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Contents

Preface

As a rule, many practical problems are studied in the situation when the input data are incomplete. For example, this is the case for a parabolic partial differential equation describing the non-stationary physical process of heat and mass transfer if it contains the unknown thermal conductivity coefficient. Such situations arising in physical problems motivated the appearance of the present work.

Coefficient inverse problems for parabolic equation are formulated as the problems of determining the unknown vector-function (Lavrent'ev, Romanov, and Shishatskii, 1980). Its components are represented by the function for which a given equation is written and the unknown coefficients of the elliptic differential operator entering this equation. Hereafter we shall assume that the unknown coefficients depend only on the space variables and do not depend on time.

Coefficient inverse problems were studied in Alifanov and Klibanov (1985), Vabishchevich and Denisenko (1990), and other works. Numerical methods of solution of coefficient inverse problems associated with their applications were developed in Abasov, Azimov, and Ibragimov (1991), Evdokimov (1995), Iskenderov (1971), Khairullin (1986, 1988), Tsirel'man (1984), Banks and Lamm (1985), Cahen (1963), Chavent (1970, 1971), Chavent, Dupuy, and Lemonnier (1975), Douglas and Jones (1962), Kravaris and Sienfeld (1982, 1985, 1986), etc.

Coefficient inverse problems are conditionally well-posed and require special methods of investigation. For example, these are the regularization method (Tikhonov and Arsenin, 1979; Tikhonov, Goncharskii, Stepanov, and Yagola, 1983; Morozov, 1987a,b), the quasi-solution method (Ivanov, Vasin, and Tanana, 1978), and the quasi-inversion method (as a version of the regularization method) (Lattès and Lions, 1967). Numerous works devoted to the method of solution of conditionally well-posed problems are well known today (see, for example, Bukhgeim, 2000; Lavrent'ev, Romanov, and

Shishatskii, 1980; Lavrent'ev, Reznitskaya, and Yakhno, 1982; Lavrent'ev, Vasil'ev, and Romanov, 1969; Lavrent'ev, 1981; Liskovets, 1981).

Although the regularization methods allow obtaining a stable solution according to the definition of well-posedness in the sense of A. N. Tihonov, this does not guarantee its uniqueness. The uniqueness of solution of conditionally well-posed problems was studied by M. M. Lavrent'ev, V. G. Romanov, and others. A wide variety of conditionally well-posed problems of mathematical physics that have practical applications are considered in Lavrent'ev, Romanov, and Shishatskii (1980), Romanov (1984), Romanov, Kabanikhin, and Pukhnacheva (1984), Romanov (1969), Belov and Lavrent'ev (1996). The specific feature of the present book is that it is based on the results obtained by M. V. Klibanov in the proofs of the corresponding uniqueness theorems (see Klibanov, 1984a,b, 1986; Klibanov and Danilaev, 1990). The main idea of these results consists consideration of the studied equation together with the overdetermined set of boundary conditions.

The quasi-inversion method developed by M. M. Lavrent'ev and J.-L. Lions was chosen as the method of solution. The developed algorithm allows reducing coefficient inverse problems to the problems on the continuation of a solution of a parabolic equation considered in Lattès and Lions, 1967. The quasi-inversion method was substantiated and developed further in Tamme (1972), Muzylev (1977), Popov and Samarskii, (1988), Vabishchevich (1991a,b,c), Samarskii and Vabishchevich (1997). M. Kh. Khairullin and M. N. Shamsiev used the quasi-inversion method to solve practical problems (see Shamsiev, 1997).

In this book we consider numerical solutions of the quasi-inversion problems, to which the solution of the original coefficient inverse problems are reduced. Numerical methods of solution of conditionally well-posed problems were developed by Bukhgeim (1986), Bakushinskii and Goncharskii (1989), Samarskii and Vabishchevich (1990), and others. The classic results obtained by A. A. Samarskii and his pupils were used to construct the algorithms of numerical solution of the quasi-inversion problems(see Samarskii, 1977; Samarskii and Nikolaev, 1978). Some studies presented in this book are associated with the numerical experiment as it was determined by A. A. Samarskii (Samarskii, 1979; Popov and Samarskii, 1988). The monographs of Alifanov (1979, 1988), Alifanov, Artyukhin, and Rumyantsev (1988), Beck, Blackwell, and Saint Clair (1989), Kozdoba and Krukovskii (1982), Kozdoba (1992), Kurpisz and Novak (1995) are devoted to applications of the methods of study of conditionally well-posed problems of mathematical physics. A wide range of practical applications of conditionally well-posed problems can be found in the papers of All-Union Seminars on Inverse problems conducted by Acad. A. N. Tihonov and Acad. V. P. Mishin, and in the proceedings of international conferences ("Identification of Dynamic Systems and Inverse Problems"), which are regularly held in the Aerospace Department of Moscow Institute of Aircraft Engineering under the direction of Prof. O. M. Alifanov.

