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Dynamical Inverse Problems of Distributed Systems

V.I. Maksimov

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VSP
P.O. Box 346
3700 AH Zeist
The Netherlands

Tel: +31 30 692 5790
Fax: +31 30 693 2081
vsppub@compuserve.com
www.vsppub.com

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Preface

In this monograph, problems of dynamical reconstruction of unknown variable characteristics (distributed or boundary disturbances, coefficients of operators etc.) for various classes of systems with distributed parameters (parabolic and hyperbolic equations, evolutionary variational inequalities etc.) are discussed. The procedures for solving the problems are established. They are based on the methods of the theory of feedback control in combination with the methods of the theory of ill–posed problems and nonlinear analysis. These procedures are oriented to the work in real time and may be realized in computers. The general constructions are illustrated on numerical examples.

The book is destined for students, post-graduate students of physical-mathematical education, and specialists in optimization theory.

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Contents

Introduction

Problems of reconstructing unknown parameters of objects and processes using available information are well-known in engineering and scientific research. Many problems of the kind are posed as static ones and solved by static algorithms. In such problems, data allowed to be used in numerical solution algorithms are given *a priori*, the solution algorithms do not take into account changes in data, which might occur during the solution process, and the latter process is not viewed as the unique one; it can be repeated. Numerous examples of problems of the kind can be found among inverse problems in mathematical physics, problems of approximations of functions, problems of open-loop (program) control and observation, etc. However, it is often necessary to reconstruct unknown parameters dynamically, i.e., synchronically with an on-going physical process, or, as engineers say, "in real time". Data used in the reconstruction process may vary, "float" in time, and be, moreover, strongly past-dependent.

A general problem of dynamical reconstruction of unknown parameters can be described as follows. A dynamical system operating on a time interval $T = [t_0, \theta]$ is given. At each point in time, t, the system's state is represented as an element $x(t)$ of a finite- or infinite-dimensional space X. A system's motion, i.e., the evolution of its state $x(t)$, develops under some input disturbance $u(\cdot)$ starting from a given initial state x_0. It is required to reconstruct the input $u(\cdot)$ using observations of the motion $x(\cdot)$.

Let us identify the system under consideration with an operator A transforming every admissible initial state x_0 and every admissible realization of the input disturbance $u(\cdot)$ into a system's motion $x(\cdot)$ being a function of time; we assume that the operator A is single-valued. We stress that we deal with dynamical systems, i.e., the systems whose state histories do not depend on future evolutions of input disturbances. Formally, this property is reflected in the requirement that A is a Volterra operator. A well-known class of systems of the kind is the class of systems described by linear parabolic equations with boundary conditions of the Dirichlet type (for example). For such systems, time-varying inputs $u(\cdot)$ are traditionally viewed as controls. However, $u(\cdot)$ can represent noncontrollable external actions, i.e., act as noises, or dynamical disturbances, or inner variable parameters of the system, or have other physical interpretations.

At any moment t, the observation result can, generally, be a function of the current state $x(t)$, $z(t) = C(x(t))$, which is called the *output* of the dynamical system. In this monograph, we study the situation where the output values $z(\cdot)$ are given not precisely, i.e., the actual observation results $\xi(\cdot) = \xi^h(\cdot)$ are connected with $z(\cdot)$ by the relation

$$\kappa(\xi^h(\cdot),\ z(\cdot)) \leq h.$$

Here κ is a nonnegative functional, and h is a level of an observation error. In this situation, the *problem of constructing the operator* B^{-1} inverse to the "input–output" operator $B : (x_0, u(\cdot)) \to z(\cdot)$ is not solvable precisely; it belongs to the class of ill-posed problems. The role of B^{-1} is played by a certain operator D_h whose set of definition is wider than that of B^{-1}; namely, D_h is defined on the set of all functions $\xi(\cdot)$ representing admissible observation results (the set of measurements). The value $v_h(\cdot)$ of the operator D_h at a function $\xi(\cdot)$ no longer equals the sought input $u(\cdot)$. However, we view D_h as a suitable approximation to the operator B^{-1} if $v_h(\cdot)$ lies sufficiently close to $u(\cdot)$ for small h. The closedness of $v_h(\cdot)$ to $u(\cdot)$ is understood in the sense that the value $\rho(u(\cdot),\ v_h(\cdot))$ of a chosen nonnegative functional (the closedness criterion) is close to zero.

For generality, we do not require that the original problem has the unique solution, i.e., we admit that the inverse operator B^{-1} may not exist. In other words, we assume that every output $z(\cdot)$ can be generated by a set of inputs. We denote this set by $U(z(\cdot))$. Let us suppose that we are not interested in finding all the inputs $u(\cdot)$ from $U(z(\cdot))$ but interested in finding only those ones that are selected by a certain selection principle. In the theory of ill-posed problems, the minimization of a certain functional acts often as a selection principle. Following this approach, we assume that we have a functional $\omega(\cdot)$ defined on the set of all inputs and reaching its minimum value $\omega_{z(\cdot)}$ on every set $U(z(\cdot))$. Then the subset

$$U_*(z(\cdot)) = \{u(\cdot) \in U(z(\cdot)) :\ \ \omega(u(\cdot)) = \omega_{z(\cdot)}\}$$

is selected in every $U(z(\cdot))$. We are interested in shifting $v_h(\cdot)$ closer to $U_*(z(\cdot))$; more accurately, our desire is to make the value

$$\beta(v_h,\ \ U_*(z(\cdot))) = \inf\{\rho(u(\cdot),\ v_h(\cdot)) :\ \ u(\cdot) \in U_*(z(\cdot))\}$$

small for small h. Finally, we require that the operator D_h possesses the Volterra property that allows us to compute $v_h(t)$ not later than at time t: if $\xi_1(\tau) = \xi_2(\tau)$ for $\tau \leq t$, then $v_{h,1}(\tau) = v_{h,2}(\tau)$ for $\tau \leq t$, where $v_{h,1}(\cdot) = D_h\xi_1(\cdot)$, $v_{h,2}(\cdot) = D_h\xi_2(\cdot)$.

Thus, the basis problem of dynamical modeling of unknown parameters, which is analyzed in this book, consists in the following: construct a Volterra operator D_h, more precisely, a family of such operators depending on the accuracy parameter h, such that for any admissible output $z(\cdot)$ and any admissible results $\xi(\cdot) = \xi_h(\cdot)$ of measurement of this output, the convergence

$$\beta(v_h(\cdot),\ \ U_*(z(\cdot))) \to 0 \text{ as } h \to 0$$

is ensured.

The above problem of constructing a family D_h falls into the scope of inverse problems of the dynamics of controlled systems. These problems consist in finding unknown inputs to the systems using observations of systems' outputs. Every input determines the unique motion of a system; usually, the inputs are either time-varying controls regulating the system, or system's initial states, or, in the general case, pairs composed of controls and initial states. The output may represent any available information on the controlled process, often such information is provided by signals on the system's trajectory (this situation is typical for practical problems). We note here that inverse problems of dynamics are used for the design of controls realizing prescribed motions.

