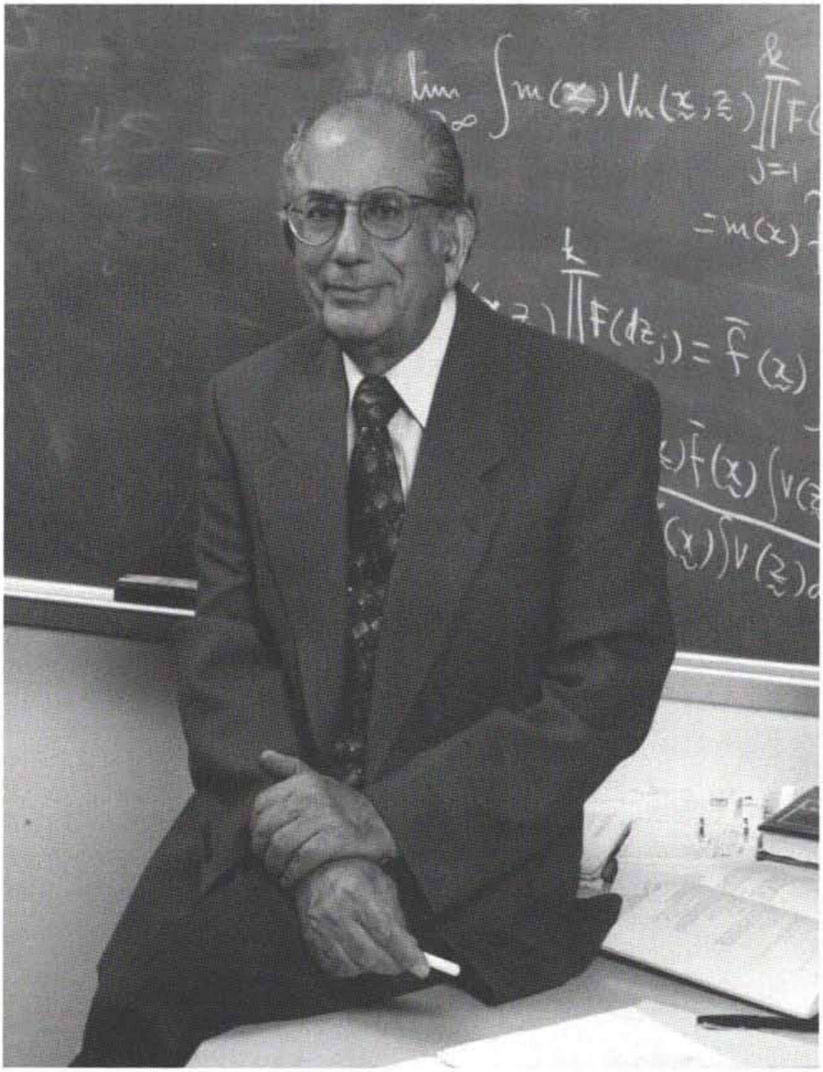
Madan Lal Puri Selected Collected Works
Volume 1



Madan Lal Puri

Dedication

These three volumes of "Selected Collected Works of Madan Lal Puri" will serve to preserve, in a unified and easily accessible form, the knowledge and wisdom conveyed in his many research papers, so as to aid its dissemination to future generations.

The Editors

MADAN LAL PURI
SELECTED COLLECTED WORKS

VOLUME I

NONPARAMETRIC METHODS IN STATISTICS AND RELATED TOPICS

Editors
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PREFACE

The volumes at hand are the outcome of a concerted effort to make Professor Puri's research works easily available to the research community. The sheer volume of the research output by him and his collaborators, coupled with the broad spectrum of the subject matters investigated, and the great number of outlets where the papers were published, attach special significance in making these works easily accessible.

In compiling these volumes, the Editors are satisfied with the feeling of discharging part of their duty to the profession, and simultaneously expressing their respect and admiration for their colleague, Madan Puri, for his immense contributions to statistics and probability literature. A generous dose of appreciation is due to the publishers of the volumes. The VSP International Science Publishers, for undertaking a less than profitable venture. Also, thanks are due to professional organizations, as well as the private publishers who graciously waived all reprinting copyright fees with a deep sense of service to the profession. An appropriate list of said entities is given at the end of this preface, whereas suitable acknowledgements are cited at the end of each paper in the three volumes.

Professor Puri's published research works number more than 3,000 pages. However, practical publishing considerations had to be taken into account, and expectedly they took their toll. Accordingly, the three volumes would have to comprise a total of about 2,000 pages. The burden of selection was not easy, and works omitted are not to be considered inferior to those selected for inclusion. The need of exclusion also explains the title assigned to this work.

The papers selected for inclusion in this work have been classified into three volumes, each consisting of several parts. Thus, Volume 1 consists of 44 papers distributed into four parts as follows: Part I of 24 papers, falling into the area of *Nonparametric Methods in Univariate Analysis*; Part II of 6 papers from *Nonparametric Methods in Multivariate Analysis*; Part III of 4 papers from *Nonparametric Methods in Design and Analysis of Experiments*; and, finally, Part IV consisting of 10 papers in *Miscellaneous Topics*, 4 of which (#'s [40]–[43]) are also from *Nonparametric Methods in Multivariate Analysis*. Volume 2 consists of 35 papers classified in three parts as follows: Part I of 18 papers under the heading of *Limit Theorems, Rates of Convergence and Related Topics* (*Independent Case*); Part II of 13 papers taken from his contributions in *Limit Theorems* (*Dependent Case*); and Part III consisting of 4 papers from the area of *Extreme Value Theory*. Volume 3 consists of a total of 48 papers distributed into three parts as described below: Part I of 18 papers under the heading of *Time Series and Related Topics*; Part II of 14 papers falling into the area of *Fuzzy Set Theory and Related Topics*; and, finally, Part III comprising 16 papers from *Miscellaneous Topics.* Also, all three volumes carry a final part consisting of the contents of the other two volumes, as well as the complete list of Professor Puri's publications.

It would be appropriate that a brief biographical sketch of Professor Puri be included in this preface. The following few paragraphs are excerpts from the preface of the book *Asymptotics, Nonparametrics, and Time Series: A Tribute to Madan Lal Puri*, edited by Subir Ghosh and published by Marcel Dekker, Inc. in 1999. One of the present Editors was a co-author of that preface.

Madan Lal Puri was born in Sialkot (then in India, now in Pakistan) on February 20, 1929. In 1947, when India gained her independence and Pakistan was created, his family migrated to Delhi as refugees. He received a B. A. degree in 1948 and an M. A. degree in 1950, both in mathematics, from Panjab University in India. From January 1951 to August 1957, he served as a Lecturer in Mathematics in different colleges of Panjab University.

In September 1957, he came to the United States as an instructor and graduate student in mathematics at the University of Colorado in Boulder. In September 1958, he moved to the University of California at Berkeley as a research assistant in the Department of Statistics and received his Ph. D. in statistics in 1962.

In 1962, Dr. Puri joined the renowned Courant Institute of Mathematical Sciences in New York University as an Assistant Professor and became an Associate Professor in 1965. He joined Indiana University at Bloomington in 1968 as a Full Professor of Mathematics and remains there to this day.

Professor Puri is one of the most versatile and prolific researchers in the world in mathematical statistics. His research areas include nonparametric statistics, order statistics, limit theory under mixing, time series, splines, tests of normality, generalized inverses of matrices and related topics, stochastic processes, statistics of directional data, random sets, and fuzzy sets and fuzzy measures. His fundamental contributions in developing new rank-based methods and precise evaluation of the standard procedures, asymptotic expansions of distributions of rank statistics, as well as large deviation results concerning them, span such areas as analysis of variance, analysis of covariance, multivariate analysis, and time series, to mention a few. His in-depth analysis has resulted in pioneering research contributions to prominent journals that have substantial impact on current research.

Professor Puri has done joint work with many researchers of different countries. To date he has collaborated with 89 scholars from 22 countries on 5 continents. He was the Alexander von Humboldt Guest Professor at the University of Göttingen in West Germany in 1974–1975 and Guest Professor at many other universities in Germany, with German National Science Foundation grants. He has been a Distinguished Visitor at the London School of Economics and Political Science, Visiting Professor at the University of Auckland in New Zealand, the Universities of Bern and Basel in Switzerland, the University of New South Wales in Australia, the University of Goteborg and Chalmers University of Technology in Sweden, Université des Sciences et

Technologies de Lille in France, Australian National University, Canberra, and University of Washington, Seattle, among other universities. In 1974, he was invited by the Japanese Society for the Promotion of Sciences to visit Japan under its Visiting Professorship Program to conduct cooperative research with Japanese scientists. He has been an invited speaker as well as a plenary speaker at many international conferences all over the world.

Professor Puri has received numerous honors and awards. He is an elected member of the International Statistical Institute, and a Fellow of the Institute of Mathematical Statistics, a Fellow of the American Statistical Association, and a Fellow of the Royal Statistical Society. In 1975, he was honored with the D. Sc. degree from Panjab University in India. He twice received the Senior U. S. Scientist Award from the Alexander von Humboldt Foundation in 1974 and 1983. In 1974, he was honored by the government of the Federal Republic of Germany, "in recognition of past achievements in research and teaching." In 1984, he received the best paper award from the Seventh European Meeting on Cybernetics and Systems Research, Vienna, Austria. In 1991, he received the Rothrock Faculty Teaching Award in recognition of outstanding teaching in the Department of Mathematics of Indiana University. He was ranked the ninth most prolific author in 1992, and the fourth most prolific author in 1997 in top statistical journals of the world.

Professor Puri has served on various committees of many international conferences in addition to those of the Institute of Mathematical Statistics and the American Statistical Association. He also served as Editor-in-Chief of the *Journal of Statistical Planning and Inference* in 1984–1988.

Professor Puri has directed 16 Ph. D. dissertations. Most of his former Ph. D. students are in research and teaching positions at respectable universities. A few hold responsible positions in industry.

Professor Puri is truly an international academician and a peripatetic scholar who works with missionary zeal. Scientists from all over the world visit him regularly and do research with him while staying at his home. His office and home have always been wide open to bright young scientists from the United States and Overseas, who were more in need of sponsorship and gentle encouragement and guidance in their professional endeavors rather than mere mathematical mentoring. He is a caring colleague with the warmest affection, an international host, a persuasive communicator, a dedicated as well as an outstanding teacher, and a versatile statistician whose work continues to inspire the scientific community.

We are editing these volumes in the hope of facilitating the availability to the research community of a substantial part of Professor Puri's work. We take great pleasure in doing so.

This project has benefited greatly from the generous financial support of Moya Andrews, Vice-Chancellor for Academic Affairs and Dean of the Faculties, Indiana University; Patrick O'Meara, Dean of the International Programs, Indiana University; Curtis R. Simic, President, Indiana University Foundation; Kumble R. Subbaswamy, Dean of the College of Arts and Sciences, Indiana University; and George Walker, Vice-President for Research and Dean of the Graduate School, Indiana University. Special thanks go to Kenneth R. R. Gros Louis, currently Chancellor-Emeritus and

Trustee Professor, Indiana University, for his never-ending enthusiastic encouragement and financial support for several of Professor Puri's research projects during his tenure as Chancellor of Indiana University, Bloomington Campus, and Vice-President for Academic Affairs, Indiana University.

The reprinting copyright fee waivers granted by professional societies and private publishers have made this undertaking financially feasible. Their deep sense of service to our profession is gratefully acknowledged here. They are Akademiai Kiado, Biometrika Trustees (The Oxford University Press), Sankhyā, SIAM, The American Mathematical Society, The American Statistical Association, The Institute of Mathematical Statistics, The Royal Society, and The Statistical Society of Canada. Also, Academic Press, Blackwell Publishing, LTD, Cambridge University Press, Elsevier Science, John Wiley & Sons, Inc., Kluwer Academic / Plenum Publisher, and Springer Verlag GmbH & Co. KG.

July 2002

The Editors
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Part I

Nonparametric Methods in Univariate Alalysis

ASYMPTOTIC EFFICIENCY OF A CLASS OF c-SAMPLE TESTS[1]

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1. Summary. For testing the equality of c continuous probability distributions on the basis of c independent random samples, the test statistics of the form

$$\mathcal{L} = \sum_{j=1}^{c} m_j[(T_{N,j} - \mu_{N,j})/A_N]^2$$

are considered. Here m_j is the size of the jth sample, $\mu_{N,j}$ and A_N are normalizing constants, and

$$T_{N,j} = (1/m_j)\sum_{i=1}^{N} E_{N,i}Z_{N,i}^{(j)}$$

where $Z_{N,i}^{(j)} = 1$, if the ith smallest of $N = \sum_{j=1}^{N} m_j$ observations is from the jth sample and $Z_{N,i}^{(j)} = 0$ otherwise. Sufficient conditions are given for the joint asymptotic normality of $T_{N,j}$; $j = 1, \cdots, c$. Under suitable regularity conditions and the assumption that the ith distribution function is $F(x + \theta_i/N^{\frac{1}{2}})$, the limiting distribution of $\mathcal{L}$ is derived. Finally, the asymptotic relative efficiencies in Pitman's sense of the $\mathcal{L}$ test relative to some of its competitors viz. the Kruskal-Wallis H test (which is a particular case of the $\mathcal{L}$ test) and the classical F test are obtained and shown to be independent of the number c of samples.

2. Introduction. One of the frequently encountered problems in statistics is to decide whether differences in various samples should be regarded as due to differences in the parent populations or due to chance variations which are to be expected among random samples from the same population. A few tests of nonparametric nature have been proposed for this c-sample problem. The Kruskal-Wallis H test [14], Terpestra's c-sample test [26], the Mood and Brown c-sample test [22] and Kiefer's K-sample analogues of the Kolmogorov-Smirnov and Cramér-von Mises tests [12] are a few of them. Tests for two-sample problems have been proposed by Wilcoxon [29], Mann and Whitney [19], Mood and Brown [22], Lehmann [15] and others. Consistency and power properties of some of these tests have been discussed by Lehmann [15], [16], [17], Mood [23], Van Dantzig [5] and others. An exhaustive review of this problem is given n Kruskal and Wallis [14] and Scheffé [25].

The H test of Kruskal and Wallis is a direct generalization of the two-sample

Received 21 February 1962; revised 4 May 1963.

[1] This paper was prepared with the partial support of the Office of Naval Research, Contract Nonr-222-(43), while the author was at the University of California, Berkeley. It was revised at the Courant Institute of Mathematical Sciences, New York University under the sponsorship of the Office of Naval Research, Contract Nonr-285(38). Reproduction in whole or in part is permitted for any purpose of the United States Government.

Wilcoxon test discussed in detail by Mann and Whitney [19], and its limiting distribution has been derived by Kruskal [13] under the null hypothesis and by Andrews [1] under an alternative hypothesis. These results are generalized by those of the present paper concerning the limiting distribution of the $\mathcal{L}$ test.

The problem discussed in this paper originated from the paper of Chernoff and Savage [2] and had its basis in the paper of Hodges and Lehmann [10]. In their paper "The efficiency of some nonparametric competitors of the t-test" [10], Hodges and Lehmann discussed the asymptotic efficiency of the Wilcoxon test with respect to all translation alternatives. In the same paper they conjectured that the normal score test which was known to be as efficient as the t-test for normal alternatives [11a] is at least as efficient as the t-test for all other alternatives. The validity of this conjecture was established by Chernoff and Savage [2], who developed a new theorem for asymptotic normality of normal score test statistics for the two-sample problem and by a variational argument proved the Hodges-Lehmann conjecture. The work presented here is an attempt toward generalizing these results to the c-sample problem.

Formally, we may state the c-sample problem as follows. Let $[X_{ij}, j = 1, \cdots, m_i; i = 1, \cdots, c]$ be a set of independent random variables and let $F^{(i)}(x)$ be the probability distribution of X_{ij}. The set of admissible hypotheses designates that each $F^{(i)}(x)$ belongs to some class of distribution functions Ω. The hypothesis to be tested, say H_0, specifies that $F^{(i)}$ is an element of Ω, for each i, and that furthermore

$$F^{(1)}(x) = \cdots = F^{(c)}(x) \text{ for all real } x. \tag{2.1}$$

The class of alternatives to H_0 can be considered to be all sets $(F^{(1)}(x), \cdots, F^{(c)}(x))$ which belong to Ω but which violate (2.1). To avoid the problem of ties, it is assumed throughout that the class Ω is the class of continuous distribution functions.

After finding sufficient conditions for the joint asymptotic normality of $T_{N,j}$; $j = 1, \cdots, c$, we study the limiting distributions of $\mathcal{L}$ under a sequence of admissible alternative hypothesis H_n^P which specifies that for each $i = 1, 2, \cdots, c$; $F^{(i)}(x) = F(x + \theta_i/n^{\frac{1}{2}})$ with $F \varepsilon \Omega$ but not specified further, and for some pair (i, j), $\theta_i \neq \theta_j$ where the θ_i's are real numbers. Limiting probability distributions of $\mathcal{L}$ will then be found as $n \to \infty$. The problem will be so formulated that $m_i(n)/n$ tends to some limit s_i, $0 < s_i < \infty$, as n tends to ∞.

3. The proposed test and its relationship to other tests. The over-all sample consists of $\sum_{i=1}^{c} m_i = N$ independent random variables X_{ij} ($i = 1, \cdots, c$; $j = 1, \cdots, m_i$), where the first subscript refers to the subsample and the second subscript indexes observations within a subsample. Under the null hypothesis all the X's have the same continuous but unknown c.d.f. (cumulative distribution function) $F(x)$.

Let $Z_{N,i}^{(j)} = 1$, if the ith smallest observation from the combined sample of size N is from the jth sample and otherwise let $Z_{N,i}^{(j)} = 0$. Denote

$$m_j T_{N,j} = \sum_{i=1}^{N} Z_{N,i}^{(j)} E_{N,i} \tag{3.1}$$

where $E_{N,i}$ are given numbers. Then we propose to consider the test statistic $\mathfrak{L}$ defined as

$$(3.2) \qquad \mathfrak{L} = \sum_{j=1}^{c} m_j[(T_{N,j} - \mu_{N,j})/A_N]^2$$

where $\mu_{N,j}$ and A_N are normalizing constants for the statistics $T_{N,j}$; $j = 1, \cdots, c$.

The $\mathfrak{L}$ test presented in this paper includes as special cases a number of well-known tests. For example, when $E_{N,i} = i/N$, it becomes the Kruskal-Wallis H test which is a direct generalization of the two-sample Wilcoxon test and is related to Terpestra's K-sample test [26]. When $c = 2$ and $E_{N,i}$ is the expected value of the ith order statistic from the standard normal distribution, then the $\mathfrak{L}$ test coincides with the symmetrical two-tail version of the normal score test, also known as the Fisher-Yates-Terry-Hoeffding c_1 test and which is asymptotically equivalent to Van der Waerden's test [30], [31]. For it is then seen that

$$\mathfrak{L} = [N/(N - m_1)]\left[\sum_{i=1}^{m_2} E(V^{(s_i)} \mid s_i)\right]^2$$

where $V^{(1)} < \cdots < V^{(N)}$ is an ordered sample of size N from a standard normal distribution, and $s_1 < \cdots < s_{m_2}$ are the ranks of $X_{21}, \cdots, X_{2m_2}$ from the combined sample. See Lehmann [17], pp. 236–237. When $c = 2$, and $E_{N,i} = |\frac{1}{2} - i/N|$, the $\mathfrak{L}$-test test reduces to the Freund-Ansari test [8] for testing the equality of dispersion of two populations.

4. Assumptions and notations. Let $X_{i1}, \cdots, X_{im_i}$ be the ordered observations of a random sample from a population with continuous c.d.f. (cumulative distribution function) $F^{(i)}(x)$; $i = 1, \cdots, c$. Let $N = \sum_{i=1}^{c} m_i$ and $\lambda_i = m_i/N$ and assume that for all N, the inequalities $0 < \lambda_0 \leqq \lambda_1, \cdots, \lambda_c \leqq 1 - \lambda_0 < 1$ hold for some fixed $\lambda_0 \leqq 1/c$.

Let

$$S_{m_i}^{(i)}(x) = m_i^{-1} \text{ (number of } X_{ij} \leqq x, j = 1, \cdots, m_i)$$

be the sample c.d.f. of the m_i observations in the ith set. We shall omit the subscript m_i whenever this causes no confusion. Define $H_N(x) = \lambda_1 S_{m_1}^{(1)}(x) + \cdots + \lambda_c S_{m_c}^{(c)}(x)$. Thus $H_N(x)$ is the combined sample c.d.f. The combined population c.d.f. is $H(x) = \lambda_1 F^{(1)}(x) + \cdots + \lambda_c F^{(c)}(x)$. Even though $H(x)$ depends on N through the λ's, our notation suppresses this fact for convenience and also because our limit theorems are uniform with respect to $F^{(1)}, \cdots, F^{(c)}$ and $\lambda_1, \cdots, \lambda_c$.

Let $Z_{N,i}^{(j)} = 1$ if the ith smallest of $N = \sum_{i=1}^{c} m_i$ observations is from the jth set and otherwise let $Z_{N,i}^{(j)} = 0$. Denote

$$(4.1) \qquad \tau_{N,j} = m_j \cdot T_{N,j} = \sum_{i=1}^{N} E_{N,i} Z_{N,i}^{(j)}$$

where the $E_{N,i}$ are given numbers. Following Chernoff and Savage [2], we shall use the representation

$$(4.2) \qquad T_{N,j} = \int_{-\infty}^{\infty} J_N[H_N(x)]\, dS_{m_j}^{(j)}(x)$$

where $E_{N,i} = J_N(i/N)$. While J_N need be defined only at $1/N, 2/N, \cdots, N/N$, we shall find it convenient to extend its domain of definition to $(0, 1]$ by letting J_N be constant on $(i/N, (i+1)/N]$.

Let

$$I_N = \{x: 0 < H_N(x) < 1\}.$$

Then I_N is a random interval, given by $I_N = [X^{(1)}, X^{(N)})$, where $X^{(1)} < \cdots < X^{(N)}$ denote the N observations arranged according to size.

Throughout, K will be used as a generic constant which may depend on J_N but will not depend on $F^{(1)}, \cdots, F^{(c)}, m_1, \cdots, m_c$ and N. The methods used in the proof for the asymptotic normality of the $T_{N,j}$'s are mainly adaptations of the methods of Chernoff and Savage [2].

