Jaroslav Lukeš, Jan Malý, Ivan Netuka, Jiří Spurný

Integral Representation Theory

Applications to Convexity, Banach Spaces and Potential Theory

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28 Painlevé Differential Equations in the Complex Plane, *Valerii I. Gromak, Ilpo Laine and Shun Shimomura*
29 Discontinuous Groups of Isometries in the Hyperbolic Plane, *Werner Fenchel and Jakob Nielsen*
30 The Reidemeister Torsion of 3-Manifolds, *Liviu I. Nicolaescu*
31 Elliptic Curves, *Susanne Schmitt and Horst G. Zimmer*
32 Circle-valued Morse Theory, *Andrei V. Pajitnov*
33 Computer Arithmetic and Validity, *Ulrich Kulisch*
34 Feynman-Kac-Type Theorems and Gibbs Measures on Path Space, *József Lörinczi, Fumio Hiroshima and Volker Betz*
35 Integral Representation Theory, *Jaroslas Lukeš, Jan Malý, Ivan Netuka and Jiří Spurný*
36 Introduction to Harmonic Analysis and Generalized Gelfand Pairs, *Gerrit van Dijk*
37 Bernstein Functions, *René Schilling, Renming Song and Zoran Vondraček*

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Introduction

In many branches of mathematics, one encounters the question of how to reconstruct a convex set from information on its vertices. This idea successfully emerged as the Krein–Milman theorem for compact convex subsets of locally convex spaces since any such set has plenty of extreme points. For any point of a compact convex set, a reformulation of the Krein–Milman theorem provides a representing measure that is concentrated in some sense on the set of extreme points. The goal of our book is to present a more general approach to integral representation theory based upon a notion of a function space and apply the obtained results to the theory of convex sets, Banach spaces and potential theory.

We point out that this approach is far from being new, but we hope that our exposition may be profitable both for students interested in the basics of integral representation theory as well as for more advanced readers. The former group could be attracted by a self-contained presentation of the Choquet theory, the latter by a substantial amount of results of fairly recent origin or appearing in a book form for the first time. We also try to incorporate more techniques from descriptive set theory into subject, which further supports our belief that the book will be worth reading even for those well acquainted with the monographs by E. M. Alfsen [5], R. R. Phelps [374], Z. Semadeni [414], L. Asimow and A. J. Ellis [24] or V. P. Fonf, J. Lindenstrauss and R. R. Phelps [179].

Let us continue by looking briefly at the contents of the book. After a prologue on the Korovkin theorem, we present basic facts on the extremal structure of finite-dimensional compact convex sets. Then we move on to infinite-dimensional spaces and prove the Krein–Milman theorem and several of its consequences. The second part of Chapter 2 studies the concept of measure convex and measure extremal sets.

Chapter 3 is devoted to cornerstones of the Choquet theory of functions spaces such as the Choquet order and its properties and integral representation theorems due to G. Choquet and E. Bishop and K. de Leeuw. Even though the results are standard, the key limiting process is established by means of the Simons lemma, which allows us to present later on several of its applications. The chapter is finished by a discussion on deeper properties of the Choquet ordering.

The next chapter studies basic properties of affine functions on compact convex sets and characterizations of functions satisfying the barycentric formula. A link between the theory of function spaces and compact convex sets starts to emerge at the end of the chapter.

Chapter 5 is crucial for the subsequent application of descriptive set theory; it describes a hierarchy of Borel sets and functions in topological spaces and proves their basic properties. The most important fact is that many descriptive properties are stable with respect to perfect mappings, which allows us to transfer abstract Borel affine functions to the setting of compact convex sets.

Simplicial function spaces are studied in Chapter 6. We discuss several classes of simplicial function spaces, namely the Bauer and Markov simplicial function spaces and spaces with boundary of type F_σ. Among other results, the abstract Dirichlet problem for continuous and non-continuous functions is considered. Choquet simplices are presented at the end of the chapter.

Next we generalize the basic concepts for function cones since they are indispensable in potential theory. We focus in particular on ordered compact convex sets.

Analogues of faces in a non-convex setting, so-called Choquet sets, are investigated in Chapter 8. The main result is a characterization of simplicial spaces by means of Choquet sets.

Suitably chosen families of closed extremal sets generate interesting boundary topologies on the set of extreme points. Chapter 9 studies these topologies and functions continuous with respect to them. It turns out that maximal measures induce measures on sets of extreme points that are regular with respect to boundary topologies. The last section is devoted to a study of a facial topology and facially continuous functions.

Chapter 10 collects several deeper results on function spaces and compact convex sets. Among others, study of Shilov and James boundaries, Lazar's improvement of the Banach–Stone theorem, results on automatic boundedness of affine and convex functions, embedding of ℓ^1 in Banach spaces, metrizability of compact convex sets and their open images and some topological properties of the set of extreme points.

The Lazar selection theorem and its consequences occupy the first part of Chapter 11. The second part is devoted to a presentation of Debs' proof of Talagrand's theorem on measurable selectors.

Chapter 12 is concerned with two methods of constructing new function spaces: products and inverse limits. We show that both operations preserve simpliciality and describe resulting boundaries. The inverse limits lead to an interesting description of metrizable simplices as inverse limits of finite-dimensional simplices. The general results are illustrated by a construction of the Poulsen simplex and a couple of compact convex sets due to Talagrand.

In Chapter 13, general results from the Choquet theory are applied to potential theory and several of its basic notions are investigated from this perspective. Important function cones and spaces appearing in potential theory are studied in detail, in particular, in connection to various solution methods for the Dirichlet problem. The functional analysis approach makes it possible to provide an interesting interpretation, for instance, of balayage and regular points in terms of representing measures and the Choquet boundary of suitable spaces and cones. The exposition covers potential theory for the Laplace equation and the heat equation as well as a more general setting (harmonic spaces, fine potential theory etc.).

The final Chapter 14 presents several applications of the integral representation theorems, such as for doubly stochastic matrices, the Riesz–Herglotz theorem, the Lyapunov theorem on the range of a vector measure, the Stone–Weierstrass theorem, positive-definite functions and invariant and ergodic measures.

Each chapter concludes with a series of exercises with sketches of proofs and with concluding notes and comments where we try to give precise references and due credits for the results presented in the main body of the text, and discuss additional material which is related to the topics of the chapter in question, but was not included with complete proofs. Open problems are also mentioned.

Since the presented material originates in an amalgamation of functional analysis, measure theory, topology, descriptive set theory and potential theory, we collect the needed notions and facts in the Appendix, sometimes even with proofs. We selected the following books for each subject as the key references: W. Rudin [403] and M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant and V. Zizler [173] for functional analysis, D. H. Fremlin [182], [181] and [183] for measure theory, R. Engelking [169] and K. Kuratowski [285] for topology, A. S. Kechris [262] and C. A. Rogers and J. E. Jayne [394] for descriptive set theory, D. H. Armitage and S. J. Gardiner [21] for classical potential theory and J. Bliedtner and W. Hansen [66] for abstract potential theory.

Next we point out what is omitted from the book. First, we focus on integral representation theorems for compact sets, and thus the readers interested in theory of sets with the Radon–Nikodym property are referred to R. D. Bourgin [82], and those interested in Choquet theory in sets of measures are referred to G. Winkler [473]. Second, although we consider several geometric aspects of simplicial spaces, they are not at the center of our attention. They are thoroughly investigated in H. E. Lacey [290] and P. Harmand, D. Werner and W. Werner [216]. Further, we do not pursue applications of integral representation theory in C^*-algebras and thus we refer the interested reader to E. M. Alfsen and F. W. Schultz [10] and [9], M. Rørdam [395] and M. Rørdam and E. Størmer [396], B. Blackadar [59] and H. Lin [303] and the references therein. And last but not least, our applications to potential theory do not require the full strength of abstract potential theory and thus we restrict ourselves to a less general framework than the one presented in J. Bliedtner and W. Hansen [66].

Except on a few explicitly stated occasions, we consider only real vector spaces and apart from Chapter 9 we deal only with Hausdorff topologies and Radon measures. We use the standard notation and terminology:

- $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the usual sets of numbers,
- $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real and imaginary part of a complex number z, respectively,

- c_A is the characteristic function of a set A (sometimes we write 1 for the characteristic function of a space),
- $A \triangle B$ is the symmetric difference of sets A and B,
- A^c is the complement of a set A,
- $f \wedge g$, $f \vee g$ denote the infimum and supremum of functions f, g, respectively, (usually they are considered pointwise),
- f^+, f^-, $|f|$ denote the positive and negative parts, and absolute value of a function f, respectively,
- $f|_A$ is the restriction of a function f to a set A,
- if $\mathcal{F}$ is a system of functions, $\mathcal{F}^b$ and $\mathcal{F}^+$ are the families of all bounded and positive elements from $\mathcal{F}$, respectively,
- ω_0 and ω_1 are the first infinite and first uncountable ordinals, respectively,
- $\overline{A}$, $\operatorname{Int} A$, ∂A are the closure, interior and boundary of a set A in a topological space, respectively,
- $\operatorname{dist}(A, B)$ denotes the distance of sets in a metric space,
- $\operatorname{diam} A$ is the diameter of a set A in a metric space,
- $U(x, r)$, $B(x, r)$ and $S(x, r)$ are the open ball, closed ball and sphere centered at x with radius $r > 0$, respectively,
- $\operatorname{co} A$ and $\operatorname{span} A$ are the convex and linear hull of a set A in a vector space, respectively, $\overline{\operatorname{co}}\, A$ is the closed convex hull of a set A in a topological vector space,
- $\ker T$ denotes the kernel of an operator between linear spaces,
- B_E, U_E and S_E are the closed unit ball, open unit ball and sphere of a normed linear space E, respectively,
- E/F is the quotient space of a locally convex space with respect to a closed subspace F,
- $E \oplus F$ is the sum of locally convex spaces E and F,
- E^* is the dual space of a topological linear space E,
- (x, y) stands for the scalar product of vectors x, y in a Hilbert space,
- c_0 is the space of sequences converging to 0,
- $\mathcal{C}(X)$ is the space of real-valued continuous functions on a topological space X,
- $\mathcal{C}^b(X)$ is the space of bounded continuous functions on a topological space X,
- ℓ^p and $L^p(\mu)$, $p \in [1, \infty]$, are the usual Lebesgue spaces (see Section A.3),
- $\mathcal{C}^n(U)$, $\mathcal{C}^n$, $\mathcal{C}^\infty(U)$, $\mathcal{C}^\infty$ stand for the space of n-times continuously differentiable functions on U or infinitely differentiable functions on U, respectively,

- $⨍_A f(y)\,dy$ is the integral mean value of f over a set A,
- $\int_{S(x,r)} f(y)\,dS(y)$ is the surface integral of f over the sphere $S(x,r) \subset \mathbb{R}^d$,
- ∇f is the gradient of f.

A function f is *positive* if $f \geq 0$, it is *strictly positive* if $f > 0$. Similarly we use *increasing*, *strictly increasing* and so on. If μ is a measure, we often write $\mu(f)$ for the integral $\int f\,d\mu$.

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Contents

Chapter 1
Prologue

1.1 The Korovkin theorem

We start with the famous Weierstrass approximation theorem.

Theorem 1.1 (The Weierstrass approximation theorem). *The space of all polynomial functions on the interval* $[0,1]$ *is uniformly dense in the space* $\mathcal{C}([0,1])$.

There are several different proofs of this result and several methods for how to associate to a given continuous function $f \in \mathcal{C}([0,1])$ a sequence of polynomials $\{P_n\}$ that converges uniformly to f on $[0,1]$.

For example, given $f \in \mathcal{C}([0,1])$ and $n \in \mathbb{N}$, we define the corresponding *Bernstein polynomial* $B_n f$ by

$$B_n f : x \mapsto \sum_{j=0}^{n} \binom{n}{j} f\left(\frac{j}{n}\right) x^j (1-x)^{n-j}, \quad x \in [0,1].$$

The task is to show that the sequence $\{B_n f\}$ converges uniformly to f on $[0,1]$. This can be easily verified in the case when

$$f(x) = 1, x \text{ or } x^2,$$

since

$$B_n 1 = 1, \quad B_n x = x \quad \text{and} \quad B_n x^2 = \frac{n-1}{n} x^2 + \frac{1}{n} x.$$

Surprisingly, this is all that we need to compute, since these three tests are enough to guarantee the uniform convergence of $B_n f$ to f for all f in $\mathcal{C}([0,1])$. Indeed, one of the current proofs of the classical Weierstrass approximation theorem is based on the Korovkin theorem about linear operators. The Weierstrass theorem is an easy consequence since the mappings

$$B_n : f \mapsto B_n f, \quad f \in \mathcal{C}([0,1]),$$

are positive linear operators on $\mathcal{C}([0,1])$.

Theorem 1.2 (Korovkin). *Let* p_j, $j = 0,1,2$, *denote the monomial function* $p_j : x \mapsto x^j$ *and let* $\{T_n\}$ *be a sequence of positive linear operators on the space* $\mathcal{C}([0,1])$. *Assume that* $T_n p_j \to p_j$ *uniformly on* $[0,1]$ *as* $n \to \infty$ *for* $j = 0,1,2$. *Then* $T_n f \to f$ *uniformly on* $[0,1]$ *as* $n \to \infty$ *for all* $f \in \mathcal{C}([0,1])$.

Proof. Pick $f \in \mathcal{C}([0,1])$ and $\varepsilon > 0$. By the uniform continuity of f, there exists $\delta \in (0,1)$ such that $|f(s) - f(t)| \le \varepsilon$ for any $s, t \in [0,1]$, $|s-t| \le \delta$. Now fix $t \in [0,1]$ and set

$$p_*(x) := f(t) - \varepsilon - (x-t)^2 \frac{2\|f\|}{\delta^2}, \quad x \in [0,1],$$

and

$$p^*(x) := f(t) + \varepsilon + (x-t)^2 \frac{2\|f\|}{\delta^2}, \quad x \in [0,1].$$

Dealing separately with the cases $|x-t| \le \delta$ and $|x-t| > \delta$, we get

$$|f(x) - f(t)| \le \varepsilon + 2(x-t)^2 \frac{\|f\|}{\delta^2}$$

for each $x \in [0,1]$. Hence,

$$p_*(x) \le f(x) \le p^*(x), \quad x \in [0,1],$$

and, therefore, for any $n \in \mathbb{N}$,

$$T_n p_* \le T_n f \le T_n p^*.$$

We find $N \in \mathbb{N}$ such that for any $n \ge N$

$$\big\|T_n p_j - p_j\big\| < \varepsilon\delta^2, \qquad j = 0, 1, 2.$$

Since

$$p^*(x) = \Big(f(t) + \varepsilon + t^2 \frac{2\|f\|}{\delta^2}\Big) - \frac{4\|f\|t}{\delta^2} x + \frac{2\|f\|}{\delta^2} x^2, \quad x \in [0,1],$$

for $n \ge N$ we have the estimate

$$\|T_n p^* - p^*\| \le C\varepsilon \quad \text{with} \quad C := 9\|f\| + \varepsilon.$$

In particular,

$$\big(T_n f\big)(t) \le \big(T_n p^*\big)(t) \le p^*(t) + C\varepsilon = f(t) + C\varepsilon + \varepsilon, \quad n \ge N,$$

and similarly

$$\big(T_n f\big)(t) \ge \big(T_n p_*\big)(t) \ge p_*(t) - C\varepsilon = f(t) - C\varepsilon - \varepsilon, \quad n \ge N.$$

Hence

$$\|T_n f - f\| \le \varepsilon(C+1), \qquad n \ge N,$$

and the proof is complete. □

In stating the Korovkin theorem, which sometimes bears the name the *first Korovkin theorem*, it is possible to go further, replacing the interval $[0, 1]$ by a suitable space, and the set of three functions $\{p_0, p_1, p_2\}$ by a more general family of functions. In the sequel, we will take a deeper look at this issue.

Definition 1.3 (Korovkin closure). Let K be a (metrizable) compact space and $\mathcal{P}$ a family of continuous functions on K (sometimes called *test functions*). We say that a sequence $\{T_n\}$ of positive operators on $\mathcal{C}(K)$ is $\mathcal{P}$*-admissible* if $\|T_n\varphi - \varphi\| \to 0$ for any $\varphi \in \mathcal{P}$, and define the *Korovkin closure of* $\mathcal{P}$ as

$$\mathrm{Kor}(\mathcal{P}) := \{f \in \mathcal{C}(K) : \|T_n f - f\| \to 0 \text{ for any } \mathcal{P}\text{-admissible sequence } \{T_n\}\} .$$

Let $\mathcal{H}$ be the linear span of $\mathcal{P}$. It is simple to check that

$$\mathcal{H} \subset \mathrm{Kor}(\mathcal{H}) = \mathrm{Kor}(\mathcal{P}).$$

Two questions immediately arise:

(a) How can $\mathrm{Kor}(\mathcal{P})$ be characterized ?

(b) Under what conditions does the equality $\mathrm{Kor}(\mathcal{P}) = \mathcal{C}(K)$ hold ?

In what follows, we will give answers to these questions and will also study analogous problems. To these ends, the framework of abstract linearity and convexity will turn out to be useful and efficient.

1.2 Notes and comments

The Korovkin theorem was proved independently by H. Bohman in [74] for a kind of special positive operators, and by P. P. Korovkin in [277] for integral-type operators. Korovkin extended his theory in [278] and we followed his proof from this monograph. The Korovkin theorem 1.2 sometimes bears the name of the Bohman–Korovkin theorem. Excellent sources for the Korovkin material are the monograph of F. Altomare and M. Campiti [13], and Chauvenet's prize paper of H. Bauer [42].

Chapter 2
Compact convex sets

We begin our exposition with classical results on convex sets in finite-dimensional spaces. After showing Carathéodory's theorem 2.6, we define extreme points and prove Minkowski's theorem 2.11 stating that any compact convex set in $\mathbb{R}^d$ is the convex hull of its extreme points. An amalgamation of these two results contained in Theorem 2.12 is a starting point leading to generalizations in infinite-dimensional spaces.

So the next section is devoted to the study of the Krein–Milman theorem and related results. In particular we are interested in its reformulation known as the Integral representation theorem. The basic idea of representing points of a compact convex set as barycenters of probability measures is a central topic of the whole book. Thus after the proof of the Krein–Milman theorem 2.22 and Bauer's minimum principle 2.24 we define the barycenter of a probability measure on a compact convex set and show its existence and uniqueness (see Theorem 2.29). Then the Integral representation theorem 2.31 and properties of the barycentric mapping are proved. We finish this part by some classical facts: Bauer's characterization 2.40 of extreme points of a compact convex set, Choquet's observation on extreme points of a compact convex set contained in Proposition 2.41 and the Milman theorem 2.43.

The aim of Subsection 2.1.C is to show that a metrizable compact convex set has abundance of exposed points, namely, that a metrizable compact convex set is the closed convex hull of its exposed points and that exposed points are dense in the set of extreme points.

Section 2.2 prepares the ground for examples concerning extremal sets and faces of compact convex sets presented in Section 2.3. We prove several facts on probability measures on compact spaces and show how they lead to a construction of affine functions on compact convex sets that do not satisfy the barycentric formula (see Proposition 2.63).

Subsection 2.3.A investigates more closely extremal sets and faces of compact convex sets. The main result contained in Proposition 2.69 shows that a closed extremal set is a union of closed faces. We generalize the concept of convexity and extremality in Subsections 2.3.B and 2.3.C by introducing measure convex and measure extremal sets. The main tool is Theorem 2.75 due to D. H. Fremlin and J. D. Pryce that characterizes measure convex sets. Then we show that convex sets of low Borel complexity are also measure convex, but that there are examples of F_σ or G_δ faces that are not measure convex. Analogous results are proved in the next section for extremal and measure extremal sets.

2.1 Geometry of convex sets

2.1.A Finite-dimensional case

Throughout this subsection, let W be a real vector space.

Definition 2.1 (Convex sets in vector spaces). A set $C \subset W$ is *convex* if $\lambda x + (1-\lambda)y \in C$ whenever $x, y \in C$ and $\lambda \in (0,1)$.

Let A be an arbitrary subset of W. The *convex hull* of A, denoted by $\operatorname{co} A$, is the intersection of all convex sets of W that contain A. Since W is a convex set and the intersection of any family of convex sets is convex, the set $\operatorname{co} A$ is the smallest convex set containing A. It is easy to check that

$$\operatorname{co} A = \Big\{ \sum_{j=1}^{n} \lambda_j x_j : n \in \mathbb{N}, x_1, \ldots, x_n \in A \text{ and } \lambda_1, \ldots, \lambda_n \geq 0, \sum_{j=1}^{n} \lambda_j = 1 \Big\}.$$

Definition 2.2 (Affine independence and n-simplices). Recall that vectors $e_0, \ldots, e_n$ of W are said to be *affinely independent* if $e_1 - e_0, \ldots, e_n - e_0$ are linearly independent. In other words, if whenever

$$\lambda_0 e_0 + \cdots + \lambda_n e_n = 0 \quad \text{and} \quad \lambda_0 + \cdots + \lambda_n = 0,$$

then $\lambda_0 = \cdots = \lambda_n = 0$. In this case, the convex hull $\operatorname{co}\{e_0, \ldots, e_n\}$ is termed an n*-simplex* with vertices $e_0, \ldots, e_n$.

