

Infinitesimal Differences

Infinitesimal Differences

Controversies between Leibniz
and his Contemporaries

Edited by
Ursula Goldenbaum and Douglas Jesseph



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Introduction

This volume had its beginnings in a conference entitled *The Metaphysical and Mathematical Discussion of the Status of Infinitesimals in Leibniz's Time* held in April 2006 at Emory University in celebration of the 50th anniversary of the university's graduate program in philosophy. Leroy E. Loemker, who initiated the graduate program as the first chair of the new Department of Philosophy at Emory, is well known as the father of North American Leibniz scholarship. Thus, the conference was dedicated as well to Loemker and his outstanding work on Leibniz, most notably his volume of Leibniz's *Philosophical Papers and Letters*, which remains one of the central works of Leibnizian scholarship in English.

The conference centered on a topic of interest for many scholars in philosophy as well as the history of mathematics, and it gave rise to many lively and interesting discussions about the nature and status of infinitesimals. Participants also had much to say about the notion of fiction, and especially the concept of a "well-founded fiction" in Leibniz's system. However, as we can now see on the basis of the largely revised papers, this conference also initiated a new effort to work out a clearer and more comprehensive understanding of these questions, some focusing particularly on *methodological* approaches to the infinitesimals in mathematics, physics and metaphysics. As a result, this volume offers a tightly focused collection of papers that address the metaphysical, physical, and mathematical treatment of infinitely small magnitudes in Leibniz's thought and that of his contemporaries, whether in the foundations of the *calculus differentialis*, the physics of forces, the theory of continuity, or the metaphysics of motion.

Although the central focus of the volume is on the development of Leibniz's calculus, the contributions provide a consistent and comprehensive overview of seventeenth and early eighteenth century discussions of the infinitesimal. In addition to addressing the role of infinitesimals in Leibniz's thought, contributors also consider the approaches of his predecessors, contemporaries, and immediate successors as Bonaventura Cavalieri, Evangelista Torricelli, Gilles Personne de Roberval, Thomas Hobbes, John Wallis, Isaac Newton, Blaise Pascal, Christiaan Huygens, Johann Bernoulli, Guillaume de L'Hôpital, Jacob Hermann, and Bernard Nieuwentijt.

The resulting collection therefore offers insight into the origins of Leibniz's conception of the infinite (and particularly the infinitely small), as well as the role this conception plays in different aspects of his mature thought on mathematics, physics, and metaphysics.

Leibniz mastered the mathematics of his day and developed his own calculus over the short span of a few years. But despite the success of his calculus in solving outstanding mathematical problems, the apparent ambiguity of Leibniz's conception of infinitesimals as fictions led to controversy at the end of the 17th century. Although urged to explain his approach more explicitly, Leibniz was generally reluctant to present the foundations of his new method. Moreover, he had offered very different accounts of the infinitesimal to different correspondents, further complicating a univocal understanding of his approach to the calculus. Even without an explicit statement of foundations, however, it is clear that Leibniz's mature view never characterized infinitesimals as real quantities, although he considered the prospects of a "realist" approach to infinitesimals in his earlier years. Although the calculus was undoubtedly successful in mathematical practice, it remained disputed precisely because its procedures seemed to lack an adequate metaphysical or methodological justification. In addition, Leibniz freely employed the language of infinitesimal quantities in the foundations of his dynamics and theory of forces, so that disputes over the very nature of infinitesimals naturally implicate the foundations of the Leibnizian science of bodies.

The fourteen essays collected here enhance and develop current scholarly understanding of the different conceptual and metaphysical issues raised by the mathematics of infinitesimals. Some essays are concerned principally with the historical origins of the mathematics of the infinitesimal, while others focus on the theoretical foundation of the calculus or on Leibniz's mature "fictionalism" about the infinite. In addition, a number of contributors seek to clarify the physics of forces Leibniz expressed in the language of the calculus. *Richard Arthur's* paper compares the Leibnizian doctrine of the infinitesimal and Newton's method of prime and ultimate ratios. He argues that these two approaches are not nearly as different as has commonly been supposed, and that both are motivated by surprisingly similar concerns about the rigorous development of a theory of continuously varying quantities. *Philip Beeley's* essay discusses John Wallis' motives for reforming Cavalieri's geometry of continua, known as indivisibles. These had already been transformed into infinitely small entities through authors such as Torricelli, Roberval and Pascal. Beeley argues that Wallis sought for the first time to combine the concepts of "infinitesimals" and

arithmetical limits, when coming up with their arithmetization. Beeley also gives an account of some of the debates which ensued with the likes of Hobbes and Fermat.

Ursula Goldenbaum argues on the basis of newly discovered marginalia of Leibniz in Hobbes' *Opera philosophica* (1668) that Leibniz embraced Hobbes' *conatus* while reading *De homine* in the end of 1669. Leibniz's great expectation toward Hobbes' theory of sensation, due to his own projected philosophy of mind, spurred him to study the conatus conception of Hobbes in *De corpore* and consequently the mathematics of indivisibles. *Siegmund Probst* presents and analyzes newly discovered material concerning Leibniz's use of indivisibles and infinitesimals in his earliest mathematical writings, shedding some light on his unknown mathematical studies of Hobbes. In particular, he draws attention to mathematical manuscripts of Leibniz that "illustrate how Leibniz operated with concepts such as indivisibles and infinitesimals," in the early 1670s.

Samuel Levey's paper analyzes the reasons for Leibniz's ultimate abandoning of his earlier commitment to actual infinitesimals in 1676. He then takes up the question of how Leibniz's fictionalism about infinitesimals should be understood, concluding that there is no single "fictionalist" treatment to which Leibniz was invariably committed, although they all can be styled "Archimedean" in their reliance on classical exhaustion techniques. *O. Bradley Bassler* distinguishes Leibniz's metaphysical concerns with infinitesimals (which concern their fictional status) from his mathematical treatment of infinitesimals as differentials. Bassler argues that the central technical issue surrounding the status of differentials concerns the specification of the "progression of variables." He then suggests some ways in which Leibniz's metaphysical and mathematical approaches to infinitesimals can be related.

Emily Grosholz emphasizes Leibniz's "productively ambiguous notation" as crucial for his development of the calculus. Leibniz's ambiguous notation, connected with the law of continuity, allowed for yoking together very unlike things and offers a means of making them mutually intelligible. Thus Leibniz's development of the infinitesimal calculus and his investigations of transcendental curves can be read as instances of ambiguity which, far from hindering understanding, makes novel mathematical objects comprehensible. *Eberhard Knobloch's* contribution focuses on Leibniz' claim for the generality of his calculus. He investigates Leibniz's declared debt to the ancients, particularly to Archimedes' emphasis on geometrical rigor. Although Leibniz made his great mathematical progress by studying the work of most recent mathematicians in Paris, Knobloch

shows how Leibniz avoided “the danger” of the method of indivisibles by a conscious turn to Archimedean methods. In addition, Knobloch draws an illuminating contrast between the Leibnizian theory of infinitesimals and the more robust (but ultimately incoherent) realism about infinitesimals embraced by Leonhard Euler. *Herbert Breger’s* essay focuses on Leibniz’ mathematical development after his departure from Paris. He emphasizes the strong influence of Pascal and Huygens on Leibniz’s approach and gives an instructive survey of their methods and arguments. The result is that, given this background to the calculus, there was in fact no genuine “foundational problem” to be addressed. Breger argues that “what was really new and what posed the actual problem of understanding the new method of calculation was the higher level of abstraction.”