5. Joint asymptotic normality. Before proving the asymptotic normality of the $T_{N,j}$'s we state a few elementary results.

$$H \geqq \lambda_i F^{(i)} \geqq \lambda_0 F^{(i)}; \qquad i = 1, \cdots, c. \tag{5.1}$$

$$1 - F^{(i)} \leqq (1 - H)/\lambda_i \leqq (1 - H)/\lambda_0; \qquad i = 1, \cdots, c. \tag{5.2}$$

$$F^{(i)}(1 - F^{(i)}) \leqq H(1 - H)/\lambda_i^2 \leqq H(1 - H)/\lambda_0^2; \qquad i = 1, \cdots, c. \tag{5.3}$$

$$dH \geqq \lambda_i \, dF^{(i)} \geqq \lambda_0 \, dF^{(i)}; \qquad i = 1, \cdots, c. \tag{5.4}$$

LEMMA 5.1. *If*

(1) $J(H) = \lim_{N\to\infty} J_N(H)$ *exists for* $0 < H < 1$ *and is not constant,*

(2) $\int_{I_N} [J_N(H_N) - J(H_N)] \, dS_{m_j}^{(j)}(x) = o_p(N^{-(\frac{1}{2})})$,

(3) $J_N(1) = o(N^{\frac{1}{2}})$

(4) $|J^{(i)}(H(x))| = |d^i J(H)/dH^i| \leqq K[H(1 - H)]^{-i-(\frac{1}{2})+\delta}$,

for $i = 0, 1, 2$, *and for some* $\delta > 0$, *and almost all* x (a.a.x), *then, for fixed* $F^{(1)}, \cdots, F^{(c)}$ *and* $\lambda_1, \cdots, \lambda_c$,

$$\lim_{N\to\infty} P\left(\frac{T_{N,j} - \mu_{N,j}}{\sigma_{N,j}} \leqq t\right) = \int_{-\infty}^{t} \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-x^2/2} \, dx, \tag{5.5}$$

where

$$\mu_{N,j} = \int_{-\infty}^{+\infty} J[H(x)] \, dF^{(j)}(x) \tag{5.6}$$

and

$$\begin{aligned} N\sigma_{N,j}^2 = {} & 2 \sum_{\substack{i=1 \\ i \neq j}}^{c} \lambda_i \iint_{-\infty<x<y<\infty} F^{(i)}(x)[1 - F^{(i)}(y)] \\ & \qquad \cdot J'[H(x)] J'[H(y)] \, dF^{(j)}(x) \, dF^{(j)}(y) \\ & + \frac{2}{\lambda_j} \sum_{\substack{i=1 \\ i \neq j}}^{c} \lambda_i^2 \iint_{-\infty<x<y<\infty} F^{(j)}(x)[1 - F^{(j)}(y)] \\ & \qquad \cdot J'[H(x)] J'[H(y)] \, dF^{(i)}(x) \, dF^{(i)}(y) \\ & + \frac{1}{\lambda_j} \sum_{\substack{i,k=1 \\ i \neq k, i \neq j, k \neq j}}^{c} \lambda_i \lambda_k \Big[\iint_{-\infty<x<y<\infty} F^{(j)}(x)[1 - F^{(j)}(y)] \\ & \qquad \cdot J'[H(x)] J'[H(y)] \, dF^{(i)}(x) \, dF^{(k)}(y) \\ & + \iint_{-\infty<y<x<\infty} F^{(j)}(y)[1 - F^{(j)}(x)] J'[H(x)] J'[H(y)] \, dF^{(i)}(x) \, dF^{(k)}(y). \end{aligned} \tag{5.7}$$

PROOF.

$$T_{N,j} = \int_{x=-\infty}^{x=+\infty} J_N[H_N(x)]\, dS_{m_j}^{(j)}(x)$$

$$= \int_{\{x:0<H_N(x)<1\}} [J_N[H_N(x)] - J[H_N(x)]]\, dS_{m_j}^{(j)}(x)$$

$$+ \int_{\{x:0<H_N(x)<1\}} J[H_N(x)]\, dS_{m_j}^{(j)}(x) + \int_{\{x:H_N(x)=1\}} J_N[H_N(x)]\, dS_{m_j}^{(j)}(x).$$

In the second integral, writing $dS_{m_j}^{(j)}(x) = d(S_{m_j}^{(j)}(x) - F^{(j)}(x) + F^{(j)}(x))$,

$$J[H_N(x)] = J[H(x)] + [H_N(x) - H(x)]J'[H(x)]$$
$$+ \tfrac{1}{2}[H_N(x) - H(x)]^2 J''[\theta H_N(x) + (1-\theta)H(x)], \text{ a.a.}x.,$$

where $0 < \theta < 1$; and $H(x) = \sum_{i=1}^{c} \lambda_i F^{(i)}(x)$, and simplifying, we obtain

$$T_{N,j} = A + B_{1N} + B_{2N} + \sum_{i=1}^{c+4} C_{iN}$$

where

(5.8) $$A = \int_{\{x:0<H(x)<1\}} J[H(x)]\, dF^{(j)}(x)$$

(5.9) $$B_{1N} = \int_{\{x:0<H(x)<1\}} J[H(x)]\, d[S_{m_j}^{(j)}(x) - F^{(j)}(x)]$$

(5.10) $$B_{2N} = \int_{\{x:0<H(x)<1\}} [H_N(x) - H(x)]J'[H(x)]\, dF^{(j)}(x)$$

(5.11) $$C_{i,N} = \lambda_i \int_{\{x:0<H(x)<1\}} [S_{m_i}^{(i)}(x) - F^{(i)}(x)]J'[H(x)]\, d[S_{m_j}^{(j)}(x) - F^{(j)}(x)]$$
$$i = 1, \cdots, c.$$

(5.12) $$C_{c+1,N} = \int_{I_N} \frac{[H_N(x) - H(x)]^2}{2} J''[\theta H_N + (1-\theta)H]\, dS_{m_j}^{(j)}(x).$$

(5.13) $$C_{c+2,N} = \int_{I_N} [J_N[H_N(x)] - J[H_N(x)]]\, dS_{m_j}^{(j)}(x).$$

(5.14) $$C_{c+3,N} = \int_{H_N=1} J_N[H_N(x)]\, dS_{m_j}^{(j)}(x).$$

(5.15) $$C_{c+4,N} = \int_{H_N=1} [-J[H(x)] - \{H_N(x) - H(x)\}J'[H(x)]]\, dS_{m_j}^{(j)}(x).$$

The proof of the lemma is accomplished by showing that (i) the A-term is nonrandom and finite, (ii) $B_{1N} + B_{2N}$ has a Gaussian distribution in the limit and (iii) the C terms are of higher order.

That the term

$$A = \int_{\{x:0<H(x)<1\}} J[H(x)]\, dF^{(j)}(x)$$

is finite and nonrandom follows from Assumption 4 of Lemma 5.1; see also in this connection [2], p. 986, and in the appendix we have shown that the C terms are of higher order. Thus, all that is required is to prove

SUB-LEMMA 5.1. *$B_{1N} + B_{2N}$ has a Gaussian distribution in the limit.*

PROOF. Integrating B_{2N} by parts, replacing $H_N(x) - H(x)$ by $\sum_{i=1}^{c} \lambda_i [S_{m_i}^{(i)}(x) - F^{(i)}(x)]$, and adding B_{1N} to it, we obtain

$$(5.16)\qquad \begin{aligned} B_{1N} + B_{2N} = &-\sum_{\substack{i=1\\ i\neq j}}^{c} \lambda_i \int_{x=-\infty}^{x=+\infty} B(x)\, d[S_{m_i}^{(i)}(x) - F^{(i)}(x)] \\ &+ \int_{x=-\infty}^{x=+\infty} [J[H(x)] - \lambda_j B(x)]\, d[S_{m_j}^{(j)}(x) - F^{(j)}(x)], \end{aligned}$$

$$(5.17)\qquad \begin{aligned} = &-\sum_{\substack{i=1\\ i\neq j}}^{c} \left[\lambda_i \cdot \frac{1}{m_i} \sum_{k=1}^{m_i} \{B(X_{ik}) - EB(X_i)\}\right] \\ &+ \frac{1}{m_j} \sum_{k=1}^{m_j} \{J[H(X_{jk})] - \lambda_j B(X_{jk}) - E[J[H(X_j)] - \lambda_j B(X_j)]\} \end{aligned}$$

where

$$(5.18)\qquad B(x) = \int_{x_0}^{x} J'[H(y)]\, dF^{(j)}(y)$$

with x_0 determined somewhat arbitrarily, say by $H(x_0) = \frac{1}{2}$; E represents the expectation and $X_1, \cdots, X_c$ have the $F^{(1)}, \cdots, F^{(c)}$ distributions respectively.

The c summations given by (5.17) involve independent samples of identically distributed random variables. Therefore, if we show that the first two moments of these random variables exist, then we can apply the central limit theorem, with the result that each sum when properly normalized will have a normal distribution in the limit and hence the sum of c summations will have a normal distribution in the limit.

First, to turn our attention to moments, note that by Assumption 4 of Lemma 5.1 and $dF^{(j)} \leqq (1/\lambda_0)\, dH$,

$$|B(x)| \leqq K \cdot [H(x)[1 - H(x)]]^{-(\frac{1}{2})+\delta}$$

and proceeding as in [2], for any δ' such that $(2 + \delta')(-\frac{1}{2} + \delta) > -1$

$$E_{F^{(i)}}\{|B(X)|\}^{2+\delta'} \leqq K; \qquad i = 1, \cdots, j-1, j+1, \cdots, c.$$

Since

$$|J(H(x)) - \lambda_j B(x)| \leqq K[H(x)(1 - H(x))]^{-(\frac{1}{2})+\delta}$$

the existence of $2 + \delta'$ absolute moments of all the terms in equation (5.17) follows.

To compute the variance of $B_{1N} + B_{2N}$, note that

$$-\lambda_i \int_{-\infty}^{+\infty} B(x)\, d[S_{m_i}^{(i)}(x) - F^{(i)}(x)]$$

$$= \lambda_i \int_{-\infty}^{+\infty} [S_{m_i}^{(i)}(x) - F^{(i)}(x)] J'[H(x)]\, dF^{(j)}(x), \quad i = 1, \cdots, j-1, j+1, \cdots, c,$$

has mean zero and variance

$$\begin{aligned}
E&\left\{\lambda_i \int_{-\infty}^{+\infty} [S_{m_i}^{(i)}(x) - F^{(i)}(x)] J'[H(x)]\, dF^{(j)}(x)\right\}^2 \\
&= E\left\{\lambda_i^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [S_{m_i}^{(i)}(x) - F^{(i)}(x)][S_{m_i}^{(i)}(y) - F^{(i)}(y)] \right. \\
&\qquad\qquad \left. \cdot J'[H(x)] J'[H(y)]\, dF^{(j)}(x)\, dF^{(j)}(y)\right\} \\
&= \frac{2\lambda_i}{N} \iint_{-\infty<x<y<\infty} F^{(i)}(x)[1 - F^{(i)}(y)] J'[H(x)] J'[H(y)]\, dF^{(j)}(x)\, dF^{(j)}(y), \\
&\qquad\qquad i = 1, \cdots, j-1, j+1, \cdots, c.
\end{aligned} \tag{5.19}$$

Note that the application of Fubini's theorem permits the interchange of integral and expectation.

By a similar argument, the variance of

$$\int_{-\infty}^{+\infty} [J(H(x)) - \lambda_j B(x)]\, d[S_{m_j}^{(j)}(x) - F^{(j)}(x)]$$

$$= -\sum_{\substack{i=1\\ i\neq j}}^{c} \lambda_i \int_{-\infty}^{+\infty} [S_{m_j}^{(j)}(x) - F^{(j)}(x)] J'(H(x))\, dF^{(i)}(x)$$

is given by

$$\begin{aligned}
&\frac{2}{N\lambda_j} \sum_{\substack{i=1\\ i\neq j}}^{c} \lambda_i^2 \iint_{-\infty<x<y<\infty} F^{(j)}(x)[1 - F^{(j)}(y)] J'[H(x)] J'[H(y)] \\
&\qquad\qquad \cdot dF^{(i)}(x)\, dF^{(i)}(y) \\
&+ \frac{1}{N\lambda_j} \sum_{\substack{i,k=1\\ i\neq k, i\neq j, k\neq j}}^{c} \lambda_i \lambda_k \iint_{-\infty<x<y<\infty} F^{(j)}(x)[1 - F^{(j)}(y)] J'[H(x)] J'[H(y)] \\
&\qquad\qquad \cdot dF^{(i)}(x)\, dF^{(k)}(y) \\
&+ \frac{1}{N\lambda_j} \sum_{\substack{i,k=1\\ i\neq k, i\neq j, k\neq j}}^{c} \lambda_i \lambda_k \iint_{-\infty<y<x<\infty} F^{(j)}(y)[1 - F^{(j)}(x)] J'[H(x)] J'[H(y)] \\
&\qquad\qquad \cdot dF^{(i)}(x)\, dF^{(k)}(y).
\end{aligned} \tag{5.20}$$

Adding the c terms given by (5.19) and (5.20) we obtain the variance result stated in (5.7).

Thus we have shown that $B_{1N} + B_{2N}$ is the sum of c independent terms, each of which has mean zero and finite absolute $2 + \delta'$ moments. Hence Sub-Lemma 5.1 follows.

We shall now extend the proof of the above lemma to the case where $F^{(1)}, \cdots, F^{(c)}$ and $\lambda_1, \cdots, \lambda_c$ are not fixed. We want to find a set of sufficient conditions under which the asymptotic normality holds uniformly with respect to $F^{(1)}, \cdots, F^{(c)}$ and $\lambda_1, \cdots, \lambda_c$. For this we need the following theorem of Esseen [6], p. 43.

THEOREM (*Esseen*) 5.1. *Let $X_1, \cdots, X_n$ be independent observations from a population with mean zero, variance σ^2 and finite absolute $2 + \delta'$ moments $\beta_{2+\delta'}$, $0 < \delta' \leqq 1$, then*

$$|F^* - \Phi^*| < c(\delta')[\rho_{2+\delta'}/n^{\delta'/2} + \rho_{2+\delta'}^{1/\delta'}/n^{\frac{1}{2}}]$$

where F^* is the c.d.f. of $\bar{X}$, Φ^* is the approximating normal c.d.f., $c(\delta')$ is a finite positive constant only depending on δ' and $\rho_{2+\delta'} = \beta_{2+\delta'}/\sigma^{2+\delta'}$. (If $\delta' = 1$, then $|F^* - \Phi^*| < c(\delta')\rho_3/n^{\frac{1}{2}}$).

To apply this theorem in our situation, it suffices, since we have shown that the A term is finite and the C terms are uniformly $o_p(N^{-(\frac{1}{2})})$, to prove the uniform convergence of $B_{1N} + B_{2N}$. For this it suffices to bound $\rho_{2+\delta'} = \beta_{2+\delta'}/\sigma^{2+\delta'}$ for $B(X_1), \cdots, B(X_c)$. Since in the above lemma we already bounded the absolute $2 + \delta'$ moments, all that is required is to bound the variances of $B(X_1), \cdots, B(X_c)$ away from zero. Thus we have

COROLLARY 5.1. *If Conditions 1 to 4 of Lemma 5.1 are satisfied, and $F^{(i)}$ and λ_i, $i = 1, \cdots, c$ (where $0 < \lambda_0 \leqq \lambda_1, \cdots, \lambda_c \leqq 1 - \lambda_0 < 1$ holds for some fixed $\lambda_0 \leqq 1/c$) are restricted to a set for which the variances of $B(X_1), \cdots, B(X_c)$ are bounded away from zero, then the asymptotic normality holds uniformly with respect to $F^{(1)}, \cdots, F^{(c)}$ and $\lambda_1, \cdots, \lambda_c$.*

Next we prove

LEMMA 5.2. *Under the assumptions of Lemma 5.1, the random vector $N^{\frac{1}{2}}(T_{N,1} - \mu_{N,1}; \cdots; T_{N,c} - \mu_{N,c})$ has a limiting normal distribution.*

PROOF. The difference $N^{\frac{1}{2}}(T_{N,j} - \mu_{N,j}) - N^{\frac{1}{2}}(B_{1N}^{(j)} + B_{2N}^{(j)})$, where $B_{1N}^{(j)} + B_{2N}^{(j)}$ is the "$B_{1N} + B_{2N}$" term for the jth component $T_{N,j} - \mu_{N,j}$, tends to zero in probability and so, by a well known theorem ([3], p. 299), the vectors $N^{\frac{1}{2}}(T_{N,1} - \mu_{N,1}; \cdots; T_{N,c} - \mu_{N,c})$ and $N^{\frac{1}{2}}(B_{1N}^{(1)} + B_{2N}^{(1)}; \cdots; B_{1N}^{(c)} + B_{2N}^{(c)})$ possess the same limiting distributions. Now since the jth component $B_{1N}^{(j)} + B_{2N}^{(j)}$ can be expressed as $\sum_{i=1}^{c} \{(1/m_i) \sum_{\alpha=1}^{m_i} B_{ij}^*(X_{i\alpha})\}$, the proof of the lemma follows by applying the Central Limit Theorem to each of the c independent vectors

$$(1/m_i) \sum_{\alpha=1}^{m_i} [B_{i1}^*(X_{i\alpha}),\ B_{i2}^*(X_{i\alpha}),\ \cdots,\ B_{ic}^*(X_{i\alpha})];\ i = 1, \cdots, c.$$

6. The Covariance of two B-Statistics. By definition

$$\begin{aligned}\text{Cov}(B_{1N}^{(j)} + B_{2N}^{(j)}, B_{1N}^{(j')} + B_{2N}^{(j')}) &= E(B_{1N}^{(j)} + B_{2N}^{(j)})(B_{1N}^{(j')} + B_{2N}^{(j')}) \\ &= E(B_{1N}^{(j)} B_{2N}^{(j')}) + E(B_{2N}^{(j)} B_{1N}^{(j')}) + E(B_{2N}^{(j)} B_{2N}^{(j')})\end{aligned} \tag{6.1}$$

where

$$B_{1N}^{(j')} = \iint_{\{x:0<H(x)<1\}} J[H(x)]\, d[S_{m_{j'}}^{(j')}(x) - F^{(j')}(x)] \tag{6.2}$$

$$B_{2N}^{(j')} = \int_{\{y:0<H(y)<1\}} [H_N(y) - H(y)]J'[H(y)]\, dF^{(j')}(y) \tag{6.3}$$

and $B_{1N}^{(j)}$ and $B_{2N}^{(j)}$ are given by (5.9) and (5.10) respectively.

Now integrating $B_{1N}^{(j)}$ by parts and using the facts that

$$\int_{-\infty}^{+\infty} d[S_{m_j}^{(j)}(x) - F^{(j)}(x)] = 0$$

$$dH(x) = \sum_{i=1}^{c} \lambda_i\, dF^{(i)}(x)$$

and

$$H_N(y) - H(y) = \sum_{i=1}^{c} \lambda_r [S_{m_r}^{(r)}(y) - F^{(r)}(y)]$$

routine computations yield, for $j \neq j'$,

$$B_{1N}^{(j)} B_{2N}^{(j')} = -\sum_{i=1}^{c} \sum_{r=1}^{c} \lambda_i \lambda_r \int_{x=-\infty}^{x=+\infty} \int_{y=-\infty}^{y=+\infty} [S_{m_j}^{(j)}(x) - F^{(j)}(x)][S_{m_r}^{(r)}(y) - F^{(r)}(y)] \cdot J'[H(x)]J'[H(y)]\, dF^{(i)}(x)\, dF^{(j')}(y).$$

Therefore,

$$\begin{aligned} E(B_{1N}^{(j)} B_{2N}^{(j')}) = &-\frac{1}{N} \sum_{i=1}^{c} \lambda_i \iint_{-\infty<x<y<\infty} F^{(j)}(x)[1 - F^{(j)}(y)]J'[H(x)] \\ &\cdot J'[H(y)]\, dF^{(i)}(x)\, dF^{(j')}(y) \\ &- \frac{1}{N} \sum_{i=1}^{c} \lambda_i \iint_{-\infty<y<x<\infty} F^{(j)}(y)[1 - F^{(j)}(x)]J'[H(x)]J'[H(y)]\, dF^{(i)}(x)\, dF^{(j')}(y). \end{aligned} \tag{6.4}$$

Proceeding analogously

$$\begin{aligned} E(B_{2N}^{(j)} B_{1N}^{(j')}) = &-\frac{1}{N} \sum_{i=1}^{c} \lambda_i \iint_{-\infty<x<y<\infty} F^{(j')}(x)[1 - F^{(j')}(y)] \cdot J'[H(x)] \\ &\cdot J'[H(y)]\, dF^{(i)}(x)\, dF^{(j)}(y) \\ &- \frac{1}{N} \sum_{i=1}^{c} \lambda_i \iint_{-\infty<y<x<\infty} F^{(j')}(y)[1 - F^{(j')}(x)] \cdot J'[H(x)] \\ &\cdot J'[H(y)]\, dF^{(i)}(x)\, dF^{(j)}(y) \end{aligned} \tag{6.5}$$

and

$$E(B_{2N}^{(j)}B_{2N}^{(j')}) = \frac{1}{N}\sum_{i=1}^{c}\lambda_i \iint_{-\infty<x<y<\infty} F^{(i)}(x)[1 - F^{(i)}(y)]\cdot J'[H(x)] \cdot J'[H(y)]\, dF^{(j)}(x)\, dF^{(j')}(y) \tag{6.6}$$
$$+ \frac{1}{N}\sum_{i=1}^{c}\lambda_i \iint_{-\infty<y<x<\infty} F^{(i)}(y)[1 - F^{(i)}(x)]\cdot J'[H(x)]\cdot J'[H(y)] \cdot dF^{(j)}(x)\, dF^{(j')}(y).$$

Thus

$$N\cdot \operatorname{Cov}(B_{1N}^{(j)} + B_{2N}^{(j)}, B_{1N}^{(j')} + B_{2N}^{(j')})$$
$$= -\sum_{i=1}^{c}\lambda_i \Big[\iint_{-\infty<x<y<\infty} F^{(j)}(x)[1 - F^{(j)}(y)]\cdot J'[H(x)]\cdot J'[H(y)] \cdot dF^{(i)}(x)\, dF^{(j')}(y)$$
$$+ \iint_{-\infty<y<x<\infty} F^{(j)}(y)[1 - F^{(j)}(x)]\cdot J'[H(x)]\cdot J'[H(y) \cdot dF^{(i)}(x)\, dF^{(j')}(y)\Big]$$
$$- \sum_{i=1}^{c}\lambda_i \Big[\iint_{-\infty<x<y<\infty} F^{(j')}(x)[1 - F^{(j')}(y)]\cdot J'[H(x)]\cdot J'[H(y)] \cdot dF^{(i)}(x)\, dF^{(j)}(y) \tag{6.7}$$
$$+ \iint_{-\infty<y<x<\infty} F^{(j')}(y)[1 - F^{(j')}(x)]\cdot J'[H(x)]\cdot J'[H(y)] \cdot dF^{(i)}(x)\, dF^{(j)}(y)\Big]$$
$$+ \sum_{i=1}^{c}\lambda_i \Big[\iint_{-\infty<x<y<\infty} F^{(i)}(x)[1 - F^{(i)}(y)]\cdot J'[H(x)]\cdot J'[H(y)] \cdot dF^{(j)}(x)\, dF^{(j')}(y)$$
$$+ \iint_{-\infty<y<x<\infty} F^{(i)}(y)[1 - F^{(i)}(x)]\cdot J'[H(x)]\cdot J'[H(y)] \cdot dF^{(j)}(x)\, dF^{(j')}(y)\Big], \qquad j \neq j'.$$

Combining the material of the previous two sections produces

THEOREM 6.1. *Under the assumptions of Lemma* 5.1, *the random vector* $T = (N^{\frac{1}{2}}(T_{N,1} - \mu_{N,1}), \cdots, N^{\frac{1}{2}}(T_{N,c} - \mu_{N,c}))$ *has a limiting normal distribution with zero mean vector and variance-covariances given by limiting forms of* (5.7) *and* (6.7) *respectively as* $N \to \infty$.

REMARK. The following theorem gives a simple sufficient condition under which Conditions 1, 2, and 3 of Lemma 5.1 hold.

THEOREM 6.2. *If $J_N(i/N)$ is the expectation of the* ith *order statistic of a sample of size N from a population whose cumulative distribution function is the inverse function of J and $|J^{(i)}[H(x)]| \leqq K[H(1-H)]^{-i-(\frac{1}{2})+\delta}$ for $i = 0, 1, 2$; for some $\delta > 0$ and* a.a. *x, then*

(i) $\lim_{N\to\infty} J_N(H) = J(H)$.

(ii) $J_N(1) = o(N^{\frac{1}{2}})$.

(iii) $\int_{I_N}[J_N(H_N) - J(H_N)]\, dS_{m_j}^{(j)}(x) = o(N^{-(\frac{1}{2})})$; $j = 1, \cdots, c$.

REMARK 1. The condition $|J^{(i)}[H(x)]| \leqq K[H(1-H)]^{-i-(\frac{1}{2})+\delta}$ a.a. x is weaker than the condition $|J^{(i)}(H)| \leqq K[H(1-H)]^{-i-(\frac{1}{2})+\delta}$ used by Chernoff and Savage [2], otherwise Theorem 6.2 is the generalization of the latter's Theorem 2.

REMARK 2. With the use of this theorem, it is easy to verify that if $J_N(i/N)$ is the expected value of the ith order statistic of a sample of size N from (i) the standard normal distribution, (ii) the logistic distribution, (iii) the double exponential distribution, (iv) the exponential distribution, then the vector $(T_{N,1}; \cdots; T_{N,c})$ has a limiting normal distribution.

7. The limiting distribution of $\mathcal{L}$ under Pitman's shift alternatives. From this section onward, we assume that $m_1, \cdots, m_c$ are nondecreasing functions of a natural number n that tends to infinity. The dependence on n is indicated when necessary, by writing $m_i(n)$, $\mu_{N,i}(n)$, etc. For convenience, it is assumed that, for all i,

$$\lim_{n\to\infty} m_i(n)/n = s_i$$

exists, and there exist two constants a and b such that $0 < a < s_i < b < \infty$.

In subsequent analysis, we shall concern ourselves with a sequence of admissible alternative hypothesis H_n^P which specifies that for each $i = 1, \cdots, c$; $F^{(i)}(x) = F(x + \theta_i/n^{\frac{1}{2}})$ with $F \varepsilon \Omega$ but not specified further, and for some pair (i, j), $\theta_i \neq \theta_j$. The letter n is used to index a sequence of situations in which H_n^P is the true hypothesis. Limiting probability distribution of $\mathcal{L}$ will then be found as $n \to \infty$.

We first prove the following

THEOREM 7.1. *If*

(1) *for all i,*

$$\lim_{n\to\infty} m_i(n)/n = s_i$$

exists,

(2) *Conditions* (1) *to* (4) *of Lemma* 5.1 *are satisfied,*

(3) *$F^{(j)}(x) = F(x + \theta_j/n^{\frac{1}{2}})$ so that for each index n, the hypothesis H_n^P is valid, then the random vector $[m_1^{\frac{1}{2}}(T_{N,1} - \mu_{N,1}), \cdots, m_c^{\frac{1}{2}}(T_{N,c} - \mu_{N,c})]$ has a limiting normal distribution with zero means and covariance matrix whose (j, j')*th *term is*

$$\left[\delta_{jj'} - (s_j s_{j'})^{\frac{1}{2}} \Big/ \sum_{i=1}^{c} s_i\right] A^2 \tag{7.1}$$

where

$$A^2 = \int_0^1 J^2(x)\,dx - \left(\int_0^1 J(x)\,dx\right)^2 \tag{7.2}$$

and the limit holds uniformly in s_i provided $0 < a < s_i < b < \infty$; $i = 1, \cdots, c$.