In $\mathbb{R}^d$, there exist at most $d+1$ affinely independent points.

Definition 2.3 (Affine hulls and subspaces, hyperplanes). For a set $A \subset W$, the *affine hull* of A, denoted by $\operatorname{aff} A$, is the set of all affine combinations of points of A. (A linear combination $\alpha_1 x_1 + \cdots + \alpha_n x_n$, where $\alpha_1 + \cdots + \alpha_n = 1$, is called an *affine combination* of points $x_1, \ldots, x_n$.)

A set $A \subset W$ is said to be an *affine subspace* of W if $\operatorname{aff} A = A$. Affine subspaces are just the translations (of type) $x + F$, where F is a linear subspace of W and $x \in W$. By definition, the *dimension* (or, the *codimension*) of $x + F$ is the dimension (or, the codimension, respectively) of F.

A set H is a *hyperplane* if there exists a nonzero linear functional f on W and $\alpha \in \mathbb{R}$ such that

$$H = \{w \in W : f(w) = \alpha\}.$$

Since a subspace F of W is a maximal proper subspace of W if and only if there exists a nonzero linear functional f on W such that $F = \ker f$, we see that H is a hyperplane if and only if there exist a maximal proper subspace F of W and $w \in W$ such that $H = w + F$. In other words, hyperplanes are exactly affine subspaces of codimension 1.

Let C be a subset of W and $H := \{w \in W : f(w) = \alpha\}$ be a hyperplane. We say that H is a *support hyperplane* of C if $C \cap H \neq \varnothing$ and either

$$C \subset \{w \in W : f(w) \leq \alpha\} \quad \text{or} \quad C \subset \{w \in W : f(w) \geq \alpha\}.$$

Any point $c \in C \cap H$ is called an *H-support point* of C. We also say that H *supports* C at c.

In the sequel, we need the following assertion.

Proposition 2.4. *Let C be a closed convex subset of $\mathbb{R}^d$ with a nonempty interior and $c \in \partial C$. Then there exists a hyperplane H such that H supports C at c.*

Proof. See, for example, A. Barvinok [31], Corollary 2.8. □

Remark 2.5. In what follows, we direct our attention to geometry of compact convex sets in the Euclidean d-dimensional space $\mathbb{R}^d$. All results of this subsection remain valid in any finite-dimensional topological vector space, since any such space is isomorphic to a suitable space $\mathbb{R}^d$.

Theorem 2.6 (Carathéodory). *Let A be an arbitrary subset of $\mathbb{R}^d$. Then each point of* co A *is a convex combination of at most $d+1$ points of A which are affinely independent.*

Proof. Assume that $x \in \operatorname{co} A$,

$$x = \lambda_1 x_1 + \cdots + \lambda_n x_n$$

where $x_1, \ldots, x_n \in A$, $\lambda_j > 0$ for all $j = 1, \ldots, n$ (which we may suppose) and $\lambda_1 + \cdots + \lambda_n = 1$. If the vectors $x_1, \ldots, x_n$ are affinely independent, then $n \leq d+1$ and we are done. Otherwise, $n > d+1$ and there is $(\alpha_1, \ldots, \alpha_n) \neq (0, \ldots, 0)$ such that

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = 0 \quad \text{and} \quad \alpha_1 + \cdots + \alpha_n = 0.$$

Let $k \in \{1, \ldots, n\}$ be such that

$$\left|\frac{\alpha_j}{\lambda_j}\right| \leq \left|\frac{\alpha_k}{\lambda_k}\right| \quad \text{for all} \quad j = 1, \ldots, n.$$

Setting

$$\eta_j := \lambda_j - \frac{\lambda_k}{\alpha_k}\alpha_j, \quad j = 1, \ldots, n,$$

we have

$$x = \sum_{j \neq k} \eta_j x_j, \quad \sum_{j \neq k} \eta_j = 1 \quad \text{and } \eta_j \geq 0 \text{ for } j = 1, \ldots, n.$$

Thus, x is a convex combination of $n-1$ points. If these points are affinely independent, the proof is finished. If not, the above argument can be repeated, and after finitely many steps x can be represented as a convex combination of affinely independent points of A. □

Corollary 2.7. *The convex hull* co A *of any set* $A \subset \mathbb{R}^d$ *is the union of all* n*-simplices* ($n \leq d$) *with vertices in* A.

Proof. Obviously, any n-simplex with vertices in A, where $n \leq d$, is a subset of co A. The reverse inclusion immediately follows from Carathéodory's theorem 2.6. □

Corollary 2.8. *The convex hull of any compact subset of* $\mathbb{R}^d$ *is compact.*

Proof. Let K be a compact subset of $\mathbb{R}^d$ and

$$D := \Big\{\lambda \in \mathbb{R}^{d+1} : \lambda = (\lambda_0, \ldots, \lambda_d), \sum_{j=0}^{d} \lambda_j = 1 \text{ and } \lambda_j \geq 0 \text{ for } j = 0, \ldots, d\Big\}.$$

The mapping $F : D \times K \times \cdots \times K \to K$ defined as

$$F : (\lambda, x_0, \ldots, x_d) \mapsto \sum_{j=0}^{d} \lambda_j x_j$$

is continuous. By Carathéodory's theorem 2.6,

$$\operatorname{co} K = F(D \times K \times \cdots \times K).$$

Hence, co K, as a continuous image of the compact set $D \times K \times \cdots \times K$, is compact. □

Definition 2.9 (Extreme points). A point z of a set $C \subset W$ is called an *extreme point* of C if z is not an internal point of any segment having its endpoints in C. In other words, z is an extreme point of C if $x, y \in C$, $\lambda \in (0, 1)$ and $z = \lambda x + (1 - \lambda)y$, implies $x = y$. It is easy to check that z is an extreme point of a convex set C if and only if the set $C \setminus \{z\}$ is convex, and this is the case if and only if z is not a midpoint of any nondegenerate segment having its endpoints in C.

We denote by ext C the set of all extreme points of C.

Lemma 2.10. *Let* S *be an* n*-simplex in* $\mathbb{R}^d$ *with vertices* $e_0, \ldots, e_n$. *Then*

$$\operatorname{ext} S = \{e_0, \ldots, e_n\}.$$

Proof. Let $x \in \operatorname{ext} S$. Since $S = \operatorname{co}\{e_0, \ldots, e_n\}$ and the set $S \setminus \{x\}$ is convex, $x = e_k$ for some $k \in \{0, 1, \ldots, n\}$. Conversely, select e_k and assume that

$$e_k = \frac{1}{2}s + \frac{1}{2}t$$

where $s, t \in S = \operatorname{co}\{e_0, \ldots, e_n\}$. Write

$$s = \sum_{j=0}^{n} \alpha_j e_j, \quad t = \sum_{j=0}^{n} \beta_j e_j,$$

with

$$\alpha_j, \beta_j \geq 0, \quad \sum_{j=0}^{n} \alpha_j = \sum_{j=0}^{n} \beta_j = 1.$$

Then

$$e_k = \sum_{j=0}^{n} \frac{1}{2}(\alpha_j + \beta_j) e_j$$

or

$$\sum_{j \neq k} \frac{1}{2}(\alpha_j + \beta_j)(e_j - e_k) = 0.$$

Consequently,

$$\alpha_j + \beta_j = 0 \quad \text{for} \quad j \in \{0, \dots, n\} \setminus \{k\},$$

thus $s = t = e_k$. □

The following assertion shows the prominent role of extreme points in finite-dimensional compact convex sets. Infinite-dimensional situation is more complicated, see Example 2.15 and the Krein–Milman theorem 2.22.

Theorem 2.11 (Minkowski). *Each point of a compact convex set $C \subset \mathbb{R}^d$ is a convex combination of extreme points of C.*

Proof. We proceed by induction on the dimension d. For the dimension $d = 0$, the set C reduces to a one-point set and the assertion holds. So assume that $d > 0$ and that the assertion is valid for compact convex sets in spaces of dimension smaller than d. We may also assume that the interior of C is nonempty, for otherwise C is a subset of an affine subspace of a smaller dimension (cf. Exercise 2.107(c)) and the assertion follows by the induction assumption.

We distinguish two cases. If x is a boundary point of C, then by Proposition 2.4 there exists a support hyperplane L of C at x. Then the compact convex set $F := C \cap L$ lies in the affine subspace L of dimension smaller than d. By the induction assumption, $x \in \operatorname{co} \operatorname{ext} F$. Since obviously $\operatorname{ext} F \subset \operatorname{ext} C$, the induction step is finished.

Now suppose that $x \in \operatorname{Int} C$. There exists a segment $[a, b] \subset C$ such that $x \in (a, b)$ and $a, b \in \partial C$. Since $a, b \in \operatorname{co} \operatorname{ext} C$ by the previous argument, we see that x can be expressed as a convex combination of extreme points of C. □

Theorem 2.12 (Minkowski–Carathéodory). *Each point of a compact convex set $C \subset \mathbb{R}^d$ is a convex combination of (at most $d + 1$) affinely independent extreme points of C.*

Proof. A consequence of Theorems 2.6 and 2.11. Indeed, given a point $c \in C$, by the Minkowski theorem 2.11, there exists a set $A \subset \operatorname{ext} C$ such that $c \in \operatorname{co} A$. Now, it suffices to apply Carathéodory's theorem 2.6. □

Corollary 2.13. *Let x be a point of a compact convex set $C \subset \mathbb{R}^d$. Then there exists an n-simplex S, $n \leq d$, such that $x \in S \subset C$ and* $\operatorname{ext} S \subset \operatorname{ext} C$.

Proof. The assertion is a rewording of the previous Minkowski–Carathéodory theorem. □

Remark 2.14. In [1] E. M. Alfsen constructed a non-simplicial compact polyhedron in ℓ^1, showing that the conclusion of the previous Corollary 2.13 in infinite-dimensional spaces fails. At the same time, he posed a question that "it would be of some interest to find sufficient conditions for a compact convex set X to admit a decomposition" as in Corollary 2.13: Given $x \in X$, there would exists a set $S \subset X$ such that $x \in S$, $\operatorname{ext} S \subset \operatorname{ext} C$, and a unique representing measure for x carried by $\operatorname{ext} S$. We present in Exercise 6.93 an example illustrating this phenomenon.

2.1.B The Krein–Milman theorem

Recall that a point z of a subset C of a vector space W is an *extreme point* of C if z is not an internal point of any nondegenerate segment having endpoints in C and that $\operatorname{ext} C$ denotes the set of all extreme points of C.

In the Euclidean space $\mathbb{R}^d$, the set $\operatorname{ext} C$ is of fundamental importance. The Minkowski theorem 2.11 says that each point of a compact convex set $C \subset \mathbb{R}^d$ is a convex combination of extreme points of C. Thus, $C = \operatorname{co}(\operatorname{ext} C)$.

The aim of this subsection is to examine an analogous result and its relatives in the framework of infinite-dimensional spaces. Note, that in infinite-dimensional spaces, a compact convex set need not be a convex hull of its extreme points, as Example 2.15 shows.

Example 2.15. Let $\{e_n\}_{n=1}^{\infty}$ be the orthonormal basis in ℓ^2 formed by the standard unit vectors e_n, and let

$$B := \{0, e_1, \frac{1}{2}e_2, \frac{1}{3}e_3, \dots\} \quad \text{and} \quad C := \overline{\operatorname{co}}\, B.$$

Since B is clearly compact, it is easy to see that C is a compact convex set (see [173], Exercise 1.56). By the Milman theorem 2.43, $\operatorname{ext} C \subset B$. (In fact, it is easy to verify that $\operatorname{ext} C = B$.) Defining

$$x_n := (1 - 2^{-n})^{-1} \sum_{k=1}^{n} 2^{-k} \frac{1}{k} e_k, \quad n \in \mathbb{N},$$

we have $x_n \in \operatorname{co} \operatorname{ext} C$ and

$$x_n \to x := \sum_{k=1}^{\infty} 2^{-k} \frac{1}{k} e_k \in C.$$

Since every element of $\operatorname{co} \operatorname{ext} C$ has only a finite number of nonzero coordinates, $x \notin \operatorname{co} \operatorname{ext} C$.

Definition 2.16 (Extremal sets and faces). A generalization of the notion of extreme points leads to an important concept: A nonempty subset F of a set $C \subset W$ is an *extremal subset* of C if $x, y \in F$ provided that $x, y \in C$ and $\lambda x + (1-\lambda)y \in F$ for some $\lambda \in (0,1)$. It is needless to say that one-point extremal sets are exactly extreme points of C, and that C itself is an extremal set.

Convex extremal sets are called *faces*.

Definition 2.17 (Affine, concave and convex functions). Let C be a convex subset of a vector space W. A real-valued function s on C is said to be *concave* if

$$s\big(\lambda x + (1-\lambda)y\big) \geq \lambda s(x) + (1-\lambda)s(y)$$

for each $x, y \in C$ and $\lambda \in [0,1]$.

A real-valued function f on C is *convex* if $-f$ is concave and f is called *affine* if both f and $-f$ are concave.

Obviously, the restriction of any linear functional on W to C is an affine function.

Lemma 2.18. *If H is an extremal subset of F and F is an extremal subset of D, then H is an extremal subset of D.*

Proof. Obvious. □

Lemma 2.19. *If X is a nonempty compact convex subset of a locally convex space E, s is a concave lower semicontinuous function on X and*

$$L := \{x \in X : s(x) = \min s(X)\},$$

then L is a compact extremal subset of X.

If K is a nonempty compact subset of E, $f \in E^*$ and

$$H := \{x \in K : f(x) = \min f(K)\},$$

then H is a compact extremal subset of K.

Proof. It is clear that L is compact and nonempty. Choose $x, y \in X$, $\lambda \in (0,1)$. If $a := \lambda x + (1-\lambda)y$ and $a \in L$, then

$$s(a) \geq \lambda s(x) + (1-\lambda)s(y) \geq \lambda s(a) + (1-\lambda)s(a) = s(a).$$

From this it easily follows that $x, y \in L$.

The proof of the second assertion is similar. □

Proposition 2.20. *Let K be a nonempty compact subset of a locally convex space E and F a compact extremal subset of K. Then $F \cap \operatorname{ext} K \neq \varnothing$. In particular,* $\operatorname{ext} K \neq \varnothing$.

Proof. Consider the family $\mathcal{F}$ of all closed extremal subsets of F ordered by the reverse inclusion. If $\mathcal{R}$ is a chain in $\mathcal{F}$, then $Y := \bigcap\{R : R \in \mathcal{R}\}$ is nonempty in view of the compactness of K. Since it is easy to check that Y is an extremal subset of K, Y is an upper bound for $\mathcal{R}$. Zorn's lemma now provides a maximal element of this family, call it D. An appeal to the Hahn–Banach theorem reveals that D is a one-point set. Indeed, assuming that D were to contain distinct points x and y, the Hahn–Banach theorem would provide $f \in E^*$ such that $f(x) < f(y)$. By Lemma 2.19, the set

$$\{z \in D : f(z) = \min f(D)\}$$

would be a proper closed extremal subset of D and, in view of Lemma 2.18, also a closed extremal subset of F. This contradicts the maximality of D. Since, as mentioned above, one-point extremal sets are exactly extreme points of F, we can find a point $x \in \operatorname{ext} F$. Again, by Lemma 2.18, $x \in \operatorname{ext} K$ and we are done. □

Theorem 2.21. *Let K be a compact subset of a locally convex space E. Then $K \subset \overline{\operatorname{co}} \operatorname{ext} K$.*

Proof. Assume that there exists a point $x \in K \setminus \overline{\operatorname{co}} \operatorname{ext} K$. Using the geometric version of the Hahn–Banach theorem, there exists $f \in E^*$ such that

$$f(t) > f(x) \quad \text{for any } t \in \overline{\operatorname{co}} \operatorname{ext} K.$$

If

$$H := \{z \in K : f(z) = \min f(K)\},$$

then H is a (nonempty) extremal subset of K by Lemma 2.19. Hence, by Proposition 2.20, $H \cap \operatorname{ext} K \neq \varnothing$. Since $H \cap \overline{\operatorname{co}} \operatorname{ext} K = \varnothing$, this is impossible. Therefore, $K \subset \overline{\operatorname{co}} \operatorname{ext} K$. □

Theorem 2.22 (Krein–Milman). *Let X be a nonempty compact convex subset of a locally convex space E. Then $X = \overline{\operatorname{co}} \operatorname{ext} X$.*

Proof. It is pretty clear that $\overline{\operatorname{co}} \operatorname{ext} X \subset X$. The reverse inclusion follows from Theorem 2.21. □

Remarks 2.23. (a) The Krein–Milman theorem also holds in locally convex spaces over complex numbers. For a proof see, for example, W. Rudin [403], Theorem 3.21.

(b) If K is a compact subset of a locally convex space, then $\overline{\operatorname{co}}\, K = \overline{\operatorname{co}} \operatorname{ext} K$. This is an immediate consequence of Theorem 2.21.

Corollary 2.24 (Bauer's concave minimum principle). *Let s be a lower semicontinuous concave function on a nonempty compact convex subset X of a locally convex space. Then there exists $z \in \operatorname{ext} X$ such that $s(z) = \min s(X)$.*

Proof. Denote
$$D := \{x \in X : s(x) = \min s(X)\}\,.$$
Then D is a nonempty compact subset of X and, by Lemma 2.19, an extremal subset of X. By Proposition 2.20, $D \cap \operatorname{ext} X \neq \varnothing$, and thus the proof is complete. □

Remarks 2.25. (a) The Krein–Milman theorem is an easy consequence of Bauer's concave minimum principle. Indeed, if X is a nonempty compact convex subset of a locally convex space E, the constant function 1 on X attains its minimum on $\operatorname{ext} X$. Thus the set $\operatorname{ext} X$ is nonempty. If $x \in X \setminus \overline{\operatorname{co}}\operatorname{ext} X$, then, by the geometric version of the Hahn–Banach theorem, there exists $f \in E^*$ such that
$$f(x) < 1 \quad \text{and} \quad f \geq 1 \quad \text{on} \quad \overline{\operatorname{co}}\operatorname{ext} X.$$
This contradicts Bauer's minimum principle since $f|_X$ is a continuous affine function.

(b) In Section 3.9 we prove a generalization of Bauer's concave minimum principle by a different method.

Integral representation. Now we would like to show how to use the Krein–Milman theorem for establishing integral-type representation theorems.

Let x be a point of a compact convex set X in a locally convex space. Our aim is to find a Radon measure μ on X so that
$$f(x) = \int_X f\,d\mu$$
for any continuous affine function f on X. Of course, the Dirac measure ε_x at x is one such measure. However, we try to find other "representing" measures, preferably concentrated on a very small part of X. More precisely, we are looking for a measure μ such that

(a) the support $\operatorname{spt}\mu$ of μ is contained in the closure $\overline{\operatorname{ext} X}$ of the set of all extreme points of X, or even

(b) μ is carried by the set $\operatorname{ext} X$.

Definition 2.26 (Barycenter of a measure). Let X be a compact convex set in a locally convex space E. Denote by $\mathfrak{A}^c(X)$ the set of all continuous affine functions on X.

A point $x \in X$ is said to be the *barycenter* of a probability Radon measure $\mu \in \mathcal{M}^1(X)$ if the following *barycentric formula*
$$f(x) = \int_X f\,d\mu$$
holds for any $f \in \mathfrak{A}^c(X)$. Since the functionals from E^* separate the points of E, and since the restrictions to X of such functionals are elements of $\mathfrak{A}^c(X)$, we see that the

barycenter $r(\mu)$ of μ (which exists by Theorem 2.29), is uniquely determined. Note that

$$r(\mu) = \int_X t \, d\mu(t),$$

where the integral is to be understood as the Pettis integral.

In the case when $r(\mu) = x$, we also say that the measure μ *represents* the point x. In other words, the equality $x = r(\mu)$ means that the Integral representation theorem holds for the point x. We denote by $\mathcal{M}_x(\mathfrak{A}^c(X))$, or for short $\mathcal{M}_x(X)$, the set of all measures representing the point x.

Obviously, as noted above, the Dirac measure ε_x always represents the point x. In what follows, we answer the following questions:

(a) Does any Radon measure have a barycenter ?

(b) Is any point of X a barycenter of a Radon measure carried by $\overline{\operatorname{ext} X}$?

Proposition 2.27. *The space $\mathcal{M}^1(K)$ consisting of all probability measures on a compact space K is a compact convex subset of $\mathcal{M}(K)$ and*

$$\operatorname{ext} \mathcal{M}^1(K) = \{\varepsilon_x : x \in K\}\,.$$

The mapping $\varepsilon : x \mapsto \varepsilon_x$, $x \in K$, is a homeomorphism of K onto $\operatorname{ext} \mathcal{M}^1(K)$.

Proof. It is easy to verify that $\mathcal{M}^1(K)$ is a convex subset of $\mathcal{M}(K)$. By Theorem A.85(a), it is compact.

If $x \in K$ and

$$\varepsilon_x = \alpha\mu + (1-\alpha)\nu \quad \text{where } \mu, \nu \in \mathcal{M}^1(K) \text{ and } \alpha \in (0,1),$$

then

$$1 = \alpha\mu(\{x\}) + (1-\alpha)\nu(\{x\}).$$

This implies that $\mu(\{x\}) = \nu(\{x\}) = 1$, and therefore $\mu = \nu = \varepsilon_x$.

