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Edited by Vladimir G. Turaev

Institut de Recherche Mathématique Avancée
Université Louis Pasteur et CNRS
7 rue René Descartes
67084 Strasbourg Cedex
France

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Locally Compact Quantum Groups and Groupoids

Proceedings of the meeting
of theoretical physicists and mathematicians
Strasbourg, February 21–23, 2002

Rencontre entre physiciens théoriciens et mathématiciens
Strasbourg, 21–23 février 2002

Editor

Leonid Vainerman

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Editor
Leonid Vainerman
Département de Mathématiques et Mechanique, Université de Caen, Campus II
Boulevard de Marechal Juin, B. P. 5186, 14032 Caen Cedex, France
e-mail: Leonid.Vainerman@math.unicaen.fr

Series Editor
Vladimir G. Turaev
Institut de Recherche Mathématique Avancée (IRMA), Université Louis Pasteur − C.N.R.S.,
7, rue René Descartes, 67084 Strasbourg Cedex, France, e-mail: turaev@math.u-strasbg.fr

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Preface of the Series Editor

This volume of IRMA Lectures in Mathematics and Theoretical Physics contains the proceedings of the workshop "Quantum Groups, Hopf Algebras and their Applications" held in Strasbourg in February 2002. The workshop was hosted by IRMA (Institute of Advanced Mathematical Research) in the framework of a longstanding wide-range program of meetings between mathematicians and theoretical physicists. This program was initially called "Cooperative Research Program" and was introduced by Jean Frenkel and Georges Reeb in 1965. Since then, these meetings between mathematicians and physicists have taken place at IRMA on the average twice a year. They are sponsored by CNRS (National Center of Scientific Research, France) and IRMA.

The proceedings of a number of these meetings have appeared as IRMA preprints, but were never published. The proceedings of the previous (68th) meeting "Deformation Quantization" appeared as the first volume of IRMA Lectures in Mathematics and Theoretical Physics. The 69-th meeting, whose proceedings constitute this volume, was organized by Leonid Vainerman and myself

The papers published in this volume concern the theory of quantum groups and quantum groupoids. The book should be useful to specialists in this area and related areas, as well as to students of quantum groups.

Préface de l'éditeur de la collection

Ce deuxième volume de "IRMA Lectures in Mathematics and Theoretical Physics" présente les actes du colloque "Groupes quantiques, algèbres de Hopf et leurs applications" qui s'est tenu à l'IRMA (Strasbourg) en février 2002. Le colloque s'est déroulé dans le cadre du programme général de rencontres entre physiciens théoriciens et mathématiciens. Ce programme intitulé initialement "Recherche Coopérative sur Programme" (RCP) a été créé en 1965 sur l'initiative de Jean Frenkel et Georges Reeb avec l'aide de Jean Leray et de Pierre Lelong. Depuis 1965 les rencontres entre physiciens et mathématiciens se déroulent à l'IRMA en moyenne deux fois par an. Ces rencontres sont soutenues financièrement par le CNRS et l'IRMA.

Les actes de plusieurs de ces rencontres avaient donné lieu aux prépublications de l'IRMA sans pour autant être publiés. Les actes de la rencontre précédente (68-ème) sur le thème "Deformation Quantization" sont parus dans le premier volume de la présente collection "IRMA Lectures in Mathematics and Theoretical Physics". La 69-ème rencontre "Groupes quantiques, algèbres de Hopf et leurs applications" – dont les actes constituent ce volume – a été organisée par Leonid Vainerman et moi-même.

Les articles de ce volume traitent de la théorie des groupes quantiques et des groupoïdes quantiques. Ce livre sera utile aux mathématiciens et physiciens travaillant sur ce sujet ainsi qu'à ceux qui étudient la théorie des groupes quantiques.

Strasbourg, novembre 2002 *Vladimir Turaev*

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Leonid Vainerman

Département de Mathématiques et Méchanique
Université de Caen, Campus II – Boulevard de Maréchal Juin
B.P. 5186, 14032 Caen Cedex, France
email: `leonid.vainerman@math.unicaen.fr`

This volume contains seven papers written by participants of the 69 th meeting of theoretical physicists and mathematicians held in Strasbourg (February 21–23, 2002). One of the main topics discussed there was "Locally compact quantum groups and groupoids" which is the title of the volume. The purpose of this introduction is to recall some motivations and ideas from which the above topic emerged and to present the above mentioned papers.

1 Locally compact quantum groups

1.1 Kac algebras

The initial motivation to introduce objects which are more general than usual locally compact groups was to extend classical results of harmonic analysis on these groups, including the Fourier transform theory and the Pontryagin duality. It is well known that the above theories work perfectly in the framework of abelian locally compact groups. If G is such a group, then the role of exponents is played by the unitary continuous characters of G, and the set $\hat{G}$ of all such characters is again an abelian locally compact group – the dual group of G. The Fourier transform maps functions on G to functions on $\hat{G}$, and the Pontrjagin duality claims that the dual of $\hat{G}$ is isomorphic to G.

If a locally compact group G is not abelian, the set of its characters is too small, and to extend the harmonic analysis and duality in a reasonable way, one should consider instead the set $\hat{G}$ of (classes of) its unitary irreducible representations and also their matrix coefficients. For compact groups, this point of view leads to the widely known Peter–Weyl theory; the duality theory for such groups was done by T. Tannaka [75] and M. G. Krein [42]. A new feature of this duality is that $\hat{G}$ does not carry a structure of a group, but can be equipped with some quite different structure (block-algebra or Krein algebra [30]); however, starting with such a structure, the initial compact group can be reconstructed. Such a non-symmetric duality was later established by W. F. Stinespring [69] for unimodular groups, and by P. Eymard [28] and T. Tatsuuma [76] for general locally compact groups.

In 1961 G. I. Kac [33], [34] proposed a completely new idea, which allowed to restore the symmetry of the duality for unimodular, not necessarily abelian, groups. Namely, he introduced a category of objects (he called them *ring groups*), containing both unimodular groups and their duals, and constructed the Fourier transform and duality within this category. His duality extended those of Pontryagin, Tannaka–Krein and Stinespring.

In algebraic terms, one can think of a ring group as of a Hopf *-algebra with an involutive antipode S (i.e., $S^2 = \mathrm{id}$). In topological terms, its algebra A is a von Neumann algebra, and the comultiplication $\Delta\colon A \to A \otimes A$ and the antipode $S\colon A \to A$ are von Neumann algebra maps. On the contrary, its counit is not a well defined von Neumann algebra map, that is why it is not present in the definition of a ring group. Instead, A is required to be equipped with a faithful normal trace φ compatible with Δ and S and playing the role of a Haar measure. Without discussing here this definition in detail, let us show two standard examples of ring groups related to an ordinary unimodular group G with a Haar measure μ.