Some of Leibniz’s contemporaries objected that his methods violated standards of mathematical rigor, and the resulting controversies are important in understanding the reception of the calculus. Two papers in this collection are directed toward these controversies. *Fritz Nagel’s* contribution investigates into the conception of the infinitesimal put forward by Hermann, which arose in response to the criticisms advanced by Nieuwentijt against Leibniz in the 1690s. Nagel notes that Herman’s approach, endorsing Leibniz’ position, has a significant degree of methodological and technical sophistication, and understanding it can shed some considerable light on the foundations of the calculus at the close of the seventeenth century. *Douglas Jesseph’s* essay deals with both early and late Leibnizian writings on the calculus. He argues that some of the fundamental notions in the *calculus differentialis* can be found in Hobbes’s concept of *conatus*. Jesseph then interprets the fictionalism espoused by Leibniz in response to criticisms as a further development of some of the key concepts that he had first encountered decades earlier in his reading of Hobbes.

The role of infinitesimals in Leibnizian physics is the focus of three of the contributions to this volume. *François Duchesneau’s* discusses the often mentioned ambiguities of Leibnizian scientific statements, arguing that such ambiguous analogies for Leibniz, when duly controlled, could become crucial means for promoting the art of discovery (*ars invenendi*). Duchesneau shows how Leibniz’ scientific methodology itself favors hypothetical constructions. With hypotheses, truths of reason may be applied to the analysis of contingent truths expressing the connection of natural phenomena. Along this line, a condition of valid hypothesizing consists in the framing of relevant mathematical models within science. *Donald Rutherford* focuses on the notion of force and the connection between the physical theory of forces and the calculus. His essay aims to reconcile two Leibni-

zian claims: first, that force as the only “real and absolute” property of bodies is an infinitesimal element of action which produces continuous change over time; and second, that the infinitesimal quantities which model forces are mere fictions rather than real entities. This reconciliation is undertaken by seeing that a substance’s transition from state to state is to be understood in terms of internal forces, which Leibniz thinks can best be modeled on the internal dynamics of the soul. *Daniel Garber’s* essay is also concerned with this tension between the physical and mathematical understanding of infinitesimals, notably the notion of “dead force” in Leibniz’s mechanics, and the connection between it and the notion of an infinitesimal magnitude. Garber argues that Leibniz distinguishes mathematics (and such fictions as infinitesimals) from the physical world in a way that allows physically real forces to be modeled or represented by mathematical devices that are not, strictly speaking, real entities.

We are grateful to the Graduate School of *Emory* having supported and generously sponsored this conference and to the colleagues of the Philosophy Department who encouraged us to organize this conference. We also thank the *Gottfried-Wilhelm-Leibniz Gesellschaft* at Hannover and the *North American Leibniz Society* for their official support and promotion of the conference. We are particularly grateful to Gertrud Grünkorn at the *de Gruyter Publishing House* at Berlin as well as to Andreas Vollmer for their supportive cooperation and the careful work on this volume, whose technical content makes it rather difficult. We would also like to thank Matt Traut, graduate student at Emory, and Stephen P. Farrelly, former graduate student at Emory (and now Assistant Professor at the Department of Philosophy at the University of Arkansas at Little Rock) for their great support in revising the papers for the publisher. Last but not least we are very grateful for the reliable cooperation with all the authors of this volume whose readiness to improve their papers mirrored the cheerful and enthusiastic atmosphere of our conference.

March 2008

Ursula Goldenbaum and Douglas Jesseph

RICHARD T. W. ARTHUR

Leery Bedfellows: Newton and Leibniz on the Status of Infinitesimals¹

1. Newtonian and Leibnizian Foundations: The Standard Contrast

As is well known, Newton did not welcome Leibniz's efforts at establishing a differential calculus: his attitude, one might say, ranged between deep suspicion, disdain and utter hostility. In his eyes, Leibniz's differential calculus was at best a sample of the new method of analysis, an unrigorous symbolic method of discovery that could not meet the standard of rigorous proof required in geometry; and at worst, not just a plagiarism of his own work, but a dressing up and masking in Leibniz's fancy new symbols of the deep truths of his method of fluxions, which did not depend on the supposition of infinitesimals but was instead founded directly in the "real geneses of things." Leibniz, for his part, while accepting many of Newton's results, harbored doubts about Newton's understanding of orders of the infinitely small, which to his way of thinking was betrayed by the unfoundedness of Newton's composition of non-uniform with uniform motions in the limit.

There are some profound differences here in the respective thinkers' philosophies of mathematics, involving differing conceptions of proof, of the utility of symbolism, and in the conceptions of how mathematics is related to the physical world. I do not want to understate them. Nevertheless, I shall contend here, there is a very real consilience between Newton's and Leibniz's conceptions of infinitesimals, and even in the foundations they provide for the method of fluxions and for the differential calculus.

Newton's own evaluation of the difference in their methods was given by him in the supposedly "neutral" report he submitted anonymously to

¹ I would like to thank Sam Levey and Niccolò Guiciardini for their helpful feedback on earlier drafts.

the Royal Society in 1715, *Account of the Commercium Epistolicum* (in MPN VIII). There he depicted his method as proceeding “as much as possible” by finite quantities, and as founded on these and the continually increasing quantities occurring in nature, in contrast to Leibniz’s, founded on indivisibles that are inadmissible in geometry and non-existent in nature:

We have no Ideas of infinitely little, & therefore Mr. Newton introduced Fluxions into his Method that it might proceed by finite Quantities as much as possible. It is more Natural & Geometrical because founded upon the *primae quantitatum nascentium rationes* w^{ch} have a Being in Geometry, whilst *Indivisibles* upon which the Differential Method is founded have no Being either in Geometry or in Nature. There are *rationes primae quantitatum nascentium* but not *quantitates primae nascentes*. Nature generates Quantities by continual Flux or Increase, & the ancient Geometers admitted such a Generation of Areas & Solids [...]. But the summing up of Indivisibles to compose an Area or Solid was never yet admitted into Geometry. (MPN VIII, 597–8)

This has been an influential account. Although it has long been recognized that Leibniz’s differential calculus is a good deal more general than the Cavalierian geometry of indivisibles, and that Newton’s characterizing of it as founded on indivisibles must be interpreted accordingly, the idea that Leibniz’s methods were committed to the existence of infinitesimals has stuck. As a result, his official position that they are to be taken as fictions has been regarded as a not very successful attempt to distance himself from the foundational criticisms brought to bear by Nieuwentijt, Rolle, and the Newtonians, when in fact his method is based upon infinite sums and infinitely small differences, and thus firmly committed to infinities and infinitesimals. Newton, on the other hand, has been seen as moving from an early purely analytic method depending on a free use of infinitesimals to a mature view, represented in his *Method of First and Ultimate Ratios* (MFUR) published in the *Principia*, where (officially, at least) there are only limiting ratios of nascent or evanescent quantities, and never infinitely small quantities standing alone.

Newton’s *Account of the Commercium Epistolicum* is a late text in his mathematical development, occurring as the culmination of a process of distancing himself from Analysis. By the 1680s he had turned away from the “moderns” in favor of Pappus and Apollonius, and an insistence on geometric demonstration. But the contrast between an early analytic Newton and the later conservative geometrician should not be overemphasized. The conception of fluxions or velocities by means of which Newton articulated what we call the *Fundamental Theorem of the Calculus* is intimately bound up with the kinematic conception of curves that he inherited from

Barrow and Hobbes. Thus although Newton's first formulations of his theory of fluxions are analytic in the sense that they are couched in terms of equations and algebraic variables, his kinematic understanding of curves and surfaces already implicitly involves a notion of the quantities represented by the variables as geometric, and as generated in time.