If $\mu \in \mathcal{M}^1(K)$ is not a Dirac measure, then there exists a compact set $F \subset K$ such that the measures $\nu := \mu|_F$ and $\lambda := \mu|_{K \setminus F}$ are nontrivial and distinct from μ. Since

$$\mu = \nu(F)\frac{\nu}{\nu(F)} + \lambda(K \setminus F)\frac{\lambda}{\lambda(K \setminus F)},$$

we see that μ is not an extreme point of $\mathcal{M}^1(K)$.

The mapping $x \mapsto \varepsilon_x$, $x \in K$, is an injective continuous mapping from K onto $\{\varepsilon_x : x \in K\}$ and hence K and $\operatorname{ext} \mathcal{M}^1(K)$ are homeomorphic. □

Corollary 2.28. *The set of all convex combinations of Dirac measures is dense in $\mathcal{M}^1(K)$.*

Proof. As an easy consequence of the Krein–Milman theorem 2.22 and the characterization of extreme points given by Proposition 2.27, we have

$$\mathcal{M}^1(K) = \overline{\mathrm{co}}\,\mathrm{ext}\,\mathcal{M}^1(K) = \overline{\mathrm{co}}\,\{\varepsilon_x : x \in K\}\,,$$

which finishes the proof. □

Theorem 2.29. *Let $X \neq \varnothing$ be a compact convex subset of a locally convex space E. Then each Radon measure from $\mathcal{M}^1(X)$ has a (unique) barycenter in X.*

Proof. With the uniqueness part already out of the way, we now concentrate on an existence proof. If a measure μ in question is molecular, $\mu = \sum_{j=1}^n \lambda_j \varepsilon_{x_j}$, where $x_j \in X$, $\lambda_j \geq 0$, $j = 1, \dots, n$, $\sum_{j=1}^n \lambda_j = 1$, then obviously $r(\mu) := \sum_{j=1}^n \lambda_j x_j \in X$ is the barycenter of μ. Given a measure $\mu \in \mathcal{M}^1(X)$, there exists a net $\{\mu_\gamma\}$ of molecular measures on X such that $\mu_\gamma \to \mu$. Since X is a compact set, there exists a subnet $\{r(\mu_\alpha)\}$ of $\{r(\mu_\gamma)\}$ converging to an element $z \in X$. Pick $f \in \mathfrak{A}^c(X)$. Then

$$f(z) = \lim_\alpha f(r(\mu_\alpha)) = \lim_\alpha \int_X f\, d\mu_\alpha = \int_X f\, d\mu,$$

and therefore z is a barycenter of μ. □

Definition 2.30 (Barycenter mapping). Let X be a compact convex subset of a locally convex space. The mapping $r : \mu \mapsto r(\mu)$, assigning to each measure $\mu \in \mathcal{M}^1(X)$ its barycenter, is called the *barycenter mapping*.

In Proposition 2.38 we show that the barycenter mapping is a continuous and affine mapping from $\mathcal{M}^1(X)$ into X. This mapping is surjective since $r(\varepsilon_x) = x$ for any $x \in X$.

Theorem 2.31 (Integral representation theorem). *Let X be a compact convex subset of a locally convex space E and let $x \in X$. Then there exists a measure $\mu \in \mathcal{M}^1(X)$ such that $r(\mu) = x$ and* $\mathrm{spt}\,\mu \subset \overline{\mathrm{ext}\,K}$.

Proof. From the Krein–Milman theorem 2.22, we can see the following fact: if $f \in \mathfrak{A}^c(X)$ and $f = 0$ on $\mathrm{ext}\,X$, then $f = 0$ on X. We denote by $\mathcal{B}$ the subspace of $\mathcal{C}(\overline{\mathrm{ext}\,X})$ consisting of all restrictions of functions in $\mathfrak{A}^c(X)$ to $\overline{\mathrm{ext}\,X}$. Then, for every $h \in \mathcal{B}$, there exists, by the above mentioned fact, a unique function $\widehat{h} \in \mathfrak{A}^c(X)$ which coincides with h on $\overline{\mathrm{ext}\,X}$. We fix $x \in X$ and set

$$\varphi : h \mapsto \widehat{h}(x), \quad h \in \mathcal{B}.$$

Evidently $\varphi \in \mathcal{B}^*$ and $\|\varphi\|_{\mathcal{B}} = 1$. The functional φ can be extended by the Hahn–Banach theorem from $\mathcal{B}$ to a functional $\Phi \in (\mathcal{C}(\overline{\mathrm{ext}\,X}))^*$ with the same norm. Since, in addition, $\Phi(1) = \varphi(1) = 1$, Φ is a positive functional. Indeed, if $f \in \mathcal{C}(\overline{\operatorname{ext} X})$, $f \geq 0$, $a = \frac{1}{2} \sup f(\overline{\operatorname{ext} X})$, then $\|a - f\| \leq a$ and so

$$a - \Phi(f) = \Phi(a) - \Phi(f) = \Phi(a - f) \leq \|a - f\| \leq a.$$

This yields $\Phi(f) \geq 0$. By the Riesz representation theorem, there exists a probability measure μ on $\overline{\operatorname{ext} X}$ such that $\Phi(f) = \int_X f \, d\mu$ for every function $f \in \mathcal{C}(\overline{\operatorname{ext} X})$. The measure μ can be regarded as a measure on X carried by the set $\overline{\operatorname{ext} X}$. Since obviously

$$\int_X g \, d\mu = \Phi(g) = \varphi(g|_{\overline{\operatorname{ext} X}}) = \widehat{g}(x) = g(x)$$

for every $g \in \mathfrak{A}^c(X)$, we see that the barycenter of the measure μ is exactly the point x. $\square$

Remarks 2.32. (a) Theorem 2.31 can be proved by an alternative manner. In fact, we can follow the proof of Proposition 2.39 step by step.

(b) In concrete applications, we are often able to characterize the set $\operatorname{ext} X$. However, the character of the elements of the set $\overline{\operatorname{ext} X} \setminus \operatorname{ext} X$ is generally rather obscure. Consequently, the information concerning the support of the measure from the theorem on integral representation is problematic, unless the set $\operatorname{ext} X$ is closed. Moreover, there is another problem. Let us imagine that the set of extreme points of a compact convex set X is dense in this set, that is, $\overline{\operatorname{ext} X} = X$. Then, naturally, the Krein–Milman theorem says nothing, and equally useless is the theorem on integral representation. Indeed, it suffices to take the Dirac measure ε_x at the point x for the measure representing the point x . This situation can actually occur. As an example, we can take the closed unit ball B in an arbitrary infinite-dimensional Hilbert space, which we of course consider to be equipped with the weak topology. The extreme points of B are then the points of the unit sphere, and its (weak) closure is equal to the whole ball B. A more sophisticated example with $\overline{\operatorname{ext} X} = X$ is the *Poulsen simplex* in the Hilbert space ℓ^2 (see Subsection 12.3.A).

However, much more is known. Namely, if we consider the so-called Hausdorff metric on the set $\mathcal{F}$ of all nonempty compact convex subsets of a given Banach space X of infinite dimension, the space $\mathcal{F}$ is complete and the set $\{C \in \mathcal{F} : \overline{\operatorname{ext} C} \neq C\}$ is merely meager in $\mathcal{F}$. This assertion was proved by V. L. Klee in [271]. Thus, in a certain sense, for the majority of compact convex sets we have $\overline{\operatorname{ext} C} = C$.

Hence the problem of whether it is possible to find a measure μ which is carried just on the set of extreme points in the theorem on integral representation is crucial. This problem was solved successfully by G. Choquet in the fifties of the 20th century and laid the foundations of the *Choquet theory.* We will devote the next chapters to it, in the more general setting of function spaces. The Choquet theory has provided many insights for abstract analysis, infinite-dimensional geometry, descriptive set theory, potential theory and other fields of mathematics. It has remained fruitful ever since and has found new applications again and again in deriving new results.

(c) In Sections 14.5 and 14.6, we exceptionally consider locally convex spaces over the field of complex numbers. Note that the Integral representation theorem 2.31 extends trivially to the complex case. Indeed, consider a compact convex subset X of a *complex* locally convex space E. Of course, E can be regarded as a locally convex space over the field of real numbers. Then, given $x \in X$, there exists, by Theorem 2.31, a measure $\mu \in \mathcal{M}^1(X)$ carried by $\overline{\operatorname{ext} X}$ such that

$$f(x) = \int_X f\, d\mu \tag{2.1}$$

for every *real* continuous functional f on E. Given a *complex* continuous functional F on E, we can apply (2.1) to $\operatorname{Re} F$ and $\operatorname{Im} F$ to conclude that

$$F(x) = \int_X F\, d\mu.$$

The Integral representation theorem will be applied several times in the sequel. For this purpose, the following easy consequence of the previous theorem will be useful.

Proposition 2.33 (Krein–Milman theorem with transfer). *Let E be a locally convex space of real-valued functions on a set M such that, for every $x \in M$, the evaluation functional $F_x : f \mapsto f(x)$ is continuous on E. Let $K \subset E$ be a compact convex set. Let Q be a compact space and $\Phi\colon y \mapsto \varphi_y$, $y \in Q$, be an injective continuous mapping of Q onto* $\operatorname{ext} K$. *Then there exists a probability measure μ on Q such that, for every $f \in K$,*

$$f(x) = \int_Q \varphi_y(x)\, d\mu(y), \quad x \in M.$$

Proof. Since $\operatorname{ext} K$ is a continuous image of a compact set, it is closed. Let $f \in K$. By Theorem 2.31, there exists a probability measure $\tilde{\mu}$ carried by $\operatorname{ext} K$ such that

$$F(f) = \int_{\operatorname{ext} K} F\, d\tilde{\mu} \tag{2.2}$$

for each continuous linear functional F on E. Let us define $\mu = \Phi^{-1}{}_{\sharp}\tilde{\mu}$. By Proposition A.92,

$$\begin{aligned} \int_{\operatorname{ext} K} F\, d\tilde{\mu} &= \int_{\operatorname{ext} K} (F \circ \Phi) \circ \Phi^{-1}\, d\tilde{\mu} = \int_Q F \circ \Phi\, d\Phi^{-1}{}_{\sharp}\tilde{\mu} \\ &= \int_Q F \circ \varphi_y\, d\mu(y). \end{aligned} \tag{2.3}$$

Applying (2.2) and (2.3) to the evaluation functional F_x, we obtain

$$f(x) = \int_{\operatorname{ext} K} F_x \, d\tilde{\mu} = \int_Q \varphi_y(x) \, d\mu(y), \quad x \in M.$$

□

Lemma 2.34. *Let X be a compact convex subset of a locally convex space E. Then the space $(E^* + \mathbb{R})|_X$ is dense in $\mathfrak{A}^c(X)$.*

Proof. Let X be a nonempty compact convex set. Fix a function $h \in \mathfrak{A}^c(X)$ and $\varepsilon > 0$. Denote

$$J_1 := \{(x,t) \in X \times \mathbb{R} : t = h(x)\}$$

and

$$J_2 := \{(x,t) \in X \times \mathbb{R} : t = h(x) + \varepsilon\}.$$

Then J_1, J_2 is a pair of disjoint nonempty compact convex subsets of $E \times \mathbb{R}$. By the Hahn–Banach theorem, there exist a functional $F \in (E \times \mathbb{R})^*$ and $\lambda \in \mathbb{R}$ such that

$$\sup F(J_1) < \lambda < \inf F(J_2).$$

There are $\varphi \in E^*$ and $\beta \in \mathbb{R}$ such that $F(t,r) = \varphi(t) + \beta r$ for any $t \in E$ and any $r \in \mathbb{R}$. From the separation of J_1 and J_2 it easily follows that $\beta \neq 0$. Put $\psi(t) := \frac{1}{\beta}(\lambda - \varphi(t))$ for $t \in E$. Since

$$\varphi(t) + \beta h(t) < \lambda < \varphi(t) + \beta h(t) + \beta\varepsilon, \quad t \in X,$$

we easily get $\|h - \psi\| < \varepsilon$. □

Remark 2.35. In general, there might exist a continuous affine function on a compact convex set X that is not of the form $(E^* + \mathbb{R})|_X$; see Exercise 2.111.

Proposition 2.36. *A Radon measure $\mu \in \mathcal{M}^1(X)$ represents $x \in X$ if and only if*

$$\varphi(x) = \int_X \varphi \, d\mu \quad \textit{for any} \quad \varphi \in E^*.$$

Proof. Recall that, by definition, μ represents $x \in X$ if $h(x) = \int_X h \, d\mu$ for any $h \in \mathfrak{A}^c(X)$. Hence the assertion is an easy consequence of Lemma 2.34 and the Lebesgue dominated convergence theorem. □

Definition 2.37 (The barycenter revisited). Proposition 2.36 enables us to extend slightly the definition of a barycenter to the case of nonconvex sets and points not belonging to this set.

Let K be a compact subset of a locally convex space E. We say that $x \in E$ is the *barycenter* of a Radon measure $\mu \in \mathcal{M}^1(K)$, or that μ *represents* x, in a symbol $x = r(\mu)$, if

$$\varphi(x) = \int_K \varphi \, d\mu \quad \text{for any} \quad \varphi \in E^*.$$

Proposition 2.38. *If X is a compact convex subset of a locally convex space, the barycenter mapping $r\colon \mathcal{M}^1(X) \to X$ is affine and continuous.*

Hint. Obviously, r is affine. If $\{\mu_\gamma\}$ is a net in $\mathcal{M}^1(X)$, $\mu_\gamma \to \mu$ and $f \in E^*$, then

$$f(r(\mu_\gamma)) = \mu_\gamma(f) \to \mu(f) = f(r(\mu)).$$

Consequently, the net $\{r(\mu_\gamma)\}$ is weakly converging in X to $r(\mu)$. Since X is a compact set, $r(\mu_\gamma) \to r(\mu)$ by Proposition A.28. □

Proposition 2.39. *If K is a compact subset of a locally convex space E and $x \in E$, then the following statements are equivalent:*

(i) $x \in \overline{\operatorname{co}}\, K$,

(ii) *there exists a Radon measure $\mu \in \mathcal{M}^1(K)$ such that $r(\mu) = x$.*

Proof. Let $\mu \in \mathcal{M}^1(K)$ satisfy $r(\mu) = x$. Assuming that $x \notin \overline{\operatorname{co}}\, K$, by the Hahn–Banach theorem there exist $\varphi \in E^*$ and $\lambda \in \mathbb{R}$ such that $\varphi(x) < \lambda \le \varphi(t)$ for any $t \in \overline{\operatorname{co}}\, K$. Then, obviously, no Radon measure $\mu \in \mathcal{M}^1(K)$ with $\varphi(x) = \int_K \varphi\, d\mu$ exists.

Conversely, assume that $x \in \overline{\operatorname{co}}\, K$. There exists a net $\{x_\alpha\}$ of points in $\operatorname{co} K$ such that $x_\alpha \to x$. We can write

$$x_\alpha = \sum_{j=1}^{n_\alpha} \lambda_j^\alpha x_j^\alpha, \quad \text{where } n_\alpha \in \mathbb{N}, x_j^\alpha \in K, \lambda_j^\alpha \ge 0, \sum_{j=1}^{n_\alpha} \lambda_j^\alpha = 1.$$

We define for each α

$$\mu_\alpha := \sum_{j=1}^{n_\alpha} \lambda_j^\alpha \varepsilon_{x_j^\alpha}.$$

Then $\mu_\alpha \in \mathcal{M}^1(K)$ and $r(\mu_\alpha) = x_\alpha$. Since the set $\mathcal{M}^1(K)$ is compact, we may assume that $\{\mu_\alpha\}$ converges to $\mu \in \mathcal{M}^1(K)$. For any $\varphi \in E^*$,

$$\varphi(r(\mu_\alpha)) = \mu_\alpha(\varphi) \to \mu(\varphi) \quad \text{and} \quad \varphi(r(\mu_\alpha)) = \varphi(x_\alpha) \to \varphi(x).$$

Hence $\varphi(x) = \mu(\varphi)$ and $r(\mu) = x$. □

Theorem 2.40 (Bauer's characterization of $\operatorname{ext} X$). *Let X be a compact convex subset of a locally convex space and $x \in X$. Then $x \in \operatorname{ext} X$ if and only if the Dirac measure ε_x is the only measure from $\mathcal{M}^1(X)$ with a barycenter x.*

Proof. If $x = \frac{1}{2}(a+b)$ where $a, b \in X$, $a \neq b$, then the measure $\frac{1}{2}(\varepsilon_a + \varepsilon_b) \in \mathcal{M}^1(X)$, which differs from ε_x, has the barycenter x.

Assume now that $x \in \operatorname{ext} X$ and $\mu \in \mathcal{M}^1(X)$ with $r(\mu) = x$ are given. It must be shown that $\mu = \varepsilon_x$. To see this, it suffices to show that $\operatorname{spt} \mu = \{x\}$. Assume that

$z \in \operatorname{spt}\mu \setminus \{x\}$. Let $U \subset X$ be a closed convex neighborhood of z in X such that $x \notin U$. The set U is compact and convex, $\mu(U) < 1$ in view of Proposition 2.39, and obviously $\mu(U) > 0$. Set

$$\mu_1 := \frac{1}{\mu(U)}\mu|_U \quad \text{and} \quad \mu_2 := \frac{1}{\mu(X \setminus U)}\mu|_{X\setminus U}.$$

Then μ_1, μ_2 are in $\mathcal{M}^1(X)$. Since $\mu_1(U) = 1$, we get $r(\mu_1) \in U$ (again from Proposition 2.39). Hence $r(\mu_1) \neq x$. We have

$$\mu = \mu(U)\mu_1 + (1 - \mu(U))\mu_2$$

and we see that

$$x = \mu(U)r(\mu_1) + (1 - \mu(U))r(\mu_2)$$

is a convex combination of $r(\mu_1)$ and $r(\mu_2)$. This contradicts the assumption that x is an extreme point of X and yields the required conclusion. □

Proposition 2.41 (Choquet). *Let X be a compact convex subset of a locally convex space E and $x \in \operatorname{ext} X$. Then the family*

$$\{y \in X : f(y) < \lambda\}, \quad f \in E^* \text{ and } \lambda \in \mathbb{R} \text{ with } f(x) < \lambda,$$

form a base of neighborhoods of x in X.

Proof. Let U be an open neighborhood of x. Then $X \setminus U$ is a compact set so that $x \notin \overline{\text{co}}(X \setminus U)$.

Indeed, if this were not true, then Proposition 2.39 would provide a probability measure μ carried by $X \setminus U$ with $r(\mu) = x$. But this would contradict our assumption that x is an extreme point because the only representing measure for x is the Dirac measure ε_x (see Theorem 2.40).

Now the Hahn–Banach separation theorem yields the existence of a continuous linear functional f and $\lambda \in \mathbb{R}$ such that $f(x) < \lambda$ and $f > \lambda$ on $\overline{\text{co}}(X \setminus U)$. Thus $x \in \{y \in X : f(y) < \lambda\} \subset U$ and the proof is finished. □

Proposition 2.42. *Let $K_1, \ldots, K_n$ be compact convex subsets of a topological vector space. Then $\operatorname{co}(K_1 \cup \cdots \cup K_n)$ is compact.*

Proof. We follow the same lines as in the proof of Corollary 2.8. Since

$$\operatorname{co}(K_1 \cup \cdots \cup K_n) = F(D \times K_1 \times \cdots \times K_n)$$

where

$$D := \Big\{\lambda \in \mathbb{R}^n : \lambda = (\lambda_1, \ldots, \lambda_n), \sum_{j=1}^{n} \lambda_j = 1 \text{ and } \lambda_j \geq 0 \text{ for } j = 1, \ldots, n\Big\}$$

and

$$F : (\lambda, x_1, \dots, x_n) \mapsto \sum_{j=1}^{n} \lambda_j x_j, \quad (\lambda, x_1, \dots, x_n) \in D \times K_1 \times \cdots \times K_n$$

is continuous, it follows that $\operatorname{co}(K_1 \cup \cdots \cup K_n)$ is compact. □

Theorem 2.43 (Milman). *Let B be a subset of a locally convex space E such that the set $X := \overline{\operatorname{co}}\, B$ is compact. Then* $\operatorname{ext} X \subset \overline{B}$.

Proof. Assume that $x \in \operatorname{ext} X \setminus \overline{B}$. By Proposition 2.41, there exist $f \in E^*$ and $\lambda \in \mathbb{R}$ such that

$$f(x) > \lambda \geq \inf f(B).$$

Since $f \in E^*$, $\inf f(B) = \inf f(\overline{\operatorname{co}}\, B)$. Therefore we get $x \notin \overline{\operatorname{co}}\, B = X$, which is a contradiction. □

Proposition 2.44. *Let X be a compact convex subset of a locally convex space E and $\varnothing \neq B \subset X$. Then the following assertions are equivalent:*

(i) $\overline{\operatorname{co}}\, B = X$,

(ii) $\inf f(B) = \min f(X)$ *for each* $f \in E^*$,

(iii) $\operatorname{ext} X \subset \overline{B}$.