Example 1.1. $A = L^\infty(G,\mu)$, $\Delta\colon f(g) \mapsto f(gh)$, $S\colon f(g) \mapsto f(g^{-1})$, $\varphi(f) = \int_G f(g)d\mu(g)$, where $g, h \in G$, $f(\cdot) \in L^\infty(G,\mu)$.

Example 1.2. $A = \mathcal{L}(G)$ – the von Neumann algebra generated by left translations L_g or by left convolutions $L_f = \int_G f(g)L_g d\mu(g)$ with continuous functions $f(\cdot) \in L^1(G,\mu)$ $\Delta\colon L_g \mapsto L_g \otimes L_g$, $S\colon L_g \mapsto L_{g^{-1}}$, $\varphi(f) = f(e)$, where $g \in G$, e is the unit of G.

G.I. Kac showed that for any commutative (resp., co-commutative) ring group $\mathcal{G}$, i.e., such that the algebra A is commutative (resp., $\sigma \circ \Delta = \Delta$, where $\sigma : a \otimes b \mapsto b \otimes a$ is the usual flip in $A \otimes A$), there is a unimodular group G such that $\mathcal{G}$ is isomorphic to the ring group of Example 1.1 (resp., 1.2) related to G. Thus, the category of unimodular groups (resp., their duals) is embedded into the category of ring groups.

The theory of ring groups used, as a technical tool, I. Segal's theory of traces on von Neumann algebras, which is a non-commutative extension of the classical theory of measure and integral. In [36], [37], [38] G. I. Kac and V. G. Paljutkin gave concrete examples of non-trivial, i.e., non-commutative and non-co-commutative, ring groups, which were neither ordinary groups nor their duals. As it was mentioned by V. G. Drinfeld [13], the Kac–Paljutkin examples were the first concrete examples of quantum groups.

The theory was completed in the early '70s, when the Tomita–Takesaki theory and the foundations of the theory of weights on operator algebras became available – our reference to these topics is [70]. Namely, G. I. Kac and L. Vainerman [39], on the one hand, and M. Enock and J.-M. Schwartz [21], on the other hand, extended the category of ring groups in order to cover all locally compact groups (certain results in this direction were obtained also by M. Takesaki [72], [73]). They allowed φ and $\varphi \circ S$ to be different weights on A playing respectively the role of a left and a right Haar measure (for ring groups $\varphi = \varphi \circ S$ was a trace), gave appropriate axioms and extended the construction of the dual.

To emphasize the importance of the pioneering work of G. I. Kac, M. Enock and J.-M. Schwartz called these more general objects *Kac algebras*. Locally compact groups and their duals were embedded in this category respectively as commutative (see [72]) and co-commutative (see [81]) Kac algebras, the corresponding duality covered all versions of duality for such groups. The standard reference to the Kac algebra theory is [22]. C^*-algebraic Kac algebras have been discussed in [63], [24] (see also [82]).

1.2 From Kac algebras to locally compact quantum groups

The discovery of quantum groups by V. G. Drinfeld [13] was accompanied by the arrival of new important examples of Hopf *-algebras, obtained by deformation either of universal enveloping algebras of Lie algebras [13], [31], or of function algebras on Lie groups [92], [93], [68]. Their operator algebra versions did not fit into the Kac algebra theory, because the antipodes were neither involutive, nor even bounded maps. This provided a strong motivation to construct a more general theory, which would be as elegant as that of Kac algebras but would also cover these new examples. The first steps in this direction were made in [92], [94], where S. L. Woronowicz constructed the theory of compact quantum groups and developed for them the Peter–Weyl theory and the Tannaka–Krein duality. Moreover, he managed to deduce the existence of a Haar measure from his set of axioms rather than assume it, as was the case in the Kac algebra theory (and, as we will see below, in some of its extensions). The last feature holds also for discrete quantum groups – see [64], [16], [15], [87].

Remark 1.3. 1) The Haar theorem for compact C^*-algebraic ring groups has been proven by V. G. Paljutkin [63] (see also [82]).

2) In [11], the Peter–Weyl theory was constructed for much more general objects than compact quantum groups, for which the comultiplication is not necessarily an algebra map.

In the case of non-compact and non-discrete quantum groups, an in-depth prior analysis of concrete examples was necessary. It was not so difficult to construct such examples in terms of generators of certain Hopf *-algebras and commutation relations between them. It was much harder to represent these generators as (typically, unbounded) operators acting on a Hilbert space and to give a meaning to the relations of commutation between these operators. Finally, it was even more difficult to associate an operator algebra with the above system of operators and commutation relations and to construct comultiplication, antipode and invariant weights as applications related to this algebra. There is no general approach to these highly nontrivial problems, and one must design specific methods in each specific case [95]–[98], [64], [1], [90].

There are other examples of operator algebraic quantum groups which are easier to construct. For example, given a non-commutative locally compact group G, one can replace the comultiplication Δ of the co-commutative Kac algebra described in Example 1.2 with the new comultiplication of the form $\Delta_\Omega(\cdot) = \Omega\Delta(\cdot)\Omega^{-1}$, where Ω is an element from $\mathcal{L}(G) \otimes \mathcal{L}(G)$ such that Δ_Ω remains co-associative. This construction (called *twisting*) was developed on a purely algebraic level by V.G. Drinfeld [14] and on an operator algebraic level in [23], [83] and [55], where numerous concrete examples were obtained as well. Note that a, in a sense dual, construction has been proposed by M. Rieffel [65].

The other construction has been developed in [35]. Given two finite groups, G_1 and G_2, viewed respectively as a co-commutative ring group $(\mathcal{L}(G_1), \Delta_1)$ (see Example 1.2) and a commutative ring group $(L^\infty(G_2), \Delta_2)$ (see Example 1.1), let us try to find a ring group (A, Δ) which makes the sequence

$$(L^\infty(G_2), \Delta_2) \to (A, \Delta) \to (\mathcal{L}(G_1), \Delta_1) \tag{1}$$

exact. G. I. Kac explained that: 1) (A, Δ) exists if and only if G_1 and G_2 are subgroups of a group G such that $G_1 \cap G_2 = \{e\}$ and $G = G_1 G_2$. Equivalently, G_1 and G_2 must act on each other (as on sets), and these actions must be compatible. 2) To get all possible (A, Δ) (they are called *extensions* of $(L^\infty(G_2), \Delta_2)$ by $(\mathcal{L}(G_1), \Delta_1)$), one must find all possible 2-cocycles for the above mentioned actions, compatible in certain sense. Under these conditions, [35] gives the explicit construction of (A, Δ) (the cocycle bicrossed product construction). The famous Kac–Paljutkin examples of non-trivial ring groups [36], [37], [38] are exactly of this type. Later on, both algebraic and analytic aspects of this construction were intensively studied by S. Majid [50]–[53] who gave also a number of examples of operator algebraic quantum groups, some of them were not Kac algebras. Very recently, the theory of extensions of the form (1), with locally compact G_1 and G_2, has been developed in [80].

