y \in \operatorname{ri}\Gamma(\mu)$ and $\xi \cdot y^* > 0$ for at least one $y^* \in \operatorname{ri}\Gamma(\mu)$. We deduce from (1.24) that $\xi \cdot y \geq 0$ holds also for all $y \in \Gamma(\mu)$. Moreover, $\xi \cdot y_0$ must be strictly positive for at least one $y_0 \in \operatorname{supp}\mu$. Hence,

$$\xi \cdot y \geq 0 \text{ for } \mu\text{-a.e. } y \in \mathbb{R}^d \quad \text{and} \quad \mu(\{y \mid \xi \cdot y > 0\}) > 0. \tag{1.27}$$

By the equivalence of μ and ν, (1.27) is also true for ν instead of μ, and so

$$\xi \cdot m = \xi \cdot \int y\, \nu(dy) = \int \xi \cdot y\, \nu(dy) > 0,$$

in contradiction to our assumption that $m = 0$. We conclude that $M(\mu) \subset \operatorname{ri}\Gamma(\mu)$. □

Remark 1.50. Note that Theorem 1.49 does *not* extend to the set

$$\tilde{M}(\mu) := \left\{ \int y\, \nu(dy) \;\middle|\; \nu \ll \mu \text{ and } \int |y|\, \nu(dy) < \infty \right\}.$$

Already the simple case $\mu := \frac{1}{2}(\delta_{-1} + \delta_{+1})$ serves as a counterexample, because here $\tilde{M}(\mu) = [-1, +1]$ while $\operatorname{ri}\Gamma(\mu) = (-1, +1)$. In this case, we have an identity between $\tilde{M}(\mu)$ and $\Gamma(\mu)$. However, also this identity fails in general as can be seen by considering the normalized Lebesgue measure λ on $[-1, +1]$. For this choice one finds $\tilde{M}(\lambda) = (-1, +1)$ but $\Gamma(\lambda) = [-1, +1]$. ◇

From Theorem 1.49 we obtain the following geometric characterization of the absence of arbitrage.

Corollary 1.51. *Let μ be the distribution of the discounted price vector $S/(1 + r)$ of the risky assets. Then the market model is arbitrage-free if and only if the price system π belongs to the relative interior* $\operatorname{ri}\Gamma(\mu)$ *of the convex hull of the support of μ.*

1.6 Contingent initial data

The idea of hedging contingent claims develops its full power only in a dynamic setting in which trading may occur at several times. The corresponding discrete-time theory is presented in Chapter 5. The introduction of additional trading periods requires more sophisticated techniques than those we have used so far. In this section we will introduce some of these techniques in an extended version of our previous market model in which initial prices, and hence strategies, are contingent on scenarios. In this context, we are going to characterize the absence of arbitrage strategies. The results will be used as building blocks in the multiperiod setting of Part II; their study can be postponed until Chapter 5.

Suppose that we are given a σ-algebra $\mathcal{F}_0 \subset \mathcal{F}$ which specifies the information that is available to an investor at time $t = 0$. The prices for our $d + 1$ assets at time 0 will be modelled as non-negative $\mathcal{F}_0$-measurable random variables $S_0^0, S_0^1, \dots, S_0^d$. Thus, the price system $\overline{\pi} = (\pi^0, \pi^1, \dots, \pi^d)$ of our previous discussion is replaced by the vector

$$\overline{S}_0 = (S_0^0, \dots, S_0^d).$$

The portfolio $\overline{\xi}$ chosen by an investor at time $t = 0$ will also depend on the information available at time 0. Thus, we assume that

$$\overline{\xi} = (\xi^0, \xi^1, \dots, \xi^d)$$

is an $\mathcal{F}_0$-measurable random vector. The asset prices observed at time $t = 1$ will be denoted by

$$\overline{S}_1 = (S_1^0, S_1^1, \dots, S_1^d).$$

They are modelled as non-negative random variables which are measurable with respect to a σ-algebra $\mathcal{F}_1$ such that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}$. The σ-algebra $\mathcal{F}_1$ describes the information available at time 1, and in this section we can assume that $\mathcal{F} = \mathcal{F}_1$.