Parabolic equations describe the processes of heat and mass transfer and are widely used in the mathematical modelling of physical processes (for example, Bolgarskii, Muhachev, and Shchyukin, 1975; Buzinov and Umrikhin, 1984; Bulygin, 1974; Charnyi, 1997; Golubev and Tumashev, 1972; Development of Research in Filtration Theory in the USSR (1917–1967), 1969; Kozdoba, 1975; Dmitriev, 1982; Sedov, 1967). In the present book, underground fluid dynamics is taken as a field of practical use of coefficient inverse problems. The significance of these problems for this application domain consists in the possibility to determine the physical fields of parameters that characterize the filtration properties of porous media (oil strata). This provides the possibility of predicting the conditions of oil-field development and the effects of the exploitation. Many authors proposed the algorithms of determining the fields of filtration parameters. In Bulygin (1958), Golubev, Danilaev, and Tumashev (1978), Khairullin (1986, 1988), Chavent (1970, 1971), Chavent, Dupuy, and Lemonnier (1975), Kravaris and Sienfeld (1982, 1985, 1986), this problem was solved using the methods of parametric identification with regularization. The role and specific character of the approach considered here as compared to the mentioned works are analyzed in Chapter 6. In the present book we generalized and developed the author's results (see Bulygin and Danilaev, 1971; Golubev, Danilaev, and Tumashev, 1978; Golubev 1992; Golubev and Danilaev, 1981, 1983, 1987, 1990, 1991a,b, 1992a,b, 1996a,b; Danilaev 1978, 1980, 1981, 1986, 1987a,b, 1988, 1989a,b, 1993, 1996, 1997, 1998a,b,; Danilaev and Golubev, 1994a,b; Danilaev, Gortyshov, and Kuz'min, 1988).

Chapter 1.

On the ill-posedness of coefficient inverse problems and the general approach to the study of them

We show the ill-posedness of coefficient inverse problems for parabolic equations following Lavrent'ev, Romanov, and Shishatskii (1980). Consider the equation

$$\frac{\partial u}{\partial t} - L_q u = f(x,t), \tag{1.1}$$

where L_q is a uniformly elliptic operator

$$L_q u = \sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_{i=1}^{2} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u,$$

$$a_{ij}(x) = a_{ji}(x), \qquad 0 < \mu_0 \le \sum_{i,j=1}^{2} a_{ij}(x)\alpha_i\alpha_j \le M_0 < \infty, \qquad \sum_{i=1}^{2}\alpha_i^2 = 1.$$

Here $q = (a_{11}, a_{12}, a_{22}, b_1, b_2, c)$ denotes the ordered set of the coefficients of the differential operator. Thus, defining the operator L_q is equivalent to specifying a vector q.

Let the elements q form some set Q associated with a set of operators L_q. From this set of operators we have to choose the one, which corresponds to the given information on the solution of equation (1.1). Let all components of the vector q be unknown. Then for their definition it is necessary to provide the data of dimension not less then that of the vector q. Consider, for example, the case where the number of solution of equation (1.1) is equal to the number of components of the vector q. Let these solutions be constructed as the solutions of initial boundary value problem corresponding to the different initial data. Introduce the vector function $U = (u_1, u_2, \dots, u_6)$ whose components consist of the solutions of equation (1.1) which conform to different initial data. For the function U we have the splitting system of Cauchy problems

$$U_t - L_q U = F(x,t), \qquad U|_{t=0} = U_0(x), \tag{1.2}$$

where the vector function U is given on some manifold $\mathcal{M}$:

$$U|_{\mathcal{M}} = \varphi(x,t). \tag{1.3}$$

It is required to find L_q, $q \in Q$, by φ.

Such statement of the problem is initial when we investigate the uniqueness and stability of solutions of coefficient inverse problems.