An important step in the generalization of the Kac algebra theory was the theory of *multiplicative unitaries*. Already W. F. Stinespring [69] mentioned an important role in the construction of the dual for a unimodular nonabelian group G played by the unitary

$$W_G(\xi)(g, h) = \xi(g, g^{-1}h) \tag{2}$$

acting on $L^2(G, \mu) \otimes L^2(G, \mu)$. G. I. Kac, in order to construct his duality for ring groups, introduced in this more general context a similar unitary

$$W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)), \tag{3}$$

where $a, b \in \mathfrak{N}_\varphi := \{x \in A : \varphi(x^*x) < \infty\}$, Λ is the GNS-mapping for φ [70]. Moreover, he was the first to point out that W verifies the *Pentagonal relation*:

$$W_{12}W_{13}W_{23} = W_{23}W_{12} \tag{4}$$

and to show that all the information about the ring group could be encoded in W.

On the contrary, S. Baaj and G. Skandalis [2] took a unitary verifying (4) (they called it a multiplicative unitary), as the starting point of their theory. They have constructed two Hopf C^*-algebras in duality out of a given multiplicative unitary, under certain regularity conditions, and gave a number of important constructions of C^*-algebraic quantum groups in this framework (including the bicrossed product construction). The investigation of the above mentioned regularity conditions and alternative *manageability* conditions [96] is one of the most important topics in the theory of multiplicative unitaries [1], [3], [96], [5]. Note that several examples of C^*-algebraic quantum groups, more general than Kac algebras, were given in [2], [67].

T. Masuda and Y. Nakagami proposed an extension of the Kac algebra theory by requiring the antipode S to have a polar decomposition consisting of a unitary part and a generator of one-parameter group of automorphisms of a von Neumann algebra A. The idea of such a polar decomposition of S is due to E. Kirchberg (unpublished). The Kac algebra case is exactly the situation when S equals its unitary part and for that reason is involutive and bounded. A certain disadvantage of this approach was the necessity for some quite complicated axioms which disappears in the Kac algebra case. A joint work by T. Masuda, Y. Nakagami and S. L. Woronowicz on the C^*-algebra version of this theory is still in progress.

To sum up, one can say that trying to extend the Kac algebra theory in order to cover important concrete examples of quantum groups, one faces a mixture of algebraic and analytic problems. That is why it was important to design a purely algebraic framework, where the main algebraic features of the future theory would be present. It was done by A. Van Daele in [88], [89] and in his joint work with J. Kustermans [46], where the notion of a multiplier Hopf *-algebra with positive integrals was proposed and a natural duality was constructed. As for analytic aspects of the story, by the end of the '90s the theory of weights on C^*-algebras had been further developed, mainly by J. Kustermans, and after that the theory of locally compact quantum groups was proposed by J. Kustermans and S. Vaes [43]–[45].

A locally compact quantum group is a collection $\mathcal{G} = (A, \Delta, \varphi, \psi)$, where A is either a C^*- or a von Neumann algebra equipped with a co-associative comultiplication $\Delta\colon A \to A \otimes A$ and two faithful semi-finite normal weights φ and ψ - right and left Haar measures. The antipode is not explicitly present in this definition, but can be constructed from the above data, as well as its polar decomposition, using the multiplicative unitary, canonically associated with $\mathcal{G}$ by means of the formula (3). Kac algebras, compact and discrete quantum groups are special cases of a locally compact quantum group, but what is even more interesting, all important concrete examples of operator algebraic quantum groups fit into this framework. One can find an exposition of this theory in [47] and [79]. In the present book, more information on locally compact quantum groups can be found in the Preliminaries of the article by S. Vaes and L. Vainerman. To simplify the notations, in what follows we denote a locally compact quantum group by (A, Δ); usually we deal with the case when A is a von Neumann algebra and $\Delta\colon A \to A \otimes A$ is a normal monomorphism of von

Neumann algebras. Let us present now the three papers on locally compact quantum groups contained in this volume. We start with a paper by J. Kustermans and E. Koelink devoted to a concrete example of a locally compact quantum group, related to $\mathrm{SU}_q(1,1)$. As a Hopf *-algebra, $\mathrm{SU}_q(1,1)$ is one of the three real forms of $\mathrm{SL}_q(2,\mathbb{C})$, the two others being $\mathrm{SU}_q(2)$ and $\mathrm{SL}_q(2,\mathbb{R})$. Remark that the quantum group $\mathrm{SU}_q(2)$ and its dual are well understood on the operator algebra level [92]–[94], [68]; such an understanding of $\mathrm{SL}_q(2,\mathbb{R})$ is still an open problem.

Concerning $\mathrm{SU}_q(1,1)$, in 1991 S. L. Woronowicz showed that this object cannot exist as a C^*-Hopf algebra, and this result was a source of pessimism for several years. Then L. Korogodsky explained that it was reasonable to deform rather the normalizer $\widetilde{\mathrm{SU}}(1,1)$ of $\mathrm{SU}(1,1)$ in $\mathrm{SL}_q(2,\mathbb{C})$ than $\mathrm{SU}_q(1,1)$ itself. The paper of J. Kustermans and E. Koelink gives a clear overview of the highly nontrivial construction of quantum $\widetilde{\mathrm{SU}}(1,1)$ and its dual as locally compact quantum groups and their theory of representations. The main tool they use is the explicit analysis of eigenfunctions of certain unbounded operators in terms of special functions of q-hypergeometric type. The paper also contains historical remarks and shows the contribution of other specialists.

The paper by A. Van Daele is a survey of the theory of algebraic quantum groups (multiplier Hopf *-algebras with positive integrals) and their relations with locally compact quantum groups. As was mentioned above, this theory provided one of the main motivations for the development of locally compact quantum groups by J. Kustermans and S. Vaes and showed almost all algebraic features of the latter. On the other hand, it is much easier technically, even if much attention is attached to the links with the corresponding operator algebraic results. The category of algebraic quantum groups contains the categories of compact and discrete quantum groups (but not all the ordinary locally compact groups), is self-dual and closed under several constructions, such as, for example, the Drinfeld double. An important tool used in the paper is the Fourier transform. Thus, algebraic quantum groups provide a good and relatively simple model for studying more general objects. So the paper will be of interest both for students and experts.

The paper by S. Vaes and L. Vainerman is devoted to extensions of *Lie groups* of the form (1). In this case, instead of the condition $G = G_1G_2$, one should require G_1G_2 to be an open dense subset of G, as in [3]. Then, for the corresponding *Lie algebras* we have $g = g_1 \oplus g_2$ – the direct sum of vector spaces. So, to construct examples of locally compact quantum groups, one can start with such a decomposition of Lie algebras and try to construct a corresponding pair of groups (G_1, G_2). But this problem proves to be not so easy to resolve (typically, one must deal with non-connected Lie groups), and often it has no solution at all. In the paper the case of complex and real Lie groups G_1 and G_2 of low dimensions is studied in detail. In particular, a complete classification of the corresponding locally compact quantum groups with two or three generators is obtained, and all the ingredients of their structure are computed, as well as their infinitesimal objects (Hopf *-algebras and Lie bialgebras).