Proof. Assertions (i) and (iii) are equivalent by the Krein–Milman and Milman theorems 2.22 and 2.43. The implication (i) $\implies$ (ii) is obvious. Conversely, the proof of the implication (ii) $\implies$ (i) follows the same lines as the end of the proof of the Krein–Milman theorem: if $\overline{\operatorname{co}}\, B \neq X$, the Hahn–Banach separation theorem yields the assertion. □

Proposition 2.45. *Any metrizable compact convex subset X of a locally convex is affinely homeomorphic to a compact convex subset of the Hilbert space ℓ^2.*

Proof. Since X is a metrizable compact set, the space $\mathcal{C}(X)$ is separable, so it is the space $\mathfrak{A}^c(X)$ of all continuous affine functions on X. Let $\{f_n : n \in \mathbb{N}\}$ be a dense subset of the closed unit ball of $\mathfrak{A}^c(X)$. If

$$T : x \mapsto \left\{ \frac{1}{n} f_n(x) \right\}, \quad x \in X,$$

then T is a continuous affine injective mapping of X into ℓ^2. Thus T is an affine homeomorphism of X onto a compact convex subset $T(X)$ of ℓ^2. □

2.1.C Exposed points

Definition 2.46 (Exposed points). If X is a compact convex set, a point $x \in X$ is *exposed* if there exists a function $f \in \mathfrak{A}^c(X)$ such that $f(x) > f(y)$ for each $y \in X \setminus \{x\}$.

We call such a function f an *exposing function for* x and the set of all exposed points of X is denoted as $\exp X$.

We point out the following simple, but important fact.

Proposition 2.47. *For any compact convex set,* $\exp X \subset \operatorname{ext} X$.

Proof. The proof follows by a straightforward verification. □

Definition 2.48 (Farthest points). Let D be a subset of a normed linear space E. A point $z \in D$ is called a *farthest point* of D if there exists $x \in E$ such that $\|x - z\| = \sup\{\|x - t\| : t \in D\}$.

The set of all farthest points of D is denoted by $\operatorname{far} D$.

Lemma 2.49. *If X is a nonempty compact convex subset of a Hilbert space H, then* $\operatorname{far} X \subset \exp X$ *and it is a nonempty set.*

Proof. The inclusion $\operatorname{far} X \subset \exp X$ follows from the fact that any point x in the closed unit ball B_H of norm 1 is an exposed point of B_H (we recall that B_H is a compact convex set in the weak topology of H). Indeed, $f(y) := (y, x)$, $y \in H$, is an exposing function for x.

If $y \in H$ is chosen arbitrarily, a simple compactness argument yields the existence of a point $z \in X$ such that $\|y - z\| = \sup\{\|y - x\| : x \in X\}$. Hence $\operatorname{far} X$ is nonempty. □

Proposition 2.50. *Let X be a nonempty compact convex subset of a Hilbert space H. Then* $X = \overline{\operatorname{co}} \operatorname{far} X = \overline{\operatorname{co}} \exp X$.

Proof. Since $\operatorname{far} X \subset \exp X$, it suffices to show that $X = \overline{\operatorname{co}} \operatorname{far} X$. Assume that $x \in X \setminus \overline{\operatorname{co}} \operatorname{far} X$. By the Hahn–Banach theorem and the Fréchet–Riesz representation theorem, there exist $h \in H$ and $\lambda \in \mathbb{R}$ such that $(x, h) < \lambda \le (y, h)$ for each $y \in \overline{\operatorname{co}} \operatorname{far} X$. Denote $s := \sup\{\|t\| : t \in X\}$. Let $\alpha > 0$ be such that

$$2\alpha\big(\lambda - (h, x)\big) > s^2 - \|x\|^2.$$

By compactness, there exists $z \in X$ such that $\|\alpha h - z\| = \sup\{\|\alpha h - t\| : t \in X\}$. Of course, $z \in \operatorname{far} X$. Since

$$\begin{aligned}\|\alpha h - z\|^2 &= \alpha^2\|h\|^2 - 2\alpha(h, z) + \|z\|^2 \le \alpha^2\|h\|^2 - 2\alpha\lambda + s^2 \\ &< \alpha^2\|h\|^2 - 2\alpha(h, x) + \|x\|^2 = \|\alpha h - x\|^2,\end{aligned}$$

we get $\|\alpha h - z\| < \|\alpha h - x\|$, which is a contradiction. □

Theorem 2.51. *Let X be a metrizable compact convex subset of a locally convex space. Then* $X = \overline{\text{co}}\,\exp X$.

Proof. By Proposition 2.45, X is affinely homeomorphic to a compact convex subset of ℓ^2. Obviously, exposed points are preserved by an affine homeomorphism and thus the assertion follows from Proposition 2.50. □

Corollary 2.52. *Let X be a metrizable compact convex subset of a locally convex space. Then* $\operatorname{ext} X \subset \overline{\exp X}$.

Proof. The assertion is an immediate consequence of Theorem 2.51 and Proposition 2.44. □

2.2 Interlude: On the space $\mathcal{M}(K)$

Let K be a compact space and $\mathcal{M}(K)$ the space of all signed Radon measures on K. We emphasize that on the space $\mathcal{M}(K)$ we always consider the weak*-topology given by the duality of $\mathcal{C}(K)$ and $\mathcal{M}(K)$. In this chapter we collect some properties of this space and its subspaces.

Example 2.53 (Image of a measure from $\mathcal{M}^1(K)$). If K is a compact space, it follows from Proposition 2.27 that the mapping $\varepsilon : x \mapsto \varepsilon_x$, $x \in K$, is a homeomorphism of K onto $\operatorname{ext} \mathcal{M}^1(K)$. Let $\Lambda := \varepsilon_\sharp \lambda$ be the image of a probability measure λ with $\operatorname{spt} \lambda \subset K$ under ε.

The following proposition will be useful in many examples.

Proposition 2.54. *The measure Λ from Example 2.53 is carried by the (closed) set $\varepsilon(\operatorname{spt} \lambda)$ and its barycenter equals λ.*

Proof. Since
$$\Lambda(\varepsilon(\operatorname{spt} \lambda)) = \lambda(\varepsilon^{-1}(\varepsilon(\operatorname{spt} \lambda))) = \lambda(\operatorname{spt} \lambda) = 1,$$
we see that Λ is carried by $\varepsilon(\operatorname{spt} \lambda)$.

Pick $\varphi \in \big(\mathcal{M}(K)\big)^*$. By duality theory, there exists $f \in \mathcal{C}(K)$ such that
$$\varphi(\mu) = \mu(f) \quad \text{for any} \quad \mu \in \mathcal{M}(K).$$
Then (cf. Proposition A.92),
$$\begin{aligned}\Lambda(\varphi) = (\varepsilon_\sharp \lambda)(\varphi) = \lambda(\varphi \circ \varepsilon) &= \int_K \varphi(\varepsilon_x)\, d\lambda(x) = \int_K f(x)\, d\lambda(x) \\ &= \lambda(f) = \varphi(\lambda),\end{aligned}$$
and $r(\Lambda) = \lambda$. □

Proposition 2.55. *Let F be a closed subset of a compact space K and $b > 0$. Then the function*

$$\varphi_b : \mu \mapsto \mu(\{x \in F : \mu(\{x\}) \geq b\}), \quad \mu \in \mathcal{M}^1(K),$$

is upper semicontinuous on $\mathcal{M}^1(K)$.

Proof. Let $c > 0$ and

$$\mu \in G := \{\nu \in \mathcal{M}^1(K) : \varphi_b(\nu) < c\}$$

be given. We will show that G contains a neighborhood W of μ.

The set $L := \{x \in F : \mu(\{x\}) \geq b\}$ is finite. Let U be an open subset of K such that $L \subset U$ and $\mu(\overline{U}) < c$. For every $x \in F \setminus U$ we find an open neighborhood V_x of x so that $\mu(\overline{V}_x) < b$. Using compactness, we select finitely many points $x_1, \ldots, x_n$ such that

$$F \setminus U \subset V_{x_1} \cup \cdots \cup V_{x_n}.$$

Since the function $\nu \mapsto \nu(H)$ is upper semicontinuous on $\mathcal{M}^1(K)$ for every closed set $H \subset K$ (see Theorem A.85(b)), the set

$$W := \{\nu \in \mathcal{M}^1(K) : \nu(\overline{U}) < c, \nu(\overline{V}_{x_i}) < b, i = 1, \ldots, n\}$$

is open and contains μ. It remains to show that $\varphi_b(\nu) < c$ for every $\nu \in W$.

Given a measure $\nu \in W$, let $L_\nu := \{x \in F : \nu(\{x\}) \geq b\}$. It follows from the choice of W that $L_\nu \subset U$. Thus

$$\varphi_b(\nu) = \nu(L_\nu) \leq \nu(\overline{U}) < c,$$

and φ_b is upper semicontinuous. □

Proposition 2.56. *Let F be a closed subset of a compact space K. Then the function*

$$\psi : \mu \mapsto \mu_d(F) := \sum_{x \in F} \mu(\{x\}), \quad \mu \in \mathcal{M}^1(K),$$

is a limit of an increasing sequence of positive upper semicontinuous functions on $\mathcal{M}^1(K)$.

Proof. For $n \in \mathbb{N}$, let ψ_n be the function φ_b from Proposition 2.55 for $b = 1/n$. Then it is easy to check that $\psi_n \nearrow \psi$ on $\mathcal{M}^1(K)$. As ψ_n are upper semicontinuous and positive functions, the proof is finished. □

Definition 2.57 (F_σ and G_δ faces). A face which is simultaneously an F_σ set is called an F_σ *face*. Analogously, a G_δ *face* is a face which is a G_δ set.

Proposition 2.58. *Let F be a closed subset of a compact space K. Then the set*

$$G := \{\mu \in \mathcal{M}^1(K) : \mu|_F \text{ is continuous}\}$$

is a G_δ face of $\mathcal{M}^1(K)$ such that $G \cap \operatorname{ext} \mathcal{M}^1(K) = \varnothing$.

Proof. Let $\psi : \mu \mapsto \mu_d(F)$, $\mu \in \mathcal{M}^1(K)$, where μ_d is the discrete part of μ. According to Proposition 2.56, there is a sequence $\{\psi_n\}$ of positive upper semicontinuous functions such that $\psi_n \nearrow \psi$ on $\mathcal{M}^1(K)$. Then

$$G = \{\mu \in \mathcal{M}^1(K) : \psi(\mu) = 0\} = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \left\{\mu \in \mathcal{M}^1(K) : \psi_n(\mu) < \frac{1}{k}\right\}.$$

It follows that G is a G_δ set which is obviously convex and extremal. Since

$$\operatorname{ext} \mathcal{M}^1(K) = \{\varepsilon_x : x \in K\}$$

(cf. Proposition 2.27) and Dirac measures are discrete, we have $G \cap \operatorname{ext} \mathcal{M}^1(K) = \varnothing$. □

Proposition 2.59. *Let*

$$H := \bigcup_{n=2}^{\infty} \{\mu \in \mathcal{M}^1([0,1]) : \operatorname{spt} \mu \subset \left[\tfrac{1}{n}, 1\right]\}.$$

Then H is an F_σ face of $\mathcal{M}^1([0,1])$ which is not of type G_δ.

Proof. Obviously, H is a convex set and extremal. Moreover, H is of type F_σ. Since both sets H and $\mathcal{M}^1([0,1]) \setminus H$ are dense in $\mathcal{M}^1([0,1])$, H is not a G_δ set (cf. Theorem A.58). □

Proposition 2.60. *Let K be a compact space and $\omega \in \mathcal{M}^1(K)$. Define*

$$\psi : \mu \mapsto \mu_s(K), \quad \mu \in \mathcal{M}^1(K),$$

where μ_s is the singular part of μ with respect to the measure ω. Then ψ is a limit of a decreasing sequence of lower semicontinuous functions on $\mathcal{M}^1(K)$.

Proof. For $n \in \mathbb{N}$, set

$$\psi_n(\mu) := \sup \left\{\mu(G) \colon G \subset K \text{ open and } \omega(G) < \frac{1}{n}\right\}, \quad \mu \in \mathcal{M}^1(K).$$

Obviously, $\{\psi_n\}$ is a decreasing sequence of lower semicontinuous functions. Recall that, by Theorem A.85(b), the function

$$\mu \mapsto \mu(G), \quad \mu \in \mathcal{M}^1(K),$$

is lower semicontinuous on $\mathcal{M}^1(K)$ for any open set $G \subset K$.

Pick $n \in \mathbb{N}$ and $\mu \in \mathcal{M}^1(K)$. There exists a Borel set $B \subset K$ such that

$$\mu_s(B) = \mu_s(K) = \psi(\mu) \quad \text{and} \quad \omega(B) = 0.$$

Let $G \subset K$ be an open set containing B for which $\omega(B) < \frac{1}{n}$. Then

$$\psi(\mu) = \mu_s(B) \leq \mu_s(G) \leq \mu(G) \leq \psi_n(\mu).$$

Hence, $\psi \leq \psi_n$ for any $n \in \mathbb{N}$.

It remains to show that $\lim_{n\to\infty} \psi_n = \psi$. To this end, pick $\mu \in \mathcal{M}^1(K)$ and $c > \psi(\mu)$. Since $\mu_{ac} \ll \omega$ (recall that μ_{ac} denotes the absolutely continuous part of μ with respect to ω, see Proposition A.65), there exists $n \in \mathbb{N}$ so that

$$\mu_{ac}(B) < c - \psi(\mu)$$

whenever B is a Borel set with $\omega(B) < \frac{1}{n}$. Now, if $G \subset K$ is an open set satisfying $\omega(G) < \frac{1}{n}$, then

$$\mu(G) = \mu_s(G) + \mu_{ac}(G) \leq \mu_s(K) + c - \mu_s(K) = c.$$

Thus, $\psi_n(\mu) \leq c$, and therefore $\psi_n \to \psi$. □

Proposition 2.61. *If λ denotes Lebesgue measure on $[0,1]$ and*

$$L := \left\{\mu \in \mathcal{M}^1([0,1]) : \mu \perp \lambda\right\},$$

then L is a G_δ face of $\mathcal{M}^1([0,1])$ which is not of type F_σ.

Proof. It is easy to check that L is convex, extremal and dense in $\mathcal{M}^1([0,1])$.

Therefore all that needs to be proved is that L is a G_δ set. Let $\{\psi_n\}$ be a sequence of functions as in Proposition 2.60 for $K := [0,1]$ and $\omega := \lambda$. The assertion then follows from the following equalities

$$\begin{aligned} L &= \left\{\mu \in \mathcal{M}^1([0,1]) : \mu_s([0,1]) = 1\right\} = \bigcap_{n=1}^{\infty} \left\{\mu \in \mathcal{M}^1([0,1]) : \psi_n(\mu) = 1\right\} \\ &= \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \left\{\mu \in \mathcal{M}^1([0,1]) : \psi_n(\mu) > 1 - \frac{1}{k}\right\}, \end{aligned}$$

because the functions ψ_n are lower semicontinuous. Since L is dense, it is not an F_σ set. □

Remark 2.62. In Exercise 2.118 we indicate another reasoning of the fact that L is a G_δ set.

Proposition 2.63 (Choquet's examples). *Let*

$$\varphi : \mu \mapsto \mu_d([0,1]), \quad \mu \in \mathcal{M}^1([0,1]),$$

and

$$\psi : \mu \mapsto \mu_s([0,1]), \quad \mu \in \mathcal{M}^1([0,1]).$$

Then φ and ψ are bounded affine functions on $\mathcal{M}^1([0,1])$ of the second Baire class and there exists a probability measure Λ on $\mathcal{M}^1([0,1])$ such that

$$\Lambda(\varphi) \neq \varphi(r(\Lambda)) \quad \textit{and} \quad \Lambda(\psi) \neq \psi(r(\Lambda)).$$

Proof. Obviously, φ and ψ are bounded and affine.

By Propositions 2.60, 2.56 and A.53, both functions φ and ψ are of the second Baire class on $\mathcal{M}^1([0,1])$ because $\mathcal{M}^1([0,1])$ is metrizable by Theorem A.85.

Let Λ be the image of Lebesgue measure λ on $[0,1]$ under ε. According to Proposition 2.54, Λ is carried by the (closed) set $\varepsilon([0,1])$ and its barycenter is equal to λ. It remains to show that the "barycentric formula"

$$\Lambda(\varphi) = \int_{\mathcal{M}^1(K)} \varphi \, d\Lambda$$

does not hold. Indeed,

$$\begin{aligned}\Lambda(\varphi) &= \int_{\mathcal{M}^1(K)} \varphi \, d\Lambda = \int_{\varepsilon([0,1])} \varphi \, d\Lambda \\ &= \int_{\varepsilon([0,1])} 1 \, d\Lambda = 1 \neq 0 = \varphi(\lambda).\end{aligned}$$

The same argument can be used in the case of the function ψ. □

2.3 Structures in convex sets

Throughout this section, X will be a compact convex subset of a locally convex space E.

2.3.A Extremal sets and faces

Recall that a nonempty set $F \subset X$ is called *extremal* if $x, y \in F$ whenever

$$x, y \in X, \lambda \in (0,1) \quad \text{and} \quad \lambda x + (1-\lambda) y \in F.$$

One-point extremal sets are just *extreme points* of X.

It is simply checked that a set F is extremal if and only if

$$\bigcup \{(\lambda F - (\lambda - 1)X) \cap X : \lambda \geq 1\} \subset F.$$

Proposition 2.64. *Let F be a subset of X. Then*
(a) $F \cap \operatorname{ext} X \subset \operatorname{ext} F$,
(b) *if F is extremal, then* $\operatorname{ext} F = F \cap \operatorname{ext} X$.

Proof. The assertion is a straightforward consequence of definitions. □

Recall that convex extremal sets are called *faces*. Closed extremal sets occasionally bear the name *absorbent sets*.

Definition 2.65 (Generated faces). It is easy to see that any intersection of faces of X is again a face. Hence, given a set $A \subset X$, there exists the smallest face of X containing A. It equals the intersection of all faces containing A and is denoted by $\operatorname{face} A$. Given $x \in X$, we will write simply $\operatorname{face} x$ instead of $\operatorname{face} \{x\}$ where no confusion can arise.

Proposition 2.66. *If F is a convex subset of X, then*

$$\operatorname{face} F = \bigcup \{(\lambda F - (\lambda - 1)X) \cap X : \lambda \geq 1\}.$$

Moreover, if F is closed, $\operatorname{face} F$ *is an F_σ set.*

Proof. Let

$$F_\lambda := (\lambda F + (\lambda - 1)X) \cap X, \quad \lambda \geq 1.$$

First we notice that, given $\lambda > 1$, $y \in F_\lambda$ if and only if there exists $x \in F$ such that

$$y + (\lambda - 1)^{-1}(x - y) \in X.$$

Hence $F_\lambda \subset F'_\lambda$ if $1 \leq \lambda \leq \lambda'$. It follows that $\bigcup_{\lambda \geq 1} F_\lambda$ is a convex set. Since it is easy to observe that $\bigcup_{\lambda \geq 1} F_\lambda$ is extremal, we have

$$\operatorname{face} F \subset \bigcup_{\lambda \geq 1} F_\lambda.$$

On the other hand, it is immediate to verify from the definition the converse inclusion. Hence $\operatorname{face} F = \bigcup_{\lambda \geq 1} F_\lambda$.

If F is closed, a routine verification yields that each F_λ is closed as well. Hence

$$\operatorname{face} F = \bigcup_{n=1}^{\infty} F_n$$

is an F_σ set. □

Corollary 2.67. *If $x \in X$, then*

$$\operatorname{face} x = \{y \in X : \text{there exists } z \in X \text{ and } \lambda \in [0,1) \text{ such that } x = \lambda z + (1 - \lambda) y\}$$

and $\operatorname{face} x$ *is an F_σ set.*

Proof. Use Proposition 2.66. □

Proposition 2.68. *Let F be a subset of X. Then the following assertions are equivalent:*

(i) *F is extremal,*

(ii) *the characteristic function c_F of F is convex,*

(iii) *F is a union of faces.*

Proof. The equivalence of (i) and (ii) is clear from the definition. Since it is easy to check that any union of extremal sets is extremal, we have (iii) $\implies$ (i).

If (i) holds, then

$$F = \bigcup_{x \in F} \operatorname{face} x,$$

since $\operatorname{face} x \subset F$ for any $x \in F$. □

Proposition 2.69. *Let F be a subset of X. The following assertions are equivalent:*

(i) *F is a closed extremal set,*

(ii) *the characteristic function c_F of F is upper semicontinuous and convex,*

(iii) *there exists a positive, lower semicontinuous and concave function f on X such that $F = \{x \in X : f(x) = 0\}$,*

(iv) *F is closed and* $\operatorname{spt} \mu \subset F$ *whenever $\mu \in \mathcal{M}^1(X)$ and $r(\mu) \in F$,*

(v) *F is closed and a union of faces,*

(vi) *F is closed and a union of closed faces.*

Proof. The equivalence of (i), (ii) and (v) follows immediately; see Proposition 2.68. Assertions (ii) and (iii) are obviously equivalent.