The first publications on this subject, which dealt with systems described by ordinary differential equations, provided criteria for unique solvability of inverse problems under the assumption that controls are sufficiently smooth, and studied the issue of the continuity of controls with respect to the observed outputs (signals on trajectories). If outputs are observed with errors, the inverse problems of dynamics become ill-posed, and the question of constructing their approximate solutions becomes equivalent to finding appropriate regularizing operators (algorithms).

Most researches on regularization address the "open-loop" setting of the problem: the regularizing algorithms process the entire history of the observation results (in this sense they are *a posteriory* algorithms). The question of constructing dynamical (Volterra-type) regularizing algorithms for finite-dimensional controlled systems was raised in Osipov, Yu.S. and Kryazhimskii, A.V., (1983). In this paper, a stable closed-loop (positional) algorithm reconstructing a minimum-norm control for a system affine in control with fully observable states was suggested.

For systems with distributed parameters, inverse problems treated within the framework of the "open-loop" setting were studied by many authors. Therefore, we do not overview this research area in detail; we only point out several important lines of research. Many publications are devoted to the issues of existence, uniqueness and stability of solutions of inverse problems for systems with distributed parameters and to development of numerical solution methods. Rather often the linearization method as well as the scheme of Newton–Kantorovich are applied for solving inverse problems. It should be noted that the methods of optimization theory are also widely used in the theory of inverse problems. This approach goes back to Tikhonov, A.N. and Marchuk, G.I. To determine unknown parameters, the least square method is also actively applied.

In the present monograph, we use the "closed-loop" (positional) approach for construction of dynamical regularizing algorithms for some classes of distributed systems. We illustrate general methods by examples and discuss problems of reconstructing distributed and boundary controls as well as coefficients in parabolic and hyperbolic (linear and nonlinear) equations and in parabolic variational inequalities.

It should be noted that the method of positional control with a model lies, as a rule, in the basis of constructions we suggest. This is one of the most effective methods of the theory of positional control; its characteristic property is robustness to informational and computational errors. This method originally suggested by Krasovskii, N.N. and developed by Ekaterinburg's scientific school is used to construct stable procedures of dynamical modeling (reconstructing) of input disturbances uniquely determining a motion of a dynamical system with distributed parameters on the basis of measurements of current system's states.

This monograph presents results of those studies only, in which the author's contribution is essential. Other algorithms and analytic methods for inverse problems of dynamics of distributed systems, which were developed within the framework of the above approach, can be found in the bibliography.

The author thanks Yuri S. Osipov and Arkadii V. Kryazhimskii for their advise and recommendations, which essentially influenced the results of research presented in this monograph, and his colleagues in the Department of Differential Equations of the Institute of Mathematics and Mechanics (the Ural Branch of the Russian Academy of Sciences) for their help in preparing the monograph.

Chapter 1.

Problems of dynamical modeling in abstract systems

In this chapter we describe an approach of construction of regularizing algorithms for solving of inverse problems of dynamics for systems with distributed parameters. This approach is based on the combination of the Lyapunov function method and the principle of positional control with a model. It was suggested by Kryazhimskii A.V. and Osipov Yu.S. (1983) and further developed by many authors (see Bibliography). In this chapter, we suggest conditions sufficiently general for the case when it is convenient to take the copies of real systems as models for solving inverse problems. These are the problems of modeling unknown inputs given approximate measurements of outputs. The conditions we obtain in this chapter are clarified for some classes of evolutionary systems.

1.1. RECONSTRUCTION OF INPUTS IN DYNAMICAL SYSTEMS. THE METHOD OF AUXILIARY CONTROLLED MODELS

In this section, a general scheme for solving the problems of dynamical reconstruction of inputs through results of approximate measurements of outputs is described. The scheme is based on the method of auxiliary positionally controlled models well-known in the theory of guaranteed control.

Further we fix an interval $T = [t_0, \theta]$, where $t_0 < \theta$. The definition below is the modified Tikhonov definition (Tikhonov, A.N., 1939; see, also, Tikhonov, A.N. and Arsenin, V.Ya., 1978).

Definition 1.1.1. An operator

$$\mathcal{A}: \quad X^{(1)} \times \ldots \times X^{(k)} \to X^{(0)},$$

where $k \in \{1, 2, \ldots\}$, $X^{(1)}, \ldots, X^{(k)}$, and $X^{(0)}$ are nonempty sets of functions from T, is called *Volterra* if for any

$$x^{(1)}(\cdot), \ y^{(1)}(\cdot) \in X^{(1)}, \ \ldots, \ x^{(k)}(\cdot), \ y^{(k)}(\cdot) \in X^{(k)}, \quad \text{and} \quad t \in T,$$

such that

$$x^{(1)}(s) = y^{(1)}(s), \ \ldots, \ x^{(k)}(s) = y^{(k)}(s) \quad \text{for all} \quad s \in [t_0, t],$$

we have

$$A(x^{(1)}(\cdot), \ \ldots, \ x^{(k)}(\cdot))(s) = A(y^{(1)}(\cdot), \ \ldots, \ y^{(k)}(\cdot))(s) \quad \text{for all} \quad s \in [t_0, t].$$

Suppose X, U, $X_0 \subset X$ are nonempty sets of elements and $X_{t_0,\theta}$, $U_{t_0,\theta}$ are nonempty sets of functions from T into X and U, respectively. Let

$$A: \quad X_0 \times U_{t_0,\theta} \to X_{t_0,\theta}$$

be an operator such that for any $x_0 \in X_0$ the operator $u(\cdot) \to A(x_0, \ u(\cdot)):$ $U_{t_0,\theta} \to X_{t_0,\theta}$ is a Volterra operator; then for any $u(\cdot) \in U_{t_0,\theta}$ we have

$$(A(x_0, u(\cdot)))(t_0) = x_0.$$

The operator A is called a *controlled dynamical system.*

Elements of the sets X_0 and $U_{t_0,\theta}$ are *an initial state* and *a control* (for the system A), respectively.

The set of elements of the product $X_0 \times U_{t_0,\theta}$ is *an input* (of the system A).

The value $A(x_0, u(\cdot))$ is *a motion* (of the system A) generated by an input $(x_0, \ u(\cdot))$.

The sets X and U are called *a phase space* and *a space of controlled parameters* (for the system A), respectively.