2 From quantum groups to quantum groupoids

2.1 Actions of locally compact quantum groups and subfactors

Classical groups are interesting first of all as groups of transformations, acting on certain spaces. Similarly, one can define a (left) action of a locally compact quantum group (A, Δ) on a von Neumann algebra N (which plays the role of a "quantum space") as a normal monomorphism $\alpha\colon N \to A \otimes N$ of von Neumann algebras such that $(\mathrm{id} \otimes \alpha)\alpha = (\Delta \otimes \mathrm{id})\alpha$. Now the *fixed point subalgebra* can be defined as

$$N^\alpha := \{x \in N : \alpha(x) = 1 \otimes x\},$$

and the *crossed product* $A \rtimes N$ as the von Neumann algebra generated by $\alpha(N)$ and $\hat{A} \otimes \mathbb{C}$, where $\hat{A}$ is the von Neumann algebra of the dual. An action is said to be *outer* if $(N^\alpha)' \cap N = \mathbb{C}$. For the motivations and details see [77], [79], [80] and Preliminaries of the article by S. Vaes and L. Vainerman in this volume. There is a series of nice results on such actions that extend classical results on actions of locally compact groups on von Neumann algebras [77], [79], but here we focus our attention on the links with subfactors.

Starting with a given inclusion $N_0 \subset N_1$ of von Neumann algebras and performing step by step the well known *basic construction* of V. Jones, one can obtain the *Jones' tower* of von Neumann algebras [32]:

$$N_0 \subset N_1 \subset N_2 \subset N_3 \subset \cdots .$$

Recall that the initial inclusion is said to be *irreducible*, if $N_0' \cap N_1 = \mathbb{C}$ (in this case all the N_i $(i = 0, 1, 2, \dots)$ are *factors*, i.e., have trivial centers), and *of depth* 2, if the triple of relative commutants

$$N_0' \cap N_1 \subset N_0' \cap N_2 \subset N_0' \cap N_3$$

is again the basic construction.

Example 2.1. Given an outer action α of a locally compact group G on a factor N_1, the inclusion $N_0 = N_1^\alpha \subset N_1$ is irreducible and of depth 2, and N_2 is isomorphic to $G \rtimes N_1$.

M. Enock and R. Nest [20], [17] showed that, conversely, for any irreducible depth 2 subfactor $N_0 \subset N_1$ satisfying a natural regularity condition, the von Neumann algebra $A = N_1' \cap N_3$ can be given the structure of a locally compact quantum group (A, Δ) with an outer action α of the commutant $(A, \Delta)'$ on N_1, such that $N_0 = N_1^\alpha$ and the triples $N_0 \subset N_1 \subset N_2$ and $\mathbb{C} \otimes N_1^\alpha \subset \alpha(N_1) \subset A' \rtimes N_1$ are isomorphic (in fact, this result was precised by S. Vaes [77]).

Remark 2.2. The idea that outer actions of Kac algebras are closely related to the structure of irreducible depth 2 subfactors, is due to A. Ocneanu (see, for example, Postface in [22]). Finite index irreducible depth 2 subfactors of type II_1 were characterized in terms of outer actions of finite-dimensional Kac algebras by R. Longo [48], W. Szymanski [71] and M. C. David [12].

This beautiful result motivates the following natural hypothesis: if we drop the irreducibility condition keeping however the depth 2 condition for a subfactor, this situation should be characterized in terms of an outer action of some more general structure than a locally compact quantum group. And this is a way to approach the notion of a locally compact quantum groupoid.

Indeed, already finite index depth 2 subfactors of type II_1 reveal the purely algebraic aspect of the story. It is shown in [59] that in this case the above mentioned result is still true, up to notations, if one gives the finite-dimensional algebra $A = N_0' \cap N_2$ a structure of a *weak C^*-Hopf algebra* (introduced in [7], [6]) acting outerly on N_1. Like a finite-dimensional Kac algebra, a weak C^*-Hopf algebra is a finite-dimensional C^*-algebra A equipped with a co-associative comultiplication, an antipode and a counit. The main difference between them is that this comultiplication is not necessarily a unital map and the counit is not necessarily a homomorphism of algebras $A \to \mathbb{C}$. This implies the existence of a canonical C^*-subalgebra R of A, called counital or *base subalgebra*, playing a fundamental role within this structure. For a weak C^*-Hopf algebra coming from subfactors we have $R = N_0' \cap N_1$; clearly, $R = \mathbb{C}$ if and only if the subfactor is irreducible. One can show that the dual vector space for a weak C^*-Hopf algebra carries the structure of the same type, i.e., this notion is self-dual.

Like in Examples 1.1 and 1.2, the algebra of functions and the groupoid algebra of a usual finite groupoid give respectively standard examples of a commutative and cocommutative weak C^*-Hopf algebra [58], [61] which justifies the usage of the term "quantum groupoid". Moreover, the notion of the base subalgebra naturally extends the function algebra on the set of units of a usual groupoid. For examples of non-trivial (i.e., non-commutative and non-cocommutative) quantum groupoids see [7], [57], [58], [59], [61], [26].

Initially, weak C^*-Hopf algebras were introduced in [7] as symmetries of certain models in algebraic quantum field theory. Another source of interest in them is their representation category, which is flexible enough to describe all rigid monoidal C^*-categories with finitely many classes of simple objects (in general, in this representation category a unit object is not a counit because the latter is not a representation, and the tensor product differs from the usual tensor product of vector spaces) [8], [57], [60], [61]. So, quantum dimensions of irreducible representations need not to be integer, and these categories have interesting applications in low-dimensional topology [57], [61]. A survey of the theory of finite quantum groupoids and their applications can be found in [61].

2.2 Multiplicative partial isometries and pseudo-multiplicative unitaries

As noted above, multiplicative unitaries are of fundamental importance in the theory of locally compact quantum group. So, it would be natural to define and to study similar objects also for quantum groupoids. Since for any weak C^*-Hopf algebra there exists a positive linear form on its C^*-algebra A playing the role of a Haar measure [6], one can define an operator W by (3). Now W is not in general a unitary, but just a partial isometry verifying the Pentagonal equation (4) [9], [86].

Like in the case of quantum groups, the inverse problem is more subtle, and in order to resolve it one should impose some regularity conditions on a given partial isometry. Namely, J. M. Vallin showed in [86] that any regular multiplicative partial isometry generates two quantum groupoids in duality, which extends the above mentioned result of S. Baaj and G. Skandalis on multiplicative unitaries. In the paper published in this book, J. M. Vallin continues the study of the structure of regular multiplicative partial isometries acting on a finite-dimensional Hilbert space, in the spirit of [4], where finite-dimensional multiplicative unitaries were studied in detail.