Let (q, U), $(\hat{u}, \hat{U})$ be two solutions of the problem (1.2) corresponding to the same functions F, U_0, but to the different data φ, $\hat{\varphi}$ in (1.3). Denote $\tilde{q} = q - \hat{q}$, $\tilde{U} = u - \hat{U}$, $\tilde{\varphi} = \varphi - \hat{\varphi}$. Then we obtain

$$\frac{\partial \tilde{U}}{\partial t} - L_q \tilde{U} = \tilde{q}(x) R_{\hat{q}}(x,t), \tag{1.4}$$

$$\tilde{U}|_{t=0} = 0, \tag{1.5}$$

$$\tilde{U}|_{\mathcal{M}} = \tilde{\varphi}(x,t), \tag{1.6}$$

where we use the notation

$$L_{\tilde{q}} \hat{U} = \tilde{q}(x) R_{\hat{q}}(x,t).$$

Lemma (Lavrent'ev, Romanov, and Shishatskii, 1980). *The study of uniqueness and stability in the inverse problem* (1.2) *with respect to the data* (1.3) *is reduced to the study of analogous problems for determination of the function* $\tilde{q}(x)$ *from relations* (1.4)–(1.6), *where* L_q *is a given operator* $(q \in Q)$; $\tilde{\varphi}(x,t)$ *and* $R_{\hat{q}}(x,t)$ *are given functions;* R *is a matrix.*

If we know the stability estimate for $\tilde{q}$ in terms of $\tilde{\varphi}$ which is uniform in Q, then this estimate is also a stability estimate for the solution of the problem (1.2), (1.3).

Thus, the problem of determining the differential operator L_q is reduced to the problem of determining the special right-hand side of the differential equation, as far as the study of uniqueness and stability is concerned. It is also an inverse problem, but it is linear in this case.

The questions of solution stability of the obtained problem depend on the function $R_{\hat{q}}(x,t)$. The requirement $\det R_{\hat{q}}(x,t) \not\equiv 0$ is intrinsic and minimal. The condition that $\det R_{\hat{q}}(x,t)$ is not equal to zero at all points of some smooth surface $t = t(x)$.

The linear problem (1.4)–(1.6) can be reduced to the Fredholm equation of the first kind using the fundamental solution $H_q(x,t,\xi,\tau)$ of equation (1.1):

$$\int_{\mathbb{R}^3} \tilde{q}(\xi) R_{q\hat{q}}(\xi,x,t)\,\mathrm{d}\xi = \tilde{\varphi}(x,t), \qquad (x,t) \in \mathcal{M}, \tag{1.7}$$

where

$$R_{q\hat{q}}(\xi,x,t) = \int_0^t R_{\hat{q}}(\xi,\tau) H_q(x,t,\xi,\tau)\,\mathrm{d}\tau.$$

Thus, the inverse problem (1.2), (1.3) is reduced to the integral Fredholm equation of the first kind (1.7).

The uniqueness theorems for solution of coefficient inverse problems for the parabolic equation (1.1) were proved by M. V. Klibanov. Now, we consider these theorems. Let $l > 0$ be a non-integer number (designations of Banach spaces conform to Ladyzenskaja, Solonnikov, and Ural'ceva (1967)), Ω be a bounded domain in $\mathbb{R}^n$, $\partial\Omega \in C^\infty$, $T = \text{const} > 0$, $Q_T = \Omega \times (0,T)$, $G_T = \Omega \times (-T,T)$, $G_{\sigma T} = \Omega \times (-\sigma, T)$, $\sigma = \text{const} \in (0,T)$, $S_T = \partial\Omega \times (0,T)$, $N_T = \partial\Omega \times (-T,T)$, L_q be a uniform elliptic operator with smooth coefficients in Q_T or G_T. The conditions of coordination of necessary order are assumed to hold in the further problems (Ladyzenskaja, Solonnikov, and Ural'ceva, 1967). Consider the linearized statements of inverse problems.

Problem 1.1. *Determine the vector function* $(u,q) \in H^{l,l/2}(\bar{Q}_T) \times H^{l-2}(\bar{\Omega})$, $l > 2$, *from the following conditions:*

$$\frac{\partial u}{\partial t} = Lu + q(x)F(x,t), \qquad (x,t) \in Q_T,$$

$$u|_{t=0} = 0, \qquad u|_{S_T} = \varphi(x,t), \quad \left.\frac{\partial u}{\partial \nu}\right|_{S_T} = \phi(x,t),$$

where L *is the uniformly elliptic operator with smooth coefficients in* Q_T *or* G_T, $L = \sum_{|\alpha| \le 2} a_\alpha D^\alpha$, $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, $D_j = \partial/\partial x_j$.