A riskless bond could be included by taking $S_0^0 \equiv 1$ and by assuming S_1^0 to be $\mathcal{F}_0$-measurable and P-a.s. strictly positive. However, in the sequel it will be sufficient to assume that S_0^0 is $\mathcal{F}_0$-measurable, S_1^0 is $\mathcal{F}_1$-measurable, and that

$$P[\, S_0^0 > 0 \text{ and } S_1^0 > 0\,] = 1. \tag{1.28}$$

Thus, we can take the 0th asset as *numéraire*, and we denote by

$$X_t^i := \frac{S_t^i}{S_t^0}, \quad i = 1, \dots, d,\ t = 0, 1,$$

the discounted asset prices and by

$$Y = X_1 - X_0$$

the vector of the discounted net gains.

Definition 1.52. An *arbitrage opportunity* is a portfolio $\overline{\xi}$ such that $\overline{\xi} \cdot \overline{S}_0 \le 0$, $\overline{\xi} \cdot \overline{S}_1 \ge 0$ P-a.s., and $P[\,\overline{\xi} \cdot \overline{S}_1 > 0\,] > 0$.

By our assumption (1.28), any arbitrage opportunity $\overline{\xi} = (\xi^0, \xi)$ satisfies

$$\xi \cdot Y \ge 0 \ P\text{-a.s.} \quad \text{and} \quad P[\, \xi \cdot Y > 0\,] > 0. \tag{1.29}$$

In fact, the existence of a d-dimensional $\mathcal{F}_0$-measurable random vector ξ with (1.29) is equivalent to the existence of an arbitrage opportunity. This can be seen as in Lemma 1.4.

The space of discounted net gains which can be generated by some portfolio is given by

$$\mathcal{K} := \{\xi \cdot Y \mid \xi \in L^0(\Omega, \mathcal{F}_0, P; \mathbb{R}^d)\}.$$

Here, $L^0(\Omega, \mathcal{F}_0, P; \mathbb{R}^d)$ denotes the space of $\mathbb{R}^d$-valued random variables which are P-a.s. finite and $\mathcal{F}_0$-measurable modulo the equivalence relation (A.23) of coincidence up to P-null sets. The spaces $L^p(\Omega, \mathcal{F}_0, P; \mathbb{R}^d)$ for $p > 0$ are defined in the same manner. We denote by $L^p_+ := L^p_+(\Omega, \mathcal{F}_1, P)$ the cone of all non-negative elements in the space $L^p := L^p(\Omega, \mathcal{F}_1, P)$. With this notation, the absence of arbitrage opportunities is equivalent to the condition

$$\mathcal{K} \cap L^0_+ = \{0\}.$$

We will denote by

$$\mathcal{K} - L^0_+$$

the convex cone of all $Z \in L^0$ which can be written as the difference of some $\xi \cdot Y \in \mathcal{K}$ and some $U \in L^0_+$.

The following definition involves the notion of the *conditional expectation*

$$E_Q[\, Z \mid \mathcal{F}_0\,]$$

of a random variable Z with respect to a probability measure Q, given the σ-algebra $\mathcal{F}_0 \subset \mathcal{F}$; see Appendix A.2. If $Z = (Z^1, \dots, Z^n)$ is a random vector, then $E_Q[\, Z \mid \mathcal{F}_0\,]$ is shorthand for the random vector with components $E_Q[\, Z^i \mid \mathcal{F}_0\,]$, $i = 1, \dots, n$.

Definition 1.53. A probability measure Q satisfying

$$E_Q[\, X_t^i\,] < \infty \qquad \text{for } i = 1, \dots, d \text{ and } t = 0, 1$$

and

$$X_0 = E_Q[\, X_1 \mid \mathcal{F}_0\,] \quad Q\text{-a.s.}$$

is called a *risk-neutral measure* or *martingale measure*. We denote by $\mathcal{P}$ the set of all risk-neutral measures P^* which are equivalent to P.