First, it is shown that, after an amplification and reduction, any regular multiplicative partial isometry is isomorphic to an *irreducible* one, i.e., verifying a certain quite strong condition. The latter condition allows to prove *quantum Markov properties*; for instance, the existence of a faithful positive linear form on the involutive algebra generated by the two quantum groupoids associated to the partial isometry (the Weyl algebra) that extends both normalized Haar measures of these quantum groupoids. In its turn, this implies that any regular multiplicative partial isometry is a composition of two very simple partial isometries. Finally, it is shown that a regular multiplicative partial isometry is completely determined by the two quantum groupoids associated and by the spaces of its fixed and cofixed vectors, and a complete characterization of quantum groupoids in duality acting on the same Hilbert space in the irreducible situation is obtained.

The notion of a locally compact quantum groupoid is much less transparent in the infinite-dimensional case, which corresponds to the infinite index depth 2 inclusions of von Neumann algebras (in fact, the development of this theory is still in progress). The reason is that in this case complicated analytical aspects play a significant role, as well as the presence of nontrivial base von Neumann algebra. In particular, instead of usual tensor products of Hilbert spaces and von Neumann algebras one should inevitably use the relative tensor product of Hilbert spaces and the "fiber" product of von Neumann algebras over a base algebra. In the finite-dimensional case these new notions reduce respectively to a subspace of the usual tensor product of Hilbert spaces and to a reduced subalgebra of the usual tensor product of von Neumann algebras. For the definitions and explanations see the paper by M. Enock on infinite-dimensional locally compact quantum groupoids published in this volume, which also outlines the nearest prospects for this field.

To approach the notion of a locally compact quantum groupoid, it is necessary first to figure out, what kind of objects can be associated with an ordinary locally compact groupoid in the spirit of Examples 1.1 and 1.2 and the formula (2). It was explained in [84], [86] that one gets this way two *Hopf bimodules* in duality – commutative and co-commutative, and a *pseudo-multiplicative unitary*. The same objects were associated with depth 2 inclusions of von Neumann algebras in [25], [18]; moreover, in both cases one can even equip the Hopf bimodules with antipodes having polar decompositions. For the definitions and explanations see the survey by M. Enock. We only remark that both these structures are defined over a base von Neumann algebra, and that in the finite-dimensional case they reduce respectively to a weak C^*-Hopf algebra and to a multiplicative partial isometry.

Like in the theory of locally compact quantum groups, it is crucial to understand exact relations between these two "faces" of a locally compact quantum groupoid. It was shown in [25], [18] that, given a pseudo-multiplicative unitary, one can construct in a natural way two Hopf bimodules in duality (as we mentioned above, in the cases related to a usual locally compact groupoid and to depth 2 inclusions of von Neumann algebras, one can even equip these objects with antipodes having polar decompositions). The work by F. Lesieur on a converse result is still in progress. Finally, in [19], the theory of quantum groupoids of compact type is developed, following the strategy of [2].

2.3 On purely algebraic quantum groupoids

Until now we have discussed quantum groups and groupoids only in the framework of operator algebras. As for purely algebraic quantum groupoids, there are several versions of them, designed from various motivations. Let us mention first the notion of a *weak Hopf algebra* [6] extending substantially the one of a Hopf algebra. Like in the C^*-case, the main difference between them is that the comultiplication of a weak Hopf algebra A is not necessarily a unital map and the counit is not necessarily a homomorphism of algebras $A \to k$ (k is the ground field), and this implies the existence of a base subalgebra R of A, which is automatically separable (if R is commutative, we get a notion of a *face algebra* [29]). The theory of these objects in the finite-dimensional case nicely extends that of Hopf algebras [6], [56], [91], [8], [57], [61]. Their representation categories cover all rigid monoidal categories with finitely many classes of simple objects, even in the case of a commutative base subalgebra [29], [62]. So, they can be used as an appropriate tool for the study of such categories [27], [57]. Dropping the antipode in a weak Hopf algebra we get a *weak bialgebra* whose representation category is monoidal, but not necessarily rigid.

The notion of a weak Hopf algebra (resp., weak bialgebra) is a partial case of that of a *Hopf algebroid* (resp., *bialgebroid*) in the sense of [49] and [99] – see [26] (resp., [66]). The definition of the latter two structures was motivated by the analogy with a usual (semi)groupoid, their base algebra naturally extends the function algebra on the set of its units. On the other hand, the notion of a bialgebroid is equivalent to that of a $\times_R$-bialgebra introduced earlier by M. Takeuchi [74] (here also, R denotes a base algebra) – see [10]. It was shown in [66] that a $\times_R$-bialgebra with a separable base is a weak bialgebra. For all the above mentioned objects, their representation category is monoidal.

Brief discussion of some other versions of quantum groupoids can be found in [61].

Now we are ready to present the two remaining papers of this volume. P. Schauenburg discusses a construction that allows to replace the base algebra R in any $\times_R$-bialgebra A with a Morita-equivalent algebra S (i.e., having equivalent representation category) in order to obtain a $\times_S$-bialgebra whose representation category is equivalent to that of A as monoidal categories. He gives a spectacular illustration: for a concrete example of a weak Hopf algebra from [60], [61] this Morita base change reduces the dimension of A from 122 to 24 without affecting the monoidal category of representations (the base algebra changes from $\mathbb{C} \oplus M_2(\mathbb{C})$ to $\mathbb{C} \oplus \mathbb{C}$).

The starting point for the paper by K. Szlachányi is a *balanced depth* 2 *extension of algebras* $N \subset M$ which is a purely algebraic generalization of the notion of finite index depth 2 von Neumann subfactors – see the definition in the text. For such an extension, the endomorphism ring $A = \operatorname{End}_N M_N$ carries a bialgebroid structure (its base R is the relative commutant of N in M) equipped with the canonical action on M, whose subalgebra of A-invariants is N [40]. This generalizes the above mentioned result of [59] in the subfactor theory.

Finally, it is explained that balanced depth 2 extensions of algebras are the proper analogues of the Galois extensions of fields (i.e., normal and separable field extensions) because they have "finite quantum automorphism groups" with subalgebra of invariants equal to N and which are characterized by a universal property, hence, unique. The role of such a "finite quantum automorphism group" is played by a bialgebroid that is finitely generated projective over its base as a left and a right module (the problem of the existence of the antipode in this bialgebroid is still open). If R is separable, then A is a weak bialgebra; if, moreover, $N \subset M$ is a Frobenius extension, then A is a weak Hopf algebra. In the special case of a separable field extension, the structure of such a universal weak Hopf algebra is written down explicitly.

References

[1] S. Baaj, Représentation régulière du groupe quantique des déplacements de Woronowicz, *Astérisque* 232 (1995), 11–48.

[2] S. Baaj and G. Skandalis, Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres, *Ann. Sci. École Norm. Sup.* (4) 26 (1993), 425–488.

[3] S. Baaj and G. Skandalis, Transformations pentagonales, *C. R. Acad. Sci. Paris Sér.* I *Math.* 327 (1998), 623–628.