PROOF. From Equation (5.7)

$$\begin{aligned} \lim_{n\to\infty} N\cdot\sigma_{N,j}^2 = &\left[\sum_{\substack{i=1\\ i\neq j}}^{c} s_i + \frac{1}{s_j}\sum_{\substack{i=1\\ i\neq j}}^{c} s_i^2 + \frac{1}{2s_j}\sum_{\substack{i,k=1\\ i\neq k, i\neq j, k\neq j}}^{c} s_i\,s_k\right] I_1 \Big/ \sum_{i=1}^{c} s_r \\ &+ \frac{1}{2s_j}\left(\sum_{\substack{i,k=1\\ i\neq k, i\neq j, k\neq j}}^{c} s_i\,s_k\right) I_2 \Big/ \sum_{i=1}^{c} s_r \end{aligned} \tag{7.3}$$

where

$$I_1 = 2\iint_{0<x<y<1} x(1-y)J'(x)J'(y)\,dx\,dy, \tag{7.4}$$

$$= \int_0^1 J^2(x)\,dx - \left(\int_0^1 J(x)\,dx\right)^2 \tag{7.5}$$

and

$$I_2 = 2\iint_{0<y<x<1} y(1-x)J'(x)J'(y)\,dx\,dy, \tag{7.6}$$

$$= \int_0^1 J^2(x)\,dx - \left(\int_0^1 J(x)\,dx\right)^2. \tag{7.7}$$

Thus, omitting the routine algebra,

$$\lim_{n\to\infty} N\cdot\sigma_{N,j}^2 = \left(-1 + \sum_{i=1}^{c} s_i \Big/ s_j\right) A^2.$$

Similarly, from equation (6.7),

$$\lim_{n\to\infty} N\,\mathrm{Cov}(T_{N,j} - \mu_{N,j}, T_{N,j'} - \mu_{N,j'}) = -A^2.$$

Hence using Theorem 6.1, we obtain the desired result.

Denoting $m_j^{\frac{1}{2}}(T_{N,j} - \mu_{N,j})/A$ by W_j, it now follows that the random vector $W = (W_1, \cdots, W_c)$ has a limiting normal distribution with zero mean vector and with covariance matrix whose (j, j')th term is

$$\left[\delta_{jj'} - (s_j\,s_{j'})^{\frac{1}{2}} \Big/ \sum_{i=1}^{c} s_i\right].$$

We now make the analysis of variance transformation

$$S_0 = \sum_{i'=1}^{c} e_{i'}^{\frac{1}{2}} W_{i'} \text{ where } e_{i'} = s_{i'}/\sum_{i=1}^{c} s_i$$

$$S_i = \sum_{i'=1}^{c} a_{ii'} W_{i'}\,;\ i = 1, 2, \cdots, c-1$$

where the a's are chosen to make the transformation orthogonal. It follows that $\sum_{i=1}^{c} W_i^2$ is asymptotically chi-square with $c - 1$ degrees of freedom.

Now recalling that

$$W_i = m_i^{\frac{1}{2}}[T_{N,i} - \mu_{N,i}(\theta)]/A$$

and letting

$$r_i = m_i^{\frac{1}{2}}[\mu_{N,i}(\theta) - \mu_{N,i}(0)]/A$$

we write $\mathcal{L}$ as $\mathcal{L} = \sum_{i=1}^{c} (W_i + r_i)^2$ and this has the same limiting distribution as $\mathcal{L}^* = \sum_{i=1}^{c} (W_i + r_i^*)^2$ where $r_i^* = \lim_{n\to\infty} r_i$ reduces to

$$r_i^* = \lim_{n\to\infty} m_i^{\frac{1}{2}}\left[\int_{-\infty}^{+\infty}\left[J\left\{\sum_{\alpha=1}^{c} \lambda_\alpha F\left(x + \frac{\theta_\alpha - \theta_i}{n^{\frac{1}{2}}}\right)\right\} - J\{F(x)\}\right] dF(x)\right]\Big/ A.$$

We assume that the above limit exists and is finite. Noting that $\sum_{i=1}^{c} s_i^{\frac{1}{2}} W_i = 0$ and $\sum_{i=1}^{c} s_i^{\frac{1}{2}} r_i^* = 0$, it follows from a theorem of Mann and Wald [20] that

THEOREM 7.3. *Suppose that for all i, $\lim_{n\to\infty} m_i/n = s_i$ exists and is positive. Then under the hypothesis H_n^P, if for any real numbers $t_1, \cdots, t_c$,*

$$\lim_{n\to\infty} m_i^{\frac{1}{2}}\left[\int_{-\infty}^{+\infty}\left[J\left\{\sum_{i=1}^{c} \lambda_i F\left(x + \frac{t_i}{n^{\frac{1}{2}}}\right)\right\} - J\{F(x)\}\right] dF(x)\right]\Big/ A$$

exists and is finite, then for $n \to \infty$, the limiting distribution of the statistic $\mathcal{L}$ is $X_{c-1}^2(\lambda^L(H_n^P))$ where $\lambda^L(H_n^P)$ is the noncentrality parameter given by

$$\lambda^{\mathcal{L}}(H_n^P) = \sum_{j=1}^{c}\left[\lim_{n\to\infty} m_j^{\frac{1}{2}} \int_{-\infty}^{+\infty}\left[J\left\{\sum_{\alpha=1}^{c} \lambda_\alpha F\left(x + \frac{\theta_\alpha - \theta_j}{n^{\frac{1}{2}}}\right)\right\} - J\{F(x)\}\right] dF(x)\right]^2 \Big/ A^2. \tag{7.8}$$

REMARK. If the function J is such that $J(u) = u$, then from (7.8), letting $m_j = n\cdot s_j$, we obtain for $\lambda^{\mathcal{L}}(H_n^P)$ the expression

$$\left[12 \Big/ \left(\sum_{i=1}^{c} s_i\right)^2\right] \sum_{j=1}^{c} s_j \left(\sum_{\alpha=1}^{c} s_\alpha \lim_{n\to\infty} \int_{x=-\infty}^{x=+\infty} n^{\frac{1}{2}} \cdot \left\{F\left(x + \frac{\theta_\alpha - \theta_j}{n^{\frac{1}{2}}}\right) - F(x)\right\} dF(x)\right)^2$$

which is the noncentrality parameter $\lambda^H(H_n^P)$ of the Kruskal-Wallis H test. (See Andrews [1], p. 726.)

In many situations, the noncentrality parameter $\lambda^{\mathcal{L}}$ can be computed easily with the aid of the following lemma which, though stated in a form appropriate to our purpose, is due to Hodges and Lehmann [11].

LEMMA 7.2 (*Hodges-Lehmann*). *If*

(i) *F is a continuous cumulative distribution function, differentiable in each of the open intervals $(-\infty, a_1)$, (a_1, a_2), $\cdots$, (a_{s-1}, a_s), (a_s, ∞) and the derivative of F is bounded in each of these intervals and*

(ii) *the function* $(d/dx)J[F(x)]$ *is bounded as* $x \to \pm\infty$ *then*

$$\begin{aligned}(7.9)\quad &\lim_{n\to\infty} n^{\frac{1}{2}} \int_{-\infty}^{+\infty} \left[J\left\{ \sum_{\alpha=1}^{c} \lambda_\alpha F\left(x + \frac{\theta_\alpha - \theta_j}{n^{\frac{1}{2}}}\right)\right\} - J\{F(x)\} \right] dF(x) \\ &= \left(1 \Big/ \sum_{i=1}^{c} s_i\right) \sum_{\alpha=1}^{c} s_\alpha(\theta_\alpha - \theta_j) \int_{-\infty}^{+\infty} \frac{d}{dx} J\{F(x)\}\, dF(x).\end{aligned}$$

The proof of this lemma follows by the methods used in Section 3 and 4 of Hodges-Lehmann (1961).

In case the conditions of Lemma 7.2 are satisfied, then

$$(7.10)\qquad \lambda^{\mathfrak{L}}(H_n^P) = \sum_{\alpha=1}^{c} s_\alpha(\theta_\alpha - \bar{\theta})^2 \left(\int_{-\infty}^{+\infty} \frac{d}{dx} J[F(x)] f(x)\, dx\right)^2 \Big/ A^2$$

where

$$(7.11)\qquad \bar{\theta} = \sum_{\alpha=1}^{c} s_\alpha \theta_\alpha \Big/ \sum_{\alpha=1}^{c} s_\alpha$$

and A^2 is defined in (7.2).

8. Asymptotic relative efficiency. The concept of asymptotic relative efficiency of one test with respect to another is due to Pitman. An exposition of his work, together with some extensions is presented by Noether [23a].

THEOREM 8.1. *If* $m_i = n \cdot s_i$ *and if the distribution function* F *is such that*

$$(1)\qquad \lim_{n\to\infty} n^{\frac{1}{2}} \int_{-\infty}^{+\infty} \left[F\left(x + \frac{\theta}{n^{\frac{1}{2}}}\right) - F(x)\right] dF(x)$$

exists

$$\begin{aligned}(2)\quad &\lim_{n\to\infty} n^{\frac{1}{2}} \int_{-\infty}^{\infty} \left[J\left\{ \left(1 \Big/ \sum_{i=1}^{c} s_i\right) \sum_{\alpha=1}^{c} s_\alpha F\left(x + \frac{\theta_\alpha - \theta_j}{n^{\frac{1}{2}}}\right)\right\} \right. \\ &\qquad \left. - J[F(x)] \right] dF(x) \Big/ A\end{aligned}$$

exists then the asymptotic relative efficiency of the H *test with respect to an arbitrary* $\mathfrak{L}$ *test for testing the hypothesis* H_0 *against* H_n^P *is given by*

$$(8.1)\quad e_{H,\mathfrak{L}}^{P}(F(x)) = \frac{12 \sum_{\alpha=1}^{c} s_\alpha \left\{ \sum_{i=1}^{c} s_i \lim_{n\to\infty} \int_{-\infty}^{+\infty} n^{\frac{1}{2}} \cdot \left[F\left(x + \frac{\theta_i - \theta_\alpha}{n^{\frac{1}{2}}}\right) - F(x)\right] dF(x)\right\}^2 A^2}{\left(\sum_{j=1}^{c} s_j\right)^2 \sum_{\alpha=1}^{c} s_\alpha \left(\lim_{n\to\infty} \int_{-\infty}^{+\infty} n^{\frac{1}{2}} \left[J\left\{ \left(1 \Big/ \sum_{j=1}^{c} s_j\right) \sum_{i=1}^{c} \cdot s_i F\left(x + \frac{\theta_i - \theta_\alpha}{n^{\frac{1}{2}}}\right)\right\} - J\{F(x)\} \right] dF(x)\right)^2}.$$

The proof of the above theorem follows by taking the ratio of the two noncentrality factors after the alternatives have been equated. The details are omitted since similar considerations have been given in several other papers, e.g., Andrews [1], Hannan [9].

COROLLARY 8.1. *If in addition to the hypotheses of Theorem* 8.1, *the hypotheses of Lemma* 7.2 *are satisfied, then*

$$(8.2)\quad e^{P}_{H,\mathfrak{L}}(F(x)) = 12A^2\left(\int_{-\infty}^{+\infty} f^2(x)\,dx \Big/ \int_{-\infty}^{+\infty} \frac{d}{dx}\{J[F(x)]\}f(x)\,dx\right)^2$$

where f is the density of F.

Here $e^{P}_{H,\mathfrak{L}}$ does not depend upon c, α, β, and is a function of F only.

It may be remarked that (8.2) agrees with the results found by Chernoff-Savage [2] and Hodges-Lehmann [11] for the two-sample case, and hence the results of this paper as well as those of [2] apply directly to the c-sample problem.

The asymptotic relative efficiency of the classical $\mathfrak{F}$ test with respect to an arbitrary $\mathfrak{L}$ test is contained in the following

THEOREM 8.2. *If*

(i) *for all i, $\lim_{n\to\infty} m_i(n)/n = s_i$ exists and is positive,*

(ii) *the distribution function $F(x)$ satisfies the assumptions of Lemma* 7.2, *and*

$$\text{(iii)}\qquad \int_{-\infty}^{+\infty} x^2\,dF(x) - \left(\int_{-\infty}^{+\infty} x\,dF(x)\right)^2 = \sigma^2$$

exists, then, the asymptotic relative efficiency of the classical $\mathfrak{F}$ test with respect to an arbitrary $\mathfrak{L}$ test for testing the hypothesis H_0 against H_n^P is

$$(8.3)\qquad e^{P}_{\mathfrak{F},\mathfrak{L}}(F(x)) = \frac{A^2}{\sigma^2}\left(1 \Big/ \int_{-\infty}^{+\infty} \frac{d}{dx} J[F(x)]\,dF(x)\right)^2.$$

PROOF. The $\mathfrak{F}$ statistic is defined as

$$\mathfrak{F} = \frac{1}{c-1}\sum_{i=1}^{c} m_i(X_{i\cdot} - \bar{X})^2 \Big/ \frac{1}{N-c}\sum_{i=1}^{c}\sum_{j=1}^{m_i}(X_{ij} - X_i)^2$$

where $X_{i\cdot} = \sum_{j=1}^{m_i} X_{ij}/m_i$ and $\bar{X} = \sum_{i=1}^{c}\sum_{j=1}^{m_i} X_{ij}/N$. It has been shown by Andrews [1] that under the hypothesis H_n^P, this has a limiting noncentral chi-square distribution with $c - 1$ degrees of freedom and noncentrality parameter $\lambda^{\mathfrak{F}}(H_n^P)$ given by

$$(8.5)\qquad \lambda^{\mathfrak{F}}(H_n^P) = \sum_{i=1}^{c} s_i[(\theta_i - \bar{\theta})/\sigma]^2.$$

Now proceeding by standard arguments, the proof follows.

In particular, when $J = \Phi^{-1}$, where Φ is the standard cumulative normal distribution function having the density ϕ,

$$(8.6)\qquad e^{P}_{\mathfrak{L},\mathfrak{F}}(F(x)) = \sigma^2\left(\int_{-\infty}^{+\infty} \frac{f^2(x)\,dx}{\phi\{\Phi^{-1}[F(x)]\}}\right)^2$$

which is known to be the asymptotic efficiency of the two sample normal scores test with respect to the student's t-test and is always $\geqq 1$. When $F(x)$ is a normal distribution function, this is 1. See in this connection Chernoff-Savage [2] and Hodges-Lehmann [11].

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APPENDIX

10. Higher order terms. Before we prove that the C terms of Lemma 5.1 are uniformly of higher order, we state the following elementary results which are used repeatedly. (Also in this connection see Chernoff and Savage [2].)

10A. *Elementary results.*

1. $H \geqq \lambda_i F^{(i)} \geqq \lambda_0 F^{(i)}$; $\quad i = 1, \cdots, c.$
2. $1 - F^{(i)} \leqq (1 - H)/\lambda_i \leqq (1 - H)/\lambda_0$. $\quad i = 1, \cdots, c.$
3. $F^{(i)}(1 - F^{(i)}) \leqq H(1 - H)/\lambda_i^2 \leqq H(1 - H)/\lambda_0^2$; $\quad i = 1, \cdots, c.$
4. $dH \geqq \lambda_i\, dF^{(i)} \geqq \lambda_0\, dF^{(i)}$; $\quad i = 1, \cdots, c.$
5. Let (a_N, b_N) be the interval $S_{N\epsilon}$ where

$$S_{N\epsilon} = \{x: H(1 - H) > \eta_\epsilon \lambda_0/N\}, \tag{10.1}$$

when η_ϵ can be chosen independent of $F^{(i)}$ and λ_i; $i = 1, \cdots, c$, such that

$$P\{X_{ij} \varepsilon S_{N\epsilon}; i = 1, \cdots, c; j = 1, \cdots, m_i\} \geqq 1 - \epsilon. \tag{10.2}$$

10B. *Detailed consideration of the C-terms of Lemma* 5.1. First, let us consider

$$\begin{aligned} C_{iN} &= \lambda_i \int_{0<H<1} [S_{m_i}^{(i)}(x) - F^{(i)}(x)]J'[H(x)]\, d[S_{m_j}^{(j)}(x) - F^{(j)}(x)]; \\ &\qquad\qquad i = 1, \cdots, j-1, j+1, \cdots, c \\ &= \lambda_i[C_{1N}^{(i)} + C_{2N}^{(i)}]; \qquad i = 1, \cdots, c; i \neq j, \end{aligned} \tag{10.3}$$

where

$$C_{1N}^{(i)} = \int_{S_{N\epsilon}} [S_{m_i}^{(i)}(x) - F^{(i)}(x)]J'[H(x)]\, d[S_{m_j}^{(j)}(x) - F^{(j)}(x)]; \qquad i = 1, \cdots, c; i \neq j, \tag{10.4}$$

and

$$C_{2N}^{(i)} = \int_{\bar{S}_{N\epsilon}} [S_{m_i}^{(i)}(x) - F^{(i)}(x)]J'[H(x)]\, d[S_{m_j}^{(j)}(x) - F^{(j)}(x)]; \qquad i = 1, \cdots, c; i \neq j. \tag{10.5}$$

First note that

$$E(C_{1N}^{(i)}) = E\{E(C_{1N}^{(i)} \mid X_{j1}, \cdots, X_{jm_j})\} = 0; i = 1, \cdots, c; i \neq j. \tag{10.6}$$

Next,

$$[C_{1N}^{(i)}]^2 = 2 \iint_{x,y \varepsilon S_{N_\epsilon}, x<y} [S_{m_i}^{(i)}(x) - F^{(i)}(x)][S_{m_i}^{(i)}(y) - F^{(i)}(y)]J'[H(x)]J'[H(y)] \cdot d[S_{m_j}^{(j)}(x) - F^{(j)}(x)]\, d[S_{m_j}^{(j)}(y) - F^{(j)}(y)]$$

$$+ \frac{1}{m_j} \int_{x \varepsilon S_{N_\epsilon}} [S_{m_i}^{(i)}(x) - F^{(i)}(x)]^2 [J'[H(x)]]^2\, dS_{m_j}^{(j)}(x); \qquad i = 1, \cdots, c; i \neq j.$$

Therefore,

$$(10.7) \quad \begin{aligned} E(C_{1N}^{(i)})^2 &= E[E\{(C_{1N}^{(i)})^2 \mid X_{j1}, \cdots, X_{jm_j}\}] \\ &= -\frac{2}{m_i m_j} \iint_{x,y \varepsilon S_{N_\epsilon}, x<y} F^{(i)}(x)[1 - F^{(i)}(y)] \cdot J'[H(x)]J'[H(y)]\, dF^{(j)}(x)\, dF^{(j)}(y) \\ &\quad + \frac{1}{m_i m_j} \int_{x \varepsilon S_{N_\epsilon}} F^{(i)}(x)[1 - F^{(i)}(x)]\{J'[H(x)]\}^2\, dF^{(j)}(x); \qquad i = 1, \cdots, c; i \neq j \\ &\leqq \frac{K}{N^2} \iint_{x,y \varepsilon S_{N_\epsilon}, x<y} H(x)[1 - H(y)][H(x)(1 - H(x))]^{-(\frac{3}{2})+\delta} \cdot [H(y)(1 - H(y))]^{-(\frac{3}{2})+\delta}\, dH(x)\, dH(y) \\ &\quad + \frac{K}{N^2} \int_{x \varepsilon S_{N_\epsilon}} H(x)[1 - H(x)][H(x)(1 - H(x)]^{(-3+2\delta)}\, dH(x) \\ &\leqq \frac{K}{N^2} + \frac{K\eta_\epsilon^{-1+2\delta}}{N^{1+2\delta}} = o\left(\frac{1}{N}\right); \qquad [K \text{ is generic}]. \end{aligned}$$

Hence from (10.6) and (10.7), we obtain, using Markoff inequality,

$$(10.8) \qquad |C_{1N}^{(i)}| = o_p(N^{-(\frac{1}{2})}).$$

We now consider $C_{2N}^{(i)}$. Let $H_1 = H(a_N)$, $H_2 = H(b_N)$. Then from (10.1) $H_1 = 1 - H_2 < K/N$. With probability greater than $1 - \epsilon$, there are no observations in $\bar{S}_{N_\epsilon}$ and

$$(10.9) \quad \begin{aligned} |C_{2N}^{(i)}| &\leqq \int_0^{H_1} F^{(i)}(x)|J'[H(x)]|\, dF^{(j)}(x) \\ &\quad + \int_{H_2}^1 (1 - F^{(i)}(x)|J'[(x)]|\, dF^{(j)}(x); \qquad i = 1, \cdots, c; i \neq j \\ &\leqq K \int_0^{H_1} \frac{H\, dH}{[H(1 - H)]^{(\frac{3}{2})-\delta}} + \int_{H_2}^1 \frac{(1 - H)\, dH}{[H(1 - H)]^{(\frac{3}{2})-\delta}} \\ &\leqq K \int_0^{H_1} H^{-(\frac{1}{2})+\delta}\, dH \leqq K \frac{1}{N^{(\frac{1}{2})+\delta}}. \end{aligned}$$

Hence

$$|C_{2N}^{(i)}| = o_p(N^{-(\frac{1}{2})}); \qquad i = 1, \cdots, c; i \neq j. \tag{10.10}$$

Consequently,

$$C_{iN} = \lambda_i[C_{1N}^{(i)} + C_{2N}^{(i)}] = o_p(N^{-(\frac{1}{2})}); \quad i = 1, \cdots, c; i \neq j. \tag{10.11}$$

The proof of $C_{jN} = o_p(N^{-(\frac{1}{2})})$ follows by first showing that

$$C_{jN} = -\tfrac{1}{2}\lambda_j[C_{11N} + C_{12N} - C_{13N}]$$

where

(a) $$C_{11N} = \int_{S_{N_\epsilon}} [S_{m_j}^{(j)}(x) - F^{(j)}(x)]^2 J''[H(x)]\, dH(x),$$

(b) $$C_{12N} = \int_{S_{N_\epsilon}} [S_{m_j}^{(j)}(x) - F^{(j)}(x)]^2 J''[H(x)]\, dH(x),$$

(c) $$C_{13N} = \frac{1}{m_j} \int J'[H(x)]\, dS_{m_j}^{(j)}(x)$$

and then showing that each C_{1KN} is $o_p(N^{-(\frac{1}{2})})$; $k = 1, 2, 3$. The proofs of the above statement are omitted since they are essentially contained in the work of Chernoff and Savage [2].

Next consider

$$C_{c+1,N} = \int_{I_N} [H_N(x) - H(x)]^2 J''[\theta H_N(x) + (1 - \theta)H(x)]\, dS_{m_j}^{(j)}(x), \qquad 0 < \theta < 1.$$

With probability $>1 - \epsilon$, the interval I_N can be replaced by S_{N_ϵ} without changing $C_{c+1,N}$. Furthermore since

$$\mathrm{Sup}_{H_N>0} |H(x)/H_N(x)| = O_p(1)$$

and

$$\mathrm{Sup}_{H_N<1} |[1 - H(x)]/[1 - H_N(x)]| = O_p(1),$$

for each $\epsilon > 0$, there exists an $\eta_\epsilon^* > 0$ such that with probability greater than $1 - \epsilon$, we have for $\{x: 0 < H_N(x) < 1\}$,

$$[\theta H_N(x) + (1 - \theta)H(x)][1 - \{\theta H_N(x) + (1 - \theta)H(x)\}] > \eta_\epsilon^* H(x)[1 - H(x)].$$

Then

$$|C_{c+1,N}| \leqq K(\eta_\epsilon^*)^{-(\frac{3}{2})+\delta} C_{aN}$$

where

$$C_{aN} = \int_{S_{N_\epsilon}} [H_N(x) - H(x)]^2 \{H(x)[1 - H(x)]\}^{-(\frac{3}{2})+\delta}\, dS_{m_j}^{(j)}(x)$$

and

$$
\begin{aligned}
E(C_{aN}) &= E[E(C_{aN} \mid X_{j1}, \cdots, X_{jm_j})] \\
&= \frac{1}{N} \int_{S_{N\epsilon}} \sum_{i=1}^{c} \lambda_i F^{(i)}(1 - F^{(i)})[H(1 - H)]^{-(\frac{1}{2})+\delta} \, dF^{(j)}(x) \\
&\qquad + \frac{1}{N^2} \int_{S_{N\epsilon}} (1 - F^{(j)})(1 - 2F^{(j)})[H(1 - H)]^{-(\frac{1}{2})+\delta} \, dF^{(j)}(x) \\
&\leqq \frac{K}{N} \int_{S_{N\epsilon}} [H(1 - H)]^{-(\frac{1}{2})+\delta} \, dH + \frac{K}{N^2} \int_{S_{N\epsilon}} [H(1 - H)]^{-(\frac{1}{2})+\delta} \, dH \\
&\leqq \frac{K}{N^{(\frac{1}{2})+\delta}}.
\end{aligned}
$$

Consequently $C_{c+1,N} = o_p(N^{-(\frac{1}{2})})$.

The negligibility of $C_{c+2,N}$ and $C_{c+3,N}$ follows from Assumptions 2 and 3 of Lemma 5.1 and the proof of the negligibility of $C_{c+4,N}$ proceeds in the same manner as given by Chernoff and Savage for the term C_{4N} and therefore is not given here.