Remark 1.54. The definition of a martingale measure Q means that for each asset $i = 0, \dots, d$, the discounted price process $(X_t^i)_{t=0,1}$ is a *martingale* under Q with respect to the σ-fields $(\mathcal{F}_t)_{t=0,1}$. The systematic discussion of martingales in a multiperiod setting will begin in Section 5.2. The martingale aspect will be crucial for the theory of dynamic hedging in Part II. ◇

As the main result of this section, we can now state an extension of the "fundamental theorem of asset pricing" in Theorem 1.7 to our present setting. In the context of Section 1.2, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$, the following arguments simplify considerably, and they yield an alternative proof of Theorem 1.7, in which the separation argument in $\mathbb{R}^d$ is replaced by a separation argument in L^1.

Theorem 1.55. *The following conditions are equivalent:*

(a) $\mathcal{K} \cap L^0_+ = \{0\}$.

(b) $(\mathcal{K} - L^0_+) \cap L^0_+ = \{0\}$.

(c) *There exists a measure $P^* \in \mathcal{P}$ with a bounded density dP^*/dP.*

(d) $\mathcal{P} \neq \emptyset$.

Proof. (d) $\Rightarrow$ (a): Suppose by way of contradiction that there exist both a $P^* \in \mathcal{P}$ and some $\xi \in L^0(\Omega, \mathcal{F}_0, P; \mathbb{R}^d)$ with non-zero payoff $\xi \cdot Y \in \mathcal{K} \cap L^0_+$. For large enough $c > 0$, $\xi^{(c)} := \mathrm{I}_{\{|\xi| \le c\}}\xi$ will be bounded, and the payoff $\xi^{(c)} \cdot Y$ will still be non-zero and in $\mathcal{K} \cap L^0_+$. However,

$$E^*[\, \xi^{(c)} \cdot Y\,] = E^*[\, \xi^{(c)} \cdot E^*[\, Y \mid \mathcal{F}_0\,]] = 0,$$

which is the desired contradiction.

(a) $\Leftrightarrow$ (b): It is obvious that (a) is necessary for (b). In order to prove sufficiency, suppose that we are given some $Z \in (\mathcal{K} - L^0_+) \cap L^0_+$. Then there exists a random variable $U \ge 0$ and a random vector $\xi \in L^0(\Omega, \mathcal{F}_0, P; \mathbb{R}^d)$ such that

$$0 \le Z = \xi \cdot Y - U.$$

This implies that $\xi \cdot Y \ge U \ge 0$, which, according to condition (a), can only happen if $\xi \cdot Y = 0$. Hence, also $U = 0$ and in turn $Z = 0$.

(b) $\Rightarrow$ (c): This is the difficult part of the proof. The assertion will follow by combining Lemmas 1.57, 1.58, 1.60, and 1.68. □

Remark 1.56. If Ω is discrete, or if there exists a decomposition of Ω in countable many atoms of $(\Omega, \mathcal{F}_0, P)$, then the martingale measure P^* can be constructed by applying the result of Theorem 1.7 separately on each atom. In the general case, the idea of patching together conditional martingale measures would involve subtle arguments of measurable selection; see [67]. Here we present a different approach which is based on separation arguments in $L^1(P)$. It is essentially due to W. Schachermayer [233]; our version uses in addition arguments by Y. Kabanov and C. Stricker [164]. ◇

We start with the following simple lemma, which takes care of the integrability condition in Definition 1.53.