[4] S. Baaj, E. Blanchard and G. Skandalis, Unitaires multiplicatifs en dimention finie et leurs sous-objets, *Ann. Inst. Fourier* 49 (1999), 1305–1344.

[5] S. Baaj, G. Skandalis and S. Vaes, Non-semi-regular quantum groups coming from number theory, preprint 2002.

[6] G. Böhm, F. Nill and K. Szlachányi, Weak Hopf algebras I. Integral theory and C^*-structure, *J. Algebra* 221 (1999), 385–438.

[7] G. Böhm and K. Szlachányi, A co-associative C^*-quantum group with nonintegral dimensions, *Lett. Math. Phys.* 35 (1996), 437–456.

[8] G. Böhm and K. Szlachányi, Weak Hopf algebras II. Representation Theory, Dimensions and the Markov Trace, *J. Algebra* 233 (2000), 156–212.

[9] G. Böhm and K. Szlachányi, Weak C^*-Hopf algebras and multiplicative isometries, *J. Operator Theory* 45 (2001), 357–376.

[10] T. Brzeziński and G. Militaru, Bialgebroids, $\times_R$-bialgebras and duality, *J. Algebra* 251 (2002), 279–294.

[11] Yu. A. Chapovsky and L. I. Vainerman, Compact Quantum Hypergroups, *J. Operator Theory* 41 (1999), 1–28.

[12] M. C. David, Paragroupe d'Adrian Ocneanu et algèbre de Kac, *Pacific J. Math.* 172 (1996), 331–363.

[13] V. G. Drinfeld, Quantum groups, *Proceedings ICM Berkeley* (1986), Amer. Math. Soc., Providence, RI, 1987, 798–820.

[14] V. G. Drinfeld, Quasi-Hopf algebras, *Leningrad. Math. J.* 1 (1990), 1419–1457.

[15] M. S. Dijkhuisen and T. H. Koornwinder, CQG algebras: a direct algebraic approach to compact quantum groups, *Lett. Math. Phys.* 32 (1994), 315–330.

[16] E. Effros and Z.-J. Ruan, Discrete quantum groups: The Haar measure, *Internat. J. Math.* 5 (1994), 681–723.

[17] M. Enock, Inclusions irréductibles de facteurs et unitaires multiplicatifs II, *J. Funct. Anal.* 154 (1998), 67–109.

[18] M. Enock, Inclusions of von Neumann algebras and quantum groupoids II, *J. Funct. Anal.* 178 (2000), 156–225.

[19] M. Enock, Quantum groupoids of compact type, *prépublication, Institut de Math. de Jussieu* 328, Mai 2002.

[20] M. Enock and R. Nest, Inclusions of factors, multiplicative unitaires and Kac algebras, *J. Funct. Anal.* 137 (1996), 466–543.

[21] M. Enock and J.-M. Schwartz, Une dualité dans les algèbres de von Neumann, *C. R. Acad. Sci. Paris* 277 (1973), 683–685; *Bull. Soc. Math. France, Suppl., Mém.* 44 (1975), 1–144.

[22] M. Enock and J.-M. Schwartz, *Kac Algebras and Duality of Locally Compact Groups*, Springer-Verlag, Berlin 1992.

[23] M. Enock and L. Vainerman, Deformation of a Kac Algebra by an Abelian Subgroup, *Comm. Math. Phys.* 66 (1993), 619–650.

[24] M. Enock and J. M. Vallin, C^*-algèbres de Kac et algèbres de Kac, *Proc. London. Math. Soc.* 172 (2000), 249–300.

[25] M. Enock and J. M. Vallin, Inclusions of von Neumann algebras and quantum groupoids, *J. Funct. Anal.* 172 (2000), 249–300.

[26] P. Etingof and D. Nikshych, Dynamical quantum groups at roots of 1, *Duke Math. J.* 108 (2001), 135–168.

[27] P. Etingof, D. Nikshych and V. Ostrik, On fusion categories, preprint `math.QA/0203060` (2002).

[28] P. Eymard, L'algèbre de Fourier d'un groupe localement compact, *Bull. Soc. Math. France* 92 (1964), 181–236.

[29] T. Hayashi, A canonical Tannaka duality for finite semisimple tensor categories, preprint `math.QA/9904073` (1999).

[30] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis* II, Grundlehren Math. Wiss. 152, Springer-Verlag, Berlin–Heidelberg–New York 1970.

[31] M. Jimbo, A q-difference analogue of $U(g)$ and the Yang-Baxter equation, *Lett. Math. Phys.* 10 (1985), 63–69.

[32] V. Jones and V. S. Sunder, *Introduction to Subfactors*, London Math. Soc. Lecture Note Ser. 234, Cambridge University Press, Cambridge 1997.

[33] G. I. Kac, Generalization of the group principle of duality, *Soviet. Math. Dokl.* 2 (1961), 581–584.

[34] G. I. Kac, Ring groups and the principle of duality, I, II, *Trans. Moscow Math. Soc.* (1963), 291–339; ibid. 13 (1965), 94–126.

[35] G. I. Kac, Extensions of groups to ring groups, *Math. USSR-Sb.* 5 (1968), 451–474.

[36] G. I. Kac and V. G. Paljutkin, Finite ring groups, *Trans. Moscow Math. Soc.* 5 (1966), 251–294.

[37] G. I. Kac and V. G. Paljutkin, Example of a ring group generated by Lie groups, *Ukraïn. Mat. Zh.* 16 (1) (1964), 99–105 (in Russian).

[38] G. I. Kac and V. G. Paljutkin, Example of a ring group of order eight, *Uspekhi Mat. Nauk* 20 (1965), 268–269 (in Russian).

[39] G. I. Kac and L. I. Vainerman, Nonunimodular ring groups and Hopf-von Neumann algebras, *Soviet. Math. Dokl.* 14 (1973), 1144–1148; *Math. USSR-Sb.* 23 (1974), 185–214.

[40] L. Kadison and K. Szlachányi, Dual bialgebroids for depth 2 ring extensions, preprint `math.RA/0108067` (2001).

[41] E. Koelink and J. Kustermans, A locally compact quantum group analogue of the normalizer of $SU(1,1)$ in $SL(2,C)$, preprint `math.QA/0105117` (2001).

[42] M. G. Krein, The principle of duality for a compact group and a square block algebra, *Soviet. Math. Dokl.* 69 (1949), 725–728; Hermitian-positive kernels on homogeneous spaces I, II, *Amer. Math. Soc. Transl. Ser.* 34 (1963), 69–108; ibid. 109–164, translated from the Russian original *Ukraïn. Mat. Zh.* 1 (4) (1949), 64–98; ibid. 2 (1) (1950), 10–59.

[43] J. Kustermans and S. Vaes, Locally compact quantum groups, *Ann. Sci. École Norm. Sup.* 33 (6) (2000), 837–934.