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ON THE ESTIMATION OF CONTRASTS IN LINEAR MODELS

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1. Summary. In linear models with several observations per cell, a class of estimates of all contrasts are defined in terms of rank test statistics such as the Wilcoxon or normal scores statistic, which extend the results of Hodges and Lehmann (1963) and Lehmann (1963). The asymptotic efficiency of these estimates relative to the standard least squares estimates, as the number of observations in each cell gets large, is shown to be the same as the Pitman efficiency of the rank tests on which they are based to the corresponding t-tests.

2. A Class of Estimates of Contrasts. Let the observable random variables be $X_{i\alpha}$, and suppose they are of the form

$$X_{i\alpha} = \xi_i + U_{i\alpha} \qquad (\alpha = 1, \cdots, m_i;\, i = 1, \cdots, c) \tag{2.1}$$

where the variables $U_{i\alpha}$ are independently distributed with common distribution F having density f, and the ξ's are unknown constants. Denote by X_i the vector $(X_{i1}, \cdots, X_{im_i})$ and suppose that the Hodges-Lehmann statistic h [(3.1) of [4]] is calculated for every pair of samples, there being $c(c-1)/2$ pairs in all. We shall write $h_{ij}(X_i, X_j)$ for the value obtained from the ith and jth samples $(i, j = 1, \cdots, c;\, i \neq j)$. Thus we have

$$h_{ij}(X_i, X_j) = \sum_{K=1}^{m_j} E_\Psi[V^{(S_K)}], \tag{2.2}$$

where $S_1 < \cdots < S_{m_j}$ denote the ranks of $X_{j1}, \cdots, X_{jm_j}$ in the combined ith and jth samples, and where $V^{(1)} < \cdots < V^{(m_i+m_j)}$ denote an ordered sample of size $(m_i + m_j)$ from a distribution Ψ. Let

$$\begin{aligned} \Delta_{ij}^* &= \sup\{\Delta_{ij} : h_{ij}(X_i, X_j - \Delta_{ij}) > \mu\}, \\ \Delta_{ij}^{**} &= \inf\{\Delta_{ij} : h_{ij}(X_i, X_j - \Delta_{ij}) < \mu\}, \end{aligned} \tag{2.3}$$

where μ is the point of symmetry of the distribution of $h_{ij}(X_i, X_j)$ when $\Delta_{ij} = 0$ i.e. when $\xi_i = \xi_j$. It was shown in [4] that the estimate $\hat{\Delta}_{ij} = (\Delta_{ij}^* + \Delta_{ij}^{**})/2$ of $\xi_i - \xi_j$ has more robust efficiency than the classical estimate $T_{ij} = X_{i\cdot} - X_{j\cdot}$, where $X_{i\cdot} = \sum_{\alpha=1}^{m_i} X_{i\alpha}/m_i$.

Since the estimates $\hat{\Delta}_{ij}$ are incompatible in the sense that they do not satisfy the linear relations satisfied by the differences they estimate [see Lehmann [5], [6]], Lehmann proposed the adjusted estimates Z_{ij} of the type

$$Z_{ij} = \hat{\Delta}_{i\cdot} - \hat{\Delta}_{j\cdot}. \tag{2.4}$$

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where

$$\hat{\Delta}_{i\cdot} = \sum_{j=1}^{c} \hat{\Delta}_{ij}/c. \tag{2.5}$$

(For a short cut method of computing $\hat{\Delta}_{ij}$, the reader is referred to [4], p. 602.)

Then for any contrast $\sum c_i\xi_i$ with $\sum c_i = 0$, which can also be written in the form

$$\theta = \sum_{i=1}^{c} \sum_{j=1}^{c} d_{ij}(\xi_i - \xi_j) \tag{2.6}$$

the estimate

$$\hat{\theta} = \sum_{i=1}^{c} \sum_{j=1}^{c} d_{ij}Z_{ij} = \sum_{i=1}^{c} \sum_{j=1}^{c} d_{ij}(\hat{\Delta}_{i\cdot} - \hat{\Delta}_{j\cdot}) \tag{2.7}$$

is proposed.

3. Asymptotic distribution and efficiency. The asymptotic distribution of the adjusted estimates Z_{ij} is given by the following theorem, where the sample sizes m_i are assumed to tend to infinity in such a way that $m_i = \rho_i \cdot N$, $N \to \infty$ and $i = 1, \cdots, c$.

THEOREM 3.1.

(i) *The joint distribution of* $(V_1, \cdots, V_{c-1})$ *where*

$$V_i = N^{\frac{1}{2}}[\hat{\Delta}_{ic} - (\xi_i - \xi_c)] \tag{3.1}$$

is asymptotically normal with zero mean and covariance matrix

$$\begin{aligned} \operatorname{Var}(V_i) &= (1/\rho_i + 1/\rho_c)A^2/B^2, \\ \operatorname{Cov}(V_i, V_j) &= A^2/(\rho_c \cdot B^2) \end{aligned} \tag{3.2}$$

where

$$A^2 = \textstyle\int_0^1 J^2(x)\,dx - (\int_0^1 J(x)\,dx)^2, \qquad J = \Psi^{-1} \tag{3.3}$$

$$B = \textstyle\int J'[F(x)]f^2(x)\,dx. \tag{3.4}$$

Here the density f of F is assumed to satisfy the regularity conditions of Lemma 7.2 *of* [8].

(ii) *For any i and j,*

$$N^{\frac{1}{2}}\hat{\Delta}_{ij} \sim N^{\frac{1}{2}}(\hat{\Delta}_{ic} - \hat{\Delta}_{jc}), \tag{3.5}$$

where $\sim$ *indicates that the difference of the two sides tend to zero in probability.*

(iii) *The difference* $N^{\frac{1}{2}}(Z_{ij} - \hat{\Delta}_{ij})$ *tends to zero in probability for all i, j.*

The proof of (i) rests on the following lemma.

LEMMA 1. *Suppose that the variables* $X_{i\alpha}$ *have the distribution specified in connection with* (2.1) *with fixed F but a sequence of means*

$$(\xi_1, \cdots, \xi_c) = (\xi_1^{(N)}, \cdots, \xi_c^{(N)})$$

satisfying

$$\xi_i^{(N)} - \xi_c^{(N)} = -a_i/N^{\frac{1}{2}}. \tag{3.6}$$

Let $h_{ij}(X_i, X_j)$ *be defined as in* (2.2) *with* Ψ *satisfying the assumptions of Theorem* 1 *of* [1], *then the variables* $(W_1, \cdots, W_{c-1})$ *given by*

$$W_i = N^{\frac{1}{2}}[h_{ic}/m_c - \mu_{ic}] \qquad i = 1, \cdots, c-1 \tag{3.7}$$

have a joint asymptotic normal distribution as $N \to \infty$, *with zero mean and covariance matrix*

$$\begin{aligned} \operatorname{Var}(W_i) &= A^2\rho_i/(\rho_i + \rho_c)\rho_c, \\ \operatorname{Cov}(W_i, W_j) &= A^2\rho_i\rho_j/\rho_c(\rho_i + \rho_c)(\rho_j + \rho_c) \end{aligned} \tag{3.8}$$

and

$$\mu_{ic} = \int J\left[\frac{m_c}{m_i + m_c}F(x) + \frac{m_i}{m_i + m_c}F(x + a_i/N^{\frac{1}{2}})\right]dF(x).$$

The proof of this lemma is given in the appendix.

Proof of Theorem 3.1. (i) By 9.1 of [4],

$$\begin{aligned} &\lim P\{N^{\frac{1}{2}}[\hat{\Delta}_{ic} - (\xi_i - \xi_c)] \leqq a_i \text{ for all } i\} \\ &\qquad = \lim P_N\{N^{\frac{1}{2}}[(1/m_c)h_{ic} - \alpha] \leqq 0 \text{ for all } i\} \end{aligned}$$

where $\alpha = \int J[F(x)]\,dF(x)$ and P_N indicates that the probability is computed for a sequence of means satisfying (3.6). Furthermore since by Lemma 7.2 of [8] $N^{\frac{1}{2}}(\mu_{ic} - \alpha) \to -a_iB\rho_i/(\rho_i + \rho_c)$ as $N \to \infty$, it follows that

$$\begin{aligned} &\lim P\{N^{\frac{1}{2}}[\hat{\Delta}_{ic} - (\xi_i - \xi_c)] \leqq a_i \text{ for all } i\} \\ &\qquad = \lim P_N\{N^{\frac{1}{2}}[(1/m_c)h_{ic} - \mu_{ic}] \leqq a_iB\rho_i/(\rho_i + \rho_c) \text{ for all } i\}. \end{aligned}$$

By Lemma 1, this is equal to $Q(a_1, \cdots, a_{c-1})$ where Q is the $(c-1)$ dimensional multivariate normal distribution with zero mean and covariance matrix (3.2).

Parts (ii) and (iii) of the theorem follow by Lemma 2 of Lehmann (1963).

The proof of the following theorem exactly parallels Lehmann's argument, see for example Theorem 3 of [6], and is therefore omitted.

Theorem 3.2. *The asymptotic efficiency of the estimate* $\hat{\theta} = \sum_{i=1}^{c}\sum_{j=1}^{c} d_{ij}Z_{ij}$ *of* $\theta = \sum_{i=1}^{c}\sum_{j=1}^{c} d_{ij}(\xi_i - \xi_j)$ *relative to the estimate* $\sum_{i=1}^{c}\sum_{j=1}^{c} d_{ij}(X_{i\cdot} - X_{j\cdot})$ *is*

$$e = \sigma^2B^2/A^2, \tag{3.9}$$

where $\sigma^2 = \operatorname{Var}(X_{i\alpha})$, *and where* A^2 *and* B^2 *are given by* (3.3) *and* (3.4) *respectively.*

In particular when $J = \Phi^{-1}$, *where* Φ *is the standard normal cumulative distribution function having the density* ϕ *then* (3.9) *is the same as the Pitman efficiency of the normal scores test relative to the t-test* [cf. 1].

4. Appendix.

Proof of Lemma 1. Let $F_{m_i}(x)$ be the cdf (cumulative distribution function) of m_i observations $X_{i1}, \cdots, X_{im_i}$ of which the population cdf is $F_i(x) = F(x - \xi_i)$. Denote $m_{ic} = m_i + m_c$ and $\lambda_{ic} = m_c/m_{ic}$; $i = 1, \cdots, c-1$. Define $H_{m_{ic}}(x) = \lambda_{ic}F_{m_c}(x) + (1 - \lambda_{ic})F_{m_i}(x)$ and $H_{ic}(x) = \lambda_{ic}F_c(x) + (1 - \lambda_{ic})F_i(x)$. Then [cf. Chernoff-Savage (1958)] we can write

$$(4.1) \qquad T_{ic} = h_{ic}/m_c = A^{(ic)} + B_{1N}^{(ic)} + B_{2N}^{(ic)} + \textstyle\sum_{K=1}^{6} C_{KN}^{(ic)},$$

where

$$(4.2) \qquad A^{(ic)} = \int J[H_{ic}(x)]\, dF_c(x),$$

$$(4.3) \qquad B_{1N}^{(ic)} = \int J[H_{ic}(x)]\, d[F_{m_c}(x) - F_c(x)],$$

$$(4.4) \qquad B_{2N}^{(ic)} = \int [H_{m_{ic}}(x) - H_{ic}(x)]J'[H_{ic}(x)]\, dF_c(x)$$

and the C-terms are all $o_pN^{-\frac{1}{2}}$.

The difference $N^{\frac{1}{2}}(T_{ic} - A^{(ic)}) - N^{\frac{1}{2}}(B_{1N}^{(ic)} + B_{2N}^{(ic)})$ tends to zero in probability and so, by a well-known theorem ([2], p. 299) the vectors $(W_1, \cdots, W_{c-1})$ and $(Z_i, \cdots, Z_{c-1})$ where $Z_i = N^{\frac{1}{2}}(B_{1N}^{(ic)} + B_{2N}^{(ic)})$ possess the same limiting distribution. Thus to prove the lemma it suffices to show that for any real δ_i ; $i = 1, \cdots, c-1$, not all zero, $\sum_{i=1}^{c-1} \delta_i Z_i$ has normal distribution in the limit. Now proceeding as in [1] or [8], we find

$$(4.5) \qquad \begin{aligned} \textstyle\sum_{i=1}^{c-1} \delta_i Z_i = &-\textstyle\sum_{i=1}^{c-1} [\delta_i \cdot ((1 - \lambda_{ic})/m_i) \sum_{\alpha=1}^{m_i} \{B_{ic}^*(X_{i\alpha}) - EB_{ic}^*(X_i)\}] \\ &+ \textstyle\sum_{i=1}^{c-1} \delta_i(1 - \lambda_{ic})\{m_c^{-1} \sum_{\alpha=1}^{m_c} B_{ic}(X_{c\alpha}) - EB_{ic}(X_c)\}, \end{aligned}$$

where

$$(4.6) \qquad B_{ic}(x) = \int_{x_{i0}}^{x} J'[H_{ic}(y)]\, dF_i(y),$$

$$(4.7) \qquad B_{ic}^*(x) = \int_{x_{i0}}^{x} J'[H_{ic}(y)]\, dF_c(y)$$

and x_{i0} is such that $H_{ic}(x_0) = \frac{1}{2}$.

The above summations involve independent samples of identically distributed random variables having finite first two moments. Hence $\sum_{i=1}^{c-1} \delta_i Z_i$ when properly normalized has normal distribution in the limit. The proof follows.

The covariance matrix (3.8) is obtained by taking limits of N Var $(B_{1N}^{(ic)} + B_{2N}^{(ic)})$ and N Cov $(B_{1N}^{(ic)} + B_{2N}^{(ic)}, B_{1N}^{(jc)} + B_{2N}^{(jc)})$ as $N \to \infty$.

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SOME DISTRIBUTION-FREE k-SAMPLE RANK TESTS OF HOMOGENEITY AGAINST ORDERED ALTERNATIVES*

MADAN L. PURI

1. Introduction and Summary

A problem which occurs frequently in statistical analysis is that of deciding whether several samples should be regarded as coming from the same population. This problem, usually referred to as the k-sample problem, when expressed formally is stated as follows: Let X_{ij}, $j = 1, \cdots, m_i$, $i = 1, \cdots, k$, be a set of independent random variables and let $F_i(x)$ be the probability distribution function of X_{ij}. The set of admissible hypotheses designates that each F_i belongs to some class of distribution functions Ω. The hypothesis to be tested, say H_0, specifies that F_i is an element of Ω, for each i, and that furthermore

$$F_1(x) = \cdots = F_k(x) \text{ for all real } x\,. \tag{1.1}$$

The class of alternatives to H_0 is considered to consist of all sets $(F_1(x), \cdots, F_k(x))$ which belong to Ω but which violate (1.1). This is the most general form of the alternative and is the basis of most of the existing work in the non-parametric theory. Reference to prior work on this problem and some of the recent work may be found in Dwass [7], Kruskal–Wallis [12], Mood [15], Terpestra [20], and the author [16].

However, in some problems, it is possible to be more precise in the specification of the alternative. When this is the case, it is advantageous to make use of this extra information to obtain more powerful tests. Thus instead of the unrestricted form of the alternative mentioned above, we shall consider in this paper the ordered alternatives

$$F_1(x) \geqq \cdots \geqq F_k(x) \tag{1.2}$$

(at least one inequality being strong).

For the case $k = 2$, the situation is met by using the single-tail test but for $k > 2$ the distinction between one- and two-tail tests is lost. The present work may therefore be regarded as generalizations of some of the single-tail non-parametric tests.

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This problem has many applications in social sciences. Jonckheere [11], for example, has mentioned an experiment to test the effect of stress on the task of manual dexterity. Here data would be obtained from groups of subjects working under high, medium, low, and minimal stress; the null hypothesis being that stress has no effect on performance, and the alternative that increasing stress produces an increasing effect. Armitage [1] discussed a similar problem in connection with $2 \times k$ contingency tables and found the applications in the medical field.

A few tests of parametric nature have been developed for this problem by Bartholomew [2], Chacko [5], Kudo [13], among others (see Bartholomew [3] for references). In non-parametric theory, attempts to meet the need for a test against ordered alternatives have only been made very recently. Jonckheere [11] discussed the one-way analysis of variance and proposed a distribution-free test which may be considered the most direct predecessor of the tests presented in this paper. Chacko [5] proposed another test similar to the one proposed by Kruskal and Wallis [12] for the unrestricted alternatives and studied its asymptotic Pitman efficiency against translation alternatives. In the present paper, we propose and develop a family V of rank tests for the equality of k probability distributions against the ordered alternatives. Limiting distributions of the proposed test statistics are derived, following the methods used in Chernoff and Savage [6] and the author [4], [16]. These results are used to derive general formulas for the asymptotic efficiencies of these tests with respect to one another and their parametric competitor, viz. the test based on the Student statistic. In some of the cases where the asymptotic efficiency cannot be used to compare the tests, the asymptotic power comparisons are made in an attempt to select the best test.

2. The Proposed Family of Tests

The over-all sample consists of $N = \sum_{i=1}^{k} m_i$ independent random variables X_{ij}, $i = 1, \cdots, k$, $j = 1, \cdots, m_i$, where the first subscript refers to the sub-sample and the second subscript indexes observations within a sub-sample. Under the null hypothesis, all the X's have the same continuous but unknown c.d.f. (cumulative distribution function) $F(x)$.

Denote by X_i the vector $(X_{i1}, \cdots, X_{im_i})$ and consider all the samples in pairs, there being $k(k-1)/2$ pairs in all. Let $\xi_\nu^{(i,j)} = 1$ if the ν-th smallest observation from the combined i-th and j-th samples is an X_i observation and, otherwise, let $\xi_\nu^{(i,j)} = 0$. Let $\eta_\nu^{(i,j)} = -1$, if the ν-th smallest observation from the combined i-th and j-th samples is an X_j observation and, otherwise, let $\eta_\nu^{(i,j)} = 0$.

Denote

$$h_{ij} = \tau_{ij}^{(i)} + \tau_{ij}^{(j)}, \tag{2.1}$$

where

$$m_i \tau_{ij}^{(i)} = \sum_{\nu=1}^{m_i+m_j} E_\nu^{(i,j)} \xi_\nu^{(i,j)} \tag{2.2}$$

and

$$m_j \tau_{ij}^{(j)} = \sum_{\nu=1}^{m_i+m_j} E_\nu^{(i,j)} \eta_\nu^{(i,j)}, \tag{2.3}$$

where the $\{E_\nu^{(i,j)}, \nu = 1, \cdots, m_i + m_j); i < j\}$ are constants satisfying certain restrictions to be stated below. Then we propose to consider the test statistics of the form

$$V = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} m_i m_j h_{ij} \tag{2.4}$$

for testing the null hypothesis against the alternative of ordered cumulative distribution functions.

Relationship to other tests. The V test presented here is a simple extension to several groups, of a class of procedures, which have been frequently recommended for the problem of deciding whether two samples come from the same population. For example, when $E_\nu^{(i,j)} = \nu/(m_i + m_j)$, the test described above coincides with the Jonckheere test [11] which is a direct generalization of the one-sided Wilcoxon test discussed in detail by Mann and Whitney [14]. When $k = 2$ and $E_\nu^{(i,j)}$ is the expected value of the k-th order statistic of a sample of size $(m_i + m_j)$ from the standard normal distribution function, then the V test is the same as the one-tail normal scores test (which is asymptotically equivalent to the Van der Waerden test) discussed in detail by Hoeffding [10], Terry [19], Chernoff and Savage [6], Hodges and Lehmann [9], and the author [16]. When $k = 2$ and $E_\nu^{(i,j)}$ is the expected value of the ν-th order statistic of a sample of size $(m_i + m_j)$ from the exponential distribution, then the V test reduces to the I. R. Savage test [18].

3. Assumptions and Notations

Let $X_{i1}, \cdots, X_{im_i}$ be the ordered observations of a random sample from a population with continuous c.d.f. $F_i(x)$. Let $N = \sum_{i=1}^{k} m_i$ and suppose that the m_i tend to infinity in such a way that $m_i = \rho_i \cdot N$, $N \to \infty$. Write $m_{ij} = m_i + m_j$. Let $F_{m_i}(x)$ be the sample c.d.f. of m_i observations X_i. Then

$$H_{m_{ij}}(x) = \frac{m_i}{m_{ij}} F_{m_i}(x) + \frac{m_j}{m_{ij}} F_{m_j}(x)$$

is the combined sample c.d.f. of the i-th and j-th samples. The combined population c.d.f. of the i-th and j-th samples is

$$H_{ij}(x) = \frac{m_i}{m_{ij}} F_i(x) + \frac{m_j}{m_{ij}} F_j(x) .$$

Then the following representation of h_{ij} is equivalent to (2.1):

$$h_{ij} = \int_{-\infty}^{+\infty} J_{(m_{ij})}[H_{m_{ij}}(x)]\, d(F_{m_i}(x) - F_{m_j}(x)) , \tag{3.1}$$

where

$$J_{(m_{ij})}[\nu/m_{ij}] = E_{\nu}^{(i,j)}, \qquad \nu = 1, \cdots, m_{ij} , \qquad i < j = 1, \cdots, k .$$

While the function $J_{(m_{ij})}$ need be defined only at $1/m_{ij}, \cdots, m_{ij}/m_{ij}$, we may extend its domain of definitions to $(0, 1]$ by letting it be constant on $(\nu/m_{ij}, (\nu + 1)/m_{ij}]$. Furthermore, we make the following assumptions:[1]

Assumption 1. $\lim_{N\to\infty} J_N(u) = J(u)$ exists for $0 < u < 1$ and is not a constant.

Assumption 2.

$$\int_{I_{m_{ij}}} [J_{(m_{ij})}[H_{m_{ij}}(x)] - J[H_{m_{ij}}(x)]]\, dF_{m_i}(x) = o_p(1/\sqrt{N}) ,^{2}$$

where

$$I_{m_{ij}} = \{x: 0 < H_{m_{ij}}(x) < 1\}, \qquad (i,j) = 1, \cdots, k, \qquad i < j .$$

Assumption 3. $J_N(1) = o(N^{1/2})$.

Assumption 4. $|J^{(i)}(u)| = |d^{(i)}J/du^{(i)}| \leqq K[u(1 - u)]^{\delta-1/2-i}$, $i = 0, 1, 2$, for some K and some $\delta > 0$.

4. Asymptotic Normality

We shall prove the following theorem.

Theorem 4.1. *Under Assumptions* 1–4,

$$\lim_{N\to\infty} P\left[N^{-3/2}\left(\frac{V - \mu_N}{\sigma_N}\right) \leqq t\right] = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx , \tag{4.1}$$

where

$$\mu_N = \sum_{i<j}\sum m_i m_j \left[\int_{-\infty}^{+\infty} J[H_{ij}(x)]\, dF_i(x) - \int_{-\infty}^{+\infty} J[H_{ij}(x)]\, dF_j(x)\right] \tag{4.2}$$

---

[1] These assumptions are analogous to those of Chernoff and Savage [6], to which paper the reader is also referred to for general background.

[2] If $\{X_n\}$ is a sequence of random variables and $\{f_n\}$ a sequence of positive numbers, we write $X_n = o_p(f_n)$ if $X_n/f_n \to 0$ in probability, or equivalently, if for each $\epsilon > 0$ there is a sequence $M_{n_\epsilon} \to 0$ such that $P\{|x_n| > M_{n_\epsilon} f_n\} < 1 - \epsilon$.

and

$$\sigma_N^2 = \sum_{i<j}\sum \rho_i^2\rho_j^2\sigma_{ij}^2 + \sum_{\substack{i<j;\\ (i,j\neq r,s)}}\sum \sum_{r<s}\sum \rho_i\rho_j\rho_r\rho_s\sigma_{ij,rs}\,, \tag{4.3}$$

σ_{ij}^2 *and* $\sigma_{ij,rs}$ *being given by* (4.12) *and* (4.13), *respectively.*

The proof of this theorem rests on the following lemma.

LEMMA 4.1. *Under Assumptions* 1–4, *the matrix with elements* $N^{1/2}(h_{ij} - \mu_{ij})$, *where*

$$\mu_{ij} = \int_{-\infty}^{+\infty} J[H_{ij}(x)]\, dF_i(x) - \int_{-\infty}^{+\infty} J[H_{ij}(x)]\, dF_j(x)\,, \tag{4.4}$$

has a limiting normal distribution with zero mean and covariance matrix given by (4.12) *and* (4.13).