Lemma 1.57. *For the proof of the implication* (b) $\Rightarrow$ (c) *in Theorem* 1.55*, we may assume without loss of generality that*

$$E[|X_t|] < \infty \quad \textit{for } t = 0, 1. \tag{1.30}$$

Proof. Define a probability measure $\tilde{P}$ by

$$\frac{d\tilde{P}}{dP} := c(1 + |X_0| + |X_1|)^{-1}$$

where c is chosen such that the right-hand side integrates to 1. Clearly, (1.30) holds for $\tilde{P}$. Moreover, condition (b) of Theorem 1.55 is satisfied by P if and only if it is satisfied by the equivalent measure $\tilde{P}$. If $P^* \in \mathcal{P}$ is such that the density $dP^*/d\tilde{P}$ is bounded, then so is the density

$$\frac{dP^*}{dP} = \frac{dP^*}{d\tilde{P}} \cdot \frac{d\tilde{P}}{dP}.$$

Therefore, the implication (b) $\Rightarrow$ (c) holds for P if and only if it holds for $\tilde{P}$. □

From now on, we will always assume (1.30). Our goal is to construct a suitable $Z \in L^\infty$ such that

$$\frac{dP^*}{dP} := \frac{Z}{E[Z]}$$

defines an equivalent risk-neutral measure P^*. The following simple lemma gives a criterion for this purpose, involving the convex cone

$$\mathcal{C} := (\mathcal{K} - L^0_+) \cap L^1.$$

Lemma 1.58. *Suppose $c \geq 0$ and $Z \in L^\infty$ are such that*

$$E[ZW] \leq c \quad \textit{for all } W \in \mathcal{C}.$$

Then:

(a) $E[ZW] \leq 0$ *for all $W \in \mathcal{C}$, i.e., we can take $c = 0$.*

(b) $Z \geq 0$ *P-a.s.*

(c) *If Z does not vanish P-a.s., then*

$$\frac{dQ}{dP} := \frac{Z}{E[Z]}$$

defines a risk-neutral measure $Q \ll P$.

Proof. (a): Note that $\mathcal{C}$ is a cone, i.e., $W \in \mathcal{C}$ implies that $\alpha W \in \mathcal{C}$ for all $\alpha \geq 0$. This property excludes the possibility that $E[\,ZW\,] > 0$ for some $W \in \mathcal{C}$.

(b): $\mathcal{C}$ contains the function $W := -\mathrm{I}_{\{Z<0\}}$. Hence, by part (a),

$$E[\,Z^-\,] = E[\,Z\,W\,] \leq 0.$$

(c): For all $\xi \in L^\infty(\Omega, \mathcal{F}_0, P; \mathbb{R}^d)$ and $\alpha \in \mathbb{R}$ we have $\alpha\xi \cdot Y \in \mathcal{C}$ by our integrability assumption (1.30). Thus, a similar argument as in the proof of (a) yields $E[\,Z\,\xi \cdot Y\,] = 0$. Since ξ is bounded, we may conclude that

$$0 = E[\,Z\,\xi \cdot Y\,] = E[\,\xi \cdot E[\,ZY \mid \mathcal{F}_0\,]].$$

As ξ is arbitrary, this yields $E[\,ZY \mid \mathcal{F}_0\,] = 0$ P-almost surely. Proposition A.12 now implies

$$E_Q[\,Y \mid \mathcal{F}_0\,] = \frac{1}{E[\,Z \mid \mathcal{F}_0\,]} E[\,ZY \mid \mathcal{F}_0\,] = 0 \quad Q\text{-a.s.},$$

which concludes the proof. □

In view of the preceding lemma, the construction of risk-neutral measures $Q \ll P$ with bounded density is reduced to the construction of elements of the set

$$\mathcal{Z} := \{Z \in L^\infty \mid 0 \leq Z \leq 1,\ P[\,Z > 0\,] > 0, \text{ and } E[\,Z\,W\,] \leq 0 \text{ for all } W \in \mathcal{C}\,\}.$$

In the following lemma, we will construct such elements by applying a separation argument suggested by the condition

$$\mathcal{C} \cap L^1_+ = \{0\},$$

which follows from condition (b) of Theorem 1.55. This separation argument needs the additional assumption that $\mathcal{C}$ is closed in L^1. Showing that this assumption is indeed satisfied in our situation will be one of the key steps in our proof; see Lemma 1.68 below.