Let Z be a nonempty set and

$$C: \quad X \to Z$$

be an operator. We introduce an operator

$$B: \quad X_0 \times U_{t_0,\theta} \to \bar{Z}_{t_0,\theta},$$

where $\bar{Z}_{t_0,\theta}$ is the set of all functions $T \to Z$ such that

$$B(x_0, u(\cdot))(t) = C(A(x_0,\ u(\cdot)))(t), \qquad t \in T.$$

Let

$$Z_{t_0,\theta} = B(X_0,\ U_{t_0,\theta}).$$

The following terminology is used below:

The operator B is *an input–output operator* (for the system A);

the value $B(x_0, u(\cdot))$ is *an output generated by an input* $(x_0,\ u(\cdot))$;

elements of the set $Z_{t_0,\theta}$ are *outputs*;

the set Z is *the set of signals*;

the operator C is *an information operator.*

For each output $z(\cdot)$ we introduce the set

$$U(z(\cdot)) = \{u(\cdot) \in U_{t_0,\theta} :\ \ z(\cdot) = B(x_0, u(\cdot))\}.$$

We fix a functional

$$\omega(\cdot) : \quad U_{t_0,\theta} \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}.$$

Suppose that for any output $z(\cdot)$ the extremal problem

$$\omega(u(\cdot)) \to \inf, \qquad u(\cdot) \in U(z(\cdot))$$

has a solution. By $U_*(z(\cdot))$ we denote the set of all solutions, by $\omega_{z(\cdot)}$ denote the optimal value.

Definition 1.1.2. We call the sets $U(z(\cdot))$ and $U_*(z(\cdot))$ *the set of controls* and *the set of* $\omega(\cdot)$ *– normal controls* compatible with an output $z(\cdot)$, respectively. The functional $\omega(\cdot)$ is *the choice criterion.*

Suppose that Ξ is a nonempty set and $\Xi_{t_0,\theta}$ is a nonempty set of functions from T into Ξ. Introduce some notation. If $G_{t_0,\theta}$ is a set of functions defined on T, then

$$G_{t_1,t_2} = \{g_{t_1,t_2}(\cdot) :\quad g(\cdot) \in G_{t_0,\theta}\} \qquad (t_1, t_2 \in T, \quad t_1 < t_2).$$

For functions $g(\cdot)$ defined in $(a, b] \subset T$ or $[a, b] \subset T$ we use, for convenience, the same symbol $g_{a,b}(\cdot)$. Below it is clear which interval is considered: $(a, b]$ or $[a, b]$. If

$$\mathcal{A}: \quad \Xi_{t_0,\theta} \times G_{t_0,\theta} \to E_{t_0,\theta}$$

is a Volterra operator ($G_{t_0,\theta}$ and $E_{t_0,\theta}$ are nonempty sets of functions defined on T), then, for each $t \in T$, we denote by $\mathcal{A}_t$ an operator acting from $\Xi_{t_0,t} \times G_{t_0,t}$ into $E_{t_0,t}$ so that

$$\mathcal{A}_t(\xi_{t_0,t}(\cdot), g_{t_0,t}(\cdot)) = \mathcal{A}(\xi_{t_0,\theta}(\cdot), g_{t_0,\theta}(\cdot))|_{[t_0,t]}.$$

Here $\xi_{t_0,\theta}(\cdot) \in \Xi_{t_0,\theta}$ and $g_{t_0,\theta}(\cdot) \in G_{t_0,\theta}$ are arbitrary elements:

$$\xi_{t_0,\theta}(\cdot)|_{[t_0,t]} = \xi_{t_0,t}(\cdot), \qquad g_{t_0,\theta}(\cdot)|_{[t_0,t]} = g_{t_0,t}(\cdot).$$

Since $\mathcal{A}_t$ is a Volterra operator, it is well defined.

A finite family of points of T

$$\Delta = (\tau_i)_{i=0}^{m}$$

such that $t_0 = \tau_0 < \ldots < \tau_m = \theta$ is called *a partition* (of the interval T). A partition Δ of the interval T is *uniform* if

$$\tau_{i+1} - \tau_i = \tau_{j+1} - \tau_j$$

for all $i, j \in [0: \ m-1]$. The value

$$\delta(\Delta) = \max\{\tau_{i+1} - \tau_i| \quad i \in [0: \ m-1]\}$$

is *the step* of the partition Δ. Further we consider only uniform partitions of the interval T. By the symbol (Δ_h) we denote a fixed family of partitions dependent on a parameter h

$$\Delta_h = \{\tau_i, \ h\}_{i=0}^{m_h}, \qquad m_h = m(\delta(h)) = (\theta - \tau_0)/\delta(h) \tag{1.1.1}$$

of the interval T so that

$$\delta(h) = \delta(\Delta_h) \to 0 \qquad \text{as} \qquad h \to 0.$$

Further, let us set for simplicity

$$h \in (0,1), \qquad \delta(h) \in (0,1).$$

Let

$$\kappa(\cdot,\cdot,\cdot): \quad \tau_{i,h} \times \Xi_{t_0,\tau_{i,h}} \times Z_{t_0,\tau_{i,h}} \to \mathbb{R}^+ = [0, +\infty)$$

be a functional.

We use the following terms:

elements of the set $\Xi_{t_0,\theta}$ are *measurements*;

the set Ξ is *the space of instantaneous results of measurements*;

the functional

$$\kappa(\xi^h(\cdot),\ z(\cdot)) = \max_{i\in[0:\,m_h-1]} \kappa(\tau_{i,h},\ \xi^h_{t_0,\tau_{i,h}}(\cdot),\ z_{t_0,\tau_{i,h}}(\cdot))$$

is *the criterion of measurement error.*

We say that a family $(\xi^h(\cdot))_{h\in(0,1)}$ of measurements *κ-approximates on output* $z(\cdot)$, if

$$\kappa(\xi^k(\cdot), z(\cdot)) \le h \quad \text{for any} \quad h \in (0,1).$$

The set of all such measurements is denoted by

$$\Xi(z(\cdot), h).$$

By (a_h) we briefly denote a family $(a_h)_{h\in(0,1)}$ of elements dependent on a positive parameter $h \in (0,1)$. For simplicity, instead of $\kappa(\tau_{i,h},\ \xi^h_{t_0,\tau_{i,h}}(\cdot),\ z_{t_0,\tau_{i,h}}(\cdot))$, we write κ_i.

Finally, let us fix a functional

$$\rho(\cdot,\cdot): \quad U_{t_0,\theta} \times U_{t_0,\theta} \to \mathbb{R}^+,$$

and, for a control $v(\cdot)$ and an output $z(\cdot)$, we set

$$\beta(v(\cdot), U(z(\cdot))) = \inf\{\rho(u(\cdot), v(\cdot)): \ u(\cdot) \in U_*(z(\cdot))\}. \tag{1.1.2}$$

Further, the value $\beta(v(\cdot),\ \ U_*(z(\cdot)))$ is called *the ρ-error of control $v(\cdot)$ for output $z(\cdot)$* and the functional $\rho(\cdot,\cdot)$ is called *the criterion of error of approximation.*

The following basic definition follows from the theory of ill-posed problems (Ivanov et al., 1978; Tikhonov A.N. and Arsenin V.Ya., 1978).

Definition 1.1.3. A family (D_h) of Volterra operators acting from $\Xi_{t_0,\theta}$ into $U_{t_0,\theta}$ is called *regularizing* if for any output $z(\cdot)$ we have

$$\lim_{h\to 0} \sup\{\beta(D_h\xi^h(\cdot),\ \ U_*(z(\cdot))): \quad \xi^h \in \Xi(z(\cdot), h)\} = 0.$$

Remark 1.1.1. Without loss of generality, we assume that the functional $\rho(\cdot,\cdot)$ is defined in the widen set $U^W_{t_0,\theta} \times U^W_{t_0,\theta}(U_{t_0,\theta} \subset U^W_{t_0,\theta})$; and the operator family (D_h) acts from $\Xi_{t_0,\theta}$ into $U^W_{t_0,\theta}$. However, mostly, it suffices to assume that $U_{t_0,\theta} = U^W_{t_0,\theta}$.