Proof of Lemma 4.1: We can rewrite h_{ij} (cf. [16]) as

$$\begin{aligned} h_{ij} &= \int_{-\infty}^{+\infty} J_{(m_{ij})}[H_{m_{ij}}(x)]\, dF_{m_i}(x) - \int_{-\infty}^{+\infty} J_{(m_{ij})}[H_{m_{ij}}(x)]\, dF_{m_j}(x) \\ &= \mu_{ij} + B_{m_{ij}} + \sum_{K=1}^{6} C_{K,m_{ij}}^{(i)} + \sum_{K=1}^{6} C_{K,m_{ij}}^{(j)}\,, \end{aligned} \tag{4.5}$$

where

$$\mu_{ij} = \int_{-\infty}^{+\infty} J[H_{ij}(x)]\, d(F_i(x) - F_j(x))\,, \tag{4.6}$$

$$\begin{aligned} B_{m_{ij}} &= \int_{-\infty}^{+\infty} J[H_{ij}(x)]\, d(F_{m_i}(x) - F_i(x)) \\ &\quad + \int_{-\infty}^{+\infty} [H_{m_{ij}}(x) - H_{ij}(x)] J'[H_{ij}(x)]\, dF_i(x) \\ &\quad - \int_{-\infty}^{+\infty} J[H_{ij}(x)]\, d(F_{m_j}(x) - F_j(x)) \\ &\quad - \int_{-\infty}^{+\infty} [H_{m_{ij}}(x) - H_{ij}(x)] J'[H_{ij}(x)]\, dF_j(x)\,, \end{aligned} \tag{4.7}$$

and the C-terms are all $o_p(N^{-1/2})$ (cf. [16]). The difference $N^{1/2}(h_{ij} - \mu_{ij}) - N^{1/2}B_{m_{ij}}$ tends to zero in probability and so the matrices with elements $N^{1/2}(h_{ij} - \mu_{ij})$ and $N^{1/2}B_{m_{ij}}$ possess the same limiting distribution if they have one at all. Thus, to prove this lemma, it suffices to show that for any real δ_{ij}, $i < j = 1, \cdots, k$, not all zero, $N^{1/2} \sum_{i<j}\sum \delta_{ij} B_{m_{ij}}$ has the normal distribution in the limit. Now

$B_{m_{ij}}$, after omitting straightforward but tedious computations, can be rewritten as

$$B_{m_{ij}} = \frac{1}{m_i} \sum_{\nu=1}^{m_i} [B_j(X_{i\nu}) - EB_j(X_i)] - \frac{1}{m_j} \sum_{\nu=1}^{m_j} [B_i^*(X_{j\nu}) - EB_i^*(X_j)] , \tag{4.8}$$

where

$$B_j(x) = \int_{x_0}^{x} J'[H_{ij}(y)] \, dF_j(y) \tag{4.9}$$

and

$$B_i^*(x) = \int_{x_0}^{x} J'[H_{ij}(y)] \, dF_i(y) \tag{4.10}$$

with x_0 determined somewhat arbitrarily, say by $H_{ij}(x_0) = \frac{1}{2}$; E represents the expectation and X_i has the F_i distribution. The rest of the proof follows by standard arguments, see for example Bhuchongkul and Puri [4].

To compute the variance-covariance matrix of $B_{m_{ij}}$, we note from (4.8) that $B_{m_{ij}}$ can be rewritten as

$$\begin{aligned} B_{m_{ij}} &= \int_{-\infty}^{+\infty} B_j(x) \, d[F_{m_i}(x) - F_i(x)] - \int_{-\infty}^{+\infty} B_i^*(x) \, d[F_{m_j}(x) - F_j(x)] \\ &= -\int_{-\infty}^{+\infty} [F_{m_i}(x) - F_i(x)] J'[H_{ij}(x)] \, dF_j(x) \\ &\quad + \int_{-\infty}^{+\infty} [F_{m_j}(x) - F_j(x)] J'[H_{ij}(x)] \, dF_i(x) . \end{aligned} \tag{4.11}$$

Since the two samples are independent and $EB_{m_{ij}} = 0$, we have

$$\sigma_{ij}^2 = \operatorname{Var}(B_{m_{ij}}) = E\left\{\int_{-\infty}^{+\infty} [F_{m_i}(x) - F_i(x)] J'[H_{ij}(x)] \, dF_j(x)\right\}^2 + E\left\{\int_{-\infty}^{+\infty} [F_{m_j}(x) - F_j(x)] J'[H_{ij}(x)] \, dF_i(x)\right\}^2 .$$

This gives (after omitting the routine computations)

$$\begin{aligned} \sigma_{ij}^2 &= \frac{2}{m_i} \iint_{-\infty < x < y < \infty} F_i(x)[1 - F_i(y)] J'[H_{ij}(x)] J'[H_{ij}(y)] \, dF_j(x) \, dF_j(y) \\ &\quad + \frac{2}{m_j} \iint_{-\infty < x < y < \infty} F_j(x)[1 - F_j(y)] J'[H_{ij}(x)] J'[H_{ij}(y)] \, dF_i(x) \, dF_i(y) . \end{aligned} \tag{4.12}$$

Note that the application of Fubini's theorem permits the interchange of integral and expectation.

Similarly,

$$\sigma_{ij,rs} = \operatorname{Cov}(B_{m_{ij}}, B_{m_{rs}})$$

$$(4.13)\quad \begin{cases} = 0 \quad \text{if } i, j, r, s \text{ are distinct,} \\[4pt] = \dfrac{1}{m_i}\Bigg[\displaystyle\iint\limits_{-\infty<x<y<\infty} F_i(x)[1 - F_i(y)]J'[H_{ij}(x)]J'[H_{is}(y)]\, dF_j(x)\, dF_s(y) \\ \quad + \displaystyle\iint\limits_{-\infty<y<x<\infty} F_i(y)[1 - F_i(x)]J'[H_{ij}(x)]J'[H_{is}(y)]\, dF_j(x)\, dF_s(y)\Bigg] \\ \hfill \text{if } i = r,\ j \neq s, \\[4pt] = \dfrac{1}{m_j}\Bigg[\displaystyle\iint\limits_{-\infty<x<y<\infty} F_j(x)[1 - F_j(y)]J'[H_{ij}(x)]J'[H_{rj}(y)]\, dF_i(x)\, dF_r(y) \\ \quad + \displaystyle\iint\limits_{-\infty<y<x<\infty} F_j(y)[1 - F_j(x)]J'[H_{ij}(x)]J'[H_{rj}(y)]\, dF_i(x)\, dF_r(y)\Bigg] \\ \hfill \text{if } i \neq r,\ j = s, \\[4pt] = -\dfrac{1}{m_i}\Bigg[\displaystyle\iint\limits_{-\infty<x<y<\infty} F_i(x)[1 - F_i(y)]J'[H_{ij}(x)]J'[H_{ri}(y)]\, dF_j(x)\, dF_r(y) \\ \quad + \displaystyle\iint\limits_{-\infty<y<x<\infty} F_i(y)[1 - F_i(x)]J'[H_{ij}(x)]J'[H_{ri}(y)]\, dF_j(x)\, dF_r(y)\Bigg] \\ \hfill \text{if } i = s,\ j \neq r, \\[4pt] = -\dfrac{1}{m_j}\Bigg[\displaystyle\iint\limits_{-\infty<x<y<\infty} F_j(x)[1 - F_j(y)]J'[H_{ij}(x)]J'[H_{js}(y)]\, dF_i(x)\, dF_s(y) \\ \quad + \displaystyle\iint\limits_{-\infty<y<x<\infty} F_j(y)[1 - F_j(x)]J'[H_{ij}(x)]J'[H_{js}(y)]\, dF_j(x)\, dF_s(y)\Bigg] \\ \hfill \text{if } i \neq s,\ j = r. \end{cases}$$

We have now proved that the set of random variables $\{N^{1/2}(h_{ij} - \mu_{ij}),\ i < j\}$ is asymptotically normally distributed. Consequently, the matrix with elements $\{N^{-3/2}U_{ij},\ i < j\}$, where $U_{ij} = m_i m_j(h_{ij} - \mu_{ij})$, has a limiting normal distribution. The theorem follows.

5. Asymptotic Distribution Under Translation Alternatives and Efficiency

In this section, we shall concern ourselves with a sequence of admissible alternative hypothesis H_N^P which specifies that, for each $i = 1, \cdots, k$, $F_i(x) = F(x + \theta_i/\sqrt{N})$, with $F \in \Omega$ but not specified further, and not all the θ's being equal.

THEOREM 5.1. *For each index N, assume that $m_i = \rho_i \cdot N$, with ρ_i a positive integer and that the hypothesis H_N^P is true. Let h_{ij} be defined as in* (3.1) *with the function J satisfying the assumptions of Lemma* 7.2 *of* [16]. *Then the matrix with elements $\{N^{-3/2}U_{ij}, i < j\}$, where $U_{ij} = m_i m_j(h_{ij} - \mu_{ij})$, has a joint asymptotic normal distribution with zero mean and convariance matrix*

$$\mathrm{Var}\,(N^{-3/2}U_{ij}) = \rho_i\rho_j(\rho_i + \rho_j)A^2,$$

$$(5.1)\qquad \mathrm{Cov}\,(N^{-3/2}U_{ij}, N^{-3/2}U_{rs}) \begin{cases} = 0 & \text{if } i, j, r, s \text{ are distinct,} \\ = \rho_i\rho_j\rho_s A^2 & \text{if } i = r,\ j \neq s, \\ = \rho_i\rho_j\rho_r A^2 & \text{if } i \neq r,\ j = s, \\ = -\rho_i\rho_j\rho_r A^2 & \text{if } i = s,\ j \neq r, \\ = -\rho_i\rho_j\rho_s A^2 & \text{if } i \neq s,\ j = r, \end{cases}$$

where

$$(5.2)\qquad A^2 = \int_0^1 J^2(x)\,dx - \left(\int_0^1 J(x)\,dx\right)^2.$$

This theorem is an immediate consequence of Theorem 4.1 and the fact that under the assumptions of Lemma 7.2 of [16],

$$\lim_{N\to\infty} N\sigma_{ij,rs} \begin{cases} = 0 & \text{if } i, j, r, s \text{ are distinct,} \\ = \dfrac{A^2}{\rho_i} & \text{if } i = r,\ j \neq s, \\ = \dfrac{A^2}{\rho_i} & \text{if } i \neq r,\ j = s, \\ = -\dfrac{A^2}{\rho_i} & \text{if } i = s,\ j \neq r, \\ = -\dfrac{A^2}{\rho_j} & \text{if } i \neq s,\ j = r, \end{cases}$$

and

$$\lim_{N\to\infty} N\sigma_{ij}^2 = (1/\rho_i + 1/\rho_j)A^2.$$

Furthermore, since under the regularity assumptions

$$N^{1/2}(\mu_{ij}(\theta) - \mu_{ij}(0)) \to (\theta_j - \theta_i)\int \{dJ[F(x)]/dx\}\,dF(x),$$

we conclude

THEOREM 5.2. *For each index N assume that $m_i = \rho_i \cdot N$, with ρ_i a positive integer, and that the hypothesis H_N^P is true. Then the statistic $N^{-3/2}V$ has a limiting distribution with mean*

$$\sum_{i<j}\sum \rho_i\rho_j(\theta_j - \theta_i)\int \{dJ[F(x)]/dx\}\,dF(x)$$

and variance

$$\frac{1}{3}\left[\left(\sum_{i=1}^{k} \rho_i\right)^3 - \sum_{i=1}^{k} \rho_i^3\right] A^2 ,$$

where A^2 is given by (5.2). *Here the function J is assumed to satisfy the regularity conditions of Lemma* 7.2 *of* [16].

We are now in a position to make large sample comparison between different members of the V test and their normal theory competitor based on Student's statistic. We shall adopt a method developed by Pitman [15a] who defined the relative asymptotic efficiency of two sequences of tests as the limiting inverse ratio of sample sizes necessary to achieve the same power against the same sequences of alternatives at the same significance level.

THEOREM 5.3. *The asymptotic efficiency of the V test relative to the normal theory test based on the statistic*

$$T = \sum_{i<j}\sum m_i m_j (X_{i\cdot} - X_{j\cdot}) ,$$

where $X_{i\cdot} = \sum_{\alpha=1}^{m_i} X_{i\alpha}/m_i$, is

$$e_{V,T}(F) = \frac{\sigma^2}{A^2}\left(\int_{-\infty}^{+\infty} \{dJ[F(x)]/dx\}\, dF(x)\right)^2 , \tag{5.3}$$

where $\sigma^2 = \mathrm{Var}\,(X_{i\alpha})$.

Proof: Let $T_{ij} = X_{i\cdot} - X_{j\cdot}$ and $V'_{ij} = N^{1/2}(T_{ij} - (\xi_i - \xi_j))$. Then the variables $\{V'_{ij}, i<j\}$ have an asymptotic normal distribution with zero mean and covariance matrix

$$\mathrm{Var}\,(V'_i) = \sigma^2(1/\rho_i + 1/\rho_j) ,$$

$$\mathrm{Cov}\,(V'_{ij,rs}) \begin{cases} = 0 & \text{if } i, j, r, s \text{ are distinct,} \\ = \dfrac{\sigma^2}{\rho_i} & \text{if } i = r,\ j \neq s , \\ = \dfrac{\sigma^2}{\rho_j} & \text{if } i \neq r,\ j = s , \\ = -\dfrac{\sigma^2}{\rho_i} & \text{if } i = s,\ j \neq r , \\ = -\dfrac{\sigma^2}{\rho_j} & \text{if } i \neq s,\ j = r . \end{cases}$$

Hence $N^{-3/2}T$ has a limiting normal distribution with zero mean and variance (after omitting the details of computation) equal to

$$\frac{\sigma^2}{3}\left[\left(\sum_{i=1}^{k} \rho_i\right)^3 - \sum_{i=1}^{k} \rho_i^3\right] .$$

Now proceeding by the standard arguments, see for example Puri [17] and Chernoff-Savage [6], the result follows.

The relative efficiency of the V test relative to the T test is the same as found by Chernoff-Savage [6] for the corresponding procedures in the two-sample problem, and shown by the author [16] to be valid also for the multi-sample problem (unrestricted alternatives).

Special cases. (i) let J be the inverse of the rectangular distribution on (0, 1), then the V test reduces to the rank-sum $V(R)$ test, better known as the Jonckheere test [11]. The efficiency (5.3) then is equal to $12\sigma^2(\int f^2(x)\,dx)^2$. This is known to satisfy $e_{V(R),T}(F) \geqq 0.864$ for all F; $e_{V(R),T}(F) = \frac{3}{\pi} \sim 0.955$ when F is normal, and $e_{V(R),T}(F) > 1$ for many non-normal distributions. (For the Gamma distribution with parameter $p = 1$, $e_{V(R),T}(F) = 3$.)

(ii) Let $J = \Phi^{-1}$, where Φ is the standard normal distribution function. The V test reduces to the normal scores $V(\Phi)$ test. The efficiency then is known to satisfy $e_{V(\Phi),T}(F) \geqq 1$ for all F and $e_{V(\Phi),T}(F) = 1$ if and only if F is normal.

Thus from the asymptotic efficiency point of view both the $V(R)$ and $V(\Phi)$ tests can appear to be advantageous compared with T test unless one can be reasonably sure of the absence of gross errors and other departures from normality. (For the asymptotic efficiency comparison of the rank-sum to the normal scores procedure, see Hodges and Lehmann [9].)

In [2], under the assumptions of normality, Bartholomew derived the likelihood ratio statistic $\bar{E}^2$ relevant to the problem treated in this paper. Chacko [5] extended the work of Bartholomew and showed that the $\bar{E}^2$ statistic has the limiting non-central chi-square distribution as $N \to \infty$. Asymptotic relative efficiency cannot be used to compare the V and $\bar{E}^2$ tests because of the fact that the forms of their limiting distributions are different. For the same reason, it is not possible to find the asymptotic relative efficiency of the V and $\mathscr{F}$ tests. However, some light can be thrown on the question of the choice between the V, $\bar{E}^2$ and $\mathscr{F}$ tests by making large sample power comparisons. Some numerical results for $V(\Phi)$, $V(R)$, $\bar{E}^2$ and $\mathscr{F}$ are given in Table 1. Some of the figures for $\bar{E}^2$ and $\mathscr{F}$ have already been given by Bartholomew [2] but they are reproduced here for ease of comparison. It must be borne in mind that they are asymptotic results and that they involve the assumptions of normality. Furthermore, it is assumed that the sample sizes m_i are all equal. Two configurations of θ's are considered and the power in each case is expressed as a function of

$$\Delta = \sqrt{\sum_{i=1}^{k} (\theta_i - \bar{\theta})^2},$$

where $\bar{\theta} = \sum_{i=1}^{k} \theta_i/k$.

TABLE 1.[5] **The asymptotic power comparisons of $V(\Phi)$, $V(R)$, E^2 and $\mathscr{F}$ when $\theta_1 \geqq \theta_2 \geqq \cdots \geqq \theta_k$.**

		Δ				
		0	1	2	3	4
$k = 3$	$V(\Phi)$	.050	.258	.637	.911	.991
		.050	.218	.532	.829	.965
	$V(R)$	.050	.252	.622	.901	.988
		.050	.212	.519	.814	.959
	E^2	.050	.239	.594	.885	.980
		.050	.221	.569	.872	.983
	$\mathscr{F}$	.050	0.130	0.402	0.776	.959
$k = 4$	$V(\Phi)$	.050	.258	.637	.911	.991
		.050	.192	.460	.749	.926
	$V(R)$	.050	0.252	0.622	0.901	0.988
		.050	.187	.448	.734	.917
	E^2	.050	.239	.594	.885	.980
		.050	.202	.531	.849	.978
	$\mathscr{F}$	0.050	0.115	0.350	0.710	0.945
$k = 8$	$V(\Phi)$	.050	.258	.637	.911	.991
		.050	.142	.311	.532	.744
	$V(R)$	.050	.252	.622	.901	.988
		.050	.140	.303	.519	.730
	E^2	.050	—	—	—	—
		.050	.191	.456	.800	.973
	$\mathscr{F}$	.050	0.090	0.249	0.535	0.853
$k = 12$	$V(\Phi)$	.050	.258	.637	.911	.991
		.050	.121	.258	.417	.606
	$V(R)$	.050	.252	.622	.901	.988
		.050	.120	.240	.406	.592
	E^2	.050	—	—	—	— —
		.050	.0178	.423	.766	.963
	$\mathscr{F}$	0.050	0.080	0.205	0.466	0.776

(I) θ's equally spaced. Let $\theta_{i-1} - \theta_i = A^*$, $i = 2, \cdots, k$, then $\theta_j - \theta_i = (i - j)A^*$, $i < j$. The asymptotic power of the $V(R)$ test (cf. Theorem 5.2) is then

$$\beta(V(R)) = 1 - \Phi[\lambda_\alpha - \Delta\sqrt{3/\pi}] , \tag{5.4}$$

and the asymptotic power of the $V(\Phi)$ test is given by (cf. Theorem 5.2)

$$\beta(V(\Phi)) = 1 - \Phi[\lambda_\alpha - \Delta] . \tag{5.5}$$

Here

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2}$$

and λ_α is the upper 100α % point of $\Phi(x)$.

[5] The upper figure of each pair corresponds to equal spacing of the θ's and the lower figure to the case when all but one of the θ's are equal.

(ii) $\theta_1 \geqq \theta_2 = \cdots = \theta_k$. The asymptotic powers of the $V(R)$ and $V(\Phi)$ tests in this case are given by

$$\beta(V(R)) = 1 - \Phi\left[\lambda_\alpha - \Delta\sqrt{\frac{3}{\pi}}\sqrt{\frac{3}{k+1}}\right] \tag{5.6}$$

and

$$\beta(V(\Phi)) = 1 - \Phi\left[\lambda_\alpha - \Delta\sqrt{\frac{3}{k+1}}\right]. \tag{5.7}$$

It is clear that, unless $k = 2$, both the $V(R)$ and $V(\Phi)$ tests are more powerful in detecting a given Δ when the means are equally spaced than when all but one are equal. Furthermore, in the latter case the power of the $V(R)$ test as well as that of the $V(\Phi)$ test decreases as k increases.

The following conclusions may be drawn from these results.

(i) The $V(\Phi)$, $V(R)$ and $\bar{E}^2$ tests are always to be preferred to the classical $\mathscr{F}$ test which assumes no prior information regarding the θ's.

(ii) The powers of the $V(\Phi)$, $V(R)$ and $\bar{E}^2$ tests for the case when all but one of the θ's are equal, are lower than the powers of the corresponding tests for the case when the θ's are equally spaced.

(iii) The $V(\Phi)$ test is superior to the $V(R)$ and $\bar{E}^2$ tests when the θ's are equally spaced; it is also superior to the $V(R)$ test but inferior to the $\bar{E}^2$ test when all but one of the θ's are equal.

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MULTI-SAMPLE ANALOGUES OF SOME ONE-SAMPLE TESTS

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Preface. The results of Part I and Part III were obtained by the second author (cf. Puri (1962)) and those of Part I and Part II by the first author by following different methods (cf. Mehra (1963)). The authors wish to express their sincere thanks to Professors Erich L. Lehmann, Jaroslav Hájek and Edward Paulson for very helpful suggestions and criticisms.

PART I

I.1. Introduction and summary. Consider K treatments in an experiment which yields paired observations, namely (X_{il}, X_{jl}), $l = 1, \cdots, N_{ij}$; $1 \leqq i < j \leqq K$, obtained by N_{ij} independent paired comparisons for each pair (i, j) of treatments and assume that N_{ij} difference scores $Z_l^{(i,j)} = X_{il} - X_{jl}$, $l = 1, \cdots, N_{ij}$, have a common continuous cdf (cumulative distribution function) $\Pi_{ij}(z)$. This is the situation, for example, if in the analysis of an incomplete blocks experiment with each block of size two, one makes the assumption of additivity in the usual analysis of variance model. Then for testing the hypothesis

$$H_0 : \Pi_{ij}(z) + \Pi_{ij}(-z) = 1 \quad \text{and} \quad \Pi_{ij}(z) = \Pi_{i'j'}(z)$$

for any two pairs (i, j) and (i', j') [which states that each of the distributions Π_{ij} of the differences $Z_{ijl} = X_{il} - X_{jl}$, $l = 1, \cdots, N_{ij}$, is symmetric with respect to the origin, and furthermore all distributions Π_{ij} are identical] some rank tests based on the generalizations of the one-sample Chernoff-Savage-Hájek type tests (cf. [9] and [3]) are proposed, their limiting distributions are derived, and their efficiency properties with respect to one another and some of their competitors, viz. the Bradley-Terry test [1], the Durbin test [6] and the classical F test are studied. (For alternative formulations of the null hypothesis, and the study of the special case of the generalization of the one-sample Wilcoxon test, the reader is referred to [16].)

Let $\{J_{N,k}; k = 1, \cdots, N\}$, be a double sequence of numbers satisfying certain conditions to be stated below (Section 2) and let $R_{N,l}^{(i,j)}$ be the rank of $|Z_l^{(i,j)}|$, when the $N = \sum_{i=1}^{k} \sum_{j>i} N_{ij}$ absolute values of the observed differences $|Z_l^{(i,j)}|$, $l = 1, 2, \cdots, N_{ij}$, $1 \leqq i < j \leqq K$, are arranged in the ascending order of

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magnitude in a *combined ranking*. Define

$$(1.1)\qquad \tau_N^{(i,j)} = \sum_{l=1}^{N_{ij}} J_N(R_{N,l}^{(i,j)}/(N+1))\cdot \operatorname{sign} Z_l^{(i,j)},$$

where $J_N(u)$ is a step function defined over $(0, 1)$ taking constant values $J_{N,k}$ over the interval $((k-1)/N, k/N]$, i.e., $J_N(u) = J_{N,k} = J(k/N+1)$ for $k - 1/N < u \leqq k/N$. (Note that $\tau_N^{(i,j)} = -\tau_N^{(j,i)}$); $\tau_N^{(i,j)}$ is also expressible as

$$(1.2)\qquad \tau_N^{(i,j)} = \tau_{ij}^{+} + \tau_{ij}^{-}$$

where

$$\tau_{ij}^{+} = \sum_{k=1}^{N} J_{N,k} a_{ij,k}^{*}, \qquad \tau_{ij}^{-} = \sum_{k=1}^{N} J_{N,k} a_{ij,k}^{**},$$

with $a_{ij,k}^{*} = 1 (a_{ij,k}^{**} = -1)$ if the kth smallest absolute Z in the combined ranking corresponds to a positive (negative) $Z_l^{(i,j)}$, $l = 1, 2, \cdots, N_{ij}$, otherwise $a_{ij,k}^{*} = 0$ $(a_{ij,k}^{**} = 0)$. Consider now, for testing the hypothesis H_0, the statistics of the form

$$(1.3)\qquad L_N = \sum_{i=1}^{K} \{\sum_{j\neq i} (\tau_N^{(i,j)}/N_{ij}^{\frac{1}{2}})\}^2 / (N^{-1} \sum_{k=1}^{N} J_{N,k}^2) K,$$

with the test consisting in rejecting H_0 at level α if L_N exceeds a predetermined number $c_{N,\alpha}$ where $P_{H_0}[L_N \geqq c_{N,\alpha}] = \alpha$. The limit distributions of these statistics as $N \to \infty$, under H_0 and "contiguous" translation alternatives, are derived in Part I under two sets of sufficient conditions—under (a) the assumptions of Hájek [9] and (b) under those of Chernoff and Savage [3] (Section 2). This enables us to determine (Section 3) the asymptotic (Pitman) efficiency of any two statistics belonging to this class relative to each other and, for that matter, relative to any other competing statistic for which the limit distribution is of the same form e.g., Bradley-Terry statistic, the classical $\mathfrak{F}$-statistic and the class of statistics $L_N{}^*$ described by (1.4).