Lemma 1.59. *Assume that $\mathcal{C}$ is closed in L^1 and satisfies $\mathcal{C} \cap L^1_+ = \{0\}$. Then for each non-zero $F \in L^1_+$ there exists some $Z \in \mathcal{Z}$ such that $E[\,FZ\,] > 0$.*

Proof. Let $\mathcal{B} := \{F\}$ so that $\mathcal{B} \cap \mathcal{C} = \emptyset$. Since the set $\mathcal{C}$ is non-empty, convex and closed in L^1, we may apply the Hahn–Banach separation theorem in the form of Theorem A.57 to obtain a continuous linear functional ℓ on L^1 such that

$$\sup_{W \in \mathcal{C}} \ell(W) < \ell(F).$$

Since the dual space of L^1 can be identified with L^∞, there exists some $Z \in L^\infty$ such that $\ell(F) = E[\,FZ\,]$ for all $F \in L^1$. We may assume without loss of generality that $\|Z\|_\infty \leq 1$. By construction, Z satisfies the assumptions of Lemma 1.58, and so $Z \in \mathcal{Z}$. Moreover, $E[\,FZ\,] = \ell(F) > 0$ since the constant function $W \equiv 0$ is contained in $\mathcal{C}$. □

We will now use an exhaustion argument to conclude that $\mathcal{Z}$ contains a strictly positive element Z^* under the assumptions of Lemma 1.59. After normalization, Z^* will serve as the density of our desired risk-neutral measure $P^* \in \mathcal{P}$.

Lemma 1.60. *Under the assumptions of Lemma* 1.59*, there exists* $Z^* \in \mathcal{Z}$ *with* $Z^* > 0$ *P-a.s.*

Proof. As a first step, we claim that $\mathcal{Z}$ is *countably convex*: If $(\alpha_k)_{k\in\mathbb{N}}$ is a sequence of non-negative real numbers summing up to 1, and if $Z^{(k)} \in \mathcal{Z}$ for all k, then

$$Z := \sum_{k=1}^{\infty} \alpha_k Z^{(k)} \in \mathcal{Z}.$$

Indeed, for $W \in \mathcal{C}$

$$\sum_{k=1}^{\infty} |\alpha_k Z^{(k)} W| \leq |W| \in L^1,$$

and so Lebesgue's dominated convergence theorem implies that

$$E[\,ZW\,] = \sum_{k=1}^{\infty} \alpha_k E[\,Z^{(k)} W\,] \leq 0.$$

For the second step, let

$$c := \sup\{P[\,Z > 0\,] \mid Z \in \mathcal{Z}\}.$$

We choose $Z^{(n)} \in \mathcal{Z}$ such that $P[\,Z^{(n)} > 0\,] \to c$. Then

$$Z^* := \sum_{n=1}^{\infty} 2^{-n} Z^{(n)} \in \mathcal{Z}$$

by step one, and

$$\{Z^* > 0\} = \bigcup_{n=1}^{\infty} \{Z^{(n)} > 0\}.$$

Hence $P[Z^* > 0] = c$.

In the final step, we show that $c = 1$. Then Z^* will be as desired. Suppose by way of contradiction that $P[Z^* = 0] > 0$, so that $W := \mathrm{I}_{\{Z^*=0\}}$ is a non-zero element of L^1_+. Lemma 1.59 yields $Z \in \mathcal{Z}$ with $E[\,WZ\,] > 0$. Hence,

$$P[\,\{Z > 0\} \cap \{Z^* = 0\}\,] > 0,$$

and so

$$P\left[\,\frac{1}{2}(Z + Z^*) > 0\,\right] > P[\,Z^* > 0\,] = c,$$

in contradiction to the maximality of $P[Z^* > 0]$. □