The problem in question is to construct a regularizing family of Volterra operators. If $Z = X$; $C = I$ (the identity operator), then the operators $\mathcal{A}^{-1}$ and $\mathcal{A}_*^{-1}: \quad X_{t_0,\theta} \to U_{t_0,\theta}$, $\mathcal{A}^{-1}(x(\cdot)) = U(x(\cdot))$, $\mathcal{A}_*^{-1} = U_*(x(\cdot))$ are Volterra operators for a sufficiently wide class of systems (this is proved below). Therefore, we should seek for a solution of the problem in the class of Volterra operators.

The approach described below is based on the well–known principle of positional control: the principle of auxiliary controlled models. Its essence is as follows. We choose an auxiliary dynamical system (we call it a model). The initial state of the system w_0 is given. The set of motions $w(\cdot)$ is given by a Volterra operator defined on pairs $(\xi(\cdot), v(\cdot))$, where $\xi(\cdot)$ is a result of measuring an output $z(\cdot)$ generated by a real input of the system A; and $v(\cdot) \in U_{t_0,\theta}$ is a control in the model. The initial state w_0 of the model is given by the value $\xi(t_0)$ of measuring at the initial moment t_0 following a rule W fixed *a priori* (this rule is called a t_0*–algorithm*). The laws of forming a model control $v(\cdot)$ are called strategies following the terminology of positional control theory (Krasovskii, N.N., and Subbotin, A.I., 1984). These strategies are identified with Volterra operators $\mathcal{Y}$, which, following the feedback principle, assign $v(\cdot)$ to a pair $(\xi(\cdot), w(\cdot))$ (measurement-motion of the model). A process $(w(\cdot), v(\cdot))$ (motion-control of the model), for the given pair $(\mathcal{W}, \mathcal{Y})$, depends on a measurement $\xi(\cdot)$. Thus, the pair $(\mathcal{W}, \mathcal{Y})$ defines an operator D acting on measurements $\xi(\cdot)$. Note that the operator D_h is a Volterra operator under some special conditions. We construct the regularizing family from such operators. The pairs $(\mathcal{W}, \mathcal{Y})$ which define these operators are called modeling algorithms.

Suppose that

W is a nonempty set;

W_0 is a subset of W;

$W_{t_0,\theta}$ is a set of functions from T into W $(W_0 \neq \emptyset,\ W_{t_0,\theta} \neq \emptyset)$,

$$\Xi_0 = \{\xi(t_0): \quad \xi(\cdot) \in \Xi_{t_0,\theta}\}, \quad M: \quad W_0 \times \Xi_{t_0,\theta} \times U_{t_0,\theta} \to W_{t_0,\theta}$$

is an operator such that for any $w_0 \in W_0$ the operator $(\xi(\cdot), v(\cdot)) \to M(w_0, \xi(\cdot), v(\cdot))$ is Volterra, and, for any $\xi(\cdot) \in \Xi_{t_0,\theta}$ and $v(\cdot) \in U_{t_0,\theta}$, we have

$$(M(w_0,\ \xi(\cdot),\ v(\cdot)))(t_0) = w_0. \tag{1.1.3}$$

Definition 1.1.4. The operator M is called *a model*; the set W is called *the phase space of the model*; W_0 is called *the set of initial states of the model*; the function

$$w(t) = M(w_0, \xi(\cdot), v(\cdot))(t) = w(t; t_0, w_0, \xi_{t_0,t}(\cdot), v_{t_0,t}(\cdot)) \in W, \quad t \in T \tag{1.1.4}$$

is called *a motion (phase trajectory) of the model.*

Definition 1.1.5. Any function $\mathcal{W}$ which may depend on h

$$\mathcal{W} = \mathcal{W}_h: \ \ w_0 = w^h(t_0) = \mathcal{W}_h(\xi(t_0)) \in W_0 \tag{1.1.5}$$

is called *an algorithm of choice of the initial state of the model.*

Definition 1.1.6. Let $w_0 \in W_0$, $\mathcal{Y}: \ \ \Xi_{t_0,\theta} \times W_{t_0,\theta} \to U_{t_0,\theta}$ be a Volterra operator, $t \in T$, and a measurement $\xi_{t_0,t}(\cdot) \in \Xi_{t_0,t}$. Then any pair $(w_{t_0,t}(\cdot), v_{t_0,t}(\cdot)) \in W_{t_0,t} \times U_{t_0,t}$ such that

$$w_{t_0,t}(\cdot) = M_t(w_0, \xi_{t_0,t}(\cdot), v_{t_0,t}(\cdot)), \qquad v_{t_0,t}(\cdot) = \mathcal{Y}_t(\xi_{t_0,t}(\cdot), w_{t_0,t}(\cdot))$$

is called *a $(\mathcal{Y}, \xi_{t_0,t}(\cdot))$-process with the initial state w_0.*

Definition 1.1.7. A Volterra operator $\mathcal{Y}: \ \ \Xi_{t_0,\theta} \times W_{t_0,\theta} \to U_{t_0,\theta}$ is called *an admissible strategy* if for any $w_0 \in W_0$, $t \in T$, and $\xi_{t_0,\theta}(\cdot) \in \Xi_{t_0,\theta}$, there exists a unique $(\mathcal{Y}, \xi_{t_0,t}(\cdot))$–process with the initial state w_0.

Definition 1.1.8. Any pair $(\mathcal{W}, \mathcal{Y})$, where $\mathcal{W}$ is a t_0–algorithm and $\mathcal{Y}$ is an admissible strategy is called *an algorithm of modeling.*

For any algorithm of modeling $(\mathcal{W}, \mathcal{Y})$, $t \in T$, and $\xi_{t_0,t}(\cdot) \in \Xi_{t_0,t}$, we denote by

$$(w(\cdot;\ \mathcal{W},\ \mathcal{Y},\ \xi_{t_0,t}(\cdot)), \qquad v(\cdot,\ \mathcal{W},\ \mathcal{Y},\ \xi_{t_0,t}(\cdot)))$$

a unique $(\mathcal{Y},\ \xi_{t_0,t}(\cdot))$–process with the initial state $\mathcal{W}(\xi(t_0))$. Let

$$D[\mathcal{W},\ \mathcal{Y}]: \ \ \Xi_{t_0,\theta} \to U_{t_0,\theta}$$

be an operator which maps each measurement $\xi(\cdot)$ into the function

$$v(\cdot,\ \mathcal{W},\ \mathcal{Y},\ \xi(\cdot)).$$

Definition 1.1.9. An operator $D[\mathcal{W},\ \mathcal{Y}]$ is called *realizing* for a modeling algorithm $(\mathcal{W},\ \mathcal{Y})$; and a pair $(w(\cdot;\ \mathcal{W},\ \mathcal{Y},\ \xi(\cdot)), v(\cdot,\ \mathcal{W},\ \mathcal{Y},\ \xi(\cdot))$, where $\xi(\cdot) \in \Xi_{t_0,\theta}$ is called *a* $(\mathcal{W},\ \mathcal{Y},\ \xi(\cdot))$*-process.*