It turns out, however, that given any statistic belonging to this family, the statistic constructed in exactly the same manner but with $\tau_N^{(i,j)}$ now based on *separate-rankings* of the absolute Z's for each pair (i, j) $(1 \leqq i < j \leqq K)$ is, in the Pitman sense, as efficient as the given statistic. This latter family of statistics is represented by

$$(1.4)\qquad L_N{}^* = \sum_{i=1}^{K} \{\sum_{j\neq i} (\tau_N^{*(i,j)}/(K\, d_{N_{ij}}^2)^{\frac{1}{2}})\}^2$$

where $d_{N_{ij}}^2 = \sum_{k=1}^{N_{ij}} J_{N_{ij},k}^2$, and

$$\tau_N^{*(i,j)} = \sum_{l=1}^{N_{ij}} J_{N_{ij}}(R_{N_{ij},l}^{*(i,j)}/(N_{ij}+1))\cdot \operatorname{sign} Z_l^{(i,j)},$$

$R_{N_{ij},l}^{*(i,j)}$ being the rank of $|Z_l^{(i,j)}|$ when the N_{ij} absolute values $|Z_l^{(i,j)}|$, $l = 1, 2, \cdots, N_{ij}$, are ranked separately for each pair (i, j) $(1 \leqq i < j \leqq K)$. The form of the hypothesis H_0 suggests that it is the "joint-ranking" procedure which is more appropriate. However, if we apply the Pitman criterion, the question as to which of the two procedures—the *joint-ranking* or the *separate-rankings*—is preferable remains unresolved. This question is partially investigated in Part II by considering the local "asymptotic" efficiency as the number of treatments tends to infinity. The results obtained suggest, that for testing against shift in location, a "joint-ranking" statistic L_N is preferable to its counterpart L_N^* based on "separate rankings" except for alternatives for which the Durbin-statistic is relatively Pitman-efficient than the given statistic L_N. It is also shown that for testing against a specified alternative, the "best" rank-order statistic (in the sense of local power) is the one based on the joint ranking procedure.

Part III contains the proof of the asymptotic joint-normality, as $N \to \infty$, of the variables $\tau_N^{(i,j)}$ $(1 \leqq i < j \leqq K)$ under fixed alternatives from which then one can easily derive the limit distribution of L for "contiguous" translation alternatives.

I.2. Limit distributions. Consider the problem of testing H_0 against the alternatives of shift in location. To investigate the asymptotic efficiency of any L_N or L_N^* (or $\mathfrak{F}$-statistic), we obtain in this section their limit distributions, assuming a sequence K_N (defined below) of translation alternatives which approach H_0, as $N \to \infty$, viz.,

$$K_N : \Pi_{ij}(z) = \Pi(z + \mu_{ij} N^{-\frac{1}{2}}) \qquad \text{for each pair} \quad (i, j), (1 \leqq i < j \leqq K), \tag{2.1}$$

where $\Pi(x)$ is a continuous cdf satisfying the symmetry condition $\Pi(z) + \Pi(-z) = 1$ and μ_{ij} are certain constants, not all zero and satisfying $\mu_{ij} = -\mu_{ji}$. Consider now the following two sets of sufficient conditions:

Hájek conditions:

Ω_1 : Assume the existence of a function $J(u)$ defined over $(0, 1)$ such that

$$\text{(i)} \quad \int_0^1 J^2(u)\, du < \infty, \qquad \text{(ii)} \quad \lim_{N\to\infty} \int_0^1 \{J_N(u) - J(u)\}^2\, du = 0.$$

Ω_2 : $\Pi(z)$ possesses a differentiable density $\pi(z)$ such that the function

$$\psi(u) = -\pi'[\Pi^{-1}((1 + u)/2)]/\pi[\Pi^{-1}((1 + u)/2)], \quad 0 < u < 1,$$

satisfies $\int_0^1 \psi^2(u)\, du < \infty$.

Chernoff-Savage type conditions: We introduce some notation. Let $c = K(K - 1)/2$ denote the number of all possible pairs and label them $\alpha = 1, 2, \cdots, c$. Let m_α, n_α be the number of positive and negative $Z^{(\alpha)}$'s (then m_α, n_α are random but $m_\alpha + n_\alpha = N_\alpha$ is non-random). Let $F^{+(\alpha)}(x)(F^{-(\alpha)}(x))$ stand for the conditional distributions of the $|Z^{(\alpha)}|$ given $Z^{(\alpha)} > 0$ $(Z^{(\alpha)} < 0)$ and $F^+_{m_\alpha}(x)(F^-_{n_\alpha}(x))$ the sample cdf's of the absolute values of the positive (negative) $Z^{(\alpha)}$'s. Further let $\lambda_\alpha = m_\alpha/N$, $\mu_\alpha = n_\alpha/N$.

$$H_N(x) = \sum_{\alpha=1}^{c} [\lambda_\alpha F^+_{m_\alpha}(x) + \mu_\alpha F^-_{n_\alpha}(x)] \tag{2.2}$$

and

$$H(x) = \sum_{\alpha=1}^{c} [\lambda_\alpha F^{+(\alpha)}(x) + \mu_\alpha F^{-(\alpha)}(x)] \tag{2.2a}$$

and denote by Ω_3 and Ω_4 the conditions

Ω_3 : (i) $J(u) = \lim_{N\to\infty} J_N(u)$ exists for $0 < u < 1$ and is not constant;

(ii) $\int_{I_N} [J_N(H_N(x)) - J(H_N(x))]\, dF^+_{m_\alpha}(x) = o_p(N^{-\frac{1}{2}})$,

$$\int_{I_N} [J_N(H_N(x)) - J(H_N(x))]\, dF^-_{n_\alpha}(x) = o_p(N^{-\frac{1}{2}}),$$

where $I_N = \{x : 0 < H_N(x) < 1\}$.

(iii) $J_N(1) = o(N^{\frac{1}{2}})$

(iv)
$$|J(u)| \leqq t[u(1-u)]^{-\frac{1}{2}+\delta},$$
$$|J^{(i)}(u)| = |d^iJ/du^i| \leqq t[u(1-u)]^{-i+\delta},$$

for $i = 1, 2$, for some t and $\delta > 0$.

Ω_4 : (i) The distribution $\Pi(z)$ admits a unimodal density $\pi(z)$ which is bounded in the neighbourhood of the origin.

(ii) $J'[\Pi(x)]\pi(x)$ is bounded.

Let $\chi_t^2(\delta^2)$ stand for the non-central χ^2-variable with t degrees of freedom and the non-centrality parameter δ^2; and let χ_t^2 stand for the corresponding central χ^2-variable. We now state

THEOREM 2.1. *Assume for each N the truth of K_N and that either* (a) *the conditions* (Ω_1, Ω_2) *or* (b) *the conditions* (Ω_3, Ω_4) *are satisfied. Assume further that* $\rho_{ij} = \lim_{N\to\infty} \{N_{ij}/N\}$ *exists and is positive for each pair* (i, j) $(1 \leqq i < j \leqq K)$. *Then the statistic L_N is distributed in the limit, as $N \to \infty$, as a $\chi^2_{K-1}(\delta^2)$ variable with*

$$\delta^2 = (B/K) \textstyle\sum_{i=1}^K \{ \sum_{j\neq i} (\rho_{ij}^{\frac{1}{2}}\mu_{ij})\}^2 \tag{2.3}$$

where

$$B = (\textstyle\int_0^1 J(u)\psi(u)\, du)^2/(\int_0^1 J^2(u)\, du) \tag{2.4}$$

under (Ω_1, Ω_2) *and*

$$B = 16(\textstyle\int_0^\infty J'(2\Pi(x) - 1)\{\pi(x)\}^2\, dx)^2/(\int_0^1 J^2(u)\, du) \tag{2.4a}$$

under (Ω_3, Ω_4).

It is easily verified that when both the conditions (Ω_1, Ω_2) and (Ω_3, Ω_4) are satisfied, the two expressions for B above coincide. This holds for most situations of applicational interest (Section 3).

For the special case when $\mu_{ij} = \theta_i - \theta_j$ where not all θ's are equal and $N_{ij} = n$ for each pair (i, j), the non-centrality parameter (2.3) takes the form

$$\delta^2 = (2B/(K-1)) \textstyle\sum_{i=1}^K (\theta_i - \bar{\theta})^2 \tag{2.5}$$

where $\bar{\theta} = \sum_{i=1}^K \theta_i/K$.

The proof of part (a) is based on the following two lemmas, the first of which is an extension of the main theorem of Hájek, based on the notion of "contiguity." This lemma, which enables us to conclude the joint-normality of the variables $\tau_N^{(i,j)}$ $(1 \leqq i < j \leqq K)$ under (Ω_1, Ω_2), is also needed for the results of Part II. The proof of part (b) is based on the more general Theorem 3.1 of Part III.

The statement of Lemma 2.1 concerns a slightly more general model described below: Let $(Z_{\nu 1} \cdots Z_{\nu N_\nu})$, $1 \leqq \nu < \infty$, be a sequence of random vectors, where $N_\nu \to \infty$ as $\nu \to \infty$ and Z's are independent, and denote by $R_{\nu k}$ the rank of $|Z_{\nu k}|$ as the totality of $|Z|$'s are ranked in ascending order of magnitude. Further, let

$$S_\nu = \sum_{k=1}^{N_\nu} d_{\nu k} J_\nu(R_{\nu k}/(N_\nu + 1)) \cdot \text{sign } Z_{\nu k} \tag{2.6}$$

where $d_{\nu k}$, $1 \leqq k \leqq N_\nu$, are certain constants satisfying

$$\lim_{\nu\to\infty} \{(\max_{1\leqq k\leqq N_\nu} d_{\nu k}^2)/(\sum_{k=1}^{N_\nu} d_{\nu k}^2)\} = 0, \tag{2.7}$$

and assume that

$$P[Z_{\nu k} \leqq z/\beta, \sigma] = \Pi((z - \beta c_{\nu k})/\sigma), \tag{2.8}$$

where $\Pi(z)$ is as defined in (2.1), $-\infty < \beta < \infty$, $\sigma > 0$, are unknown parameters and $c_{\nu k}$ are again certain constants satisfying the condition (2.7) with d's replaced by c's and

$$\sup_\nu (\sum_{k=1}^{N_\nu} c_{\nu k}^2) < \infty. \tag{2.8a}$$

Let $\mathcal{L}(Y_\nu/P_\nu) \to N(a_\nu, b_\nu^2)$ denote that the distribution of $b_\nu^{-1}(Y_\nu - a_\nu)$ converges, as $\nu \to \infty$, to $N(0, 1)$ distribution.

LEMMA 2.1. *Suppose that the conditions* (Ω_1, Ω_2) *are satisfied. Then under the model* (2.8), *the statistic* (2.6) *satisfies* $\mathcal{L}(S_\nu) \to N(\eta_\nu, t_\nu^2)$ *where*

$$\eta_\nu = (\beta/\sigma)(\int_0^1 J(u)\psi(u)\, du) \cdot \sum_{k=1}^{N_\nu} d_{\nu k} c_{\nu k}, \tag{2.9}$$
$$t_\nu^2 = (\int_0^1 J^2(u)\, du) \sum_{k=1}^{N_\nu} d_{\nu k}^2.$$

PROOF. Consider a particular distribution Π and let Q_ν and P_ν stand for the probability distributions under $\beta = \beta_0$, $\sigma = \sigma_0$ and $\beta = 0$, $\sigma = \sigma_0$ respectively. The proof below is simply a reconstruction of certain essential steps in Hájek's proof. Let T_ν denote

$$T_\nu = \sum_{k=1}^{N_\nu} d_{\nu\alpha} J[T(|Y_{\nu k}|)] \text{ sign } Z_{\nu\alpha}$$

where $Y_{\nu k} = (Z_{\nu k} - \beta_0)/\sigma_0$ and $T(x) = 2\Pi(x) - 1$, if $x \geqq 0$, and $T(x) = 0$ otherwise. Then, as in [9], one obtains that if after proper normalization, one of the limit exists,

$$\lim_{\nu\to\infty} \mathcal{L}(S_\nu/Q) = \lim_{\nu\to\infty} \mathcal{L}(T_\nu/Q). \tag{2.10}$$

To apply Lemma 4.2 of [9] we have to show now that $\mathcal{L}(T_\nu, L_\nu/P_\nu)$ converges to some bivariate normal distribution. The equation (2.10) and part (iii) of Lemma 4.2 [9] would then give the result forthwith. For this it suffices, on account of the arguments of Section 7 of Wald-Wolfowitz [24], to prove the asymptotic normality of an arbitrary linear combination of T_ν and L_ν, where L_ν is as defined by (4.16) of [9], viz.,

$$\mu_1 T_\nu + \mu_2 L_\nu. \tag{2.11}$$

From equation (5.21) of [9] we know that $P_\nu - \lim_{\nu\to\infty} \{L_\nu + (\gamma^2 d_\nu^2/2) - \gamma S_\nu^*\} = 0$, where $\gamma = (\beta_0/\sigma_0)$, d_ν^2 is defined by (7.6) of [9], and $S_\nu^* = -\sum_{k=1}^{N_\nu} c_{\nu k}\{\pi'(Y_{\nu k})/\pi(Y_{\nu k})\}$, so that (2.11) is asymptotically equivalent in distribution, after proper normalization, to the statistic

$$(2.12) \qquad \sum_{k=1}^{N_\nu} (r_{1k} + r_{2k}) - \tfrac{1}{2}\mu_2\gamma^2 d_\nu^2$$

where $r_{1k} = \mu_1 d_{\nu k} J\{T(|Y_{\nu k}|)\}\cdot \text{sign}\, Z_{\nu k}$ and $r_{2k} = -\mu_2 c_{\nu k}\gamma\cdot\{\pi'(Y_{\nu k})/\pi(Y_{\nu k})\}$. It is easy to see that the variance σ_ν^2 of (2.12) is given (since the summands r_{1k} and r_{2k} have zero expectations under P_ν) by

$$\sigma_\nu^2 = \mu_1^2\left(\int_0^1 J^2(u)\,du\right)\sum_{k=1}^{N_\nu} d_{\nu k}^2 + \mu_2^2\gamma^2\left(\int_0^1 J^2(u)\,du\right)\sum_{k=1}^{N_\nu} c_{\nu k}^2 + 2\mu_1\mu_2\gamma\left(\int_0^1 J(u)\psi(u)\,du\right)\sum_{k=1}^{N_\nu} c_{\nu k} d_{\nu k}\,.$$

We may assume that σ_ν^2 is bounded away from zero (for otherwise the result trivially holds).

Letting now I_A denote the indicator function of the set A, we have for every $\epsilon > 0$

$$(2.13) \qquad \begin{aligned}\sigma_\nu^{-2}\sum_{k=1}^{N_\nu} E\{I_{[|r_{1k}+r_{2k}|\geq\epsilon\sigma_\nu]}(r_{1k}+r_{2k})^2\} &\leq \sigma_\nu^{-2}\sum_{k=1}^{N_\nu} E\{I_{[|r_{1k}|\geq\frac{1}{2}\epsilon\sigma_\nu]} r_{1k}^2\} \\ &\quad + \sigma_\nu^{-2}\sum_{k=1}^{N_\nu} E\{I_{[|r_{2k}|\geq\frac{1}{2}\epsilon\sigma_\nu]} r_{1k}^2\} + \sigma_\nu^{-2}\sum_{k=1}^{N_\nu} E\{I_{[|r_{1k}|\geq\frac{1}{2}\epsilon\sigma_\nu]} r_{2k}^2\} \\ &\quad + \sigma_\nu^{-2}\sum_{k=1}^{N_\nu} E\{I_{[|r_{2k}|\geq\frac{1}{2}\epsilon\sigma_\nu]} r_{2k}^2\}\end{aligned}$$

where each summation on the right of (2.13) converges to zero on account of conditions $\Omega_1(1)$, Ω_2, (2.7) and (2.8). Thus the Lindeberg-Feller condition is satisfied, which establishes the asymptotic normality of (2.11); and the proof is complete.

LEMMA 2.2. *Under the conditions of Theorem* 2.1 *with either* (a) (Ω_1, Ω_2) *or* (b) (Ω_3, Ω_4), *the* $c = K(K-1)/2$ *random variables* $\{\tau_N^{(i,j)}/N_{ij}^{\frac{1}{2}}\}$, $(1 \leq i < j \leq K)$, *are distributed in the limit, as* $N \to \infty$, *as independent* $N(\eta^{(i,j)}, A^2)$ *variables, where* $A = [\int_0^1 J^2(u)\,du]^{\frac{1}{2}}$ *and*

$$(2.14) \qquad \begin{aligned}\eta^{(i,j)} &= \rho_{ij}^{\frac{1}{2}}\,\mu_{ij}\left(\int_0^1 J(u)\psi(u)\,du\right) \quad \textit{under} \quad (\Omega_1, \Omega_2), \\ \eta^{(i,j)} &= 4\rho_{ij}^{\frac{1}{2}}\,\mu_{ij}\left(\int_0^\infty J'[2\Pi(x)-1]\pi(x)\,d\Pi(x)\right) \quad \textit{under} \quad (\Omega_3, \Omega_4).\end{aligned}$$

PROOF. The proof of part (a) of this lemma is based on Lemma 2.1, and that of part (b) is given in Part III. Under a labelling $\alpha = 1, 2, \cdots, c$ of the $c = K(K-1)/2$ pairs (i, j) $(1 \leq i < j \leq K)$, the statistic (2.6) can be expressed in the present context as

$$(2.15) \qquad \begin{aligned} S_N &= \sum_{\alpha=1}^{c}\sum_{l=1}^{N_\alpha} d_{N,l}^{(\alpha)} J_N(R_{N,l}^{(\alpha)}/(N+1))\cdot\text{sign}\, Z_l^{(\alpha)} \\ &= \sum_{i=1}^{K}\sum_{j>i}\left\{\sum_{l=1}^{N_{ij}} d_{N,l}^{(i,j)} J_N(R_{N,l}^{(i,j)}/(N+1))\cdot\text{sign}\, Z_l^{(i,j)}\right\}.\end{aligned}$$

For a given pair (i, j), the statistic $\tau_N^{(i,j)}/N^{\frac{1}{2}}$ is obtained from (2.15) by setting $d_{N,l}^{(i,j)} = N^{-\frac{1}{2}}$, $l = 1, 2, \cdots, N_{ij}$, and all other d's equal to zero. The condition (2.7) for this choice of d's is satisfied, so that by Lemma 2.1 $\mathcal{L}(\tau_N^{(i,j)}/N_{ij}^{\frac{1}{2}}) \to N(\eta^{(i,j)}, A^2)$ under K_N. Furthermore, a similar argument shows that any arbitrary linear combination of $\{\tau_N^{(i,j)}/N_{ij}^{\frac{1}{2}}, 1 \leq i < j \leq K\}$ has normal distribution in the limit. The proof follows.

PROOF OF THEOREM 2.1. It follows from Lemma 2.1 that the variables

$$W_{N,i} = [\sum_{j\neq i} \{N_{ij}^{-\frac{1}{2}} \tau_N^{(i,j)} - \eta^{(i,j)}\}]/AK^{\frac{1}{2}},$$

$i = 1, 2, \cdots, K$, have in the limit a multivariate normal distribution $N(\mathbf{0}, \mathbf{\Lambda})$ where $\mathbf{\Lambda} = \| \delta_{ii'} - 1/K \|$. Now making the analysis of variance transformation

$$U_0 = \sum_{i'=1}^{K} (K^{-\frac{1}{2}}) W_{N,i},$$
$$U_i = \sum_{i'=1}^{K} A_{ii'} W_{N,i'}, \qquad i = 1, 2, \cdots, K-1,$$

where A's are chosen to make the transformation orthogonal and proceeding exactly as in [18], the proof follows.

The following theorem concerns the limiting distribution of the separate-rankings statistic L_N^* defined by (1.4).

THEOREM 2.2. *Under the assumptions of Theorem* 2.1, *the statistic* L_N^* *is distributed in the limit, as* $N \to \infty$, *as a* $\chi_{K-1}^2(\delta^2)$ *variable with* δ^2 *given by* (2.3).

PROOF. Similar to that of Theorem 2.1.

From Theorems 2.1 and 2.2 it follows by letting $\mu_{ij} = 0$ for all pairs (i, j) that L_N and L_N^* are asymptotically distributed, under H_0 as χ_{K-1}^2 variables. This provides a large sample approximation to the critical points $c_{N,\alpha}$ and $c_{N,\alpha}^*$.

I.3. Asymptotic efficiency. In this section we consider some interesting special cases of the statistics L_N and L_N^* and discuss their asymptotic efficiencies relative to each other and the $\mathfrak{F}$-test. If we now let

(3.1)

(i) $J(u) = u$, $0 < u < 1$, then L_N reduces to the rank-sum version of L_N discussed in [16].

(ii) $J(u) = \chi^{-1}(u)$, where χ is the cdf of the chi-distribution with one degree of freedom, we get the multi-sample analogues of the Fisher-Yates-Fraser and Van der Waerden tests of symmetry respectively.

(iii) If we let $J(u) =$ constant, $L_N(L_N^*)$ reduces to the Durbin-statistic.

Let these statistics be denoted by W_N, $L_{N,1}$, $L_{N,2}$ and D_N respectively. Similarly one obtains the counterparts of the above statistics from L_N^*. Let these be denoted by the corresponding starred letters.

Now it is well known [10] that in the situations we are considering the asymptotic efficiency of one test relative to the other is equal to the ratio of their noncentrality parameters. Hence we have (e.g. when $\mu_{ij} = \theta_i - \theta_j$ and $N_{ij} = n$) the efficiencies of $L_{N,1}$, $L_{N,2}$, W_N, D_N and $\mathfrak{F}$-statistics as follows:

$$E_{L_1,\mathfrak{F}} = E_{L_2,\mathfrak{F}} = \sigma_\Pi^2 [\textstyle\int_0^1 \chi^{-1}(u)\psi(u)\, du]^2$$
$$(3.2) \qquad = \sigma_\Pi^2 [\textstyle\int_{-\infty}^{\infty} \pi(x)\, dx/\phi[\Phi^{-1}(\Pi(x))]]^2;$$
$$E_{L_1,W} = E_{L_2,W} = [\textstyle\int_{-\infty}^{\infty} \pi^2(x)\, dx/\phi[\Phi^{-1}(\Pi(x)]]^2/12[\int_{-\infty}^{\infty} \pi^2(x)\, dx]^2$$
$$E_{L_1,D} = E_{L_2,D} = [\textstyle\int_{-\infty}^{\infty} \pi^2(x)\, dx/\phi[\Phi^{-1}(\Pi(x))]]^2/4\pi^2(0).$$

TABLE 1

Distribution	E_{L_1}	E_{W,L_1}	$E_{L_1,D}$
Normal	1	$3/\pi \sim .955$	$\pi/2 \sim 1.571$
Uniform	∞	0	∞
Double exponential	$4/\pi \sim 1.273$	$3\pi/8 \sim 1.18$	$2/\pi \sim .637$

Table 1 gives the efficiency comparisons for different densities of the L_1 test, the W test, the D test and the $\mathfrak{F}$ test.

For distributions $\Pi(x)$ which are not covered under the conditions Ω_2, one may define

$$E^*_{S_1,S_2}(\Pi) = \lim_{\sigma\to 0} E_{S_1,S_2}(\Pi_\sigma),$$

if it exists, to be the asymptotic efficiency of S_1 relative to S_2, where Π_σ denotes the convolution of $\Pi(z)$ with $N(0, \sigma^2)$. For $\Pi_\sigma(z)$ the condition Ω_2 is satisfied. This covers the case, for example, of uniform distribution over $[-\theta, \theta]$. It is also interesting to observe that if the form of $\Pi(z)$ is specified, one can be letting $J(u) = \psi(u)$ obtain from the family L_N (or L_N^*) a statistic which is most (Pitman) efficient for the given distribution $\Pi(z)$—for example, by letting

$$(3.3)\qquad \begin{aligned} J(u) &= \chi^{-1}(u) && \text{if } \Pi(z) \text{ is normal;}\\ J(u) &= u && \text{if } \Pi(z) \text{ is logistic;}\\ J(u) &= \text{constant} && \text{if } \Pi(z) \text{ is double exponential.}\end{aligned}$$

Finally, we observe on account of Theorems 2.1 and 2.2 that $E_{L,L^*} = 1$.

Discussion. On account of the last remark above, the question of preference between the *joint-ranking* and *separate-ranking* procedures remains unresolved. It is worth observing that the Pitman efficiency, although satisfactory in most situations, is a rather narrow criterion for comparing the expected performance of two tests, being just a limiting number which compares only their local asymptotic powers as the number of observations tends to infinity. A more comprehensive definition of asymptotic efficiency is discussed by Hodges and Lehmann [11]; but such a comprehensive comparison is often too difficult to carry out in more complex situations. The considerations of Part II, however, based on a comparison of the "asymptotic" efficiencies of the statistics L_N and L_N^* as the number of treatments is allowed to increase, do throw some light on this question.

PART II

II.1. Local "asymptotic" efficiency. In view of the result that the joint-ranking statistic L_N and the separate-ranking statistic L_N^* are equally efficient in the Pitman sense (I.3), the question of the relative merits of these two statistics remains undecided. This part is devoted to an investigation of this question. For reasons stated in the last paragraph of Part I, however, we shall attempt to throw some light on this question by a comparison only of their local "asymptotic" powers, as K, the number of treatments, is allowed to increase indefinitely.