I shall argue, accordingly, that there is not such a huge gulf between Newton's analytic method of fluxions and the synthetic methods he later developed. Moreover, I contend, when Newton comes to secure the foundations of his synthetic method in the *Method Of First and Ultimate Ratios*, he appeals to Lemma 1, which is a synthetic version of the *Archimedean axiom*: "Quantities, and also ratios of quantities, which in any finite time constantly tend to equality, and which before the end of that time approach so close to one another that their difference is less than any given quantity, become ultimately equal" (Newton, 1999, 433). The axiom then serves to justify Newton's appeals to infinitesimal moments in supposedly geometric proofs such as that of Proposition 1 of the *Principia*, since these moments can be understood as finite but arbitrarily small geometric quantities in accordance with the *Archimedean axiom*. Furthermore, although Newton himself is careful to apply Lemma 1 only to *ratios of quantities*, the lemma as stated by him also applies directly to *quantities*; and Leibniz will appeal to a very similar principle applied to quantities as the foundation of his own method. In fact, the principle Leibniz appeals to, which takes differences smaller than any assignable to be null, is stated independently by Newton in his analytic method of fluxions (1971), and is a straightforward application of the *Archimedean axiom*.

Contrary to the standard depiction of their methods, then, there is a great similarity in the foundations of Newton's and Leibniz's approaches to the calculus. In fact, as I show by a detailed analyses of Newton's proof of Lemma 3 of his MFUR, and of Leibniz's proof of his Proposition 6 of *De quadratura arithmetica* (1676; DQA), their (contemporary and independent) attempts to provide rigorous foundations for their infinitesimalist methods by an appeal to the *Archimedean axiom* are in detailed correspondence, and perfectly rigorous. The rigor of Leibniz's approach to proving proposition 6 has already been stressed by Eberhard Knobloch (Knobloch, 2002). Here I extend that analysis to show its compatibility with the *syn-categorematic interpretation of infinitesimals* attributed to Leibniz by Hidé Ishiguro.

2. Newton's Moments and Fluxions

The paper that is generally taken as containing Newton's first full statement of his analytic method of fluxions is *To Resolve Problems by Motion*, written in October 1666 as the culmination of several redraftings (MPN I, 400–448). The commitment to the kinematical representation of curves is evident in its title, and this is so also for the earlier drafts out of which it develops: two drafts of *How to draw tangents to Mechanicall lines* [30? October 1665 and 8 November 1665, resp.], a third draft titled *To find y^e velocity of bodys by y^e lines they describe*, [November 13th 1665], and a fourth titled *To resolve these & such like Problems these following propositions may bee very usefull*, [May 14, 1666].² Thus Newton's recipe in Proposition 7 for what we, after Leibniz, call differentiation, is couched by him in terms of the velocities of bodies:

[Prop.] 7. Haveing an Equation expressing y^e relation twixt two or more lines x, y, z &c: described in y^e same time by two or more moving bodies A, B, C, &c [Fig. 1]: the relation of their velocitys *p*, *q*, *r*, &c may bee thus found, viz:

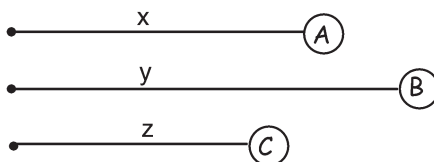


Figure 1.

Set all y^e termes on one side of y^e Equation that they may become equall to nothing. And first multiply each terme by so many times p/x as *x* hath dimensions in y^t terme. Secondly multiply each terme by so many times q/y as *y* hath dimensions in it. Thirdly (if there be 3 unknowne quantitys) multiply each terme by so many times r/z as *z* hath dimensions in y^t terme, (& if there bee still more unknowne quantitys doe like to every unknowne quantity). The summe of all these products shall be equall to nothing. w^{ch} Equation gives y^e relation of y^e velocitys *p*, *q*, *r*, &c. (MPN I, 402)

The first thing to notice about this algorithm is that it is not purely analytic: the equations are given a geometrical interpretation in terms of lines traced by moving bodies. Second, what results from the algorithm is not a veloc-

² These drafts are given in MPN I, 369–377, 377–382, 382–389, and 390–392, resp. The last draft was subsequently cancelled and rewritten as *To resolve Problems by motion ye 6 following prop. are necessary and sufficient*, dated May 16, 1666 (MPN I, 392–399).

ity but the *ratio* of two velocities, and these velocities (say, p and q) are the instantaneous velocities of two bodies at the beginning of the moment o for which they are assumed to travel with that velocity.

A very simple example of applying this algorithm is provided by the result Newton quotes in his demonstration of Proposition 1 of this tract – this being perhaps the very first application of the method of fluxions in physics.³ Proposition 1 is a statement of the resolution of velocities, and its demonstration depends on finding the relation between the velocities of the body A in two directions, towards d and towards f , as it travels along the line gc below, with $df \perp ac$, at the very point a when it reaches the perimeter of the circle. Letting $df = a$, $fg = x$, and $dg = y$, we have (since Δadf is a right triangle)

$$a^2 + x^2 - y^2 = 0.$$

According to Newton's algorithm given in Proposition 7 above, we must multiply each term in x in the equation by $2p/x$ and each term in y by $2q/y$, yielding

$$2xp - 2yq = 0.$$

This result is quoted by Newton in his demonstration as follows:

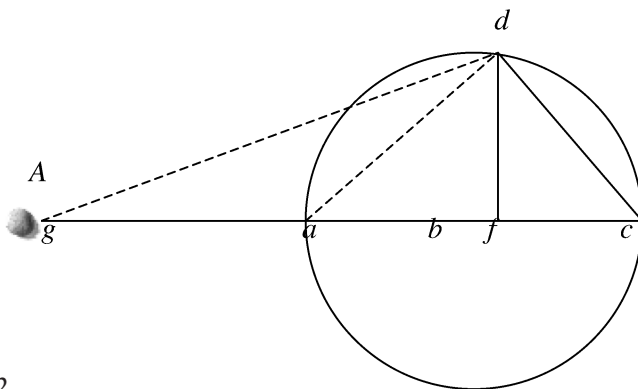


Figure 2.

Now by Prop 7th, may y^e proportion of (p) y^e velocity of y^t body towards f ; to (q) its velocity towards d be found viz: $2px - 2pq = 0$. Or $x:y::q:p$. That is $gf:gd::$ its velocity to d : its velocity towards f or c . & when y^e body A is at a , y^t is when y^e points g & a are coincident then is $ac:ad::ad:af::$ velocity to c : velocity to d . (MPN I, 415)

³ Newton first gives the demonstration of Proposition 1 immediately after stating all 8 propositions (MPN I, 415), but as Whiteside notes, Newton alludes to the fact that it can be so demonstrated in the draft of May 14th, 1666 (MPN I, 390).