To see that (i) $\implies$ (iv), suppose that F is a closed extremal set and $\mu \in \mathcal{M}^1(X)$ such that $r(\mu) \in F$. If $\operatorname{spt} \mu$ is not a subset of F, then there exists a closed convex set $C \subset X \setminus F$ such that $\mu(C) > 0$. We have $\mu(C) < 1$, since otherwise $r(\mu) = r(\mu|_C) \notin F$. Set

$$\mu_1 := \frac{1}{\mu(C)} \mu|_C \quad \text{and} \quad \mu_2 := \frac{1}{1 - \mu(C)} \mu|_{X \setminus C}.$$

Now,

$$\mu = \mu(C)\mu_1 + (1 - \mu(C))\mu_2,$$

hence

$$r(\mu) = \mu(C) r(\mu_1) + (1 - \mu(C)) r(\mu_2) \in F.$$

Since F is extremal, $r(\mu_1), r(\mu_2) \in F$, which is a contradiction.

Now let F be a closed set and let $x \in F$, $x = \lambda y + (1-\lambda)z$ where $y, z \in X$, $\lambda \in (0,1)$. If

$$\mu := \lambda\varepsilon_y + (1-\lambda)\varepsilon_z,$$

then $r(\mu) = x$ and $\operatorname{spt}\mu = \{y, z\}$. Hence $y, z \in F$, which shows that F is extremal and proves that (iv) $\implies$ (i).

Since (vi) obviously implies (v), all that remains to be proved is that (i) $\implies$ (vi). So let F be a closed extremal subset of X and $x \in X$. The set

$$\mathcal{F} := \{C \subset F : x \in C, C \text{ convex}\}$$

ordered by inclusion satisfies the assumptions of Zorn's lemma. Indeed, if $\mathcal{R} \subset \mathcal{F}$ is a chain, then the set

$$\bigcup\{R : R \in \mathcal{R}\}$$

belongs to $\mathcal{F}$ and it is an upper bound of $\mathcal{R}$. Hence there exists a maximal element $C \in \mathcal{F}$. Since $x \in C \subset \overline{C} \subset F$ and $\overline{C} \in \mathcal{F}$, the maximality of C implies that $C = \overline{C}$. It remains to show that C is a face.

Since C is convex, we must verify only that C is extremal. So, we are given $a, b \in X$ and $\lambda \in (0,1)$ such that $z := \lambda a + (1-\lambda)b \in C$, and we wish to show that $a, b \in C$. In order to prove this, let $\widetilde{C}$ be the convex hull of C and $\{a, b\}$. The proof that $a, b \in C$ will be achieved by showing that $\widetilde{C} \subset F$. Then, due to the maximality of C, $\widetilde{C} = C$.

Choose $\widetilde{c} \in \widetilde{C}$. Then

$$\widetilde{c} = \lambda_1(\lambda_2 a + (1-\lambda_2)b) + (1-\lambda_1)c,$$

where $c \in C$ and $\lambda_1, \lambda_2 \in [0,1]$. It is clearly sufficient to assume that $\lambda_1 \in (0,1)$. (If $\lambda_1 = 0$, $\widetilde{c} = c$. If $\lambda_1 = 1$, $\widetilde{c}$ belongs to the segment joining a and b, and thus $\widetilde{c} \in F$ by the extremality of F.) Moreover, we may assume that $\lambda_2 \geq \lambda$ (otherwise, $1 - \lambda_2 \geq \lambda$). With λ_1 and λ_2 chosen in this manner, we set

$$\alpha := \frac{\lambda_1\lambda_2}{\lambda_1\lambda_2 + \lambda(1-\lambda_1)} \quad \text{and} \quad \beta := \frac{\lambda}{\lambda_1\lambda_2 + \lambda(1-\lambda_1)}.$$

Since $\beta > 0$,

$$\beta\widetilde{c} + (1-\beta)b = \alpha z + (1-\alpha)c \in C \subset F$$

and since F is extremal, we get $\widetilde{c} \in F$. □

Remarks 2.70. (a) Let F be an extremal set and $x \in F$. If $\mu \in \mathcal{M}^1(X)$ and $x = r(\mu)$, then $\operatorname{spt}\mu \subset \overline{F}$. The proof of this assertion can be obtained as a slight modification of the proof of the implication (i) $\implies$ (iv) in Proposition 2.69.

(b) In general, the closure of a face need not be a face. An example can be found in E. M. Alfsen [1], Theorem 1. In Exercise 4.52 we present an example of a compact convex set X and a point $x \in X$ such that $\overline{\operatorname{face} x}$ is not a face.

Corollary 2.71. *Let F be a closed convex subset of X. Then F is a face if and only if for every measure $\mu \in \mathcal{M}^1(X)$ with barycenter $r(\mu)$ in F, we have* $\operatorname{spt}\mu \subset F$.

Proof. A closed convex set is a face if and only if it is extremal. Hence the assertion follows immediately from Proposition 2.69, (i) $\Longleftrightarrow$ (iv). □

Proposition 2.72. *Let $\varphi : X \to Y$ be a continuous affine surjection of a compact convex set X onto a compact convex set Y.*

(a) *If $H \subset Y$ is extremal, then $\varphi^{-1}(H)$ is an extremal set of X.*

(b) *If $H \subset Y$ is a face, then $\varphi^{-1}(H)$ is a face of X.*

(c) $\varphi(\operatorname{ext} X) \supset \operatorname{ext} Y$.

Proof. Let $H \subset Y$ be an extremal set and $\alpha x_1 + (1-\alpha)x_2 \in \varphi^{-1}(H)$, where x_1, $x_2 \in X$, $\alpha \in [0,1]$. Then $\alpha\varphi(x_1) + (1-\alpha)\varphi(x_2) \in H$, which yields $\varphi(x_1), \varphi(x_2) \in H$. Hence $x_1, x_2 \in \varphi^{-1}(H)$, concluding the proof of (a).

Since (b) is a straightforward consequence of (a), we proceed to the proof of (c). Given a point $y \in \operatorname{ext} Y$, the set $\varphi^{-1}(y)$ is a closed extremal set by (a). By Proposition 2.20, it intersects $\operatorname{ext} X$. Hence $y \in \varphi(\operatorname{ext} X)$, and we are done. □

2.3.B Measure convex sets

In the sequel, stronger versions of convexity and extremality are investigated. Constructions of counterexamples use properties of sets of probability measures studied in Section 2.2.

As above, throughout this subsection, X will be a compact convex subset of a locally convex space E.

Definition 2.73 (Measure convex sets). A universally measurable set $F \subset X$ is *measure convex* if the barycenter $r(\mu)$ belongs to F for any measure $\mu \in \mathcal{M}^1(X)$ carried by F.

We will show that, for a universally measurable set $F \subset X$, the relations between "F measure convex" (labelled as MC) and "F convex" (labelled as C) are as follows:

MC $\Rightarrow$ C		(2.74)
MC $\not\Leftarrow$ C		(2.81)
MC $\Leftarrow$ C	F is closed or open	(2.74, 2.76)
MC $\Leftarrow$ C	F is resolvable	(2.80)
MC $\Leftarrow$ C	$\dim E < \infty$	(2.77)
MC $\Leftarrow$ C	F is a resolvable face	(2.91)
MC $\not\Leftarrow$ C	F is F_σ face	(2.82, 2.83)
MC $\not\Leftarrow$ C	F is G_δ face	(2.84)

Proposition 2.74. *Every measure convex universally measurable subset of X is convex, and every closed convex subset of X is measure convex.*

Proof. Let $F \subset X$ be measure convex, $x, y \in F$, $\lambda \in [0,1]$ and $z = \lambda x + (1-\lambda)y$. Since the measure $\mu := \lambda\varepsilon_x + (1-\lambda)\varepsilon_y$ is carried by F and $r(\mu) = z$, we get $z \in F$. Therefore, F is convex.

If F is a closed convex set and if a measure $\mu \in \mathcal{M}^1(X)$ has its support contained in F, then the implication (ii) $\implies$ (i) of Proposition 2.39 tells us that $r(\mu) \in F$. Accordingly, F is measure convex. $\square$

Theorem 2.75. *Let A be a universally measurable subset of X. Then A is measure convex if and only if $\overline{\text{co}}\, K \subset A$ for any compact set $K \subset A$.*

Proof. Let A be a measure convex subset of X and $K \subset A$ a compact set. If $x \in \overline{\text{co}}\, K$, then there exists a measure $\mu \in \mathcal{M}^1(K)$ such that $x = r(\mu)$ (see Proposition 2.39). By the assumption, $x \in A$.

For the proof of the converse implication, let $\mu \in \mathcal{M}^1(X)$ be a probability measure with $\mu(A) = 1$. If $\mu(K) = 1$ for some compact set $K \subset A$, then by Proposition 2.39 and by the assumption, $r(\mu) \in \overline{\text{co}}\, K \subset A$.

So assume that $\mu(K) < 1$ for each compact set $K \subset A$. In this case, there exists an increasing sequence $\{K_n\}_{n=0}^{\infty}$ of compact sets in A satisfying

$$K_0 = \varnothing, \quad \alpha_n := \mu(K_{n+1} \setminus K_n) > 0,\ n \geq 0, \quad \text{and} \quad \mu(K_n) \to 1.$$

Since $\sum_{n=1}^{\infty} \alpha_n = 1 - \alpha_0 < 1$, there is a sequence $\{\beta_n\}_{n=1}^{\infty}$ of real numbers in $(0,1]$ such that

$$\beta_n \to 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\alpha_n}{\beta_n} = 1.$$

Put

$$L := K_1 \cup \bigcup_{n=1}^{\infty} \left(\beta_n K_{n+1} + (1-\beta_n)K_1\right).$$

Then $L \subset A$. Since each K_n is compact and $\beta_n \to 0$, L is compact as well. Pick $f \in E^*$ and set

$$s_n := \sup f(K_{n+1}), \quad n \geq 0.$$

Then

$$\begin{aligned} s := \sup f(L) &= \max\left(s_0, \sup_{n\in\mathbb{N}} \big(\beta_n s_n + (1-\beta_n)s_0\big)\right) \\ &= s_0 + \sup_{n\in\mathbb{N}} (s_n - s_0)\beta_n. \end{aligned}$$

Hence $s_n \le s_0 + \beta_n^{-1}(s - s_0)$ for each $n \in \mathbb{N}$. Since $f \le s_n$ on $K_{n+1} \setminus K_n$, we get

$$\begin{aligned} f(r(\mu)) &= \int_X f\,d\mu = \sum_{n=0}^{\infty} \int_{K_{n+1}\setminus K_n} f\,d\mu \\ &\le \sum_{n=0}^{\infty} \alpha_n s_n \le \alpha_0 s_0 + \sum_{n=1}^{\infty} \alpha_n (s_0 + \beta_n^{-1}(s - s_0)) \\ &= \alpha_0 s_0 + (s - s_0) \sum_{n=1}^{\infty} \frac{\alpha_n}{\beta_n} + s_0 \sum_{n=1}^{\infty} \alpha_n \\ &= s_0 + s - s_0 = s. \end{aligned}$$

Thus $f(r(\mu)) \le \sup f(L)$. As f is arbitrary, Proposition 2.44 gives $r(\mu) \in \overline{\text{co}}\, L$. Since our assumption ensures that $\overline{\text{co}}\, L \subset A$, the proof is finished. □

Proposition 2.76. *Any open convex subset of X is measure convex.*

Proof. Let $G \subset X$ be an open convex set. By Theorem 2.75, it is enough to show that $\overline{\text{co}}\, K \subset G$ whenever $K \subset G$ is a compact set. So fix such K. For every $x \in K$ there exists a compact convex neighborhood V_x such that $x \in V_x \subset G$. By compactness, the set K can be covered by finitely many compact convex sets $V_{x_1}, \dots, V_{x_n}$. Then, by Proposition 2.42,

$$\overline{\text{co}}\, K \subset \overline{\text{co}}\,(V_{x_1} \cup \cdots \cup V_{x_n}) = \text{co}\,(V_{x_1} \cup \cdots \cup V_{x_n}) \subset G,$$

which is the required inclusion. □

Proposition 2.77. *Let X be a subset of a finite-dimensional space. Then any universally measurable convex set $A \subset X$ is measure convex.*

Proof. We again use Theorem 2.75. If $K \subset A$ is a compact set, then $\text{co}\, K \subset A$ is compact by Theorem 2.8 (see also Remark 2.5). □

Lemma 2.78. *Let λ be a probability measure on X. If*

$$\mathcal{T} := \{\mu \in \mathcal{M}^+(X) : \mu \le \lambda, \mu \ne 0\}$$

and

$$S := \{r(\frac{\mu}{\|\mu\|}) : \mu \in \mathcal{T}\},$$

then the closure of S equals $\overline{\text{co}}\, \text{spt}\, \lambda$.

Proof. It is easy to see that

$$S = \{r(\mu) : \mu \in \mathcal{M}^1(X), \text{ there exists } c \in \mathbb{R} \text{ so that } \mu \le c\lambda\},$$

from which it follows that S is convex.

Set $L := \overline{\text{co}}\,\text{spt}\,\lambda$. To show that $\overline{S} \subset L$, let μ be a nontrivial measure on X with $\mu \le \lambda$. Then μ is carried by L and thus $r(\frac{\mu}{\|\mu\|}) \in L$ because L is a closed convex set. Thus $S \subset L$ and consequently $\overline{S} \subset L$.

Conversely, assuming that $\lambda(\overline{S}) < 1$, we can find a compact set $K \subset X \setminus \overline{S}$ such that $\lambda(K) > 0$. For every $x \in K$ we choose its closed convex neighborhood V_x not intersecting $\overline{S}$. Using a compactness argument we select finitely many points $x_1, \dots, x_n$ of K so that $V_{x_1} \cup \cdots \cup V_{x_n}$ covers K. As $\lambda(K) > 0$, there is $i \in \{1, \dots, n\}$ so that $\lambda(V_{x_i}) > 0$. We set $V := V_{x_i}$ and $\mu := \lambda|_V$. Then μ is nontrivial and $\mu \le \lambda$. Hence the barycenter of $\frac{\mu}{\|\mu\|}$ belongs to S. On the other hand, $r(\frac{\mu}{\|\mu\|}) \in V$ because V is a closed convex set. This contradiction shows that $\lambda(\overline{S}) = 1$. Thus $\text{spt}\,\lambda \subset \overline{S}$ which gives $L \subset \overline{S}$. □

We recall that resolvable sets are defined and their basic properties presented in Section A.5. (In particular we note that any resolvable set in a compact space is universally measurable by Proposition A.118.)

Lemma 2.79. *Let $F \subset X$ be a resolvable convex set and let $\lambda \in \mathcal{M}^1(X)$ be carried by F. Then there exists a nonempty set $G \subset F \cap \overline{\text{co}}\,\text{spt}\,\lambda$ which is open in $\overline{\text{co}}\,\text{spt}\,\lambda$.*

Proof. Let $L := \overline{\text{co}}\,\text{spt}\,\lambda$. In order to find the required set G we note that $L = \overline{F \cap L}$ because the latter set is a closed convex set containing the support of λ. In particular, $F \cap L$ is a dense resolvable set in L. Due to Proposition A.117(c), $F \cap L$ has a nonempty interior (relative to L). Hence, the interior of $F \cap L$ is the sought set G. □

Proposition 2.80. *Any resolvable convex subset of X is measure convex.*

Proof. Let F be a resolvable convex subset of X and let λ be a probability measure on X carried by F. We set $\lambda_0 := \lambda$ and let $L_0 := \overline{\text{co}}\,\text{spt}\,\lambda_0$. Let S_0, $\mathcal{T}_0$ and G_0 be sets obtained from Lemma 2.78 and Lemma 2.79 when applied to the measure λ_0. Since S_0 is dense in L_0 and G_0 is nonempty and open in L_0, there is a measure $\mu_0 \in \mathcal{T}_0$ with

$$r(\frac{\mu_0}{\|\mu_0\|}) \in G_0 \subset F.$$

We set $\lambda_1 := \lambda_0 - \mu_0$ and construct by transfinite induction a sequence $\{\lambda_\alpha\}$ of positive measures on X such that, for every ordinal number $\alpha \ge 1$,

(i) $\lambda_{\alpha+1} \le \lambda_\alpha$,

(ii) either $\lambda_\alpha = 0$ or $\|\lambda_{\alpha+1}\| < \|\lambda_\alpha\|$,

(iii) if $\lambda_\alpha - \lambda_{\alpha+1} \neq 0$, then

$$r\left(\frac{\lambda_\alpha - \lambda_{\alpha+1}}{\|\lambda_\alpha - \lambda_{\alpha+1}\|}\right) \in F.$$

Suppose that the construction has been completed up to an ordinal α. If $\lambda_\alpha = 0$, we set $\lambda_{\alpha+1} := 0$. If λ_α is nontrivial, we apply Lemma 2.78 and Lemma 2.79 to the measure $\frac{\lambda_\alpha}{\|\lambda_\alpha\|}$ (which is carried by F) and get relevant sets $L_\alpha, \mathcal{T}_\alpha$, S_α and G_α with the properties described there. In particular, we have $G_\alpha \subset F \cap L_0$. As in the first step of the proof we choose a nontrivial measure $\nu \in \mathcal{T}_\alpha$ such that

$$r\Big(\frac{\nu}{\|\nu\|}\Big) \in G_\alpha.$$

By setting $\lambda_{\alpha+1} := \lambda_\alpha - \nu$ we finish the inductive step for an isolated ordinal number.

Let α be a limit ordinal number. Assume that λ_β has been defined for every $\beta < \alpha$. Since $\{\lambda_\beta\}_{\beta<\alpha}$ is a decreasing sequence of positive measures, by the Riesz representation theorem, the mapping

$$\lambda_\alpha : g \mapsto \inf_{\beta<\alpha} \lambda_\beta(g), \quad g \in \mathcal{C}(X), g \geq 0,$$

defines the measure λ_α. This step finishes the inductive construction.

Let γ be the first ordinal number for which $\lambda_\gamma = 0$. Since $\{\|\lambda_\alpha\| : \alpha < \gamma\}$ is a strictly decreasing transfinite sequence, the ordinal number γ is countable. We enumerate $\{\lambda_\alpha - \lambda_{\alpha+1}\}_{1\leq\alpha<\gamma}$ into a (possibly finite) sequence $\{\mu_n\}$, and obtain that

$$\lambda = \mu_0 + \sum_{n\geq 1} \mu_n$$

and

$$\|\lambda\| = \|\mu_0\| + \sum_{n\geq 1} \|\mu_n\|.$$

If the sequence $\{\mu_n\}$ is finite, the equality

$$\lambda = \|\mu_0\| \cdot \frac{\mu_0}{\|\mu_0\|} + \sum_{n\geq 1} \|\mu_n\| \cdot \frac{\mu_n}{\|\mu_n\|}$$

yields that λ is a finite convex combination of probability measures having their barycenters in F. Thus, in this case, $r(\lambda) \in F$.

Now, assume that the sequence $\{\mu_n\}$ is infinite. For every $k \in \mathbb{N}$ we set

$$c_0 := \|\mu_0\|, \quad c_k := \sum_{n\geq k} \|\mu_n\|$$

and

$$\omega_k := \frac{c_0}{c_0 + c_k} \cdot \frac{\mu_0}{c_0} + \frac{c_k}{c_0 + c_k} \cdot \frac{\sum_{n\geq k} \mu_n}{c_k}.$$

Then $\{\omega_k\}$ is a sequence of probability measures tending to $c_0^{-1}\mu_0$. Moreover, $\mu_0 + \sum_{n\geq k}\mu_n$ is obviously an element of $\mathcal{T}_0$, and thus the barycenter $r(\omega_k)$ of ω_k is contained in L_0. As $r(\frac{\mu_0}{c_0}) \in G_0$, which is a relatively open subset of L_0, we can find a sufficiently large $k \in \mathbb{N}$ such that $r(\omega_k) \in G_0 \subset F$. Then

$$\begin{aligned}\lambda &= c_0\frac{\mu_0}{c_0} + \sum_{n=1}^{k-1}\|\mu_n\|\frac{\mu_n}{\|\mu_n\|} + \sum_{n\geq k}\|\mu_n\|\frac{\mu_n}{\|\mu_n\|}\\ &= \sum_{n=1}^{k-1}\|\mu_n\|\frac{\mu_n}{\|\mu_n\|} + (c_0 + c_k)\cdot\omega_k,\end{aligned}$$

and the last formula shows that λ is a finite convex combination of measures which have their barycenters in F. Since F is convex, the barycenter of λ belongs to F as well. □

There are convex Borel subsets of a compact convex set which are not measure convex. We present some of them.

Proposition 2.81. *Let $X := \mathcal{M}^1([0,1])$ and*

$$B := \left\{\mu \in \mathcal{M}^1([0,1]) : \mu \text{ is discrete}\right\}.$$

Then B is a convex Borel set containing $\overline{\operatorname{ext} X}$ which is not measure convex.