It is shown below that if the number of comparisons $N_{ij}(=n)$ is kept fixed for each pair (i, j), but instead K tends to infinity, both the statistic L_N and the statistic L_N^*, after proper normalization, converge in distribution to the $N(0, 1)$ variable. This enables us to compare their local "asymptotic" powers for each fixed N. We observe that (since $N = n\binom{K}{2}$) as $K \to \infty$, $\{K_N\}$ again provides a sequence of translation alternatives approaching H_0. Let $E(\cdot)$ and $\sigma^2(\cdot)$ stand in the sequel for the expectation and the variance, respectively, with any subscripts indicating the conditions under which these quantities are obtained. We need the following:

LEMMA 1.1. *Let $\chi_r^2 = \chi_r^2(\Delta_r)$ denote the non-central chi-square variable with r* df *and the non-centrality parameter Δ_r, and assume that $\Delta_r = o(r)$, as $r \to \infty$. Then, as $r \to \infty$, $\mathcal{L}([\chi_r^2 - E(\chi_r^2)]/\sigma(\chi_r^2)) \to N(0, 1)$.*

PROOF. The density of χ_r^2 is given by $p_{\Delta_r}(x) = \sum_{k=0}^{\infty} p_k(\Delta_r) f_{r+2k}(x)$, where $p_k(\Delta_r) = (\Delta_r/2)^k \exp\{-(\Delta_r/2\}/k!$ and $f_{r+2k}(x)$ is the probability density of the central χ_{r+2k}^2 variable, so that the characteristic function of $[\chi_r^2 - E(\chi_r^2)]/\sigma(\chi_r^2)$ is given by

$$
\begin{aligned}
f(t) &= \exp\left(-it(r+\Delta_r)(2r+4\Delta_r)^{-\frac{1}{2}}\right) \sum_{k=0}^{\infty} p_k(\Delta_r)(1-2it(2r+4\Delta_r)^{-\frac{1}{2}})^{-(r/2+k)} \\
&\sim \{(1-it(2/r)^{\frac{1}{2}})^{-r/2}\exp(-it(r/2)^{\frac{1}{2}})\} \\
&\quad \cdot \exp(-it(\Delta_r(2r)^{-\frac{1}{2}}) \sum_{k=0}^{\infty} p_k(\Delta_r)(1-it(2/r)^{\frac{1}{2}})^{-k} \\
&= \{(1-it(2/r)^{\frac{1}{2}})^{-r/2}\exp(-it(r/2)^{\frac{1}{2}})\} \\
&\quad \cdot \exp\{-\tfrac{1}{2}\Delta_r(1+it(2/r)^{\frac{1}{2}}) + \tfrac{1}{2}\Delta_r(1-it(2/r)^{\frac{1}{2}})^{-1}\}
\end{aligned}
$$

where the first term converges to $\exp\{-t^2/2\}$ and the second to unity, as $r \to \infty$, on account of the condition $\Delta_r = o(r)$; the proof is complete.

REMARK. In the statement of Lemma 1.1 above we may replace $E(\chi_r^2)$ and $\sigma(\chi_r^2)$ by $r + \Delta_r$ and $(2r + \Delta_r)^{\frac{1}{2}}$ respectively.

THEOREM 1.1. *Assume, for each index N, the truth of K_N with $\mu_{ij} = \theta_i - \theta_j$ (where not all θ's are equal) and $N_{ij} = n$ for all pairs (i, j) $(1 \leq i < j \leq K)$. Further, assume that*

$$\sup_K K^{-2} \sum_{i<j} (\theta_i - \theta_j)^2 < \infty. \tag{1.1}$$

Then under the conditions Ω_1 and Ω_2 of Part I, *$\mathcal{L}(L_N) \to N(\eta, 2(K-1))$ as $K \to \infty$, where*

$$\eta = (K-1) + \delta_K^2, \tag{1.2}$$

with δ_K^2 given by (2.5) of Part I.

PROOF. Let the $C = K(K-1)/2$ pairs (i, j) $(1 \leq i < j \leq K)$ be labelled $\alpha = 1, 2, \cdots, C$ (as in the proof of Lemma 2.2 of Part I) in some convenient manner, where if α corresponds to the pair (i, j), $\mu_\alpha = \mu_{ij} = \theta_i - \theta_j$. Then, the

$N = nK(K-1)/2$ c_N's defined for each N and α by

$$c_{N,\alpha,l} = \mu_\alpha/N^{\frac{1}{2}} = (\theta_i - \theta_j)/N^{\frac{1}{2}},$$

$l = 1, 2, \cdots, n$, satisfy the condition (2.7) I, as $K \to \infty$ (and consequently $N \to \infty$). This is easily seen by observing that, for the above c_N's, the left hand side of (2.7)I reduces to

$$\text{(1.3)} \quad \lim_{k\to\infty} \{\max_{1\leq i<j\leq K} (\theta_i - \theta_j)^2/n \textstyle\sum_{i<j} (\theta_i - \theta_j)^2\}$$
$$\leqq (4/n) \lim_{k\to\infty} \{\max_{1\leq i\leq K} (\theta_i - \bar{\theta})^2/K \textstyle\sum_{i=1}^K (\theta_i - \bar{\theta})^2\} = 0,$$

where $\bar{\theta} = \sum_{i=1}^K \theta_i/K$. The last inequality follows since $\sum_{i<j} (\theta_i - \theta_j)^2 = K\sum_{i=1}^K (\theta_i - \bar{\theta})^2$. On account of (1.1)II and (1.3)II, the conditions (2.7)I and (2.8a)I are satisfied, so that by applying Lemma 2.1 of Part I one obtains the asymptotic normality, after proper normalization, of any statistic of the type (2.15)I (or (2.6)I) for which (2.7)I is satisfied. Consider now any arbitrary linear combination of the variables $v_N^{(i)} = \sum_{j\neq i} V_N^{(i,j)}$, $i = 1, 2, \cdots, K$; viz.,

$$S_N = \textstyle\sum_{i=1}^K \lambda_i V_N^{(i)} = \sum_{i=1}^K (\sum_{j\neq i} \lambda_i V_N^{(i,j)})$$
$$= \textstyle\sum_{i=1}^K \sum_{j>i} (\lambda_i - \lambda_j) V_N^{(i,j)},$$

(using $V_N^{(i,j)} = -V_N^{(j,i)}$), where not all λ's are equal and zero values are permissible. The statistic S_N is obtainable from (2.15)I by letting for each $i = 1, 2, \cdots, K$, $d_{N,l}^{(i,j)} = \lambda_i - \lambda_j$ for $j > i$ and $l = 1, 2, \cdots, n$. With the above choice of d's the left hand side of (2.7)I takes the form

$$\lim_{K\to\infty} \{\max_{1\leq i<j\leq K} (\lambda_i - \lambda_j)^2/n \textstyle\sum_{i<j} (\lambda_i - \lambda_j)^2\}$$

which equals zero by the same arguments as used in (1.3)II. Accordingly, by applying Lemma 2.1 of Part I and using, for any K, however large, the same arguments as in Section 7 of Wald and Wolfowitz [24], it follows that, for sufficiently large K, the variables $(nK)^{-\frac{1}{2}}\{V_N^{(i)} - m^{(i)}\}$, $i = 1, 2, \cdots, K$, where $m^{(i)} = \{-(2n)^{\frac{1}{2}}(\theta_i - \bar{\theta})^2(\int_0^1 J(u)\psi(u)\,du)\}$, are approximately jointly normally distributed with mean vector zero and the covariance matrix $\mathbf{\Sigma} = \| \delta_{ii'} - (1/K \| \cdot(\int_0^1 J^2(u)\,du)$. Arguments similar to those used in the proof of Theorem 2.1 of Part I, coupled with an application of Lemma 1.1 above and Theorem 5 of Mann and Wald [24] gives the result forthwith; and the proof is complete.

A similar result also holds for the statistic L_N^* defined by (1.4)I:

THEOREM 1.2. *Assume, for each index N, the truth of K_N with $\mu_{ij} = \theta_i - \theta_j$ and $N_{ij} = n$ for all pairs (i, j) $(1 \leqq i < j \leqq K)$. Then, under the conditions of Theorem* 1.1, $\mathcal{L}(L_N^*) \to N(\eta', 2(K-1))$, *as* $K \to \infty$, *where*

$$\text{(1.4)} \qquad \eta' = (K-1) + (\delta_K^{*2}/n\, d_n^2)$$

with δ_K^{*2} and d_n^2 are given by (1.8)II *and* (1.9)II *respectively.*

PROOF. The proof of this theorem can be accomplished by using the central limit theorem for random vectors, Lemma 1.1 and arguments similar to those used in the proof of Theorem 1.1 above.

Theorems 1.1 and 1.2 can be used to compare the local powers of the two statistics L_N and L_N^*, as $K \to \infty$. Let $\theta = (\theta_1, \theta_2, \cdots, \theta_K)$ and let $\{l_{N\alpha}\}$ and $\{l_{N\alpha}^*\}$ be two sequences of numbers determined such that $\lim_{N\to\infty} P_{H_0}[L_N > l_{N\alpha}] = \lim_{N\to\infty} P_{H_0}[L_N^* > l_{N\alpha}^*] = \alpha$. The local powers of the statistics L_N and L_N^* under K_N at level α, are given respectively by

$$\beta_L(n, \alpha, \theta) = P_{K_N}[L_N > l_{N\alpha}] \sim 1 - \Phi\{(l_{N\alpha} - \eta)/(2(K-1))^{\frac{1}{2}}\},$$

$$\beta_{L^*}(n, \alpha, \theta) = P_{K_N}[L_N^* > l_{N\alpha}^*] \sim 1 - \Phi\{(l_{N\alpha}^* - \eta')/(2(K-1))^{\frac{1}{2}}\},$$

for sufficiently large K, on account of Theorems 1.1 and 1.2 above, where $\Phi(x)$ represents the standard normal cdf, $\phi(x)$ the corresponding density and the symbol $\sim$ denotes that the ratio of the two sides tends to one, as $K \to \infty$. Accordingly, from Theorem 1.1 above it follows that

$$(1.5) \quad \beta_L(n, \alpha; \theta) \sim \alpha + 2^{\frac{1}{2}}\{(\textstyle\int_0^1 J(u)\psi(u)\,du)^2/(\int_0^1 J^2(u)\,du)\} \cdot K^{-\frac{1}{2}} \sum_{i=1}^K (\theta_i - \bar{\theta})^2 \cdot \phi(t_\alpha)$$

for sufficiently large K, where t_α is the upper α point of $N(0, 1)$ distribution. To obtain a similar expression for $\beta_{L^*}(n, \alpha, \theta)$, set

$$(1.6) \qquad a_n^{(i,j)} = \lim_{N\to\infty} N^{\frac{1}{2}} E_{K_N}(V_N^{*(i,j)};$$

then following the above reasoning again we obtain

$$(1.7) \qquad \beta_{L^*}(n, \alpha, \theta) \sim \alpha + (\delta_K^{*2}/n\, d_n^2)\phi(t_\alpha)$$

where

$$(1.8) \qquad \delta_K^{*2} = (2^{\frac{1}{2}}/K^{\frac{1}{2}}) \sum_{i=1}^K \{\textstyle\sum_{j\neq i} (a_n^{(i,j)}/K)\}^2,$$

and

$$(1.9) \qquad d_n^2 = \sigma_{H_0}^2(V_n^{*(i,j)}) = \sum_{k=1}^n J_{n,k}^2,$$

$J_{n,k} = J_n(k/(n+1))$, $k = 1, 2, \cdots, n$, being the scores on which the definition of the function $\xi_n(u)$, $0 < u < 1$, is based. From (1.5) and (1.7), it follows that for large K the local power for shift alternatives $\beta_L(n, \alpha, \theta)$ will tend to be larger than $\beta_{L^*}(N, \alpha, \theta)$ if and only if

$$\begin{aligned} e_{L,L^*}^{(n)} &= \lim_{K\to\infty} \{(\beta_L(n, \alpha, \theta) - \alpha)(\beta_{L^*}(n, \alpha, \theta) - \alpha)^{-1}\} \\ (1.10) \qquad &= \lim_{K\to\infty} \{(\textstyle\int_0^1 J(u)\psi(u)\,du)^2/(\int_0^1 J^2(u)\,du)\} \sum_{i=1}^K (\theta_i - \bar{\theta})^2 \\ &\quad \cdot [\textstyle\sum_{i=1}^K \{\sum_{j\neq i} (a_n^{(i,j)}/K)\}^2/n\, d_n^2]^{-1} \end{aligned}$$

is larger than unity. The expression $e_{L,L^*}^{(n)}$ may be called the *local ("asymptotic") efficiency* of L_N and L_N^*, as $K \to \infty$, and may be used to throw some light on the question of comparison of L_N and L_N^*. It may be pointed out, however, that for the ratio $e^{(n)}_{L,L^*}$ a meaningful interpretation as in the case of Pitman's formula cannot be given.

II.2. The explicit evaluation of $e^{(n)}_{L,L^*}$. We shall derive in this section an explicit expression for the local "asymptotic" efficiency $e^{(n)}_{L,L^*}$ by evaluating $a_n^{(i,j)}$:

$$a_N^{(i,j)} = \lim_{N\to\infty} N^{\frac{1}{2}} E_{K_N}\{\textstyle\sum_{l=1}^n J_n(R_{n,l}^{(i,j)}/(n+1))\ \operatorname{sign} Z_l^{(i,j)}\}$$

$$= \lim_{N\to\infty} N^{\frac{1}{2}} \textstyle\sum_{k=1}^n (n!/(k-1)!(n-k)!) J_{n,k} \int_0^1 [T_N^{(i,j)}(x)]^{k-1}$$

$$\cdot [1 - T_N^{(i,j)}(x)]^{n-k}\, d[\Pi(x - (\theta_i - \theta_j)N^{-\frac{1}{2}}) - \Pi(x + (\theta_i - \theta_j)N^{-\frac{1}{2}})]$$

where $T_N^{(i,j)}(x) = \Pi(x - (\theta_i - \theta_j)N^{-\frac{1}{2}}) - \Pi(-x - (\theta_i - \theta_j)N^{-\frac{1}{2}})$, if $x \geqq 0$ and $T_N^{(i,j)}(x) = 0$ if $x < 0$. In evaluating the above limit, it is permissible to interchange the operations of limit and integration as is shown by the following:

LEMMA 2.1. *If the distribution $\Pi(x)$ possesses a differentiable density $\pi(x)$ and the condition Ω_2 is satisfied, then the expression $a_n^{(i,j)}$ is given by*

$$(2.1)\quad a_n^{(i,j)} = (\theta_i - \theta_j) \textstyle\sum_{k=1}^n (n!/(k-1)!(n-k)!)$$

$$\cdot\, J_{n,k} \int_0^1 \psi(u) u^{k-1}(1-u)^{n-k} du.$$

PROOF. Since $\Pi(x)$ possesses a differentiable density $\pi(x)$, $a_n^{(i,j)}$ can be written as

$$\lim_{N\to\infty} (\theta_i - \theta_j) \textstyle\sum_{k=1}^n (n!/(k-1)!(n-k)!) J_{n,k}(A_{k,N} + B_{k,N})$$

where, setting $t_N = (\theta_i - \theta_j)N^{-\frac{1}{2}}(\max_{i<j}(t_N) \to 0$, as $K \to \infty)$.

$$A_{k,N} = \int_0^\infty [T_N(x)]^{k-1}[1 - T_N(x)]^{n-k}(\pi(x - t_N) - \pi(x))(2t_N\cdot\pi(x))^{-1}\, dT(x)$$

and

$$B_{k,N} = \int_0^\infty [T_N(x)]^{k-1}[1 - T_N(x)]^{n-k}(\pi(x) - \pi(x + t_N))(2t_N\cdot\pi(x))^{-1}\, dT(x);$$

(in $A_{k,N}$ and $B_{k,N}$ we have suppressed the index (i,j) for convenience). The proof of the lemma will be complete if we show that

$$\lim_{N\to\infty} A_{k,N} = \lim_{N\to\infty} B_{k,N}$$

$$= \tfrac{1}{2} \int_0^\infty [T(x)]^{k-1}[1 - T(x)]^{n-k}(-\pi'(x)/\pi(x))\, dT(x) = D_k \text{ (say)}$$

where $D_k = \int_0^1 \psi(u) u^{k-1}(1-u)^{n-k}\, du$, and $T(x) = 2\Pi(x) - 1$, if $x \geqq 0$ and $T(x) = 0$ if $x < 0$. To see this, note that

$$|A_{k,N} - D_k| \leqq \tfrac{1}{2}\Big|\int_0^\infty [T_N(x)]^{k-1}[1 - T_N(x)]^{n-k}$$

$$\cdot\{(\pi(x - t_N) - \pi(x))(t_N\cdot\pi(x))^{-1} - (-\pi'(x)/\pi(x)\}\, dT(x)$$

$$+ \tfrac{1}{2}\Big|\int_0^\infty \{[T_N(x)]^{k-1}[1 - T_N(x)]^{n-k} - [T(x)]^{k-1}[1 - T(x)]^{n-k}\}$$

$$\cdot\pi'(x)/\pi(x)\, dT(x)\Big|$$

where the second term on the right $\to 0$, as $N \to \infty$, using the dominated convergence theorem and the condition Ω_2. Consider now the first term, which cannot exceed

$$\begin{aligned}
&\tfrac{1}{2}\left|\int_0^\infty [(\pi(x - t_N))^{\frac{1}{2}} - (\pi(x)^2)^{\frac{1}{2}}](t_N \cdot \pi(x))^{-1}\, dT(x)\right| \\
&\quad + \left|\int_0^\infty \{(\pi(x - t_N))^{\frac{1}{2}} - (\pi(x))^{\frac{1}{2}}](-t_N(\pi(x))^{\frac{1}{2}})^{-1} - \pi'(x)/2\pi(x)\}\, dT(x)\right| \\
&\leqq |t_N|[\int_0^\infty \{[(\pi(x - t_N))^{\frac{1}{2}} - (\pi(x))^{\frac{1}{2}}]t_N^{-1}\}^2\, dx] \\
&\quad + 2^{\frac{1}{2}}[\int_0^\infty \{[(\pi(x - t_N))^{\frac{1}{2}} - (\pi(x))^{\frac{1}{2}}](-t_N^{-1}) - \pi'(x)/(2(\pi(x))^{\frac{1}{2}})\}^2\, dx]^{\frac{1}{2}}.
\end{aligned}$$

The last inequality follows by applying Schwartz inequality to the second term. Both terms on the right tend to zero, as $N \to \infty$, on account of Lemma 4.3 of Hájek [9], since the condition Ω_2 implies the quadratic integrability of the derivative of $(\pi(x))^{\frac{1}{2}}$. This establishes that $\lim_{N\to\infty} A_{k,N} = D_k$. The same argument shows that $\lim_{N\to\infty} B_{k,N} = D_k$. The proof is complete.

Now substituting (2.1)II and the expression for d_n^2 in (1.10)II, we obtain

$$(2.2)\quad e_{L,L^*}^{(n)} = \left(\int_0^1 J(u)\psi(u)\, du\right)^2\left(\int_0^1 J^2(u)\, du\right)^{-1} \left[\left(\sum_{k=1}^n J_{n,k}\binom{n-1}{k-1} \int_0^1 \psi(u)u^{k-1}(1-u)^{n-k}\, du\right)^2\left(n^{-1}\sum_{k=1}^n J_{n,k}^2\right)^{-1}\right]^{-1}.$$

One naturally expects the local efficiency $e_{L,L^*}^{(n)}$ to converge to the asymptotic efficiency $E_{L,L^*} = 1$, as $n \to \infty$. However, despite the plausibility of the above statement, we are able to prove it only for the case when $\xi(u)$ is monotone.

THEOREM 2.1. *Suppose that the conditions Ω_1 and Ω_2 of Part* I *are satisfied. Then under the assumption of monotonicity of $J(u)$, $\lim_{n\to\infty} e_{L,L^*}^{(n)} = 1$.*

PROOF. Clearly we need to prove the theorem for non-constant $J(u)$, for otherwise L_N and L_N^* are identical and the result is trivially true. First we observe that, on account of the conditions Ω_1,

$$(2.3)\qquad n^{-1}\sum_{k=1}^n J_{n,k}^2 = \int_0^1 J_n^2(u)\, du \to \int_0^1 J^2(u)\, du < \infty,$$

as $n \to \infty$. Further, if we let $p_k(u) = \binom{n-1}{k-1}u^{k-1}(1-u)^{n-k}$,

$$\begin{aligned}
&\left(\int_0^1 \psi(u)\left[\sum_{k=1}^n J_{n,k}p_k(u)\right] du - \int_0^1 \psi(u)J(u)\, du\right)^2 \\
(2.4)\qquad &\leqq \left(\int_0^1 \psi^2(u)du\right)\left(\int_0^1 \sum_{k=1}^n [J_{n,k} - J(u)]^2 p_k(u)du\right) \\
&\leqq 2\left(\int_0^1 \psi^2(u)\, du\right)\left(\int_0^1 \sum_{k=1}^n [J_n(k/(n+1)) - J_n(u)]^2 p_k(u)\, du\right. \\
&\qquad + \int_0^1 [J_n(u) - J(u)]^2\, du,
\end{aligned}$$

by substituting $J_{n,k} = J_n(k/(n+1))$. The proof of the theorem will be complete if we show that the right hand side of (2.4)II tends to zero as $n \to \infty$, and use (2.3)II and (2.4)II in (2.2)II. For this it suffices, on account of Ω_1 and Ω_2, to prove that

$$(2.5)\qquad \lim_{n\to\infty} \int_0^1 \sum_{k=1}^n [J_n(k/(n+1)) - J_n(u)]^2 p_k(u)\, du = 0.$$

In order to prove (2.5)II we observe that on account of monotonicity of $J(u)$

and Ω_1, there is no loss of generality in assuming that $J_{n,1} \leqq J_{n,2} \leqq \cdots \leqq J_{n,n}$. Thus, using Lemma 2.1 of Hájek [9] it follows that left hand side of (2.5)II does not exceed

$$2 \cdot 2^{\frac{1}{2}} n^{-\frac{1}{2}} \max_{1 \leqq k \leqq n} |J_{n,k} - \bar{J}_n| [\textstyle\sum_{k=1}^{n} (J_{n,k} - \bar{J}_n)^2/n]^{\frac{1}{2}} \tag{2.6}$$

where $\bar{J}_n = \sum_{k=1}^{n} J_{n,k}/n$. Now the expression (2.6)II tends to zero as $n \to \infty$, since

$$n^{-1} \textstyle\sum_{k=1}^{n} (J_{n,k} - \bar{J}_n)^2 \leqq n^{-1} \sum_{k=1}^{n} J_{n,k}^2 \to \int_0^1 J^2(u)\, du < \infty$$

on account of (2.3) and

$$n^{-1} \max_{1 \leqq k \leqq n} |J_{n,k}|^2 = \max_{1 \leqq k \leqq n} \int_{[(k-1)/n, k/n]} J_n^2(u)\, du \to 0$$

as $n \to \infty$, on account of uniform integrability of the functions $J_n^2(u)$, a consequence of the conditions Ω_1; and the proof is complete.

For distribution functions $\Pi(z)$ not satisfying the differentiability conditions of Theorem 2.1 of Part I, one may define the local efficiency of L_N relative to L_N^* (as $K \to \infty$) in the same manner as the asymptotic relative efficiency $E^*_{S_1, S_2}(\pi)$ was defined in Part I, viz.,

$$e^{*(n)}_{L, L^*}(\pi) = \lim_{\sigma \to 0} e^{(n)}_{L, L^*}(\pi_\sigma), \tag{2.7}$$

provided the limit exists. It is interesting to note that $\lim_{n\to\infty} e^{*(n)}_{L, L^*}(\pi)$ may or may not be equal to 1, as is illustrated by the case when $\pi(z)$ is the cdf of the uniform distribution over $(-t, t)$ (see II.3).