Or, as we would say in more Leibnizian terms, differentiating $a^2 + x^2 - y^2 = 0$ yields $2xp - 2yq = 0$, with p and q the derivatives of x and y respectively. Thus the velocities p and q are in the inverse ratio of x and y . Now when the body A reaches the point a we have $x = af$, $y = ad$, $q = v_{ad}$ and $p = v_{ac}$, yielding

$$v_{ac} : v_{ad} = ad : af$$

and since by similar triangles $ad:af = ac:ad$, we obtain finally

$$v_{ac} : v_{ad} = ac : ad \quad \text{or} \quad v_{ad} : v_{ac} = ad : ac$$

which, in modern notation, is the correct formula for the resolution of velocities in an oblique direction:

$$v_{ad} = v_{ac} \cos \phi, \text{ where } \phi = \angle dac$$

Of interest to us here is the justification Newton gives in 1666 for Proposition 7. The demonstration he provides is by reference to a specific equation, $x^3 - abx + a^3 - dy^2 = 0$. There is no loss in generality in our substituting for it the above equation for Proposition 1, $a^2 + x^2 - y^2 = 0$. Newton first supposes two bodies A and B moving uniformly, the one from a to c , d , e , f , the other from b to g , h , k , l , in the same time. Then the pairs of lines ac and bg , cd and gh , de and hk etc. are “as their velocitys” p and q (MPN I, 414):

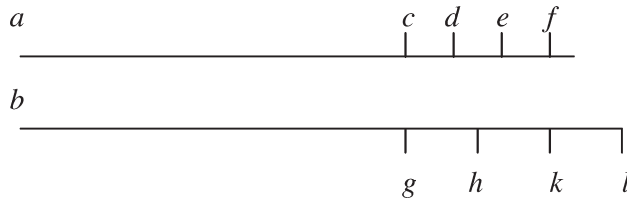


Figure 3.

He then reasons that:

And though they move not uniformly, yet are y^e infinitely little lines w^{ch} each moment they describe, as their velocitys w^{ch} they have while they describe y^m . As if y^e body A w^{th} y^e velocity p describe y^e infinitely little line ($cd =$) $p \times o$ in one moment, in y^t moment y^e body B w^{th} y^e velocity q will describe y^e line ($gh =$) $q \times o$. For $p:q :: po:qo$. Soe y^t if y^e described lines be $(ac =) x$, & ($bg =$) y , in one moment, they will be $(ad =) x + po$, & ($bh =$) $y + qo$ in y^e next. (MPN I, 414)

Now he claims that “I may substitute $x + po$ & $y + qo$ into y^e place of x & y ; because (by y^e lemma) they as well as x & y , doe signify y^e lines described by y^e bodys A & B ” (414). Thus for the equation $a^2 + x^2 - y^2 = 0$ we get

$$a^2 + x^2 + 2pox + p^2o^2 - y^2 - 2qoy - q^2o^2 = 0$$

Subtracting the original equation gives

$$2pox + p^2o^2 - 2qoy - q^2o^2 = 0$$

“Or dividing it by o tis [$2px + p^2o - 2qy - q^2o = 0$]. Also those terms are infinitely little in w^{ch} o is. Therefore omitting them there rests [$2xp - 2yq = 0$]. The like may bee done in all other equations” (MPN I, 414–15).

Here Newton’s division by o prior to omitting terms in o because they are “infinitely little” is, of course, lacking in rigor. Either, one may object, adding po to x takes body A to “y^e next” point on the line representing its path, and one is committed to composing that line out of successive infinitesimal linelets (and thus succumbing to the paradoxes of the continuum); or indeed, $x + po$ does not at all differ from x , in which case division by o is completely illegitimate. And yet Newton’s algorithm is framed in terms of *ratios* of quantities and their velocities in the moment o . Of course, there is no way to represent an instantaneous velocity geometrically save by showing the line segment (cd in figure 3) that a body would cover if it continued with that velocity for a time o . From this point of view, the moment o is more nearly a device enabling instantaneous velocities to be geometrically represented: po is the distance the body A would have covered if it had proceeded with the velocity p for some time o . The ratio $po:qo$ is of course equal to the ratio of p and q for any finite o . Moreover, it is implicit in the kinematic representation that the velocities p and q are the velocities at the very beginning of the moment o , so that the term for $po:qo$ calculated by Newton’s algorithm, which will still generally contain terms in o , will be closer to $p:q$ the closer o is to 0. The justification for neglecting the remaining terms in o is therefore not so much that they are conceived as “infinitely little” in the sense of absolutely infinitely small, but in the sense that the ratio $p:q = po:qo$ represents the ratio of p and q right at the beginning of the moment, so that the smaller o is made, the smaller will be the terms still containing o , and the more nearly will the resulting expression represent the ratio.

Thus in the context of a kinematic and geometric interpretation of the quantities involved, Newton’s early appeal to the infinitely small cannot simply be taken as committing him to a composition of quantities out of infinitesimals. In fact, his procedure already implicitly involves a kind of limiting process: to find the ratio of the velocities precisely at the beginning of the moment o (e.g. at the instant the moving body A reaches the point a in the above diagram), o must be shrunk to zero, so that the extra terms in the expression of this ratio still depending on the quantity o will therefore also vanish, with the resulting expression yielding the “first ratio” of these velocities.

Newton himself recognized this soon enough, and proceeded to make the limit concept implicit in the kinematical representation the foundation of the synthetic method of fluxions. He drew up these early results, as well as those outlined in his *De Analysi per æquationes numero terminorum infinitas* (1669; publ. 1711), into a formal Latin treatise intended for publication, the *Tractatus de methodis serierum et fluxionum* (1671; publ. 1736; MPN III, 32–328), or *Treatise on Fluxions* for short, where the terminology of “fluxions” was first introduced. But he remained unsatisfied with the foundations of his methods, and in an Addendum on *The Theory of Geometrical Fluxions* made just after completing the latter, he developed a wholly synthetic approach, “based on the genesis of surfaces by their motion and flow” (MPN III, 328–31; Guicciardini, 2003, 315). Axiom 4 of this Addendum was “Contemporaneous moments are as their fluxions” (MPN III, 330), or more perspicuously perhaps, “Fluxions are as the contemporaneous moments generated by those fluxions” (draft). Whiteside observes: “This fundamental observation opens the way to subsuming limit-increment arguments as fluxional ones, and conversely so” (MPN III, 330, fn 7).

As Guicciardini has noted (Guicciardini, 2002, 414–17), these foundations are synthetic in two distinct senses: they are based on explicit axioms from which propositions are derived, “synthesis” as opposed to “analysis”; and the quantities are not the symbols but fluent geometrical figures, synthetic in the sense of flowing, increasing, staying constant, or decreasing continuously in time. The emphasis on synthesis (in this dual sense) is a symptom of Newton’s progressive disenchantment with analysis in the 1670s, and a growing respect for the geometry of the ancients. This process is taken further in *Geometria curvilinea*, written some time between 1671 and 1684, where Newton stresses the generation of geometric quantities in time:

Those who have measured out curvilinear figures have usually viewed them as consisting of infinitely many infinitely small parts. But I will consider them as generated by growing, arguing that they are greater, equal or smaller according as they grow more swiftly, equally swiftly or more slowly from the beginning. And this swiftness of growth I shall call the fluxion of a quantity. (MPN IV, 422–23)

This interpretation of his mathematics explains the contrast Newton draws between the ontological foundation of his methods (“This Method is derived immediately from Nature her self”) and the lack of such a foundation in the analysis of Leibniz. It is emphasized even more strongly in the *De quadratura curvarum* of 1693, where Newton writes:

I don't here consider Mathematical Quantities as consisting of *indivisibles*, whether least possible parts or infinitely small ones, but as described by a continual motion. Lines are described, and by describing are generated, not by any apposition of Parts, but by the continuous motion of Points, Surfaces by the motions of Lines, Solids by the motion of Surfaces, Angles by the Rotation of their Legs, Time by a continual Flux, and so on in all the rest. These *Geneses* are founded upon Nature, and are every Day seen in the motions of Bodies. (Newton, 1964, 141)

In these passages Newton not only claims that geometric quantities are founded *in rerum natura*, he also explicitly repudiates their composition out of infinitely small parts (infinitely small quantities have “no Being either in Geometry or in Nature”). As he had come to recognize, the moments of quantities do not have to be supposed as infinitely small quantities, falling outside the scope of geometry based on the *Archimedean axiom*, but can instead stand for finite quantities that can be taken as small as desired. This is the foundation of his synthetic method of fluxions Newton presents in the *Geometria curvilinea*, and which he will publish in the *Principia* under the new moniker the *Method of First and Ultimate Ratios*. Although infinitely small quantities still occur in Newton's mature work, they are interpreted as standing for finite but small quantities that are on the point of vanishing, with the ratio between two such quantities remaining finite in this temporal limit.