Proof. It is clear that B is convex. Further, by Proposition 2.27, $\operatorname{ext} X = \overline{\operatorname{ext} X} = \{\varepsilon_x : x \in [0,1]\} \subset B$, and B is a Borel set by Proposition 2.56, since

$$B = \left\{\mu \in \mathcal{M}^1([0,1]) : \mu_d([0,1]) = 1\right\}.$$

Assume that B is measure convex. Pick $\omega \in X$ and find a measure $\Omega \in \mathcal{M}^1(X)$ with barycenter ω carried by $\overline{\operatorname{ext} X}$ (see Theorem 2.31). Since B is supposed to be measure convex, $\omega \in B$. Hence $B = X$, which is a contradiction. □

Proposition 2.82. *Let*

$$X := \left\{\{x_n\} \in \ell^1 : 0 \leq x_n \leq \frac{1}{n^2} \text{ for any } n \in \mathbb{N}\right\}$$

and

$$B := \big\{\{x_n\} \in X : \text{the set } \{n \in \mathbb{N} : x_n \neq 0\} \text{ is finite}\big\}.$$

Then X is a compact convex set and B is an F_σ face of X which is not measure convex.

Proof. Since

$$B = \bigcup_{j=1}^{\infty} \big\{ \{x_n\} \in X : x_n = 0 \text{ for all } n \geq j \big\},$$

B is an increasing countable union of closed faces. Therefore, B is an F_σ face. If $a_n := \frac{1}{n^2} e_n$, $n \in \mathbb{N}$, where e_n is the standard unit vector in ℓ^1, then $a_n \in B$ for each n. Setting

$$\mu := \sum_{n=1}^{\infty} \frac{1}{2^n} \varepsilon_{a_n},$$

μ is carried by B whereas $r(\mu) = \sum_{n=1}^{\infty} \frac{1}{2^n} a_n \notin B$. □

Proposition 2.83. *If*

$$H := \bigcup_{n=2}^{\infty} \left\{ \mu \in \mathcal{M}^1([0,1]) : \operatorname{spt} \mu \subset [\tfrac{1}{n}, 1] \right\},$$

then H is an F_σ face of $\mathcal{M}^1([0,1])$ which is not measure convex.

Proof. By Proposition 2.59, H is an F_σ face of $\mathcal{M}^1([0,1])$. Define the measure ω on $[0,1]$ as

$$\omega := \sum_{n=1}^{\infty} \frac{1}{2^n} \varepsilon_{\frac{1}{n}}.$$

If $\Omega := \varepsilon_\sharp \omega$, then $\Omega \in \mathcal{M}^1\big(\mathcal{M}^1([0,1])\big)$, $\Omega(H) = 1$ and $r(\Omega) = \omega \notin H$. This shows that H is not measure convex. □

Proposition 2.84. *There exists a G_δ face which is not measure convex.*

Proof. Let λ be Lebesgue measure on $[0,1]$ and

$$L := \left\{ \mu \in \mathcal{M}^1([0,1]) : \mu \perp \lambda \right\}.$$

By Proposition 2.61, L is a G_δ face. If $\Lambda := \varepsilon_\sharp \lambda$ (cf. Example 2.53), then $\Lambda \in \mathcal{M}^1\big(\mathcal{M}^1([0,1])\big)$, $\Lambda(L) = 1$ and $r(\Lambda) = \lambda \notin L$. Hence, L is not measure convex. □

2.3.C Measure extremal sets

Definition 2.85 (Measure extremal sets). Recall (see condition (iv) of Proposition 2.69) that a closed set F is extremal if and only if $\operatorname{spt} \mu \subset F$ for every measure $\mu \in \mathcal{M}^1(X)$ having its barycenter $r(\mu)$ in F. Hence, the following definition seems to be quite natural.

A universally measurable set $F \subset X$ is *measure extremal* if every measure $\mu \in \mathcal{M}^1(X)$ with barycenter $r(\mu)$ in F is carried by F.

We will show that, for a universally measurable set $F \subset X$, the relations between "F measure extremal" (labelled as ME) and "F extremal" (labelled as E) are as follows:

ME $\Rightarrow$ E		(2.86)
ME $\Leftarrow$ E	F is closed or open	(2.86, 2.88)
ME $\Leftarrow$ E	F is resolvable	(2.92)
ME $\Leftarrow$ E	$\dim E < \infty$	(2.89)
ME $\not\Leftarrow$ E	F is F_σ face	(2.94, 2.96)
ME $\not\Leftarrow$ E	F is G_δ face	(2.93, 2.95)

Proposition 2.86. *Every measure extremal universally measurable subset of X is extremal, and every closed extremal subset of X is measure extremal.*

Proof. Let $F \subset X$ be measure extremal. Suppose that $x, y \in X$, $\lambda \in (0,1)$ and $z := \lambda x + (1-\lambda)y \in F$. Then the barycenter of the measure $\mu := \lambda\varepsilon_x + (1-\lambda)\varepsilon_y$ is z, hence in F. Since F is measure extremal, μ is carried by F. Therefore, $x, y \in F$.

As has been already mentioned, any closed extremal set is measure extremal. □

Proposition 2.87. *Let F be a universally measurable extremal subset of X. Then F is measure extremal if and only if $X \setminus F$ is measure convex.*

Proof. Assume that F is measure extremal and $\mu \in \mathcal{M}^1(X)$ is carried by $X \setminus F$. According to the hypothesis, $r(\mu) \in X \setminus F$, which gives that $X \setminus F$ is measure convex.

Conversely, let $X \setminus F$ be measure convex. Pick $\mu \in \mathcal{M}^1(X)$ with $r(\mu) \in F$. Note that $\mu(F) > 0$ since otherwise $r(\mu)$ would be contained both in F and in $X \setminus F$. Assume that $\mu(X \setminus F) > 0$ and set

$$\mu_1 := \frac{1}{\mu(F)}\mu|_F \quad \text{and} \quad \mu_2 := \frac{1}{\mu(X \setminus F)}\mu|_{X\setminus F}.$$

Then

$$r(\mu_2) \in X \setminus F$$

and

$$r(\mu) = \mu(F)r(\mu_1) + \mu(X \setminus F)r(\mu_2).$$

This is a contradiction since F is assumed to be extremal. Hence $\mu(X \setminus F) = 0$ and F is measure extremal. □

Since $X \setminus F$ is convex if F is extremal, Propositions 2.77, 2.74 and 2.76 yield using Proposition 2.87 the following two corollaries.

Corollary 2.88. *Every open or closed extremal subset of X is measure extremal.*

Corollary 2.89. *If A is a universally measurable extremal subset of a compact convex set in a finite-dimensional space, then A is measure extremal.*

Lemma 2.90. *If X is a compact convex subset of a locally convex space E and F a proper extremal subset of X, then it has empty interior in X.*

Proof. Assume that $z \in \operatorname{Int} F$ and let x be any point of $X \setminus F$. By the continuity of the vector operations in E, there is $\alpha \in (0,1)$ so that $y := \alpha x + (1-\alpha)z \in \operatorname{Int} F$. Since F is extremal, $x \in F$, which is a contradiction. □

Proposition 2.91. *A resolvable face is closed and, consequently, it is measure convex.*

Proof. Let F be a nonempty resolvable face such that $\overline{F} \setminus F \neq \varnothing$. Notice that $\overline{F}$ is a convex compact set. By Lemma 2.90, F has empty interior in $\overline{F}$. Thus $\overline{F} \setminus F$ is dense in $\overline{F}$. Since F and $\overline{F} \setminus F$ are disjoint nonempty dense subsets of the compact space $\overline{F}$, Proposition A.117(e) and Theorem A.58 yield a contradiction. □

Proposition 2.92. *Any resolvable extremal set is measure extremal.*

Proof. This follows from Proposition 2.80 and Proposition 2.87. □

Proposition 2.93. *There exists a G_δ face which is not measure extremal.*

Proof. Let
$$G := \left\{\mu \in \mathcal{M}^1([0,1]) \colon \mu = \mu_c\right\}.$$
By Proposition 2.58, G is a G_δ face of $\mathcal{M}^1([0,1])$ such that $G \cap \operatorname{ext} \mathcal{M}^1([0,1]) = \varnothing$. Let λ denote Lebesgue measure on $[0,1]$. If $\Lambda := \varepsilon_\sharp \lambda$ (cf. Example 2.53), then by Proposition 2.54, $r(\Lambda) = \lambda \in G$ whereas $\Lambda(G) = 0$ since Λ is carried by $\varepsilon([0,1])$. Whence, G is not measure extremal. □

Proposition 2.94. *There exists an F_σ face which is not measure extremal.*

Proof. Let again λ denote Lebesgue measure on $[0,1]$ and $\Lambda = \varepsilon_\sharp \lambda$ (see Example 2.53). If $F := \operatorname{face} \lambda$ is the face generated by λ, then by Corollary 2.67, F is an F_σ face.

Assume that $\mu \in F \cap \varepsilon([0,1])$. Hence, $\mu = \varepsilon_x$ for some $x \in [0,1]$ and by Corollary 2.67, there exist $\nu \in \mathcal{M}^1([0,1])$ and $\alpha \in [0,1)$ so that
$$\lambda = \alpha\nu + (1-\alpha)\varepsilon_x.$$
Then
$$0 = \lambda(\{x\}) = \alpha\nu(\{x\}) + (1-\alpha),$$
which implies that $\alpha = 1$, a contradiction. Therefore, $F \cap \varepsilon([0,1]) = \varnothing$. We see that $\Lambda(F) = 0$ while $r(\Lambda) = \lambda \in F$. Therefore F is not measure extremal. □

Proposition 2.95. *There exists a G_δ face which is neither measure convex nor measure extremal.*

Proof. We combine Examples 2.84 and 2.93. Let λ be Lebesgue measure on $[0,1]$ and C the Cantor ternary set. Set $G := G_1 \cap G_2$ where

$$G_1 := \left\{\mu \in \mathcal{M}^1([0,1])\colon\, \mu \perp \lambda\right\}$$

and

$$G_2 := \left\{\mu \in \mathcal{M}^1([0,1])\colon\, \mu|_C \text{ is continuous}\right\}.$$

It follows from Propositions 2.58 and 2.61 that G is a G_δ set in $\mathcal{M}^1([0,1])$. Further, as an intersection of faces, G is a face.

Let Λ denote again the image $\varepsilon_\sharp\lambda$ of Lebesgue measure λ on $[0,1]$ (see Example 2.53). Then $r(\Lambda) = \lambda$ by Proposition 2.54 and the barycenter $r(\Lambda)$ does not belong to G, although

$$\Lambda(G) = \lambda(\varepsilon^{-1}(G)) = \lambda([0,1] \setminus C) = 1.$$

Thus G is not measure convex.

Let $\Omega := \varepsilon_\sharp\nu$, where ν is a continuous probability measure carried by C. Then Ω is carried by $\varepsilon(C)$, and consequently $\Omega(G) = 0$. On the other hand, $r(\Omega) = \nu \in G$, and consequently G is not measure extremal. □

Proposition 2.96. *There exists an F_σ face which is neither measure convex nor measure extremal.*

Proof. Here we combine examples constructed in Propositions 2.83 and 2.94. Set $F := F_1 \cap F_2$, where

$$F_1 := \bigcup_{n=2}^{\infty} \left\{\mu \in \mathcal{M}^1([0,1])\colon\, \operatorname{spt}\mu \subset [\tfrac{1}{n}, 1]\right\} \quad \text{and} \quad F_2 := \operatorname{face}\lambda$$

(here, λ is again Lebesgue measure on $[0,1]$ and face λ denotes the face generated by λ). According to the aforementioned examples, F is an F_σ face in $\mathcal{M}^1([0,1])$. Let

$$\omega := 2\lambda|_{[\frac{1}{2},1]} \quad \text{and} \quad \Omega := \varepsilon_\sharp\omega.$$

Then $\operatorname{spt}\Omega = \varepsilon\big([\tfrac{1}{2},1]\big)$, thus $\operatorname{spt}\Omega \cap F = \varnothing$. Hence $\Omega(F) = 0$, but $r(\Omega) = \omega$ is contained in F. Hence F is not measure extremal.

For the proof of the second statement, we define for $n \in \mathbb{N}$

$$\lambda_n := \frac{n}{n-1}\lambda|_{[\frac{1}{n},1]} \quad \text{and} \quad \Omega := \sum_{n=1}^{\infty} \frac{1}{2^n}\varepsilon_{\lambda_n}.$$

Since $\lambda_n \in F$ for every $n \in \mathbb{N}$, $\Omega(F) = 1$. On the other hand,

$$r(\Omega) = \sum_{n=1}^{\infty} \frac{1}{2^n} \lambda_n$$

is not contained in F. Thus F is not measure convex and the proof is finished. $\square$

2.4 Exercises

Exercise 2.97. Let $\{C_\alpha\}_{\alpha \in A}$ be a collection of nonempty convex subsets of a vector space W. Prove that

$$\operatorname{co}\Big(\bigcup_{\alpha \in A} C_\alpha\Big) = \bigcup_{n=1}^{\infty} \Big\{ x \in W \colon x = \sum_{j=1}^{n} \lambda_j x_{\alpha_j}, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, x_{\alpha_j} \in C_{\alpha_j} \text{ for } j = 1, \dots, n \text{ and } \alpha_j \neq \alpha_k \text{ for } j \neq k \Big\}.$$

Exercise 2.98. Prove that x is an extreme point of a convex set X if and only if $x = x_1 = x_2 = \cdots = x_m$ whenever $x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m$ with $m \in \mathbb{N}$, $\sum_{j=1}^{m} \lambda_j = 1$, $x_j \in X$ and $\lambda_j > 0$ for each $j = 1, 2, \dots, m$.

Exercise 2.99. Let $X \neq \varnothing$ be a compact convex subset of $\mathbb{R}^d$. Prove directly (do not use the Minkowski theorem 2.11) that $\operatorname{ext} X \neq \varnothing$.

Hint. Find $x, y \in X$ such that $|x - y| = \operatorname{diam} X$ (here $|x - y|$ denotes the Euclidean distance between points x and y). It easily follows that $x, y \in \operatorname{ext} X$. Indeed, assume that

$$x = \frac{1}{2}(x_1 + x_2) \quad \text{where } x_1, x_2 \in X, x_1 \neq x_2.$$

Since the vectors $x_1 - y$ and $x_2 - y$ are linearly independent, we get

$$\begin{aligned} \operatorname{diam} X = |x - y| &= \left| \frac{1}{2}(x_1 - y) + \frac{1}{2}(x_2 - y) \right| \\ &< \frac{1}{2}\big(|x_1 - y| + |x_2 - y|\big) \leq \operatorname{diam} X. \end{aligned}$$

$\square$

Another hint. Prove that any point $z \in X$ having the property that $|z| \geq |x|$ for any $x \in X$ is an extreme point of X. $\square$

Exercise 2.100 (Radon). *Assume that a set $M \subset \mathbb{R}^d$ contains at least $d + 2$ points. Then $M = M_1 \cup M_2$ where $M_1 \cap M_2 = \varnothing$ and $\operatorname{co} M_1 \cap \operatorname{co} M_2 \neq \varnothing$.*

Hint. Suppose that $x_1, \ldots, x_n \in M$, $n \geq d+2$. Then there exists a nontrivial solution $(\alpha_1, \ldots, \alpha_n)$ to the system of $d+1$ equations

$$\sum_{j=1}^{n} \alpha_j = 0 \quad \text{and} \quad \sum_{j=1}^{n} \alpha_j x_j = 0.$$

Set

$$I^+ := \{j : \alpha_j \geq 0\} \quad \text{and} \quad I^- := \{j : \alpha_j < 0\}$$

and

$$M_1 := \left\{x_j : j \in I^+\right\} \quad \text{and} \quad M_2 := \left\{x_j : j \in I^-\right\}.$$

Then $M_1 \cap M_2 = \varnothing$. Since

$$\lambda := \sum_{j \in I^+} \alpha_j > 0,$$

we get

$$\operatorname{co} M_1 \ni \sum_{j \in I^+} \frac{\alpha_j}{\lambda} x_j = \sum_{j \in I^-} -\frac{\alpha_j}{\lambda} x_j \in \operatorname{co} M_2.$$

□

Exercise 2.101 (Helly). *Assume that $\mathcal{K}$ is a family of at least $d+1$ convex sets in $\mathbb{R}^d$ such that either $\mathcal{K}$ is finite or the sets of $\mathcal{K}$ are in addition closed and one of them is compact. If each $d+1$ sets of $\mathcal{K}$ have nonempty intersection, then*

$$\bigcap \{K : K \in \mathcal{K}\} \neq \varnothing.$$

Hint. For a finite family $\mathcal{K} = \{K_1, \ldots, K_n\}$, suppose first that $n = d+2$. By our assumption, there exist

$$x_i \in \bigcap_{j \in \{1,\ldots,n\} \setminus \{i\}} K_j, \quad 1 \leq i \leq n.$$

If there exist indices $i \neq k$ with $x_i = x_k$, then $x_i \in \bigcap_{j=1}^n K_i$. Otherwise we use Exercise 2.100 to find disjoint sets M_1, M_2 such that $M_1 \cup M_2 = \{x_1, \ldots, x_n\}$ and $\operatorname{co} M_1 \cap \operatorname{co} M_2$ contains a point y. Let $i \in \{1, \ldots, n\}$ be arbitrary. If $x_i \in M_1$, then $K_i \supset M_2$. Hence $y \in \operatorname{co} M_2 \subset K_i$. Analogously we get that $y \in K_i$ in the case when $x_i \in M_2$. Hence $y \in \bigcap_{i=1}^n K_i$.

Assume now that the assertion has been proved for each family $\mathcal{K}$ in $\mathbb{R}^d$ consisting of $n-1$ sets, where $n \geq d+2$. Let $\mathcal{K} = \{K_1, \ldots, K_n\}$. By the first part, each family $\mathcal{K}' \subset \mathcal{K}$ of at most $d+2$ elements has nonempty intersection. Thus the family

$$\{K_1, \ldots, K_{n-2}, K_{n-1} \cap K_n\}$$

satisfies that any subfamily with $d+1$ elements has nonempty intersection. Hence the inductive assumption yields that the latter, and, consequently, the former family has nonempty intersection.

Assume now that the family $\mathcal{K}$ consisting of closed convex sets is infinite and a set $Z \in \mathcal{K}$ is compact. By the first part, every finite subfamily of $\mathcal{K}$ has nonempty intersection. Hence $\mathcal{K}$, and consequently $\{Z \cap K : K \in \mathcal{K}\}$, has the finite intersection property. By compactness,

$$\bigcap\{Z \cap K : K \in \mathcal{K}\} = \bigcap\{K : K \in \mathcal{K}\}$$

is nonempty. □

Exercise 2.102. (a) Let G be an open subset of $\mathbb{R}^d$. Prove that $\operatorname{co} G$ is an open set.

(b) If F is a closed subset of $\mathbb{R}^d$, the convex hull $\operatorname{co} F$ need not be closed.

Hint. Consider, for example, the following sets

$$\{(x, y) \in \mathbb{R}^2 : x > 0, xy = 1 \text{ or } xy = -1\}$$

or $\{(x, y) \in \mathbb{R}^2 : x = 0\} \cup (1, 0)$. □

Exercise 2.103. (a) Let C be a compact convex subset of $\mathbb{R}^2$. Prove that the set $\operatorname{ext} C$ is closed.

(b) The set $\operatorname{ext} C$ need not be closed if C is a compact convex subset of $\mathbb{R}^d$ for $d \geq 3$.

Hint. Let $\{x_n\}$ be a sequence of points in $\operatorname{ext} C$ converging to x. Assuming that x is not an extreme point of C, let $x = \frac{1}{2}(a + b)$ for some points $a, b \in C$, $a \neq b$. By passing to a subsequence, we may assume that all points x_n are contained in the same open halfplane determined by the line passing through a and b. Then the interior of the triangle $\operatorname{co}\{x_1, a, b\}$ contains x_n for a suitable $n \in \mathbb{N}$, a contradiction with $x_n \in \operatorname{ext} C$.

For (b), consider the convex hull in $\mathbb{R}^3$ of the set

$$\left\{(x, y, z) : (x - 1)^2 + y^2 = 1, z = 0\right\} \cup \{(0, 0, 1) \cup (0, 0, -1)\}.$$

□

Exercise 2.104. Let C be a closed convex subset of $\mathbb{R}^d$ containing a line. Prove that $\operatorname{ext} C = \varnothing$. If C contains no line, then $\operatorname{ext} C \neq \varnothing$.

Hint. If C contains a line L and $x \in C$, then C contains also a line passing through x parallel to L.

If C contains no line then use induction on the dimension d. Find a boundary point of C and follow the reasoning of the proof of the Minkowski theorem 2.11. □

Exercise 2.105 (Closed convex hulls). Let C be a subset of a locally convex space E. Prove that the *closed convex hull* $\overline{\text{co}}\, C$ defined as

$$\overline{\text{co}}\, C := \bigcap\{F \colon F \text{ is a convex closed subset of } E, F \supset C\}$$

is the closure of the convex hull $\text{co}\, C$, that is, $\overline{\text{co}}\, C = \overline{\text{co}\, C}$.

Exercise 2.106 (Exposed points in $\mathbb{R}^d$). (a) Construct a compact convex set $K \subset \mathbb{R}^2$ for which $\text{ext}\, K \setminus \exp K \neq \varnothing$.

(b) Construct an example of a nonempty closed convex subset C of $\mathbb{R}^d$ such that the set $\exp C$ is not closed and $\text{co}\exp C \neq C$.