[44] J. Kustermans and S. Vaes, Locally compact quantum groups in the von Neumann algebraic setting, to appear in *Comm. Math. Phys.* (2000).

[45] J. Kustermans and S. Vaes, A simple definition for locally compact quantum groups, *C. R. Acad. Sci. Paris Sér.* I 328 (10) (1999), 871–876.

[46] J. Kustermans and A. Van Daele, C^*-algebraic quantum groups arising from algebraic quantum groups, *Internat. J. Math.* 8 (1997), 1067–1139.

[47] J. Kustermans, S. Vaes, L. Vainerman, A. Van Daele and S. L. Woronowicz, Locally Compact Quantum Groups, in *Lecture Notes for the School on Noncommutative Geometry and Quantum Groups in Warsaw (*17–29 *September* 2001), Banach Center Publ., Warsaw, to appear.

[48] R. Longo, A duality for Hopf algebras and subfactors I, *Comm. Math. Phys.* 159 (1994), 133–150.

[49] J.-H. Lu, Hopf algebroids and quantum groupoids, *Internat. J. Math.* 7 (1996), 47–70.

[50] S. Majid, Physics for algebraists: Non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction, *J. Algebra* 130 (1990), 17–64.

[51] S. Majid, Hopf-von Neumann algebra bicrossproducts, Kac algebra bicrossproducts, and the classical Yang-Baxter equations, *J. Funct. Anal.* 95 (1991), 291–319.

[52] S. Majid, More examples of bicrossproduct and double cross product Hopf algebras, *Israel J. Math.* 72 (1990), 133–148.

[53] S. Majid, Foundations of quantum group theory, Cambridge University Press, Cambridge 1995.

[54] T. Masuda and Y. Nakagami, A von Neumann algebraic framework for the duality of the quantum groups, *Publ. Res. Inst. Math. Sci.* 30 (1994), 799–850.

[55] D. Nikshych, K_0-ring and twisting of finite dimensional Hopf algebras, *Comm. Algebra* 26 (1998), 321–342.

[56] D. Nikshych, On the structure of weak Hopf algebras, to appear in *Adv. Math.*, preprint `math.QA/0106010` (2001).

[57] D. Nikshych, V. Turaev and L. Vainerman, Invariants of Knots and 3-manifolds from Quantum Groupoids, to appear in *Topology and its applications*; preprint `math.QA/0006078` (2000).

[58] D. Nikshych and L. Vainerman, Algebraic versions of a finite-dimensional quantum groupoid, in *Hopf algebras and quantum groups* (S. Caenepeel et al., eds.), Lecture Notes in Pure and Appl. Math. 209 , Marcel Dekker, New York 2000, 189–221.

[59] D. Nikshych and L. Vainerman, A characterization of depth 2 subfactors of II_1-factors, *J. Funct. Anal.* 171 (2000), 278–307.

[60] D. Nikshych and L. Vainerman, A Galois correspondence for II_1-factors and quantum groupoids, *J. Funct. Anal.* 178 (2000), 113–142.

[61] D. Nikshych and L. Vainerman, Finite Quantum Groupoids and Their Applications, in

New Directions in Hopf Algebras (S. Montgomery and H.-J. Schneider, eds.), Math. Sci. Res. Inst. Publ. 43, Cambridge University Press, Cambridge 2002, 211–262.

[62] V. Ostrik, Module categories, weak Hopf algebras, and modular invariants, preprint `math.QA/0111139` (2001).

[63] V. G. Paljutkin, Invariant measure of a compact ring group, *Trans. Amer. Math. Soc.* 84 (1969), 89–102.

[64] P. Podleś and S. L. Woronowicz, Quantum deformation of Lorentz group, *Comm. Math. Phys.* 130 (1990), 381–431.

[65] M. Rieffel, Some solvable quantum groups. Operator algebras and topology, Proceedings of the OATE 2 conference, Bucarest, Romania, 1989 (W. B. Arveson et al., eds.), *Pitman Res. Notes Math. Ser.* 270 (1992), 146–159.

[66] P. Schauenburg, Weak Hopf algebras and quantum groupoids, preprint `math.QA/0204180` (2002).

[67] G. Skandalis, Duality for locally compact 'quantum groups' (joint work with S. Baaj), Mathematisches Forschungsinstitut Oberwolfach, Tagungsbericht 46/1991, C^*-algebren, 20.10–26.10.1991, 20pp.

[68] L. L. Vaksman and Y. S. Soibel'man, Algebra of functions on the quantum group $SU(2)$, *Func. Anal. Appl.* 22 (1988), 170–181.

[69] W. F. Stinespring, Integration theorem for gauges and duality for unimodular locally compact groups, *Trans. Amer. Math. Soc.* 90 (1959), 15–56.

[70] S. Stratila, *Modular Theory in Operator Algebras*, Editura Academiei/Abacus Press, Bucuresti/Tunbridge Wells, Kent, 1981.

[71] W. Szymanski, Finite index subfactors and Hopf algebra crossed products, *Proc. Amer. Math. Soc.* 120 (1994), 519–528.

[72] M. Takesaki, A characterization of group algebras as a converse of Tannaka-Stinespring-Tatsuuma duality, *Amer. J. Math.* 91 (1969), 529–564.

[73] M. Takesaki, Duality and von Neumann algebras, in *Lectures on operator algebras*, Tulane University Ring and Operator Theory Year 1970–1971, Vol. II, Lecture Notes in Math. 247, Springer-Verlag, Berlin–Heidelberg–New York 1972, 665–785.

[74] M. Takeuchi, Groups of algebras over $A \times \overline{A}$, *J. Math. Soc. Japan* 29 (1977), 459–492.

[75] T. Tannaka, Über den Dualitätssatz der Nichtkommutativen Topologischen Gruppen, *Tôhoku. Math. J.* 45 (1938), 1–12.

[76] N. Tatsuuma, A duality theorem for locally compact groups, I, II, *Proc. Japan Acad.* 41 (1965), 878–882; ibid. 42 (1966), 46–47.

[77] S. Vaes, The unitary implementation of a locally compact quantum group action, *J. Funct. Anal.* 180 (2001), 426–480.

[78] S. Vaes, Examples of locally compact quantum groups through the bicrossed product construction, in *Proceedings of the XIIIth Int. Conf. Math. Phys., London* 2000 (A. Grigoryan, A. Fokas, T. Kibble and B. Zegarlinsky, eds.), International Press of Boston, Sommerville, MA, 2001, 341–348.

[79] S. Vaes, Locally compact quantum groups, Ph. D. Thesis. K.U. Leuven, 2001.

[80] S. Vaes and L. Vainerman, Extensions of locally compact quantum groups and the bicrossed product construction, to appear in *Adv. Math.*, preprint `math.QA/0101133` (2001).

[81] L. I. Vainerman, Characterization of Dual Objects to Locally Compact Groups, *Funct. Anal. Appl.* 8 (1974), 75–76.