An example of this synthetic method of fluxions, I claim, is provided by Newton's demonstration of Proposition 1, in Book 1 of the *Principia*. In fact, this proposition provides a particularly good specimen of the advantages of the synthetic method of fluxions. For not only is the proof extremely economical compared to any analytic derivation of Kepler's *Area Law*, it also depends on no assumptions about the nature of the force except that it is continuous and centrally directed.⁴ Newton's demonstration goes as follows:

Let the time be divided into equal parts, and in the first part of the time let a body by its inherent force describe the straight line *AB*. In the second part of the time, if nothing hindered it, this body would (by law 1) go straight on to *c*, describing line *Bc* equal to *AB*, so that – when radii *AS*, *BS* and *cS* are drawn to the centre – the equal areas *ASB* and *BSc* would be described. But when the body comes to *B*, let a centripetal force act with a single but great impulse and make

⁴ Also, of course, as explained by Nauenberg, 2003, 450, the curvature of the curve must remain finite, and the radius vector cannot become tangential to it.

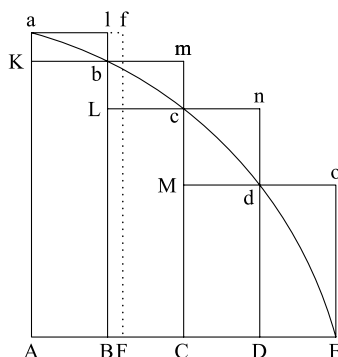


Figure 5

Lemma 3 itself is: “the ultimate ratios [which the inscribed figure $AKbLcMdD$, the circumscribed figure $AalbmncdoE$, and the curvilinear figure $AabcdE$ have to one another] are also ratios of equality when the widths AB , BC , CD , [...] of the parallelograms are unequal and are all diminished indefinitely” (Newton, 1999, 433). Newton uses this result to argue in Corollary 1 that “the ultimate sum of the vanishing parallelograms coincides with the curvilinear figure in its every part,” in Corollaries 2 and 3 that the figure comprehended by the chords or the tangents of the vanishing arcs “coincides ultimately with the curvilinear figure,” and in Corollary 4 that “therefore these figures (with respect to their perimeters acE) are not rectilinear, but curvilinear limits of rectilinear figures” (Newton, 1999, 434). Thus by a similar argument the triangles in Figure 4 are not rectilinear, but curvilinear limits of rectilinear figures, the ratios between any two of which are equal.

Let us now turn to Newton’s justification of this Lemma. He demonstrates it by reference to the same figure used for all the first four Lemmas. Having proved Lemma 2 on the supposition of equal intervals AB , BC , DE , etc., he now supposes them unequal, and lets “ AF be equal to the greatest width” of any of the rectangles. Hence $FAaf$ is at least as wide as any of the rectangles, and its total height will be the sum of the heights of the differences between the circumscribed and inscribed figures. “This parallelogram will therefore be greater than the difference of the inscribed and circumscribed figures; but if its width AF is diminished indefinitely, it will become less than any given rectangle. Q.E.D.” (Newton, 1999, 434)

The last step of this proof is an application of Lemma 1 of the *Method of First and Ultimate Ratios*, which I quote here in its original wording from the First Edition:

Quantities, and also ratios of quantities, which in a given time constantly tend to equality, and which before the end of that time approach so close to one another that

their difference is less than any given quantity, become ultimately equal. (Newton, 1999, 434)

Here one might object that an infinitesimal is precisely a quantity that is “less than any given quantity,” so that if there exist non-zero infinitesimals then such a difference will be non-zero. In default of some further premise, the argument therefore seems to assume what it needs to prove. The missing premise is that in order for the quantities to count as geometrical quantities, they must obey the *Archimedean axiom*:

If a and b are two line segments (or other continuous geometric quantities) with $a < b$, we can always find a (finite) number n such that $na > b$.

This axiom rules out the existence of an actual infinitesimal quantity, such as the “difference less than any given quantity” mentioned in Lemma 1. As Newton argues in his demonstration of the Lemma:

If you deny this, let their ultimate difference be D . Then they cannot approach so close to equality that their difference is less than the given difference D , contrary to the hypothesis. (Newton, 1999, 433)

The “hypothesis” in question here is that they can always “approach so close to one another that their difference is less than any given quantity.” This is simply an expression in synthetic form of the *Archimedean axiom*: given two quantities whose difference D is less than some quantity a , we can always find a number n such that $nD > a$, so that $c = a/n < D$.

In fact, if we explore the origins of Lemma 1 of the MFUR we can trace a direct line of descent from the “Treatise on Fluxions.” Two paragraphs of this are rewritten into the “Addendum on Geometrical Fluxions,” the latter is reworked into the *Geometria curvilinea*, and it is from this that the *Method of First and Ultimate Ratios* is derived. The first of the two paragraphs of the *Treatise on Fluxions* runs:

This method for proving that curves are equal or have a given ratio by the equality or given ratio of their moments, I have used because it has an affinity with methods usually employed in these cases; but a method based on the genesis of surfaces from the motion of their flowing seems more natural [...]. (MPN III, 282)

This is transcribed to the Addendum, with the addition “[...], one which will prove to be more perspicuous and elegant if certain foundations are laid out in the style of the synthetic method; such as the following” (MPN III, 328–330), and this introduces the axioms and theorems that constitute the synthetic method. But the previous method referred to in this paragraph,

that of proving “through the equality of moments,” is described in the immediately preceding paragraph of the *Treatise* as follows:

In demonstrations of this sort it should be observed that I take those quantities to be equal whose ratio is one of equality. And a ratio of equality is to be regarded as one which differs less from equality than any unequal ratio that can be assigned. Thus in the preceding demonstration I set the rectangle $Ep \times Ac$, that is, $Feqf$, equal to the space $FEef$ since (because their difference Eqe is infinitely smaller than them, i.e. nothing with respect to them), they have no ratio of inequality. And for the same reason, I set $Dp \times HI = Hlib$, and likewise in the others. (MPN III, 282)

The principle appealed to here is this:

If an inequality is such that its difference from a strict equality can be made smaller than any that can be assigned, it can be taken for an equality.