Remark 1.1.2. Further, we meet the situation when a model depends on a parameter $h \in (0,1)$: $M = M(w_0,\ \xi(\cdot),\ v(\cdot),\ h)$. However, we show later that the essential role in solving the problems is played by values of M only on elements dependent on h: $w_0 = w_{0,h}$, $\xi(\cdot) = \xi_h(\cdot)$, $v(\cdot) = v^h(\cdot)$. Therefore, introducing an operator M, we omit the argument h mentioning that M depends on h, i.e., $M = M(w_{0,h},\ \xi_h(\cdot),\ v^h(\cdot),\ h)$.

Lemma 1.1.1. *For any modeling algorithm* $(\mathcal{W},\ \mathcal{Y})$, *the operator* $D[\mathcal{W},\ \mathcal{Y}]$ *realizing this algorithm is a Volterra operator.*

Proof. We suppose that $\mathcal{W}$ is a t_0–algorithm and $\mathcal{Y}$ is an admissible strategy. We take an arbitrary $t \in T$, $\xi_1(\cdot)$, $\xi_2(\cdot) \in \Xi_{t_0,\theta}$ such that

$$\xi_1(s) = \xi_2(s) \quad \text{for} \quad s \in [t_0,\ t]. \tag{1.1.6}$$

We need to show that

$$v_1(s) = v_2(s) \quad \text{for} \quad s \in [t_0,\ t]. \tag{1.1.7}$$

Here

$$v_j(\cdot) = D[\mathcal{W},\ \mathcal{Y}], \qquad \Big(v_j(\cdot) = v(\cdot;\ \mathcal{W},\ \mathcal{Y},\ \xi_j(\cdot))\Big).$$

Suppose that

$$w_j(\cdot) = w(\cdot;\ \mathcal{W}_0,\ \mathcal{Y},\ \xi_j(\cdot)), \qquad j = 1,\ 2, \qquad \xi_0 = \xi_1(t_0) = \xi_2(t_0)$$

and set

$$\xi^{(j)}_{t_0,t}(\cdot) = \xi_j(\cdot)|_{[t_0,t]}, \qquad j = 1,\ 2.$$

Since $(w_j(\cdot),\ v_j(\cdot))$ is a $(\mathcal{Y},\ \xi_j(\cdot))$–process with the initial state $w_0 = \mathcal{W}(\xi_0)$, we have

$$w_j(\cdot) = M_t(w_0,\ \xi_j(\cdot),\ v_j(\cdot)), \qquad v_j(\cdot) = \mathcal{Y}_t(\xi_j(\cdot),\ w_j(\cdot)), \qquad j = 1,\ 2.$$

Hence

$$\begin{aligned} w_j(\cdot)|_{[t_0,t]} &= M_t(w_0,\ \xi_{t_0,t}(\cdot),\ v_j(\cdot)|_{[t_0,t]}), \\ v_j(\cdot)|_{[t_0,t]} &= \mathcal{Y}_t(\xi^{(j)}_{t_0,t}(\cdot),\ w_j(\cdot)|_{[t_0,t]}), \qquad j = 1,\ 2, \end{aligned} \tag{1.1.8}$$

i.e.,

$$p_j = (w_j(\cdot)|_{[t_0,t]}, \quad v_j(\cdot)|_{[t_0,t]})$$

is the $(\mathcal{Y},\ \xi^{(j)}_{t_0,t}(\cdot))$-process with the initial state w_0, $j = 1,\ 2$. It follows from (1.1.6) and from the Volterra property of $M(w_0, \cdot)$ that the operators

$$v_{t_0,t}(\cdot) \to M_t(w_0,\ \xi^{(j)}_{t_0,t}(\cdot),\ v_{t_0,t}(\cdot)) : \quad U_{t_0,t} \to W_{t_0,t} \quad \text{for} \quad j = 1,\ 2$$

coincide. The operators

$$w_{t_0,t}(\cdot) \to \mathcal{Y}_t(\xi^{(j)}_{t_0,t}(\cdot),\ w_{t_0,t}(\cdot)) : \quad W_{t_0,t} \to U_{t_0,t}$$

have the similar property. Therefore, in (1.1.8), for $j = 1$, we can replace the function $\xi^{(1)}_{t_0,t}(\cdot)$ by $\xi^{(2)}_{t_0,t}(\cdot)$. This means that the pair p_1 is the $(\mathcal{Y}, \xi^{(2)}_{t_0,t}(\cdot))$-process with the initial state w_0. As this process is unique, it coincides with p_2. Therefore, (1.1.7) holds. The lemma is proved. □

Now, we seek for a solution of the problem in question in the class of families $(D[\mathcal{W}_h,\ \mathcal{Y}_h])$ of Volterra operators which are realizing for certain modeling algorithms $(\mathcal{W}_h,\ \mathcal{Y}_h)$.

Definition 1.1.10. A family $(\mathcal{W}_h,\ \mathcal{Y}_h)$ of modeling algorithms is called *regularizing* if the family of operators $(D[\mathcal{W}_h,\ \mathcal{Y}_h])$ is regularizing.

The problem that we discuss consists in finding a regularizing family of modeling algorithms (and in the choice of a model M).

Problem 1.1.1. It is necessary to find a model M and construct modeling algorithms $(\mathcal{W}_h,\ \mathcal{Y}_h)$ such that the correspondent family of realizing operators $D_h = (D[\mathcal{W}_h,\ \mathcal{Y}_h])$ is regularizing.

In the end of this section, we describe a class of modeling algorithms which is used below. The following condition plays an important role.

Condition 1.1.1. For any t_*, $t^* \in T$, $t_* < t^*$, $u_{t_0,t_*}(\cdot) \in U_{t_0,t_*}$, and $u_{t_*,t^*}(\cdot) \in Ut_*, t^*$ the function $u_{t_0,t_*}(\cdot)$: $[t_0, t^*] \to U$ of the form

$$u_{t_0,t^*}(t) = \begin{cases} u_{t_0,t_*}(t),\ t \in [t_0, t_*], \\ u_{t_*,t^*}(t),\ t \in (t_*, t^*] \end{cases}$$

belongs to U_{t_*,t^*}.