Hint. For the proof of (a) consider K to be the convex hull of the union of two circles

$$\{(x,y) \in \mathbb{R}^2 : (x+1)^2 + y^2 = 1\} \cup \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 = 1\}$$

then $(-1,1) \in \text{ext}\, K \setminus \exp K$.

To show (b) consider again the example from (a). □

Exercise 2.107. (a) Let S be a d-simplex in $\mathbb{R}^d$ determined by affinely independent points $e_0, \dots, e_d$. Prove that S has nonempty interior.

(b) Using (a) prove that there are no $d+1$ affinely independent points in any convex subset of $\mathbb{R}^d$ with empty interior.

(c) Let C be a convex subset of $\mathbb{R}^d$ with empty interior. Prove that there is an affine subspace A of $\mathbb{R}^d$ containing C such that $\dim A < d$.

Hint. For (a) show that

$$\frac{e_0 + \cdots + e_d}{d+1} \in \text{Int}\, S.$$

To verify (c), let $e_0, \dots, e_n$ be affinely independent points in C where $n < d+1$ is the maximum number of affinely independent points in C. Consider now the affine hull of $e_0, \dots, e_n$. □

Exercise 2.108. Let X be a nonempty compact convex subset of a locally convex space E and $x \in X$. Prove that $\mathcal{M}_x(X)$ is a nonempty convex compact subset of $\mathcal{M}^1(X)$.

Hint. A straightforward verification. □

Exercise 2.109. Let F be a closed subset of a compact convex set X such that $\text{ext}\, X \subset F$. Assume that for any $x \in X$ there exists a unique measure $\mu \in \mathcal{M}^1(F)$ such that $r(\mu) = x$. Prove that $\text{ext}\, X = F$.

Hint. Let $x \in F \setminus \operatorname{ext} X$. Then $x = \frac{y+z}{2}$, $y \neq z$, for some $y, z \in X$. The Integral representation theorem 2.31 yields measures $\mu_y, \mu_z \in \mathcal{M}^1(\overline{\operatorname{ext} X})$ such that $r(\mu_y) = y$ and $r(\mu_z) = z$. If $\mu = \frac{1}{2}(\mu_y + \mu_z)$, then

$$\mu \in \mathcal{M}^1(\overline{\operatorname{ext} X}), \quad r(\mu) = x \quad \text{and} \quad \mu \neq \varepsilon_x.$$

This contradicts the assumption, hence $\operatorname{ext} X = F$. □

Exercise 2.110 (Proof of the Milman theorem 2.43). Verify the following indication of an alternative proof of Theorem 2.43.

Hint. Let $x \in \operatorname{ext} X$. We note that $\overline{\operatorname{co}}\, B = \overline{\operatorname{co}}\, \overline{B}$. By Proposition 2.39 applied to $K = \overline{B}$, there exists a measure $\mu \in \mathcal{M}^1(\overline{B})$ representing the point x. Since $x \in X$, Bauer's characterization in 2.40 asserts that $\mu = \varepsilon_x$. The measure μ is carried by the (closed) set $\overline{B}$, which yields that $x \in \overline{B}$. □

Exercise 2.111. Find an example of a compact convex set X in a locally convex space E such that $(E^* + \mathbb{R})|_X \neq \mathfrak{A}^c(X)$.

Hint. Let $E := \ell^2$, $X := \{x \in E : 0 \leq x_n \leq 4^{-n}, n \in \mathbb{N}\}$ and $f(x) := \sum_{n=1}^{\infty} 2^n x_n$, $x \in X$. □

Exercise 2.112. Prove that in any infinite-dimensional Banach space E, there exists a compact convex set X such that $(E^* + \mathbb{R})|_X \neq \mathfrak{A}^c(X)$.

Hint. Find inductively points $x_n \in S_E$ and functionals $\varphi_n \in E^*$, $n \in \mathbb{N}$, such that

$$\varphi_n(x_m) = \begin{cases} 1, & n = m, \\ 0, & n \neq m, \end{cases} \qquad n, m \in \mathbb{N}.$$

Set $X := \overline{\operatorname{co}}\{4^{-n} x_n : n \in \mathbb{N}\}$ and $f := \sum_{n=1}^{\infty} 2^n \varphi_n$. Since $0 \leq \varphi_n \leq 4^{-n}$ on X, $n \in \mathbb{N}$, f is well defined.

If $f = \varphi + c$ on X for some $\varphi \in E^*$ and $c \in \mathbb{R}$, then $c = 0$, because $f(0) = 0$. Further, since $f(4^{-n} x_n) = 2^{-n}$, we get $\varphi(x_n) = 2^n$ for each $n \in \mathbb{N}$. But this is impossible as φ is bounded on B_E. □

Exercise 2.113. Find a compact convex set $X \subset \mathbb{R}^2$ such that $\operatorname{far} X \neq \exp X$.

Hint. Let
$$X = \{(x, y) \in \mathbb{R}^2 : |x|^3 \leq y \leq 1, -1 \leq x \leq 1\}.$$

Then the point $(0, 0)$ is obviously exposed. However, $(0, 0) \notin \operatorname{far} X$ as an easy geometrical argument shows. □

Exercise 2.114. Find an example of a compact convex subset X of a locally convex space E and a point $z \in X$ that is exposed but there is no $f \in E^*$ exposing z.

First hint. Let H be a Hilbert space, M be a dense proper subspace of H^* and let $\Phi : H^* \to H$ be the mapping assigning to each $\varphi \in H^*$ a point $y \in H$ such that $\varphi(h) = (h, y)$ for any $h \in H$. Let $X := B_H$ be equipped with the $\sigma(H, M)$-topology. Since B_H is w-compact and $\sigma(H, M)$ is weaker and Hausdorff, $w = \sigma(H, M)$ on X. Choose $x \in S_H \setminus \Phi(M)$ and denote $\psi := \Phi^{-1}(x)$. Show that ψ is not $\sigma(H, M)$-continuous. Further show that $\psi|_X \in \mathfrak{A}^c((X, \sigma(H, M))$ and that ψ exposes x. On the other hand, no functional from $(H, \sigma(H, M))^* = M$ exposes x. (This easily follows from the fact that, given $x, y \in S_H$, $(x, y) = 1$ if and only if $y = x$.) □

Second hint. Let $\{e_n\}$ be a sequence of standard unit vector in c_0. For $1 \le i \le j$ denote $u_{i,j} := -e_i + 2e_j$. Let further

$$X := \overline{\text{co}}^{\|\cdot\|}\, C \quad \text{where} \quad C := \left\{2^{-i-j} u_{i,j} : 1 \le i \le j\right\}.$$

Define the function h on X as

$$h : x \mapsto \sum_{n=1}^{\infty} x_n, \quad x = \{x_n\} \in X.$$

It is easy to show that h is an affine continuous function on X. Obviously, $C \cup \{0\}$ is a compact subset of c_0 and $C \subset \{\{x_n\} \in c_0 : |x_n| \le 2^{-n}\}$. It easily follows that X is compact.

Let $x \in X \setminus \{0\}$. Since $\operatorname{ext} X \subset C \cup \{0\}$, by the Krein–Milman theorem 2.22 we have $x = \sum_{i \le j} \alpha_{i,j} 2^{-i-j} u_{i,j}$, where $\alpha_{i,j} \ge 0$ and their sum is smaller or equal to 1. There exists a pair $i \le j$ such that $\alpha_{i,j} > 0$. Hence

$$h(x) = \sum_{i \le j} \alpha_{i,j} 2^{-i-j} h(u_{i,j}) = \sum_{i \le j} \alpha_{i,j} 2^{-i-j} > 0.$$

It follows that 0 is an exposed point of X.

On the other hand, 0 is not exposed by any functional from $(c_0)^* = \ell^1$. Indeed, assume that $0 \ne f = \{f_n\} \in (c_0)^* = \ell^1$. There exists i such that $f_i \ne 0$. If $f_i < 0$, then $f(u_{i,i}) = f(e_i) = f_i < 0$. If $f_i > 0$, then $f(u_{i,j}) = f(-e_i + 2e_j) = -f_i + 2f_j$. Since $\lim_{j \to \infty} f_j = 0$, we have $\lim_{j \to \infty} f(u_{i,j}) = -f_i$. Hence, there exists $j \ge i$ such that $f(u_{i,j}) < 0$. If $x := 2^{-i-j} u_{i,j}$, then $x \in X$, $x \ne 0$, and $f(x) < 0 = f(0)$. □

Exercise 2.115. Keeping the notation of the first hint in Exercise 2.114, consider locally convex spaces $E_1 := (H, w)$ and $E_2 := (H, \sigma(H, M))$. Prove that the compact convex sets $(B_H, \sigma(H, M))$ and (B_H, w) are affinely homeomorphic (by the identity mapping), and that the point x is exposed by a functional from E_1^* whereas it is not exposed by a functional from E_2^*. (Compare with the proof of Theorem 2.51.)

Hint. Follow the reasoning of Exercise 2.114. □

Exercise 2.116. Let X be a nonempty compact convex set and $K \subset X \setminus \operatorname{ext} X$ be compact. Then $\overline{\text{co}}\ K \cap \operatorname{ext} X = \varnothing$.

Hint. For any $x \in \overline{\text{co}}\, K \setminus K$ there exists $\mu \in \mathcal{M}_x(K)$ (see Proposition 2.39). Obviously, $\mu \neq \varepsilon_x$, and thus $x \notin \operatorname{ext} X$ by Theorem 2.40. □

Exercise 2.117. Let F be a closed face of a compact convex set X and let $U \subset X$ be an open set containing F. Then $F \cap \overline{\text{co}}(X \setminus U) = \varnothing$.

Hint. Proceed as in Exercise 2.116. □

Exercise 2.118. Let ν be a Radon probability measure on a compact space K. Let

$$L := \left\{\mu \in \mathcal{M}^1(K) \colon \mu \perp \nu\right\}.$$

Prove that L is a G_δ set.

Hint. For each $n \in \mathbb{N}$ and each open subset G of K, let

$$L(n, G) := \left\{\mu \in \mathcal{M}^1(K) : \mu(G) > 1 - 2^{-n}\right\}$$

and

$$L_n := \bigcup \left\{L(n, G) \colon G \subset K \text{ open}, \nu(G) < 2^{-n}\right\}.$$

Since each set $L(n, G)$ is open in $\mathcal{M}^1(K)$ by Theorem A.85(b), the set $\bigcap_{n=1}^\infty L_n$ is a G_δ set. The proof will be complete once we show that

$$L = \bigcap_{n=1}^\infty L_n.$$

Pick an arbitrary index $n \in \mathbb{N}$ and $\mu \in L$. As $\mu \perp \nu$, there is a Borel set $A \subset K$ such that $\nu(A) = 0$ and $\mu(A) = 1$. Due to the regularity of ν it follows that there exists an open set $G \supset A$ such that $\nu(G) < 2^{-n}$. Since $\mu(G) \geq \mu(A) = 1 > 1 - 2^{-n}$, we get $\mu \in L_n$.

Conversely, assume that $\mu \in \bigcap_{n=1}^\infty L_n$. There is a sequence $\{G_n\}$ of open subsets of K such that

$$\nu(G_n) < 2^{-n} \quad \text{and} \quad \mu(G_n) > 1 - 2^{-n}$$

for each $n \in \mathbb{N}$. Set $G := \bigcap_{m=1}^\infty \bigcup_{n=m}^\infty G_n$. Then $\nu(G) = 0$. Since

$$1 \geq \mu(G) = \lim_{m \to \infty} \mu\Big(\bigcup_{n=m}^\infty G_n\Big) \geq \limsup_{m \to \infty} \mu(G_m) \geq \limsup_{m \to \infty} (1 - 2^{-m}) = 1,$$

it follows that $\mu \perp \nu$. Hence, $\mu \in L$, as required. □

Exercise 2.119. If F is a face of X and G is a face of F, then G is a face of X.

Hint. A straightforward verification. □

Exercise 2.120. Prove that a set $A \subset X$ is convex if $X \setminus A$ is extremal.

Hint. A straightforward verification. □

Exercise 2.121. Find an F_σ face that is not a countable union of closed faces.

Hint. Use the set F from Proposition 2.94. If F were a countable union of closed faces, F would be measure extremal by Proposition 2.86. But this is not the case. □

Exercise 2.122. If F is a closed face of X, then there exists a set $A \subset \operatorname{ext} X$ such that $F = \overline{\operatorname{co}}\, A$. In particular, if F is a nonempty closed face of X, then $F \cap \operatorname{ext} X \neq \varnothing$ (cf. Theorem 2.20).

Hint. Set $A := F \cap \operatorname{ext} X$. Then $A = \operatorname{ext} F$ by Proposition 2.64(b) and it suffices to apply the Krein–Milman theorem 2.22. □

Exercise 2.123. Let X be a convex subset of a locally convex space E. Prove that the boundary ∂X is an extremal subset of X.

Hint. If $\operatorname{Int} X = \varnothing$, the assertion is obvious. Hence, assume that $\operatorname{Int} X \neq \varnothing$ and that ∂X is not extremal. In this case, there exist $z \in \partial X$, $x \in X$ and $y \in \operatorname{Int} X$ so that $z = \lambda x + (1-\lambda)y$ for some $\lambda \in (0,1)$. Find a neighborhood V of 0 such that $y + V \subset \operatorname{Int} X$. For every $\varepsilon > 0$, there exists z_ε such that $z_\varepsilon \in (z + \varepsilon V) \setminus X$. Let

$$w_\varepsilon := \frac{z_\varepsilon - \lambda x}{1-\lambda}.$$

Show that we can choose ε small enough such that $w_\varepsilon \in y + V$. Since $z_\varepsilon = \lambda x + (1-\lambda)w_\varepsilon$, this implies that $z_\varepsilon \in X$, which is a contradiction. □

Exercise 2.124. Let X be a compact convex set and $x \in X$. Prove that

$$D := \bigcap \{A \subset X \colon A \text{ is a closed extremal set containing } x\}$$

is convex.

Hint. The set D is obviously closed and extremal. Then use characterization (vi) of Proposition 2.69. □

Exercise 2.125. Find a continuous affine surjection of a compact convex set X onto a compact convex set Y such that $\operatorname{ext} Y \neq \varphi(\operatorname{ext} X)$.

Hint. Consider a triangle given as the convex hull of points $(0,0), (1,0), (\frac{1}{2}, 1)$ and its projection onto the unit segment $[0,1]$. □

Exercise 2.126. Prove that the set

$$G := \{\mu \in \mathcal{M}^1([0,1]) : \mu = \mu_c\}$$

of all continuous Radon measures on $[0,1]$ is measure convex (cf. also Proposition 2.93).

Hint. Let $\Omega \in \mathcal{M}^1\big(\mathcal{M}^1([0,1])\big)$, $\Omega(G) = 1$ with $r(\Omega) = \omega$ be given. For a bounded Borel function f on $[0,1]$, define an affine function If on $\mathcal{M}^1([0,1])$ by the formula

$$If(\mu) := \mu(f), \quad \mu \in \mathcal{M}^1([0,1]).$$

Obviously, If is continuous whenever f is. Thus

$$\Omega(If) = If(r(\Omega)) = \omega(f) \quad \text{for any} \quad f \in \mathcal{C}[0,1]. \tag{2.4}$$

Now, use the Lebesgue dominated convergence theorem to show that equality (2.4) holds for any bounded Borel function f on $[0,1]$. Pick $x \in [0,1]$. Then

$$\omega(\{x\}) = Ic_{\{x\}}(\omega) = \int_G Ic_{\{x\}}(\mu)\, d\Omega(\mu) = 0.$$

Thus $\omega(\{x\}) = 0$ for any $x \in [0,1]$, and therefore $\omega \in G$. □

Exercise 2.127. Let K be a compact space, $X := \mathcal{M}^1(K)$, $\omega \in \mathcal{M}^1(K)$,

$$A_1 := \Big\{\frac{\mu}{\mu(K)} : \mu \in \mathcal{M}^+(K), \mu \leq \omega \text{ and } \mu(K) > 0\Big\}$$

and

$$A_2 := \Big\{\frac{f\omega}{\omega(f)} : 0 \leq f \leq 1 \text{ Borel}, \omega(f) > 0\Big\}.$$

Prove that

(a) $\operatorname{face}\omega = A_1 = A_2$,

(b) $\overline{\operatorname{face}\omega} = \mathcal{M}^1([0,1])$ if $K = [0,1]$ and ω is Lebesgue measure on $[0,1]$.

Hint. (a) Let $\mu \in \operatorname{face}\omega$ be given. Then there exist $\alpha \in (0,1]$ and $\nu \in \mathcal{M}^1(K)$ such that $\omega = \alpha\mu + (1-\alpha)\nu$. Then $0 < \alpha\mu \leq \omega$.

If $\mu \leq \omega$, by the Radon–Nikodym theorem there exists a Borel measurable function f such that $\mu = f\omega$. Then $0 \leq f \leq 1$ ω-almost everywhere and $\omega(f) = \mu(K) > 0$. Hence

$$\frac{\mu}{\mu(K)} = \frac{f\omega}{\omega(f)}$$

and $\mu \in A_2$.

Finally, let $\mu = \frac{f\omega}{\omega(f)}$ for some Borel function f with $0 \le f \le 1$ and $\omega(f) > 0$. We may assume that $\omega(f) < 1$. Then

$$\omega = \omega(f)\frac{f\omega}{\omega(f)} + \omega(1-f)\frac{(1-f)\omega}{\omega(1-f)},$$

and $\mu \in \operatorname{face}\omega$.

(b) Assume that λ is Lebesgue measure on $[0,1]$. Given a point $x \in [0,1]$, it is easy to find a sequence of measures in face λ that converges to ε_x. Hence face λ is a convex set containing all extreme points of $\mathcal{M}^1([0,1])$, and thus $\overline{\operatorname{face}\lambda} = \mathcal{M}^1([0,1])$. □

2.5 Notes and comments

The material of the Subsection 2.1.A on convexity in finite-dimensional spaces is standard and can be found in many textbooks; see, for example, Barvinok's monograph [31]. Besides Minkowski's result, at the beginning of 20$^{\text{th}}$ century, three important theorems on convex sets in Euclidean spaces associated with the names of Carathéodory, Helly and Radon appeared. H. Minkowski proved Theorem 2.11 in the period 1901-03. The result appeared for the first time in a chapter on convex bodies (where the origin of the notion of extreme points can be traced) included in his collected works published in 1911; see J. J. Saccoman [405]. An exhaustive survey on Helly's theorem 2.101 and its proofs, variants and applications is presented in the paper by L. Dantzer, B. Grünbaum and V. Klee [130].

The Krein–Milman theorem 2.22 was proved by M. Krein and D. Milman in [283] for the case of w^*-compact convex sets in the dual to a Banach space using the transfinite induction. The proof of a more general version contained in Theorem 2.22, based on an application of Zorn's lemma, goes back to J. L. Kelley [266]. A similar proof was given by E. Artin (a letter from Artin to his former student M. Zorn published in the Picayune Sentinel of Indiana University in 1950; cf. [23]), A. Hotta [240] and by K. Yosida and M. Fukamiya [477]. The Krein–Milman theorem is one of the fundamental theorems of functional analysis and has rich applications. For instance, recall its use in de Branges' proof of the Stone–Weierstrass theorem, Lindenstrauss' proof of the Lyapunov theorem on the range of a vector measure, and in the proof of the Banach–Stone theorem on isometrically isomorphic spaces of continuous functions. See Chapter 14 for more applications. Bauer's concave minimum principle 2.24 (even in a more general form) appeared in H. Bauer [36].

The Integral representation theorem 2.31 is a reformulation of the Krein–Milman theorem. It has wide applications in several areas of analysis. We gave two different proofs of this theorem (besides Theorem 2.31 it is its more general form in Proposition 2.39). Still another proof of Theorem 2.29, avoiding the notion of a net, is presented in Phelp's book [374], Proposition 1.1.

Bauer's characterization of ext X in Theorem 2.40 presented in [37] is used later on as the definition of the Choquet boundary of function spaces in 3.4. Proposition 2.41 can be found as Proposition 25.13 in G. Choquet [108] and it is used several times in our text. For example, Milman's converse 2.43 of the Krein–Milman theorem, which goes to V. P. Milman [346], is an easy consequence of it.