Let us call this the *Principle of Unassignable Difference*. This principle, clearly, is the analytic equivalent of the chief synthetic axiom, Lemma 1 of the *Method of First and Ultimate Ratios*. And like that Lemma, it derives its warrant from the *Archimedean axiom*. This common warrant underwrites the equivalence between the analytic and synthetic methods of fluxions, allowing the translatability of statements given in terms of “indivisibles” (i.e. infinitesimals) into fluxional terminology, thus justifying Newton’s claim in the *Principia* that having reduced the propositions there to the limits of the sums and ratios of First and Ultimate ratios of nascent and evanescent quantities, he had thereby “performed the same thing as by the method of indivisibles.” He continues:

Accordingly, whenever in what follows [...] I use little curved lines in place of straight ones, I wish it always to be understood that I mean not indivisibles but evanescent divisible quantities, and not the sums and ratios of determinate parts, but the limits of such sums and ratios; and that the force of such demonstrations always rests on the method of the preceding lemmas. (Newton, 1999, 441–2; trans. slightly modified)

3. Leibniz’s Syncategorematic Infinitesimals

Now let us turn to Leibniz. During the same period (1671–1684) in which Newton was perfecting his synthetic interpretation of the results he had obtained in 1666, Leibniz was independently developing the algorithms and techniques he was to present as the differential and integral calculus. In his

approach to the development and application of his calculus, Leibniz often stressed the pragmatic utility of his techniques, and how they could be exploited by mathematicians without their having to trouble themselves with foundational problems. These comments, together with the lack of clarity regarding foundations in his early publications, and his late pronouncements on the nature of infinitesimals precipitated by the controversies involving Rolle, Nieuwentijt and Varignon, have conspired to produce the impression that Leibniz developed his calculus without much attention to its foundations.

But this impression is entirely mistaken. For just as Newton had attempted to strengthen the foundations of his methods in his Latin treatise *De methodis serierum et fluxorum* in 1671, and again in *Geometria curvilinea* not long afterwards, so in 1675–76 Leibniz had also written a comprehensive Latin treatise on his infinitesimal methods, *De quadratura arithmetica*, which has only recently been edited and published by Eberhard Knobloch (DQA); and in this treatise, as Knobloch has shown, “Leibniz laid the rigorous foundation of the theory of infinitely small and infinite quantities” (Knobloch, 2002, 59). I have argued elsewhere (Arthur, 2008a) that Knobloch’s interpretation of Leibniz’s foundational work is fully in keeping with Hidé Ishiguro’s attribution to Leibniz of an interpretation of infinitesimals as “syncategorematic.” That is, as I have tried to show, Leibniz’s mature interpretation of infinitesimals as “fictions” has a precise mathematical content, perfectly consistent with his philosophy of the infinite and solution to the continuum problem (Arthur, 2001, 2008b). Moreover, I shall argue here, this content is given by the foundation of the method on the *Archimedean axiom*. Thus Leibniz’s justification of his infinitesimal methods will be seen to be in surprising conformity with Newton’s.

As regards foundations, the nub of the *De quadratura arithmetica* occurs in Proposition 6 (DQA, 28–36), as Eberhard Knobloch has explained. Leibniz himself describes it as

spinosissima in qua morose demonstratur certa quaedam spatia rectilinea gradiformia itemque polygona eousque continuari posse, ut inter se vel a curvis differant quantitate minore quavis data, quod ab aliis plerumque assumi solet. Praeteriri initio ejus lectio potest, servit tamen ad fundamenta totius Methodi indivisibilium firmissime jacienda.⁶ (DQA, 24).

⁶ “[...] most thorny; in it, it is demonstrated in fastidious detail that the construction of certain rectilinear and polygonal step spaces can be pursued to such a degree that they differ from one another or from curves by a quantity smaller than any given, which is something that is most often [simply] assumed by other authors. Even though one can skip over it at

The “thorniness” is evident from *Figure 6* (fig. 3 in the DQA):

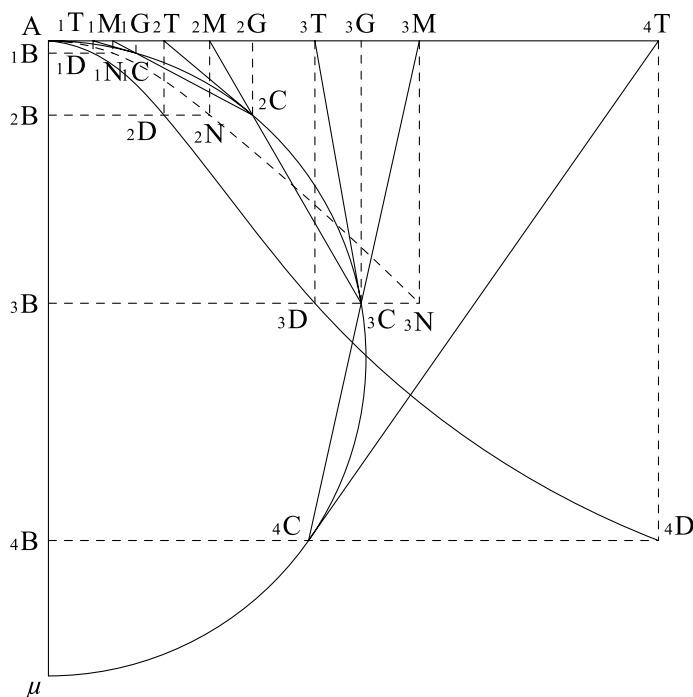


Figure 6.

In this figure, the x-axis is vertical, and the y-axis is the horizontal axis across the top. The curve considered here is a circular arc C, the tangents to which at successive points on this curve ($_1C$, $_2C$, $_3C$, $_4C$) cut the y-axis at the points $_1T$, $_2T$, $_3T$, $_4T$. Now a second, auxiliary curve D is defined by the points of intersection of these tangents to C with the ordinates $_1B$, $_2B$, $_3B$, $_4B$, yielding the points $_1D$, $_2D$, $_3D$, $_4D$, on this new curve. The secants joining successive pairs of points on the original curve, $_1C_2C$, etc., are extended to cut the y-axis in the points $_1M$, $_2M$, $_3M$. The points of intersection of the perpendiculars from these points M down through the ordinates B of the original curve define another set of points $_1N$, $_2N$, $_3N$. Provided certain conditions are satisfied – continuity, no point of inflection, no point with a vertical tangent – this construction is always possible, and as Knobloch

first reading, it serves to lay the foundations for the whole method of indivisibles in the soundest possible way”.

$|{}_1B_1D_2D_2B - {}_1B_1N_1P_2B| < {}_1D\alpha_2D_1E$, etc. This is proved in stage 3 by subtracting from each their common part, ${}_1B_1D_1F_1P_2B$, etc., leaving a difference of two trilinear areas. Even the sum of these two areas is less than the complementary rectangle, so their difference certainly is. Thus $|{}_1B_1D_2D_2B - {}_1B_1N_1P_2B| = |{}_1D_1N_1F - {}_1F_2D_1P| < {}_1D\alpha_2D_1E$, etc. In step 4, it is shown that this inequality holds for all such differences between curvilinear areas and their corresponding elementary rectangles. As Knobloch has shown, Leibniz is here implicitly appealing to the triangle inequality $||A| - |B|| \leq |A - B|$ (Knobloch, 2002, 65).

Therefore (stages 5 and 6) the absolute value of the difference between the sum M of all the mixtilinear areas (the area under the curve, called by Leibniz the “total Quadrilineal”) and the sum E of all the elementary rectangles approximating the area under the curve (the Riemannian sum, called by Leibniz the “step space [*spatium gradiforme*]”) is less than the sum C of all the complementary rectangles: $|M - E| < C$. But the sum C of all the complementary rectangles ${}_1D\alpha_2D_1E$, ${}_2D\beta_3D_2E$, etc. would be less than the sum of all their bases times their common height, if all the ordinates were equally spaced. Since by hypothesis they are not, let the greatest height (say, the difference between successive ordinates ${}_3B$ and ${}_4B$) be h_m . The sum of all the bases is the difference between the greatest and smallest ordinate, ${}_1L_3D$. Therefore C is smaller than the rectangle equal to the product ${}_3B_4B \times {}_1L_3D$, i.e. $C < {}_1L_3D h_m$. Hence, since $|M - E| < C$, we have

$$M - E < {}_1L_3D \cdot h_m,$$

where h_m is the greatest height of any of the elementary rectangles.