Definition 1.1.11. (Krasovskii, N.N. and Subbotin, A.I., 1984). Any pair

$$\mathcal{Y} = (\Delta,\ \mathcal{U}),$$

where Δ is a partition (1.1.1), $\mathcal{U}$ is the function mapping each triple $(\tau_i, \xi_{t_0,\tau_i}(\cdot), w_{t_0,\tau_i}(\cdot))$, $i \in [0:\ m-1]$, $\xi_{t_0,\tau_i}(\cdot) \in \Xi_{t_0,\tau_i}$, $w_{t_0,\tau_i}(\cdot) \in W_{t_0,\tau_i}$, $(\xi_{t_0,t_0}(\cdot) = \xi(t_0))$ into an element

$$\mathcal{U}(\tau_i, \xi_{t_0,\tau_i}(\cdot), w_{t_0,\tau_i}(\cdot)) \in U_{\tau_i,\tau_{i+1}} \tag{1.1.9}$$

is called *a positional strategy.*

If Condition 1.1.1 holds, then an arbitrary positional strategy $\mathcal{Y} = (\Delta,\ \mathcal{U})$ is identified with the operator acting from $\Xi_{t_0,\theta} \times W_{t_0,\theta}$ into $U_{t_0,\theta}$ whose value $v(\cdot)$ on an arbitrary element $(\xi(\cdot),\ w(\cdot)) \in \Xi_{t_0,\theta} \times W_{t_0,\theta}$ is defined from the condition

$$v_{d_i}(\cdot) = \mathcal{U}(\tau_i,\ \xi(\cdot)\|_{[t_0,\ \tau_i]},\quad w(\cdot)|_{[t_0,\tau_i]}),\qquad i \in [0:\ m-1]. \tag{1.1.10}$$

Here $d_0 = [t_0,\ \tau_1]$, $d_i = (\tau_i,\ \tau_{i+1}]$ for $i > 0$. This operator is also denoted by $\mathcal{Y}$. A positional strategy $\mathcal{Y} = (\Delta,\ \mathcal{U})$ represents the convenient and feasible way of control of the model. At each moment τ_i, on the basis of the history $\xi_{t_0,\tau_i}(\cdot)$ of measurement and the history $w_{t_0,\tau_i}(\cdot)$ of motion, following rule (1.1.9), we construct a control fed onto the model on the half-interval $(\tau_i,\ \tau_{i+1}]$. The process of forming a model control is subdivided into the finite number of steps $(m-1)$.

So, the quadruple $(M,\ \mathcal{W}_h,\ \Delta_h,\ \mathcal{U}_h)$, for each $h \in (0,1)$, defines a certain algorithm D_h in the space of measurements $\xi(\cdot) \in \Xi(z(\cdot),\ h)$, $(D_h:\ \Xi_T \to U_T)$ which forms the output $v^h(\cdot) = D_h\xi(\cdot)$ following the feedback principle (1.1.4), (1.1.5), (1.1.9), (1.1.10). Note that the algorithm D_h is a Volterra algorithm. We construct the regularizing family D_h, $h \in (0,1)$ from such algorithms; we identify each algorithm D_h with the quadruple $(M,\ \mathcal{W}_h,\ \Delta_h,\ \mathcal{U}_h)$. So, we consider the problem of construction of regularizing families of algorithms $D_h = (M,\ \mathcal{W}_h,\ \Delta_h,\ \mathcal{U}_h)$, $h \in (0,1)$ of the form (1.1.4), (1.1.5), (1.1.9), (1.1.10).

The algorithm D_h (if h is fixed) has the following work scheme. Before the moment t_0, we choose and fix a partition

$$\Delta = \Delta_h = \{\tau_i\}_{i=0}^{m},\qquad (\tau_i = \tau_{i,h},\quad m = m_h)$$

of the interval T. The next i-th step of the algorithm is carried out in the interval $[\tau_i,\ \tau_{i+1})$. We measure (with an error) the output $z_{t_0,\tau_i}(\cdot)$, i.e., an element $\xi_{t_0,\tau_i}(\cdot) = \xi_{t_0,\tau_i}(\cdot) \in \Xi(z(\cdot), h)_{t_0,\tau_i}$ with the property

$$\kappa(\tau_{i,h},\ \xi_{t_0,\tau_{i,h}}(\cdot),\ z_{t_0,\tau_{i,h}}(\cdot)) \leq h$$

is calculated. Then, following rule (1.1.9), (1.1.10), we define the control in the model. After that, we make the correction of the memory: the part of the model trajectory $w(t)$, $t \in (\tau_i, \tau_{i+1}]$, is formed instead of $w(\tau_i)$ following (1.1.5). The procedure stops at the moment θ. The quadruple

$$D_h = (M,\ \mathcal{W}_h,\ \Delta_h,\ \mathcal{U}_h)$$

defined by relations (1.1.4), (1.1.5), (1.1.9), (1.1.10) is called further *the positional algorithm of modeling* (PAM). We seek for solutions of the problems under consideration in the class of such algorithms.

Note that in some cases we cannot use the history of motion of the system A. In this connection it is convenient to set

$$v_i^h = \mathcal{U}_h(\tau_i,\ \xi(\tau_i),\ w(\tau_i)), \tag{1.1.11}$$

$$w(\tau_{i+1}) = w(\xi(\tau_i),\ w(\tau_i),\ v_i^h). \tag{1.1.12}$$

The first function defines the law of approximation of the control, and the second function is the law of memory correction. The result of the work of the algorithm in the interval T is a piecewise-constant control $v^h(\cdot)$ of the form

$$v^h_{\tau_i,\tau_{i+1}}(t) = v_i^h, \qquad t \in [\tau_i,\ \ \tau_{i+1}). \tag{1.1.13}$$

Relation (1.1.12) may be interpreted as a controlled discrete dynamical system (a model) with the control v_i^h. Its initial state $w(t_0)$ is given *a priori* by (1.1.5). Thus, the modeling algorithm in this case is given by the quadruple $(M,\ \mathcal{W}_h,\ \Delta_h,\ \mathcal{U}_h)$: discrete model (1.1.12), the mapping $\mathcal{W}_h$, the family of partitions Δ_h, and the strategy $\mathcal{U}_h$ of the form (1.1.11), (1.1.13). We call this algorithm *finite-step dynamic modeling algorithm* (FSDAM).

In some cases (see, for example, Chapter 3) a model may be absent. Then, a positional strategy $\mathcal{Y}$ is a pair

$$(\Delta,\ \mathcal{U}), \qquad \mathcal{U}:\ \ (\tau_i,\ \ \xi_{t_0,\tau_i}(\cdot)) \to U_{\tau_i,\tau_{i+1}};$$

an operator

$$D = D[\mathcal{Y}]:\ \ \Xi_{t_0,\theta} \to U_{t_0,\theta}$$

is identified with $\mathcal{Y}$; the value of D on an element $\xi(\cdot) \in \Xi_{t_0,\theta}$ is defined from the condition

$$v(\cdot)|_{d_i} = \mathcal{U}(\tau_i, \xi(\cdot)|_{[t_0,\tau_i]}), \qquad i \in [0:\ \ m-1].$$

In this case, Problem 1.1.1 is reduced to the following problem.

Problem 1.1.2. It is necessary to construct a family of positional strategies $\mathcal{Y}_h$ such that the correspondent family (D_h) of Volterra operators is regularizing.