[82] L. I. Vainerman, Relation between Compact Quantum Groups and Kac Algebras, *Adv. Soviet Math.* 9 (1992), 151–160.

[83] L. Vainerman, 2-Cocycles and Twisting of Kac Algebras, *Comm. Math. Phys.* 191 (1998), 697–721.

[84] J. M. Vallin, Bimodules de Hopf et poids opératoriels de Haar, *J. Operator Theory* 35 (1996), 39–65.

[85] J. M. Vallin, Unitaire pseudo-multiplicatif associé à un groupoïde; applications à la moyennabilité, *J. Operator Theory* 44 (2000), 347–368.

[86] J. M. Vallin, Groupoïdes quantiques finis, *J. Algebra* 26 (2001), 425–488.

[87] A. Van Daele, Discrete quantum groups, *J. Algebra* 180 (1996), 431–444.

[88] A. Van Daele, Multiplier Hopf Algebras, *Trans. Amer. Math. Soc.* 342 (1994), 917–932.

[89] A. Van Daele, An algebraic framework for group duality, *Adv. Math.* 140 (1998), 323–366.

[90] A. Van Daele and S. L. Woronowicz, Duality for the quantum $E(2)$ group, *Pacific J. Math.* 173 (1996), 375–385.

[91] P. Vecsernyes, Larson-Sweedler theorem, grouplike elements, and irreducible modules in weak Hopf algebras, preprint `math.QA/0111045` (2001).

[92] S. L. Woronowicz, Compact matrix pseudogroups, *Comm. Math. Phys.* 111 (1987), 613–665.

[93] S. L. Woronowicz, Twisted $SU(2)$ group. An example of a non-commutative differential calculus, *Publ. Res. Inst. Math. Sci.* 23 (1987), 117–181.

[94] S. L. Woronowicz, Tannaka-Krein duality for compact matrix pseudogroups, Twisted $SU(N)$ groups. *Invent. Math.* 93 (1988), 35–76.

[95] S. L. Woronowicz, Quantum $E(2)$ group and its Pontryagin dual, *Lett. Math. Phys.* 23 (1991), 251–263.

[96] S. L. Woronowicz, From multiplicative unitaries to quantum groups, *Internat. J. Math.* 7 (1996), 127–149.

[97] S. L. Woronowicz, Quantum $'az + b'$ group on complex plane, *Internat. J. Math.* 12 (2001), 461–503.

[98] S. L. Woronowicz and S. Zakrzevski, Quantum $'ax + b'$ group, preprint KMMF (1999).

[99] P. Xu, Quantum groupoids, *Comm. Math. Phys.* 216 (2001), 539–581.

Quantum groupoids and pseudo-multiplicative unitaries

Michel Enock

Institut de Mathématiques de Jussieu
Unité Mixte Paris 6 / Paris 7 / CNRS de Recherche 7586, Case 191
Université Pierre et Marie Curie, 75252 Paris Cedex 05, France
email: `enock@math.jussieu.fr`

Abstract. To any groupoid, equipped with a Haar system, Jean-Michel Vallin had associated several objects (pseudo-multiplicative unitary, Hopf bimodule) in order to generalize, up to the groupoid case, the classical notions of a multiplicative unitary and a Hopf-von Neumann algebra, which were intensively used in the quantum group theory, in the operator algebra setting. In two recent articles (one of them in collaboration with Jean-Michel Vallin), starting with a depth 2 inclusion of von Neumann algebras, we have constructed the same objects, which allowed us to study two "quantum groupoids" dual to each other. Here is a survey of these notions and results, including the announcement of new results about pseudo-multiplicative unitaries.

1 Introduction

The quantum group theory in the operator algebra setting has recently reached a new viewpoint from which the landscape is greater.

First of all, in their theory of "locally compact quantum groups", Kustermans and Vaes [KV] have obtained a beautiful and efficient axiomatisation of quantum groups. Their axioms are simple, easy to verify and cover all known examples. Many results in harmonic analysis seem now to be obtainable in that new setting and this article seems to be the new keystone of the theory.

Secondly, the links between quantum group theory and subfactor theory are now completely clarified ([EN], [E1], [V]): up to some regularity condition, every depth 2 irreducible inclusion of factors is given by an action of a locally compact quantum group on a factor, and vice-versa.

This situation leads several mathematicians to face two new questions:

– How to modify Kustermans and Vaes axioms in order to catch also locally compact groupoids? How does it correspond to what was done by several theoretical physicists ([BSz1], [BSz2], [Sz])?

– What is to be obtained if we deal with non-irreducible depth 2 inclusions of von Neumann algebras?

Of course, the answers to these two questions are closely linked, and many results were found in this direction. It turned out that the tools are completely different in finite- and in infinite-dimensional situation.

In the finite-dimensional situation, after some early work by Yamanouchi [Y2], the most important work is due to Böhm, Nill and Szlachányi [BNS] and Nikshych and Vainerman [NV1], who gave there a general setting to "finite quantum groupoids" and constructed several examples (see also earlier papers [BSz1], [BSz2], [Sz]). In [NV2], the links of this theory with depth 2 non-irreducible inclusions of type II_1 von Neumann factors are given. Another point of view, with multiplicative partial isometries, is due to Vallin ([Val3], [Val 4]) and Böhm and Szlachányi [BSz3].

In the infinite-dimensional situation, Vallin had associated with any locally compact groupoid, equipped with a left Haar system, two objects (Hopf bimodule structure, pseudo-multiplicative unitary), which generalize the usual coproduct and multiplicative unitary associated with a locally compact group ([Val1], [Val 2]). It appeared then clear that, for going from locally compact groups to locally compact groupoids, it was necessary to use the Hilbert space relative tensor product (Connes–Sauvageot tensor product) instead of the usual Hilbert space tensor product, and the "fiber product" of von Neumann algebras instead of the usual von Neumann algebra tensor product.

In [EV], [E2] the structures of the same kind have been obtained starting with non-irreducible depth 2 inclusions of von Neumann algebras.

New results in that theory can be found in [E3] and in [L], the latter will appear soon.

Here we give a survey on "quantum groupoids of infinite dimension", and announce some results, still unpublished. In Section 2 we recall all the preliminaries required, in particular a description of the Connes–Sauvageot tensor product (2.4) and of the fiber product of von Neumann algebras (2.5). In Section 3 we give the definitions of Hopf bimodules (3.1) and of pseudo-multiplicative unitaries (3.2), as well as examples coming from groupoids (3.1, 3.2) and from depth 2 inclusions (3.3). We also discuss the first properties of these objects. In Section 4, inspired by [BS], we develop the theory of quantum groupoids of compact type. Examples are given in Section 5.

For the sake of simplicity, all von Neumann algebras are supposed to be σ-finite.

This article is mostly inspired by the talk I gave at the conference on Quantum groups, Hopf algebras and their applications, which held in Strasbourg on February 21–23, 2002. I would like to thank the organizers of this conference.