But (stage 7) the abscissa representing this greatest height, “tametsi caeteris majus, aut certe non minus sit assumtum intervallis, tamen assignata quantitate minus assumi potest; nam ipso sumto utcunque parvo caetera sumi possunt adhuc minora”.⁸ (DQA, 31–32) Therefore “sequetur ut rectangulum ψ ${}_4D$ ${}_1L$, altitudinem habens quae data recta minor sumi posit, etiam data aliqua superficie reddi posse minus.”⁹ (DQA, ibid.).

It therefore follows (stage 8) that “*Differentia hujus Quadrilineae, (de quo et proposition loquitur) et spatii gradiformis data quantitate minor reddi po-*

⁸ “[...] even though it is greater than, or at any rate not less than, any of the other intervals assumed, can nevertheless be assumed smaller than any assigned quantity; for however small it is assumed to be, others can be assumed still smaller.”

⁹ “[...] it will follow that the rectangle ψ ${}_4D$ ${}_1L$ [${}_3B_4B$ ${}_1L_3D$], having a height which can be assumed smaller than any given line, also can be made smaller than any given surface”.

test”¹⁰ (DQA, 32). That is, the difference between the Riemannian sum and the area under the curve is smaller than any assignable, and therefore null.

As Leibniz points out, the prolixity of this proof is due in part to the fact that it is considerably more general than the “communi methodo indivisibilium”¹¹ (DQA, 32), where one is “securitatis causa cognimur, ut Cavalierius, ad ordinatas parallelas methodum restringere, et aequalia semper duarum proximarum ordinatarum intervalla postulare”¹² (DQA, 69). In that case the points N and the points D coincide and “longe facilius fuisset demonstratio”¹³ (DQA, 32), as he proceeds to show.

Several things about this demonstration are worthy of note. As Leibniz observes in the Scholium to Proposition 7: “Demonstratio illud habet singulare, quod rem non per inscripta ac circumscripta simul, sed per sola inscripta absolvit.”¹⁴ (DQA, 35) More accurately though, the step figure is, as Knobloch says, “something in between” an inscribed and a circumscribed one (Knobloch, 2002, 63). Leibniz’s method, in fact, is extremely general and rigorous; the same construction of elementary and complementary rectangles could be constructed for any curve whatsoever satisfying the three conditions outlined. It amounts in modern terms to a demonstration of “the integrability of a huge class of functions by means of Riemannian sums which depend on intermediate values of the partial integration intervals” (Knobloch, 2002, 63).

Second, it is strictly finitist. As Leibniz observes, the traditional Archimedean method of demonstration was by a double *reductio ad absurdum*. But his preference is instead to proceed by a direct *reductio* to prove that “inter duas quantitates nullam esse differentiam”.¹⁵ (DQA, 35) As he explains in the continuation of the Scholium to Prop. 7,

Equidem fateor nullam hactenus mihi notam esse viam, qua vel unica quadratura perfecte demonstrari possit sine deductione ad absurdum; imo rationes habeo, cur verear ut id fieri possit per naturam rerum sine quantitatibus fictitiis, infinitis

¹⁰ “[...] the difference between this Quadrilineal (which is the subject of this proposition) and the step space [i.e. $M - E$] can be made smaller than any given quantity”.

¹¹ “[...] common method of indivisibles”.

¹² “[...] compelled for safety’s sake, as was Cavalieri, to restrict the method to parallel ordinates, and to suppose that the intervals between any two successive ordinates are always equal”.

¹³ “[...] the demonstration is far easier”.

¹⁴ “[...] the demonstration has the singular feature that the result is achieved not by inscribed and circumscribed figures taken together, but by inscribed ones alone”.

¹⁵ “[...] the difference between two quantities is nothing”.

scilicet vel infinite parvis assumtis: ex omnibus tamen ad absurdum deductionibus nullam esse credo simplicem magis et naturalem, ac directae demonstratione propiorem, quam quae non solum simpliciter ostendit, inter duas quantitates nullam esse differentiam, adeoque esse aequales, (cum alioquin alteram altera neque maiorem neque minorem esse ratiocinatione duplici probari soleat) sed et quae uno tantum termino medio, inscripto scilicet circumscripto, non vero utroque simul, utitur.¹⁶ (DQA, 35)

We see here a distinction between the method of integration using infinitely many infinitely small elements, which Leibniz characterizes as fictitious, and the direct *reductio ad absurdum method* just exploited in the demonstration above. As we saw there, this involves an inference from the fact that a difference between two quantities can be made smaller than any that can be assigned, to their difference being null. This is a *reductio* in the sense that whatever minimum difference one supposes there to be, one can prove that the difference is smaller. As we have seen, that is the very same reasoning Newton appeals to in his *Principia* to demonstrate Lemma 1 of his *Method of First and Ultimate Ratios*.

Third, Leibniz's demonstration of Proposition 6, just like Newton's Lemmas 1–4, licenses his infinitesimal techniques in quadratures, “servit tamen ad fundamenta totius Methodi indivisibilium firmissime jacienda.”¹⁷ (DQA, 24). The term “indivisible” here needs to be taken with a pinch of salt: Leibniz is clear that “plurimum interest inter indivisibile et infinite parvum”,¹⁸ and that “Fallax est indivisibilium Geometria, nisi de infinite parvis explicetur; neque enim puncta vere indivisibilia tuto adhibentur, sed lineis utendum est, infinite quidem parvis, lineis tamen, ac proinde divisibilibus.”¹⁹

¹⁶ “For my part I confess that there is no way that I know of up till now by which even a single quadrature can be perfectly demonstrated without an inference ad absurdum. Indeed, I have reasons for doubting that this would be possible through natural means without assuming fictitious quantities, namely, infinite and infinitely small ones; but of all inferences ad absurdum I believe none to be simpler and more natural, and more proper for a direct demonstration, than that which not only simply shows that the difference between two quantities is nothing, so that they are then equal (whereas otherwise it is usually proved by a double reductio that one is neither greater nor smaller than the other), but which also uses only one middle term, namely either inscribed or circumscribed, rather than both together.”

¹⁷ “[...] laying the foundations of the whole method of indivisibles in the soundest possible way”.

¹⁸ “[...] there is a profound difference between the indivisible and the infinitely small”.

¹⁹ “The Geometry of Indivisibles is fallacious unless it is explicated by means of the infinitely small; for truly indivisible points may not safely be applied, and instead it is necessary to use lines which, although infinitely small, are nevertheless lines, and therefore divisible.”

(Scholium to Proposition 11, DQA, 133)²⁰ In Proposition 7, explaining that “Per Summam Rectarum ad quondam axem applicatarum”²¹ (DQA, 39) he means “figurae perpetua applicatione factae aream”,²² he comments:

Quicquid enim de tali summa demonstrari poterit, sumto intervallo, utcunque parvo, id quoque de areae curvilineae ${}_0C_0B_3B_3C_0C$ magnitudine demonstratum erit, cum summa ista (intervallo satis exiguo sumto) talis esse posit, ut ab ista summa rectangulorum differentiam habeat data quavis minorem. Et proinde si quis assertiones nostras neget facile convinci posit ostendendo errorem quovis assignabili esse minorem, adeoque nullum.²³ (DQA, 39)

This is precisely the same as the principle appealed to by Newton to found his analytic method of fluxions, which I called above the *Principle of Unassignable Difference*; it is simply an application of the *Archimedean axiom*.