Lemma 1.1.2. *If Condition 1.1.1 holds, then any positional strategy is admissible.*

Proof. Suppose that $\mathcal{Y} = (\Delta, \mathcal{U})$ is a positional strategy and Δ is a partition of T. From the definition of $\mathcal{Y}$ we see that it is a Volterra operator. Let $t \in T$, $w_0 \in W_0$, and $\xi_{t_0,t}(\cdot) \in \Xi_{t_0,t}$ be arbitrary. We need to show (see Definition 1.1.7) that the process $(\mathcal{Y},\ \xi_{t_0,t}(\cdot))$ with the initial state w_0 exists and is unique. For this purpose, we apply the method of induction. Consider the case $t \in [t_0, \tau_1]$. Let $\overline{w}_{t_0,t}(\cdot) \in W_{t_0,t}$ be such that $\overline{w}_{t_0,t}(t_0) = w_0$ (as follows from (1.1.3), such element $\overline{w}_{t_0,t}(\cdot)$ exists). We introduce the functions

$$v_{t_0,t}(\cdot) = \mathcal{U}(t_0,\ \xi_{[t_0,t]}(\cdot)|_{[t_0,t_0]},\ \ \overline{w}_{t_0,t}(\cdot)|_{[t_0,t_0]}), \tag{1.1.14}$$

$$w_{t_0,t}(\cdot) = M_t(w_0,\ \xi_{t_0,t}(\cdot),\ v_{t_0,t}(\cdot)). \tag{1.1.15}$$

Following (1.1.3), $w_{t_0,t}(t_0) = w_0$; therefore, in (1.1.14), $\overline{w}_{t_0,t}(\cdot)|_{[t_0,t_0]}$ can be replaced by $w_{t_0,t}(\cdot)|_{[t_0,t_0]}$. By the definition of $\mathcal{Y}$, this means that

$$v_{t_0,t}(\cdot) = \mathcal{Y}_t(\xi_{t_0,t}(\cdot),\ \ w_{t_0,t}(\cdot)). \tag{1.1.16}$$

Relations (1.1.15), (1.1.16) show that $(w_{t_0,t}(\cdot),\ v_{t_0,t}(\cdot))$ is a $(\mathcal{Y}, \xi_{t_0,t}(\cdot))$–process with the initial state w_0 (see Definition 1.1.6). We prove now that it is unique. Suppose that $(w_{t_0,t}(\cdot),\ v_{t_0,t}(\cdot))$ and $(\overline{w}_{t_0,t}(\cdot),\ v_{t_0,t}(\cdot))$ are two $(\mathcal{Y}, \xi_{t_0,t}(\cdot))$–processes with the initial state w_0. Since

$$\overline{v}_{t_0,t}(\cdot) = \mathcal{Y}_t(\xi_{t_0,t}(\cdot),\ \overline{w}_{t_0,t}(\cdot)),$$

we see that, by the definition of $\mathcal{Y}$, $\overline{v}_{t_0,t}(\cdot)$ coincides with the right–hand side of equality (1.1.14). We show above that this equality holds true. Therefore, $\overline{v}_{t_0,t}(\cdot) = v_{t_0,t}(\cdot)$. Then $\overline{w}_{t_0,t}(\cdot)$ coincides with the right–hand side of (1.1.15), i.e., $\overline{w}_{t_0,t}(\cdot) = w_{t_0,t}(\cdot)$. Uniqueness is proved. Now, if we assume that a $(\mathcal{Y}, \xi_{t_0,t}(\cdot))$–process with the initial state w_0 exists and is unique for each $t \in [t_0, \tau_i]$, where $i \in [0 :\ m-1]$, we can show that it exists and is unique for $t \in [\tau_i, \tau_{i+1}]$. We omit the detailed proof of this fact. So, the theorem is proved. $\square$

Definition 1.1.12. If Condition 1.1.1 holds, a modeling algorithm $(\mathcal{W},\ \mathcal{Y})$, where $\mathcal{Y} = (\Delta, \mathcal{U})$ is a positional strategy, is called *positional* and is denoted by $(\mathcal{W},\ \Delta,\ \mathcal{U})$.

1.2. METHOD OF STABILIZATION OF LYAPUNOV FUNCTIONALS

We describe here sufficiently general method for constructing a regularizing family of modeling algorithms. This method consists in stabilizing an appropriate functional of Lyapunov type along the model motion.

Suppose that for each $h \in (0,1)$ we fix a functional

$$\Lambda_h : \quad Z_{t_0,\theta} \times U_{t_0,\theta} \times W_{t_0,\theta} \to \mathbb{R}.$$

Definition 1.2.1. A family (Λ_h) is called *estimating* if for any output $z(\cdot)$, family $(w_{0,h})$ of elements from W_0, family $(v_h(\cdot))$ of controls and family $(\xi^h(\cdot))$ of measurements κ-approximating $z(\cdot)$, from the convergence

$$\Lambda(z(\cdot),\ v_h(\cdot),\ M(w_{0,h},\ \xi^h(\cdot),\ v_h(\cdot))) \to 0 \qquad \text{as} \qquad h \to 0 \qquad (1.2.1)$$

it follows that the convergence (see (1.1.2))

$$\beta(v_h(\cdot),\ U_*(z(\cdot))) \to 0 \qquad \text{as} \qquad h \to 0 \qquad (1.2.2)$$

takes place.

Definition 1.2.2. A family $(\mathcal{W}_h,\ \mathcal{Y}_h)$ of modeling algorithms is Λ_h-stable if for any output $z(\cdot)$ and family $(\xi^h(\cdot))$ of measurements κ-approximating $z(\cdot)$ the convergence

$$\Lambda_h(z(\cdot),\ v(\cdot;\ \mathcal{W}_h,\ \mathcal{Y}_h,\ \xi^h(\cdot)),\ w(\cdot;\ \mathcal{W}_h,\ \mathcal{Y}_h,\ \xi^h(\cdot))) \to 0 \qquad \text{as} \qquad h \to 0 \qquad (1.2.3)$$

holds.