Proposition 2.45 enables to solve certain problems concerning metrizable compact convex sets reducing them to subsets of Hilbert spaces. It was shown by O. -H. Keller [265] that any infinite-dimensional compact convex subset of a Hilbert space is homeomorphic to the Hilbert cube $[0,1]^{\mathbb{N}}$ (which is affinely homeomorphic to the set $\big\{\{x_n\} \in \ell^2 : |x_n| \le \frac{1}{n}, n \in \mathbb{N}\big\}$). Moreover, V. L. Klee observed that any compact (convex) subset of a Banach space is affinely homeomorphic to a subset of a Hilbert space. An infinite-dimensional compact convex subset C of a topological vector space is said to be a *Keller set* if C is affinely homeomorphic to a compact convex subset of a Hilbert space ℓ^2. Hence, Proposition 2.45 shows that any infinite-dimensional metrizable compact convex set is a Keller set. Moreover, it has been shown by C. Bessaga and T. Dobrowolski in [56] that *any locally compact convex subset C of a topological vector space with a countable family of continuous affine functions on C separating points of C can be affinely embedded into ℓ^2.*

The concept of exposed points in the case of Euclidean spaces was introduced by S. Straszewicz [441] in 1935. In this case, Corollary 2.52 and example in Exercise 2.103(b) are due to him. The proof of Proposition 2.50 is taken from V. P. Fonf, J. Lindenstrauss and R. R. Phelps [179]. In the paper [53] by S. K. Berberian, the Krein–Milman theorem in Hilbert spaces is derived. This result can be deduced from more general statements concerning the exposed points in normed linear spaces; see, for example, the paper of V. L. Klee [272]. In fact, from the proof of Klee's result it follows that in any smooth and strictly convex normed linear space any compact convex set is the closed convex hull of its set of so-called bare points (cf. a review of the paper [53] in Mathematical Reviews). See also the paper by M. V. Balashov [30]. We also refer the reader to the paper [151] by M. Edelstein and J. E. Lewis on exposed and farthest points.

In [18], R. F. Arens and J. L. Kelley described extreme points of the unit ball of the space $\big(\mathcal{C}(K)\big)^*$ as Dirac measures ε_x and their antipodes $-\varepsilon_x$ (cf. Proposition 2.27). For an alternative proof of Corollary 2.28 see, for example, Theorem 30.4 and Corollary 30.5 in Bauer's monograph [45]. In Proposition 2.56 we present a simplified version of the proof of [5], Proposition I.2.8. Examples described in Proposition 2.63 are due to G. Choquet [106]; see also Alfsen's monograph [5], Example I.2.10.

Any union of faces was labelled in Goullet de Rugy in [200] as a *σ-face*. Closed extremal sets were studied by E. M. Alfsen in [1] under the name "stable sets". The equivalence of (i) and (v) in Proposition 2.69 was proved by E. M. Alfsen in [1]; the idea of the proof (i) $\Longrightarrow$ (vi) is from D. P. Milman [346] (§ 4, Theorem 7).

In part, the material Subsections 2.3.B and 2.3.C concerning a more detailed study of measure convex and measure extremal sets is taken from the paper [146] by P. Dostál, J. Lukeš and J. Spurný. Theorem 2.75 and its proof are taken from D. H. Fremlin and J. D. Pryce [185]. Alfsen's example 2.81 is taken from [5], p. 130. The proof of Proposition 2.92 can also be proved by means of a result of J. Saint Raymond 10.75 without recourse to Proposition 2.80. Counterexamples contained in Propositions 2.93 and 2.94 are just suitable modifications of the example by G. Choquet in [106]. The examples of Propositions 2.95 and 2.96 partially use a construction of H. v. Weizsäcker (see [463]). We also refer the reader to Lecture Notes [473] by G. Winkler where a thorough investigation of measure convex sets is given.

The result of Exercise 2.103(a) is due to G. B. Price [377]. A characterization of continuous affine functions on X (cf. Lemma 2.34 and Exercise 2.112) belonging to $\overline{E^* + \mathbb{R}}|_X$ even for noncompact convex sets X is given in a paper [281] by M. Kraus. Examples of Exercises 2.114 and 2.115 are due to M. Kraus and O. Kurka.

Chapter 3

Choquet theory of function spaces

This chapter lays the groundwork for the rest of the book by presenting the foundations of the Choquet theory of function spaces. The central concept of a function space is defined and its basic properties investigated in Section 3.1. We generalize the framework of spaces of affine continuous functions on compact convex sets by taking a subspace $\mathcal{H}$ of the space $\mathcal{C}(K)$ of continuous functions on a compact space K such that $\mathcal{H}$ contains constants and separates points of K. Then we introduce $\mathcal{H}$-representing measures, $\mathcal{H}$-affine and $\mathcal{H}$-convex functions, and so on. A suitable substitute for the set of extreme points turns out to be the Choquet boundary and we show in Proposition 3.15 its nonemptiness and prove a minimum principle in Theorem 3.16.

A crucial notion for obtaining integral representation theorems is the Choquet ordering introduced in Definition 3.19. This ordering somehow indicates how close to the Choquet boundary a measure is situated. A key tool for handling function spaces is Lemma 3.21 which serves as a substitute for the Hahn–Banach theorem. Several of its applications are shown afterwards, along with Bauer's characterization 3.24 of the Choquet boundary.

These abstract results are then applied to a reexamination of Korovkin's theorems; Theorems 3.32, 3.34 and 3.36 are proved by means of the Choquet theory. After generalizing the concept of the barycenter mapping in Section 3.3, we turn our attention to the Choquet representation theorem 3.45 for function spaces on metrizable compact spaces. Our approach is to use the existence of a "strictly convex" function.

Next, Section 3.5 indicates how the Key lemma 3.21 enables us to prove analogues of classical results on approximation of semicontinuous convex functions on compact convex sets. More precisely, we show that semicontinuous $\mathcal{H}$-convex functions can be approximated by continuous $\mathcal{H}$-convex functions (see Propositions 3.48 and 3.54). An important corollary is the fact that the Choquet ordering of measures can be extended to semicontinuous $\mathcal{H}$-convex functions (see Proposition 3.56).

Measures maximal with respect to the Choquet ordering are investigated in Section 3.6. First we prove Mokobodzki's characterization in Theorem 3.58. Then we show that the set of maximal measures is rich enough (see Theorem 3.65) and we finish the section with Theorem 3.70, which describes the space of boundary measures.

In order to prove the most important properties of maximal measures, namely, that they are carried by any Baire set containing the Choquet boundary, we need some kind of Fatou's lemma (see Lemma 3.77). This task is accomplished in Section 3.7 by presenting the important Simons inequality 3.74 and a couple of its applications.

Then we can prove the integral representation theorem for nonmetrizable spaces in Theorem 3.81.

The existence of representing measures "carried" by the Choquet boundary opens the way for the proof of several variants of the minimum principle, as shown in Section 3.9. The last section is devoted to a characterization of the fact that a pair of measures μ, ν is related in the Choquet ordering. Theorem 3.92 shows that the important notion of a dilation plays the key role here.

3.1 Function spaces

Definition 3.1 (Function spaces). By a *function space* $\mathcal{H}$ on a compact topological space K we mean a (not necessarily closed) linear subspace of $\mathcal{C}(K)$ containing the constant functions and separating points of K.

We introduce some main examples of function spaces.

Examples 3.2. (a) *Continuous functions.* The space $\mathcal{C}(K)$ of all continuous functions on a compact space K represents a simple example of a function space. Clearly, the space $\mathcal{C}(K)$ separates points of K.

(b) *Quadratic polynomials.* The space $\mathcal{P}_2([0,1])$ of all quadratic polynomials on the interval $[0,1]$ is a further example of a function space. We considered this example in Chapter 1.

(c) *Convex case – affine functions.* Let X be a compact convex subset of a locally convex space E. The linear space $\mathfrak{A}^c(X)$ consisting of all continuous affine functions on X is a function space.

(d) *Harmonic case – harmonic functions.* Let U be a bounded open subset of the Euclidean space $\mathbb{R}^d$. The function space $\mathbf{H}(U)$ consisting of all continuous functions on $\overline{U}$ which are harmonic on U is another example of a function space.

More generally, we can consider a relatively compact open subset U of a Bauer harmonic space (cf. Section A.8) and the function space $\mathbf{H}(U)$, the linear subspace of $\mathcal{C}(\overline{U})$ of functions which are harmonic on U. We tacitly assume that constant functions are harmonic and $\mathbf{H}(U)$ separates points of $\overline{U}$.

(e) If K is a compact subset of $\mathbb{R}^d$, we define

$$\mathbf{H}_0(K) := \bigcup\{\mathbf{H}(U)|_K : U \text{ is relatively compact open set}, K \subset U\}.$$

Generally, the function space $\mathbf{H}_0(K)$ is not a closed subspace of $\mathcal{C}(K)$ (see Exercise 13.155).

(f) Further examples can be found in 3.47, 3.82, 3.83, 3.103, 3.106, 3.111, 3.119, 6.67, 6.76, 6.77, 6.94, 7.65, 8.11(a), 8.29, 8.80, 9.11, 9.56, 10.97, 10.98, 10.99 and 12.3.B.

(g) Let K be a compact space and T be a *Markov operator* on K (cf. Subsection 6.6.B). Let

$$\mathcal{H}_T := \{h \in \mathcal{C}(K) : T(h) = h\}.$$

If $\mathcal{H}_T$ separates points of K, then $\mathcal{H}_T$ is a function space.

Convention. In what follows, $\mathcal{H}$ denotes a function space on a compact space K.

Definition 3.3 ($\mathcal{H}$-representing measures). Recall that $\mathcal{M}^1(K)$ denotes the set of all probability Radon measures on K. We denote by $\mathcal{M}_x(\mathcal{H})$ the set of all *$\mathcal{H}$-representing measures* for $x \in K$, that is,

$$\mathcal{M}_x(\mathcal{H}) := \{\mu \in \mathcal{M}^1(K) : f(x) = \int_K f \, d\mu \text{ for any } f \in \mathcal{H}\}.$$

Of course, the Dirac measure ε_x at the point x always belongs to $\mathcal{M}_x(\mathcal{H})$.

Definition 3.4 (Choquet boundary). Define the *Choquet boundary* $\mathrm{Ch}_{\mathcal{H}}(K)$ of a function space $\mathcal{H}$ as the set of those points $x \in K$ for which the Dirac measure ε_x is the only $\mathcal{H}$-representing measure for x; that is,

$$\mathrm{Ch}_{\mathcal{H}}(K) := \{x \in K : \mathcal{M}_x(\mathcal{H}) = \{\varepsilon_x\}\}.$$

Examples 3.5. We describe the Choquet boundary of our main examples from 3.2. We postpone the proofs to later sections.

(a) *Continuous functions.* In the case when $\mathcal{H} = \mathcal{C}(K)$ (K is a compact space), the equality $\mathrm{Ch}_{\mathcal{H}}(K) = K$ follows immediately from the definition.

(b) *Quadratic polynomials.* Let $\mathcal{H} = \mathcal{P}_2([0,1])$ be the function space of all quadratic polynomials on the interval $[0,1]$. Then $\mathrm{Ch}_{\mathcal{H}}([0,1]) = [0,1]$, as easily follows by Proposition 3.7.

(c) *Convex case – affine functions.* If $\mathcal{H}$ is the linear space $\mathfrak{A}^c(X)$ of all continuous affine functions on a compact convex set X, then $\mathrm{Ch}_{\mathfrak{A}^c(X)}(X) = \operatorname{ext} X$, by Bauer's characterization of $\operatorname{ext} X$ in Theorem 2.40.

(d) *Harmonic case – harmonic functions.* If $\mathbf{H}(U)$ consists of all continuous functions on $\overline{U} \subset \mathbb{R}^d$ which are harmonic on U, then $\mathrm{Ch}_{\mathbf{H}(U)} \overline{U} = \partial_{\mathrm{reg}} U$ (cf. Theorem 13.35).

In the general situation of an abstract Bauer harmonic space, the situation is more delicate, cf. Theorem 13.41.

(e) The Choquet boundary of $\mathbf{H}_0(K)$ (see Example 3.2(e)) consists of stable points of K (see Definition 13.4 and Theorem 13.35).

(f) Let
$$\mathcal{H} := \{f \in \mathcal{C}([-1,1]) : 2f(0) = f(-1) + f(1)\}.$$
Then $\mathrm{Ch}_{\mathcal{H}}([-1,1]) = [-1,1] \setminus \{0\}$.

Further examples of Choquet boundaries can be found in the examples of function spaces mentioned in Examples 3.2(f).

Definition 3.6 ($\mathcal{H}$-exposing functions and $\mathcal{H}$-exposed points). Let $x \in K$. A function $h \in \mathcal{H}$ such that $0 = h(x) < h(t)$ for any $t \in K$, $t \neq x$, is said to be an $\mathcal{H}$-*exposing* function for x. A point $x \in K$ is called an $\mathcal{H}$*-exposed point* if there exists an $\mathcal{H}$-exposing function for x.

The set of all $\mathcal{H}$-exposed points of K will be denoted by $\exp_{\mathcal{H}}(K)$.

Proposition 3.7. *Any $\mathcal{H}$-exposed point belongs to the Choquet boundary of $\mathcal{H}$.*

Proof. Let $x \in K$ and let $h \in \mathcal{H}$ for which $0 = h(x) < h(t)$ for any $t \in K \setminus \{x\}$. If $\mu \in \mathcal{M}_x(\mathcal{H})$, then $0 = h(x) = \mu(h)$. Hence $\operatorname{spt}\mu \subset \{x\}$, and therefore $\mu = \varepsilon_x$. We see that $x \in \mathrm{Ch}_{\mathcal{H}}(K)$. □

Definition 3.8 ($\mathcal{H}$-affine, $\mathcal{H}$-convex and $\mathcal{H}$-concave functions). We define the family $\mathcal{A}(\mathcal{H})$ of all $\mathcal{H}$*-affine functions* as the family of all universally measurable functions $f : K \to [-\infty, \infty]$ such that $\mu(f)$ exists for every $\mu \in \mathcal{M}_x(\mathcal{H})$, $x \in K$, and the following *barycentric formula* holds:
$$f(x) = \int_K f\,d\mu \quad \text{for each} \quad x \in K,\ \mu \in \mathcal{M}_x(\mathcal{H}).$$
Further, let $\mathcal{A}^c(\mathcal{H})$ be the family of all continuous $\mathcal{H}$-affine functions on K.

Similarly, we say that a universally measurable function $f : K \to [-\infty, \infty]$ is $\mathcal{H}$*-convex* if $\mu(f)$ exists for every $\mu \in \mathcal{M}_x(\mathcal{H})$, $x \in K$, and $f(x) \leq \mu(f)$. A function f is $\mathcal{H}$*-concave* if $-f$ is $\mathcal{H}$-convex. We denote by $\mathcal{K}^c(\mathcal{H})$, $\mathcal{K}^{usc}(\mathcal{H})$, and $\mathcal{K}^{lsc}(\mathcal{H})$ the family of all continuous, upper semicontinuous, and lower semicontinuous $\mathcal{H}$-convex functions, respectively. We write $\mathcal{S}^c(\mathcal{H})$, $\mathcal{S}^{usc}(\mathcal{H})$ and $\mathcal{S}^{lsc}(\mathcal{H})$ for the analogous families of $\mathcal{H}$-concave functions.

Remark 3.9. As we will see later in Chapter 4, in the convex case of Example 3.2(c), a continuous function is $\mathfrak{A}^c(X)$-concave if and only if it is concave in the usual sense.

Definition 3.10 (Cone $\mathcal{W}(\mathcal{H})$). We denote by $\mathcal{W}(\mathcal{H})$ the smallest min-stable cone generated by $\mathcal{H}$, that is, $\mathcal{W}(\mathcal{H})$ consists of all functions of the form $h_1 \wedge \cdots \wedge h_n$, $h_1, \dots, h_n \in \mathcal{H}$, $n \in \mathbb{N}$.

The following proposition collects several easy facts.

Proposition 3.11.

(a) *$\mathcal{A}^c(\mathcal{H})$ is a closed function space on K containing $\overline{\mathcal{H}}$.*

(b) *The families* $\mathcal{W}(\mathcal{H})$, $\mathcal{S}^c(\mathcal{H})$, $\mathcal{S}^{usc}(\mathcal{H})$ *and* $\mathcal{S}^{lsc}(\mathcal{H})$ *form min-stable convex cones.*

(c) *The space* $\mathcal{W}(\mathcal{H}) - \mathcal{W}(\mathcal{H})$ *is dense in* $\mathcal{C}(K)$.

Proof. Since (a) is obvious, we proceed to the proof of (b). Let $\mathcal{F}$ be any of the considered families. Obviously, $\mathcal{F}$ is a convex cone and the required topological property is stable with respect to taking finite minima (for $\mathcal{F} = \mathcal{W}(\mathcal{H})$ we use identity

$$f_1 \wedge \cdots \wedge f_n + g_1 \wedge \cdots \wedge g_k = \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{k} (f_i + g_j).)$$

If $k_1, k_2 \in \mathcal{F}$ and $\mu \in \mathcal{M}_x(\mathcal{H})$, then $\mu(k_1 \wedge k_2)$ exists and

$$\mu(k_1 \wedge k_2) \le \mu(k_1) \wedge \mu(k_2) \le (k_1 \wedge k_2)(x).$$

For the proof of (c), we just use the lattice version of the Stone–Weierstrass theorem from Proposition A.31. □

Definition 3.12 ($\mathcal{H}$-extremal sets). Let $\mathcal{H}$ be a function space on a compact space K. A universally measurable set $F \subset K$ is called $\mathcal{H}$-*extremal* if any measure representing a point in F is carried by F.

If X is a compact convex set in a locally convex space and $\mathcal{H} = \mathfrak{A}^c(X)$, then a closed set $F \subset X$ is $\mathfrak{A}^c(X)$-extremal if and only if F is extremal (see Proposition 2.69).

Lemma 3.13. *Let F be a closed $\mathcal{H}$-extremal set and $f \in \mathcal{S}^{lsc}(\mathcal{H})$. Then the set*

$$H := \{x \in F : f(x) = \min f(F)\}$$

is $\mathcal{H}$-extremal.

Proof. Obviously, H is closed. To show that H is $\mathcal{H}$-extremal, pick $x \in H$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. Since F is $\mathcal{H}$-extremal, $\operatorname{spt}\mu \subset F$. Then from the inequalities

$$\min f(F) = f(x) \ge \int_F f(t)\, d\mu(t) \ge \min f(F),$$

it follows that $\operatorname{spt}\mu \subset H$. □

Proposition 3.14. *The family $\mathcal{F}$ of all closed $\mathcal{H}$-extremal sets is stable under finite unions and arbitrary intersection.*

Proof. Obviously, $\mathcal{F}$ is stable under finite unions.

If $\{F_a\}_{a\in A}$ are closed $\mathcal{H}$-extremal sets, then their intersection $F := \bigcap_{a\in A} F_a$ is a closed set as well. Pick $x \in F$ and $\mu \in \mathcal{M}_x(\mathcal{H})$ and choose a compact set

$$H \subset K \setminus F = \bigcup_{a\in A} (K \setminus F_a)$$

arbitrarily. There are sets $F_{a_1}, \dots, F_{a_n}$ so that

$$H \subset (K \setminus F_{a_1}) \cup \cdots \cup (K \setminus F_{a_n}).$$

Since $\mu(K \setminus F_{a_j}) = 0$ for each $j = 1, \dots, n$, we get $\mu(H) = 0$. From the inner regularity of μ we obtain $\mu(K \setminus F) = 0$. □

The proof of Proposition 3.15 follows along the lines to that of the Krein–Milman theorem 2.22.

Proposition 3.15. *The Choquet boundary* $\mathrm{Ch}_{\mathcal{H}}(K)$ *intersects any nonempty closed* $\mathcal{H}$*-extremal set. In particular, the Choquet boundary* $\mathrm{Ch}_{\mathcal{H}}(K)$ *is nonempty if* $K \neq \varnothing$.

Proof. Let F be a nonempty closed $\mathcal{H}$-extremal set. We partially order the family $\mathcal{S}$ of all nonempty closed $\mathcal{H}$-extremal subsets of F by the reverse inclusion. If $\mathcal{Z}$ is a chain in $\mathcal{S}$, then $\bigcap\{C : C \in \mathcal{Z}\}$ is nonempty since it is the intersection of a down-directed family of nonempty compact sets. Moreover, it is $\mathcal{H}$-extremal according to the previous Proposition 3.14. Zorn's lemma now provides a maximal element $H \in \mathcal{S}$. Assume that H contains two distinct points x and y. Since $\mathcal{H}$ is a function space, there exists a function $h \in \mathcal{H}$ such that $h(x) \neq h(y)$. According to Lemma 3.13, the set

$$\{z \in H : h(z) = \min h(H)\}$$

is a closed $\mathcal{H}$-extremal set strictly contained in H, which contradicts the fact that H is maximal. Hence, $H = \{x\}$ for some $x \in F$. Since one-point $\mathcal{H}$-extremal sets are in $\mathrm{Ch}_{\mathcal{H}}(K)$, it follows that $x \in \mathrm{Ch}_{\mathcal{H}}(K)$. □

Theorem 3.16 (Minimum principle for $\mathcal{S}^{lsc}(\mathcal{H})$). *Let* $f \in \mathcal{S}^{lsc}(\mathcal{H})$ *be a lower semicontinuous* $\mathcal{H}$*-concave function,* $f \geq 0$ *on* $\mathrm{Ch}_{\mathcal{H}}(K)$. *Then* $f \geq 0$ *on* K.

Proof. Let $K \neq \varnothing$ and $f \in \mathcal{S}^{lsc}(\mathcal{H})$ satisfy $f \geq 0$ on $\mathrm{Ch}_{\mathcal{H}}(K)$. Then

$$F := \{x \in K : f(x) = \min f(K)\}$$

is a closed $\mathcal{H}$-extremal set by Lemma 3.13. By Proposition 3.15, $F \cap \mathrm{Ch}_{\mathcal{H}}(K) \neq \varnothing$, which implies that $f \geq 0$ on K. □