Fourth, Leibniz is explicit that the equivalence between a proof effected by infinitesimals and the corresponding rigorous kind of proof from first principles given in Proposition 6, means that infinitesimals can always be taken as a kind of shorthand for the arbitrarily small finite lines occurring in the latter. Acknowledging his free use of infinite and infinitely small quantities in proving his results concerning the circle, the ellipse and the infinite hyperboloid, Leibniz writes in the Scholium to Proposition 23:

Quae de infinitis atque infinite parvis huc usque diximus, obscura quibusdam videbuntur, ut omnia nova; sed mediocri meditatione ab unoquoque facile percipiuntur: qui vero perceperit, fructum agnoscat. Nec refert an tales quantitates sint in rerum natura, sufficit enim fictione introduci, cum loquendi cogitandique, ac proinde inveniendi pariter ac demonstrandi compendia praebeant, ne semper inscriptis vel circumscriptis uti, et ad absurdum ducere, et errorem assignabili quovis minorem ostendere necesse sit. Quod tamen ad modum eorum quae *prop. 6. 7. 8.* diximus facile fieri posse constat. Imo si quidem possibile est directas de his rebus exhiberi demonstrationes, ausim asserere, non posse eas dari, nisi

²⁰ This Scholium to Proposition 11 is recorded as deleted by Knobloch (DQA, 132–33), but is included in the main text without comment in the edition of Parmentier (Leibniz, 2004, 96–101).

²¹ “[...] by the sum of the straight lines applied to a certain axis”.

²² “[...] the area of the figure formed by this continued application”.

²³ “For whatever properties of such a sum could be demonstrated by taking the interval arbitrarily small, will also be demonstrated of the curvilinear area ${}_0C_0B_3B_3C_0C$, since, if the interval is taken sufficiently small, this sum could be such that its difference from the sum of the rectangles will be smaller than any given. And so anyone contradicting our assertion could easily be convinced by showing that the error is smaller than any assignable, and therefore null.”

his quantitatibus fictitiis, infinite parvis, aut infinitis, admissis, adde supra prop. 7 schol.²⁴ (DQA, 69)

An infinitesimal, therefore, is simply a shorthand for a quantity that may be taken as small as desired; likewise an infinite quantity is a quantity “qualibet a nobis assignabili, numerisve designabili, majorem.”²⁵ (DQA, 133; Leibniz, 2004, 98) Both are, with respect to geometry, *fictions*. On whether they can be found in nature, Leibniz is here agnostic; but “Geometrae sufficit, quid ex ipsis positis sequatur, demonstrare (Scholium to Prop. 11; DQA, 133; Leibniz, 2004, 98).²⁶

This interpretation, as I have argued elsewhere (Arthur, 2001, 2008a, b), is completely in accord with the insightful presentation of Leibniz’s mature interpretation of infinitesimals given by Hidé Ishiguro in the second edition of her *Leibniz’s Philosophy of Logic and Language* (1990). According to Ishiguro, Leibniz held, analogously to Russell’s position regarding definite descriptions,

that one can have a rigorous language of infinity and infinitesimal while at the same time considering these expressions as being syncategorematic (in the sense of the Scholastics), i.e. regarding the words as not designating entities but as being well defined in the proposition in which they occur (Ishiguro, 1990, 82).

As she goes on to argue, “Leibniz denied that infinitesimals were fixed magnitudes, and claimed that [in our apparent references to them] we were asserting the existence of variable finite magnitudes that we could choose as small as we wished” (Ishiguro, 1990, 92). This is indeed the case, as we have seen.

²⁴ “The things we have said up to now about infinite and infinitely small quantities will appear obscure to some, as does anything new; nevertheless, with a little reflection they will be easily comprehended by everyone, and whoever comprehends them will recognize their fruitfulness. Nor does it matter whether there are such quantities in the nature of things, for it suffices that they be introduced by a fiction, since they allow economies of speech and thought in discovery as well as in demonstration. Nor is it necessary always to use inscribed or circumscribed figures, and to infer by *reductio ad absurdum*, and to show that the error is smaller than any assignable; although what we have said in Props. 6, 7 & 8 establishes that it can easily be done by those means. Moreover, if indeed it is possible to produce direct demonstrations of these things, I do not hesitate to assert that they cannot be given except by admitting these fictitious quantities, infinitely small or infinitely large (see above, Scholium to Prop 7).”

²⁵ “[...] greater than any assignable by us, or greater than any number that can be designated”.

²⁶ “[...] for Geometry it suffices to demonstrate what follows from their supposition”.

There is, of course, much more to say on Leibniz's syncategorematic interpretation, in particular, concerning the philosophical status of infinitesimals as fictions. Other contributors to this volume will have more to say here on such issues. But I think it will be instructive for me to close by showing how Leibniz's use of infinities and infinitesimals can be justified by the Archimedean foundation he shared with Newton. Eberhard Knobloch has identified twelve rules occurring in his treatise that may be said to constitute Leibniz's "arithmetic of the infinite" (Knobloch, 2002, 67–8). In the interests of space I shall just consider a small sample. The first of these rules is "Finite + infinite = infinite". Rule 2.1 is "Finite \pm infinitely small = finite," and Rule 2.2 is "If $x = y +$ infinitely small, then $x - y \approx 0$ (is unassignable)" where x and y are finite quantities.

Let us take 2.2 first. If $x = y + dy$, where dy is an arbitrarily small finite variable quantity, and D is any pre-assigned difference between x and y , no matter how small, then dy may always be taken so small that $dy < D$. In particular, if D is supposed to be some fixed ultimate difference between them, then dy can be supposed smaller: so long as D and dy are quantities obeying the *Archimedean axiom*, the variability of dy means that it can always take a value such that $dy < D$ for any assigned D . Therefore, since the difference between x and y is smaller than any assignable, it is unassignable, and effectively null. The same reasoning justifies 2.1.

Leibniz gives such an argument explicitly in a short paper dated 26 March, 1676:

Videndum exacte an demonstrari possit in quadraturis, quod differentia non tamen sit infinite parva, sed quod omnino nulla, quod ostendetur, si constet eousque inflecti semper posse polygonum, ut differentia assumpta etiam infinite parva minor fiat error. Quo posito sequitur non tantum errorem non esse infinitum parvum, sed omnino esse nullum. Quippe cum nullus assumi possit.²⁷ (A VI, 3, 434)

Notable here is his claim that this argument works even if the difference D is assumed infinitely small; it does so, of course, only if the variable dy obeys the *Archimedean axiom*.²⁸

²⁷ "We need to see exactly whether it can be demonstrated in quadratures that a difference is not after all infinitely small, but nothing at all. And this will be shown if it is established that a polygon can always be inflected to such a degree that even when the difference is assumed infinitely small, the error will be smaller. Granting this, it follows not only that the error is not infinitely small, but that it is nothing at all – since, of course, none can be assumed." (DLC, 64–65)

²⁸ As Sam Levey has pointed out to me, this will also entail that the n in the Archimedean axiom would have to be allowed to range over infinite numbers. In that case, by the same