Lemma 1.2.1. *Suppose that a family (Λ_h) of functionals is estimating. Then any (Λ_h)-stable family of modeling algorithms is regularizing.*

Proof. We suppose that $(\mathcal{W}_h,\ \mathcal{Y}_h)$ is an arbitrary (Λ_h)–stable family of modeling algorithms. We take an arbitrary output $z(\cdot)$ and a family $(\xi^h(\cdot))$ of measurements κ–approximating $z(\cdot)$. We set

$$v_h(\cdot) = v(\cdot;\ \mathcal{W}_h,\ \mathcal{Y}_h,\ \xi^h(\cdot)),$$
$$w_h(\cdot) = w(\cdot;\ \mathcal{W}_h,\ \mathcal{Y}_h,\ \xi^h(\cdot)), \quad w_{0,h} = \mathcal{W}_h(\xi^h(t_0)).$$

Then $M(w_{0,h},\ \xi^h(\cdot),\ v_h(\cdot)) = w_h(\cdot)$. Hence, it follows that convergence (1.2.3) (which holds since the family of modeling algorithms $(\mathcal{W}_h,\ \mathcal{Y}_h)$ is (Λ_h)–stable) can be written in the form (1.2.1). Since the family of functionals Λ_h is estimating, convergence (1.2.1) yields convergence (1.2.2). By the definition of the operator $D[\mathcal{W}_h,\ \mathcal{Y}_h]$, we have

$$D[\mathcal{W}_h,\ \mathcal{Y}_h]\xi^h(\cdot) = v_h(\cdot),$$

so the last convergence can be written in the form

$$\beta(D[\mathcal{W}_h,\ \mathcal{Y}_h]\xi^h(\cdot),\ U_*(z(\cdot))) \to 0 \quad \text{as} \quad h \to 0.$$

The lemma is proved. □

Further we consider, as a rule, only mean square criterions of approximation error. Suppose that the space U of controlled parameters of the dynamical system A is a uniformly convex Banach space with a norm $|\cdot|_U$; the set $U_{t_0,\theta}$ of all controls of the system A belongs to the space $L_2(T;U)$ (the Lebesque measure in T is fixed); and the criterion of approximation error has the form

$$\rho(u(\cdot),\ v(\cdot)) = |u(\cdot) - v(\cdot)|_{L_2(T;U)} \qquad (u(\cdot),\ v(\cdot) \in U_{t_0,\theta}). \tag{1.2.4}$$

Suppose also that the choice criterion has the form

$$\omega(\cdot) = |\cdot|_{L_2(T;U)}.$$

Let the dynamical system A be such that for any $x_0 \in X_0$, $u_1(\cdot)$, $u_2(\cdot) \in U_{t_0,\theta}$,

$$A(x_0,\ u_1(\cdot)) = A(x_0,\ u_2(\cdot)), \quad \text{if} \quad u_1(t) = u_2(t) \quad \text{almost everywhere in} \quad T.$$

Therefore, taking into account (1.2.4), we identify any two controls $u_1(t)$ and $u_2(t)$ such that $u_1(t) = u_2(t)$ almost everywhere in T.

Below we use the following two conditions.

Condition 1.2.1. For any output $z(\cdot)$, the set $U_*(z(\cdot))$ of ω–normal controls compatible with $z(\cdot)$ is a singleton:

$$U_*(z(\cdot)) = \{u_*(\cdot;\ z(\cdot))\}.$$

Condition 1.2.2. The set $U_{t_0,\theta}$ is weakly closed and weakly compact in $L_2(T;U)$.

We show how to construct an estimating family of functionals. Suppose that for each $h \in (0,1)$ a functional

$$\overline{\Lambda}_h : \quad Z_{t_0,\theta} \times W_{t_0,\theta} \to \mathbb{R}$$

is fixed.

Definition 1.2.3. A family $\overline{\Lambda}_h$ is called *weakly estimating* if for any output $z(\cdot)$, family $(w_{0,h})$ of elements from the family W_0, family of controls $(v_h(\cdot))$, family $(\xi^h(\cdot))$ of measurements κ–approximating $z(\cdot)$, and sequence (h_k), $h_k \to 0$ as $k \to \infty$ such that

$$\overline{\Lambda}_{h_k}(z(\cdot),\ M(w_{0,h_k},\ \xi^{h_k}(\cdot),\ v_{h_k}(\cdot))) \to 0,$$

$$v_{h_k}(\cdot) \to v(\cdot) \quad \text{weakly in} \quad L_2(T;U) \qquad \text{as} \qquad k \to \infty,$$

the inclusion

$$v(\cdot) \in U(z(\cdot))$$

holds.

Lemma 1.2.2. *Suppose that*

a) Conditions 1.2.1 and 1.2.2 hold;

b) a family $(\overline{\Lambda}_h)$ is weakly estimating;

c)

$$\Lambda_h(z(\cdot),\ v(\cdot),\ w(\cdot)) = \Phi(h,\ \overline{\Lambda}_h(z(\cdot),\ w(\cdot)),\ |v(\cdot)|_{L_2(T;U)}, \omega_{z(\cdot)}) \tag{1.2.5}$$

$$(z(\cdot) \in Z_{t_0,\theta}, \quad v(\cdot) \in U_{t_0,\theta}, \quad w(\cdot) \in W_{t_0,\theta}),$$

where the function $\Phi: \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \overline{\mathbb{R}}$ *is such that for* $h_k \to 0$ *the condition*

$$\Phi(h_k,\ a_k,\ b_k,\ c_k) \to 0$$

yields

$$a_k \to 0 \qquad \text{and} \qquad \overline{\lim_{k\to 0}}(b_k - c_k) \le 0.$$

Then the family (Λ_h) *is estimating.*

Proof. Assume the opposite: there exist an output $z(\cdot)$, a family of elements $(w_{0,h})$ from W_0, a family of controls $(v_h(\cdot))$, a family of measurements $(\xi^h(\cdot))$ κ–approximating $z(\cdot)$, and a sequence $\{h_k\}$, $h_k \to 0$ as $k \to \infty$ such that

$$\Lambda_{h_k}(z(\cdot),\ v_{h_k}(\cdot),\ M(w_{0,h_k},\ \xi^{h_k}(\cdot),\ v_{h_k}(\cdot))) \to 0.$$

However,

$$\beta(v_{h_k}(\cdot),\ U_*(z(\cdot))) = \inf\{|u(\cdot) - v_{h_k}(\cdot)|_{L_2(T;U)}|\ u(\cdot) \in U_*(z(\cdot))\} = \varepsilon > 0. \tag{1.2.6}$$

Without loss of generality, taking into account Condition 1.2.2, we can write

$$v_{h_k}(\cdot) \to v(\cdot) \ \text{ weakly in } L_2(T;U) \qquad \text{as} \qquad k \to \infty,\ \ v(\cdot) \in U_{t_0,\theta}. \tag{1.2.7}$$

Besides, taking into account c), we have

$$\overline{\Lambda}_{h_k}(z(\cdot),\ \ M(w_{0,h_k},\ \xi^{h_k}(\cdot),\ v_{h_k}(\cdot))) \to 0, \qquad \text{as} \qquad k \to \infty, \tag{1.2.8}$$

$$\overline{\lim_{k\to\infty}}|v_{h_k}(\cdot)|_{L_2(T;U)} \le \omega_{z(\cdot)}. \tag{1.2.9}$$

Hence, taking (1.2.7), (1.2.8), and b) into account, we obtain

$$v(\cdot) \in U(z(\cdot)). \tag{1.2.10}$$

In turn, (1.2.10) and the properties of weak limit yield the inequalities:

$$\underline{\lim_{k\to\infty}}\ |v_{h_k}(\cdot)|_{L_2(T;U)} \ge |v(\cdot)|_{L_2(T;U)} \ge \omega_{z(\cdot)}. \tag{1.2.11}$$

By Condition 1.2.1, the set $U_*(z(\cdot))$ is a singleton:

$$U_*(z(\cdot)) = \{u_*(\cdot;\ z(\cdot))\}.$$