II.3. Special cases. In this section, we shall evaluate the local efficiency $e^{(n)}_{L, L^*}$ for some well known distributions and the special choices of the functions $J_n(u)$ and $J(u)$ considered in Section I.3:

Wilcoxon-statistics. By substituting $\xi(u) = u$, $0 < u < 1$, and $J_{n,k} = (k/(n+1))$, $k = 1, 2, \cdots, n$, in (2.2)II we obtain

$$e^{(n)}_{W, W^*}(\pi) = \tfrac{1}{2}(n+1)(2n+1)\left(\int_0^1 u\psi(u)\, du\right)^2 \cdot \left(\int_0^1 \psi(u)[(n-1)u + 1)\, du\right)^{-2}, \tag{3.1}$$

so that from (3.3)I it follows that

$$\begin{aligned} e^{(n)}_{W, W^*}(\text{Normal}) &= (n+1)(2n+1)/2(n-1+2^{\frac{1}{2}})^2 > 1, \\ e^{(n)}_{W, W^*}(\text{Logistic}) &= (2n+2)/(2n+1) > 1, \\ e^{(n)}_{W, W^*}(\text{Double exponential}) &= (2n+1)/(2n+2) < 1, \\ e^{n}_{W, W^*}(\text{Cauchy}) &= (2n+1)/(2n+2) < 1. \end{aligned} \tag{3.2}$$

For evaluating $e^{*(n)}_{W, W^*}$ (uniform), defined by (2.7)II, we note that the density $\pi_\sigma(z)$ of the distribution $\Pi_\sigma(z)$, the convolution of $R(-\frac{1}{2}, \frac{1}{2})$ and $N(0, 1)$ distributions, is given by

$$\pi_\sigma(z) = \Phi((2z+1)/2\sigma) - \Phi((2z-1)/2\sigma), \tag{3.3}$$

where Φ is the standard normal cdf, so that from (3.1)II

$$
\begin{aligned}
e^{*(n)}_{W,W^*}\text{ (uniform)} &= \lim_{\sigma\to 0} e^{(n)}_{W,W^*}(\pi) \\
&= \lim_{\sigma\to 0} \tfrac{1}{2}(n+1)(2n+1)\left(\textstyle\int_0^1 u\psi_\sigma(u)\,du\right)^2 \\
(3.4)\qquad &\quad\cdot\left[\left(\textstyle\int_0^1 \psi_\sigma(u)[(n-1)u+1]\,du\right)^2\right]^{-1} \\
&= \lim_{\sigma\to 0} \tfrac{1}{2}(n+1)(2n+1)\left(\textstyle\int_{-\infty}^{\infty} \psi_\sigma^2(x)\,dx\right)^2 \\
&\quad\cdot\left(\pi_\sigma(0) + (n-1)\textstyle\int_{-\infty}^{\infty} \pi_\sigma^2(x)\,dx\right)^{-2} \\
&= (n+1)(2n+1)(2n^2)^{-1} > 1;
\end{aligned}
$$

the last equality follows by interchanging the limit and integration, which is permissible since $|\pi_\sigma(x)| < 2$ for $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$ and $|\pi_\sigma(x)|$ is bounded by a Lebesgue integrable function for $x < -\frac{1}{2}$ and $x > \frac{1}{2}$. We note that the local efficiency expressions (3.2)II and (3.4)II converge to 1, as $n \to \infty$.

Absolute-normal-score statistics. By letting $J(u) = \chi^{-1}(u)$, $0 < u < 1$, in (2.2)II we obtain the local efficiencies $e^{(n)}_{L_1,L_1^*}$ and $e^{(n)}_{L_2,L_2^*}$ for the absolute-normal-score statistics defined in Section 3I, namely,

$$
\begin{aligned}
(3.5)\quad e^{(n)}_{L,L^*}(\Pi) &= n^{-1}\textstyle\sum_{k=1}^n J^2_{n,k}\left(\int_0^1 \chi^{-1}(u)\psi(u)\,du\right)^2 \\
&\quad\cdot\left[\left(\textstyle\sum_{k=1}^n J_{n,k}\binom{n-1}{k-1}\int_0^1 \psi(u)u^{k-1}(1-u)^{n-k}\,du\right)^2\right]^{-1}
\end{aligned}
$$

which yields $e^{(n)}_{L_1,L_1^*}(\Pi)$ if the scores $J_{n,k} = J_n(k/(n+1))$, $k = 1, \cdots, n$, correspond to the Fisher-Yates type and $e^{(n)}_{L_2,L_2^*}(\Pi)$, if these scores correspond to the Van der Waerden type. From (3.3)II and (3.5)II we obtain

$$
\begin{aligned}
e^{(n)}_{L,L^*}(\text{Normal}) &= \left(\left(\textstyle\sum_{k=1}^n J^2_{n,k}\right)/n\right)\left(\textstyle\sum_{k=1}^n J_{n,k}\binom{n-1}{k-1}\right. \\
&\quad\left.\textstyle\int_0^1 \Phi^{-1}((1+u)/2)u^{k-1}(1-u)^{n-k}\,du\right)^{-2}, \\
e^{(n)}_{L,L^*}(\text{Logistic}) &= n(n+1)^2\left(\textstyle\sum_{k=1}^n J^2_{n,k}\right) \\
(3.6)\qquad &\quad\cdot\left(\pi\left(\textstyle\sum_{k=1}^n kJ_{n,k}\right)^2\right)^{-1}, \\
e^{(n)}_{L,L^*}(\text{Double Exponential}) &= 2n\left(\textstyle\sum_{k=1}^n J^2_{n,k}\right)\left(\pi\left(\textstyle\sum_{k=1}^n J_{n,k}\right)^2\right)^{-1}, \\
e^{(n)}_{L,L^*}(\text{Cauchy}) &= \left(\textstyle\sum_{k=1}^n J^2_{n,k}/n\right)\left(\textstyle\int_0^1 \Phi^{-1}((1+u)/2)\right. \\
&\quad\left.\cdot(\sin \pi u)\,du\right)^2\left(\textstyle\sum_{k=1}^n J_{n,k}\binom{n-k}{k-1}\right. \\
&\quad\left.\textstyle\int_0^1 (\sin \pi u)u^{k-1}(1-u)^{n-k}\,du\right)^{-2}.
\end{aligned}
$$

For evaluating $e^{*(n)}_{L,L^*}(\text{Uniform})$, we note from (3.5)II that

$$
\begin{aligned}
e^{*(n)}_{L,L^*}(\text{Uniform}) &= \lim_{\sigma\to 0} e^{(n)}_{L,L^*}(\Pi_\sigma) \\
(3.7)\qquad &\geqq \lim_{\sigma\to 0}\left(\textstyle\int_0^1 \Phi^{-1}((1+u)/2)\psi_\sigma(u)\,du\right)^2 \\
&\quad\cdot\left(n^2\left(\textstyle\int_0^1 \psi_\sigma(u)\,du\right)^2\right)^{-1}
\end{aligned}
$$

where, on account of Fatou's lemma,

TABLE 2

$e^{(n)}_{L_1,L_1^*}$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = \infty$
Normal	1.571	1.34	1.24	1.20	1.14	1
Logistic	1.273	1.152	1.102	1.075	1.058	1
Double exponential	.637	.746	.803	.839	.862	1
$e^{(n)}_{L_2,L_2^*}$						
Normal	1.571	1.34	1.24	1.20	1.14	1
Logistic	1.273	1.148	1.096	1.068	1.051	1
Double exponential	.637	.730	.781	.814	.835	1

$$\begin{aligned}\lim_{\sigma\to 0}&\left(\int_0^1 \Phi^{-1}((1+u)/2)\psi_\sigma(u)\,du\right)\\ &= \liminf_{\sigma\to 0}\int_0^1 (\pi_\sigma[\Pi_\sigma^{-1}(u)]/\phi[\Phi^{-1}(u)])\,du\\ &\geqq \int_0^1 \liminf_{\sigma\to 0}(\pi_\sigma[\Pi_\sigma^{-1}(u)]/\phi[\Phi^{-1}(u)])\,du\\ &= \int_{-\frac12}^{\frac12}\{\phi[\Phi^{-1}(u+\tfrac12)]\}^{-1}\,du = \infty.\end{aligned} \tag{3.8}$$

Also, from (3.3)II we have

$$\int_0^1 \psi_\sigma(u)\,du = 2[\Phi(1/2\sigma) - \Phi(-1/2\sigma)] \to 2$$

as $\sigma \to \infty$. From (3.7), (3.8), and (3.9) of this section, it follows that $e^{*(n)}_{L,L^*}$(Uniform) $= \infty$ for both version $L_{N,1}$ and $L_{N,2}$ of the absolute-normal-score statistics.

The *approximate* numerical values of the local efficiency expressions (3.6) II are tabled in Table 2 for both versions of the absolute-normal-score statistics.

We observe that for the cases considered in Table 2, the numerical values of $e^{(n)}_{L,L^*}(\Pi)$ seem to converge monotonically to 1, as $n \to \infty$.

II.4. Conclusion. The local efficiency expressions and their numerical values obtained in the preceding section indicate the superiority of the "joint-ranking" procedure against shift alternatives with normal, uniform or logistic as the underlying distribution; whereas against a double exponential or Cauchy distribution the "separate-ranking" statistic L_N^* seems to have better local power. These observations however seem merely incidental to a presumably more basic pattern suggested by the following: (a) First, we note that for $n = 1$ the local efficiency $e^{(n)}_{L,L^*}$ reduces to $E_{L,D}$, the asymptotic efficiency of L_N relative to the Durbin statistic, and (b) secondly, that for the special cases considered above the local efficiency seems to converge monotonically to 1, as $n \to \infty$. Thus if we consider, for a given choice of function $J_N(u)$ and $J(u)$, the class of all distributions satisfying (b), it follows that $e^{(n)}_{L,L^*} > 1$ or $<$ for all n, according as $E_{L,D}(\Pi) > 1$ or <1. These considerations suggest the following heuristic conclusion (for the class of distributions satisfying the condition (b)): *For a given functions $J(u)$ and $J_N(u)$, the "joint-ranking statistic" $L_N(J, J_N)$ is preferable to its counterpart $L_N^*(J, J_N)$ based on "separate-rankings", except for alternative distributions for which the Durbin-statistic is relatively Pitman-efficient than the statistic $L_N(J, J_N)$* i.e., *for which $E_{L(J,J_N),D}(\Pi) > 1$*. It would be of interest to characterize for a given function $J(u)$ the class of distributions satisfying the condition (b). For example, for the Wilcoxon-statistics W and W^*, a simple characterization of such a class would be: The class of all cdf's $\Pi(z)$ for which

$$t(\pi) = \pi(0)/\int_{-\infty}^{\infty} \pi^2(z)\, dz$$

is either $<(12/7)$ or $>(7/4)$. In fact, $e_{W,W^*}^{(n)}$ decreases monotonically to 1 if $t(\pi) < (12/7)$ (e.g., normal, logistic, uniform distributions) and increases monotonically to 1 if $t(\pi) > (7/4)$ (e.g. Cauchy and double exponential). Accordingly, since $E_{W,D} = e_{W,W^*}^{(1)}$, for this class of distributions the above heuristic conclusion clearly holds for the Wilcoxon-statistics.

A strong argument in favour of the "joint-ranking" procedure, however, is the following: Consider the problem of testing H_0 against the alternatives of shift in location and assume that the underlying distribution $\Pi(z)$ is specified. Then, one can select a most Pitman-efficient rank-order statistic by letting $J(u) = \psi(u)$ in L_N or L_N^*. However, since $E_{L,L^*} = 1$, the choice is still to be made between the "joint-ranking" and the "separate-ranking" procedures. Now one can easily show that, for the above choice of the function $J(u)$,

$$\begin{aligned} e_{L,L^*}^{(n)}(\pi) &= \left(\int_0^1 \psi^2(u)\, du\right)\left(n^{-1} \sum_{k=1}^{n} \psi_{n,k}^2\right) \\ &\quad \cdot \left(\sum_{k=1}^{n} \psi_{n,k} \binom{n-1}{k-1} \int_0^1 \psi(u) u^{k-1}(1-u)^{n-k}\, du\right)^{-2} \\ &\geqq \left(\left(\sum_{k=1}^{n} \psi_{n,k}^2\right)/n\right)\left(\int_0^1 \left[\sum_{k=1}^{n} \psi_{n,k}\binom{n-1}{k-1} u^{k-1}(1-u)^{n-k}\right]^2 du\right)^{-1} \geqq 1, \end{aligned}$$

with equality sign only if $J(u) = \psi(u) = \text{const.}$, $0 < u < 1$, in which case obviously the statistics L_N, L_N^* are identical. This leads us to the conclusion that, against a specified alternative distribution $\Pi(z)$, the "best" rank-order statistic (in the sense of local-power) is the one based on the "joint-ranking" procedure.

Finally, it seems worth mentioning that the form of the hypotheses H_0 favours the "joint-ranking" procedure. The "separate-ranking" statistic is essentially a test of symmetry about zero for each of the distributions $\Pi_{ij}(z)$ i.e., $\Pi_{ij}(z) + \Pi_{ij}(-z) = 1$, $(1 \leqq i < j \leqq K)$. It does not take into consideration the second part of the hypothesis H_0, namely, that $\Pi_{ij}(z) = \Pi_{i'j'}(z)$ for any two pairs (i, j) and (i', j'), whereas the "joint-ranking" statistic L_N does take this into consideration.

PART III

III.1. Summary. Let (ξ_{il}, η_{jl}), $l = 1, \cdots, N_{ij}$; $1 \leqq i < j \leqq K$ be independent samples from populations with absolutely continuous cdf's $D_{ij}(u, v)$. Denote $a_{ij,r}^* = +1$, if the rth smallest observation from the ordered absolute values $|Z_{ijl}|$ where $Z_{ijl} = \xi_{il} - \eta_{il}$, in the combined sample of size $N = \sum\sum_{i<j} N_{ij}$ is from a positive Z_{ij}, and otherwise let $a_{ij,r}^* = 0$. Denote $a_{ij,r}^{**} = -1$, if the rth smallest observation from the ordered absolute values $|Z_{ijl}|$ in the combined sample of size N is from a negative Z_{ij} and otherwise let $a_{ij,r}^{**} = 0$.

Denote

$$(1.1)\qquad \tau_N^{(i,j)} = \tau_{ij}^{+} + \tau_{ij}^{-}$$

where

$$(1.2)\qquad \tau_{ij}^{+} = m_{ij}T_{ij}^{+} = \sum_{r=1}^{N} E_{N,r}a_{ij,r}^{*}$$

and

$$(1.3)\qquad \tau_{ij}^{-} = n_{ij}T_{ij}^{-} = \sum_{r=1}^{N} E_{N,r}\, a_{ij,r}^{**}.$$

The $E_{N,r}$ are given numbers satisfying certain restrictions to be stated below; and m_{ij} and n_{ij} are the number of positive and negative Z's among $Z_{ij1}, \cdots, Z_{ijN_{ij}}$. The purpose of this part is to find a set of sufficient conditions for the joint asymptotic normality of the statistics $\tau_N^{(i,j)}$. Various applications of these statistics are given in Part I where the problem of testing the hypothesis of no difference among several different treatments is considered for the case when the comparison between the treatment is possible only in pairs. (However, in Part I, the joint asymptotic normality of the statistics $\tau_N^{(i,j)}$'s which can be obtained as a special case of the more general theorem (Theorem 3.1 below), is obtained by following the methods of Hájek [9] so as to present a different approach to the reader).

III.2. Assumptions and notations. Let $c = \binom{K}{2}$ denote the number of all possible pairs and label them $\alpha = 1, \cdots, c$. Let m_α and n_α be the number of positive and negative Z's respectively for the αth pair. m_α and n_α are random but $m_\alpha + n_\alpha = N_\alpha$ is non-random. For given m_α, let $X_{\alpha 1}^{+}, \cdots, X_{\alpha m_\alpha}^{+}$ denote the positive Z's and $\bar{X}_{\alpha 1}, \cdots, \bar{X}_{\alpha n_\alpha}$ denote the absolute values of negative Z's among $Z_{\alpha 1}, \cdots, Z_{\alpha N_\alpha}$; $\alpha = 1, \cdots, c$. Let $F^{+(\alpha)}(x)$ and $F^{-(\alpha)}(x)$ denote the cdf's of X_α^{+}'s and X_α^{-}'s respectively. Let $F_{m_\alpha}^{+}(x)$ and $F_{n_\alpha}^{-}(x)$ denote the sample cdf's of X_α^{+}'s and X_α^{-}'s respectively. Define

$$(2.1)\qquad H_N(x) = \sum_{\alpha=1}^{c} \rho_\alpha F_{n_\alpha}^{-}(x) + \sum_{\alpha=1}^{c} \rho_\alpha \nu_\alpha (F_{m_\alpha}^{+}(x) - F_{n_\alpha}^{-}(x))$$

and

$$(2.2)\qquad H(x) = \sum_{\alpha=1}^{c} \rho_\alpha F^{-(\alpha)}(x) + \sum_{\alpha=1}^{c} \rho_\alpha \nu_\alpha \Delta_\alpha(x)$$

where

$$(2.3)\qquad \rho_\alpha = N_\alpha/N, \quad \nu_\alpha = m_\alpha/N_\alpha, \quad \Delta_\alpha(x) = F^{+(\alpha)}(x) - F^{-(\alpha)}(x).$$

Denote

$$(2.4)\qquad H^{*}(x) = \sum_{\alpha=1}^{c} \rho_\alpha F^{-(\alpha)}(x) + \sum_{\alpha=1}^{c} \rho_\alpha p_\alpha \Delta_\alpha(x)$$

where

$$(2.5)\qquad p_\alpha = E(\nu_\alpha) \quad \text{and} \quad E \quad \text{denotes the expectation.}$$

Let

(2.6) $$\mu_{2,\alpha} = E(\nu_\alpha - p_\alpha)^2; \qquad s_\alpha = (\nu_\alpha - p_\alpha)/\mu_{2,\alpha}^{\frac{1}{2}}.$$

Define

$$\tilde{E}(\cdot) = E[(\cdot) \mid |s_\alpha| \leqq \omega], \qquad \alpha = 1, \cdots, c,$$

where ω is a fixed positive constant, and similarly $\tilde{\text{v}}\text{ar}(\cdot)$ and $\tilde{\text{c}}\text{ov}(\cdot)$. Note that $(\ ) \mid [\ ]$ stands for $(\ )$ given $[\ ]$. Denote

(2.7) $$a^*_{\alpha,r} = +1,$$

if the rth smallest observation from the ordered absolute values $|Z_{\alpha j}|$, $j = 1, \cdots, N_\alpha$; $\alpha = 1, \cdots, c$, is an X_α^+ observation and otherwise denote $a^*_{\alpha,r} = 0$. Denote $a^{**}_{\alpha,r} = -1$, if the rth smallest observation from the ordered absolute values $|Z_{\alpha j}|$, $j = 1, \cdots, N_\alpha$; $\alpha = 1, \cdots, c$, is an X_α^- observation and otherwise denote $a^{**}_{\alpha,r} = 0$. Then [cf. (1.1), (1.2), (1.3)], we can rewrite $\tau_N^{(\alpha)}, \tau_\alpha^+, \tau_\alpha^-, T_\alpha^+, T_\alpha^-$ as

(2.8) $$\tau_N^{(\alpha)} = \tau_\alpha^+ + \tau_\alpha^-$$

where

(2.9) $$\tau_\alpha^+ = m_\alpha T_\alpha^+ = \textstyle\sum_{r=1}^N E_{N,r} a^*_{\alpha,r} = m_\alpha \int J_N[H_N(x)]\, dF^+_{m_\alpha}(x),$$

(2.10) $$\tau_\alpha^- = n_\alpha T_\alpha^- = \textstyle\sum_{r=1}^N E_{N,r} a^{**}_{\alpha,r} = -n_\alpha \int J_N[H_N(x)]\, dF^-_{n_\alpha}(x),$$

and where

(2.11) $$E_{N,r} = J_N(r/N), \qquad r = 1, \cdots, N.$$

While J_N need be defined only at $1/N, \cdots, N/N$, it will be convenient to extend its domain of definition to $(0, 1]$ by letting it have constant value on $(r/N, (r + 1)/N]$. Let

$$J(H(x)) = \lim_{N\to\infty} J_N(H(x)).$$

Denote

(2.12) $$a_\alpha^+ = \int J[H(x)]\, dF^{+(\alpha)}(x), \qquad a_\alpha^- = -\int J[H(x)]\, dF^{-(\alpha)}(x);$$

(2.13) $$\tilde{a}_\alpha^+ = \tilde{E}(m_\alpha a_\alpha^+), \qquad \tilde{a}_\alpha^- = \tilde{E}(n_\alpha a_\alpha^-);$$

(2.14) $$L_0^{+(\alpha)} = \int J[H^*(x)]\, dF^{+(\alpha)}(x); \qquad L_0^{-(\alpha)} = \int J[H^*(x)]\, dF^{-(\alpha)}(x);$$

(2.15) $$L_{1,i}^{+(\alpha)} = \int \Delta_i(x) J'[H^*(x)]\, dF^{+(\alpha)}(x);$$

$$J'[H^*(x)] = dJ[H^*(x)]/dH^*(x);$$

(2.16) $$L_{1,i}^{-(\alpha)} = \int \Delta_i(x) J'[H^*(x)]\, dF^{-(\alpha)}(x);$$

(2.17) $$d_\alpha^+ = N_\alpha p_\alpha L_0^{+(\alpha)}, \qquad d_\alpha^- = -N_\alpha q_\alpha L_0^{-(\alpha)}, \qquad q_\alpha = 1 - p_\alpha;$$

(2.18) $$d_N^{(\alpha)} = d_\alpha^+ + d_\alpha^-;$$

(2.19) $$I_{(+\alpha;+i,+k)}(x, y) = F^{+(\alpha)}(x)[1 - F^{+(\alpha)}(y)] J'[H(x)] J'[H(y)];$$

$I^*_{(+\alpha;+i,+k)}(x, y)$ = the expression for $I_{(+\alpha;+i,+k)}(x, y)$ with H changed to H^*;

(2.20) $$U_{(+\alpha;+i,+k)} = \int\int_{-\infty<x<y<\infty} I_{(+\alpha;+i,+k)}(x, y)\, dF^{+(i)}(x)\, dF^{+(k)}(y);$$
$$V_{(+\alpha;+i,+k)} = \int\int_{-\infty<y<x<\infty} I_{(+\alpha;+i,+k)}(y, x)\, dF^{+(i)}(x)\, dF^{+(k)}(y);$$

$U^*_{(+\alpha;+i,+k)}$ = the expression of $U_{(+\alpha;+i,+k)}$ with I changed to I^*, and $V^*_{(+\alpha;+i,+k)}$ = the expression for $V_{(+\alpha;+i,+k)}$ with I changed to I^*.

(2.21) $$\begin{aligned} Nb^2_{\alpha^+} = {} & 2\sum_{i=1,i\neq\alpha}^{c} \lambda_i U_{(+i;+\alpha,+\alpha)} + 2\sum_{i=1}^{c} \mu_i U_{(-i;+\alpha,+\alpha)} \\ & + (2/\lambda_\alpha)\sum_{i=1,i\neq\alpha}^{c} \lambda_i^2 U_{(+\alpha;+i,+i)} + (2/\lambda_\alpha)\sum_{i=1}^{c} \mu_i^2 U_{(\alpha;-i,-i)} \\ & + (1/\lambda_\alpha)\sum_{(3)} \lambda_i\lambda_k W_{(+\alpha;+i,+k)} + (1/\lambda_\alpha)\sum_{(2)} \mu_i\mu_k W_{(+\alpha;-i,-k)} \\ & + (2/\lambda_\alpha)\sum_{(1)} \lambda_i\mu_k W_{(+\alpha;+i,-k)}\,; \qquad \lambda_i = m_i/N, \quad \mu_i = n_i/N, \end{aligned}$$

where (1) indicates the summation over all (i, k) with $i \neq \alpha$, the (2) over all (i, k) with $i \neq k$, and (3) over all (i, k) with $i \neq k, i \neq \alpha, k \neq \alpha$, and where $W = U + V$ with U and V having the same subscripts as W.

(2.22) $Nb^2_{\alpha^-}$ = the expression of $Nb^2_{\alpha^+}$ with λ's and μ's interchanged, and the subscripts of U's and V's written with opposite signs.

(2.23) $$\begin{aligned} Nb^{++}_{\alpha,\alpha'} = {} & -\sum_{i=1}^{c} \lambda_i[W_{(+\alpha;+i,+\alpha')} + W_{(+\alpha';+i,+\alpha)} - W_{(+i;+\alpha,+\alpha')}] \\ & - \sum_{i=1}^{c} \mu_i[W_{(+\alpha;+i,+\alpha')} + W_{(+\alpha';+i,+\alpha)} - W_{(-i;+\alpha,+\alpha')}] \end{aligned}$$

and a similar expression for $Nb_{\alpha-,\alpha'-}$.

(2.24) Nb_{α^+,α'^-} = $-$(the right hand side of (2.23) with $+\alpha'$ changed to $-\alpha'$), and similar expressions for Nb_{α^+,α^-} and $Nb_{\alpha'^+,\alpha^-}$.

(2.25) $$a_N^{(\alpha)} = m_\alpha a_\alpha^{+} + n_\alpha a_\alpha^{-}$$

(2.26) $$b_N^{(\alpha)2} = m_\alpha^2 b_\alpha^2 + n_\alpha^2 b_\alpha^2 + 2m_\alpha n_\alpha b_{\alpha^+,\alpha^-}$$

(2.27) $$b_N^{(\alpha,\alpha')} = m_\alpha m_{\alpha'} b_{\alpha^+,\alpha'^+} + m_\alpha n_{\alpha'} b_{\alpha^+,\alpha'^-} + m_{\alpha'} n_\alpha b_{\alpha'^+,\alpha^-} + n_\alpha n_{\alpha'} b_{\alpha^-,\alpha'^-}\,.$$

(2.28) $$\begin{aligned} \beta^2_{\alpha^+} = {} & 2N_\alpha\rho_\alpha p_\alpha^2 \sum_{i=1,i\neq\alpha}^{c} \rho_i p_i U^*_{(+i;+\alpha,+\alpha)} \\ & + 2N_\alpha\rho_\alpha p_\alpha^2 \sum_{i=1}^{c} \rho_i q_i U^*_{(-i;+\alpha,+\alpha)} \\ & + 2N_\alpha p_\alpha \sum_{i=1,i\neq\alpha}^{c} \rho_i^2 p_i^2 U^*_{(+\alpha;+i,+i)} + 2N_\alpha p_\alpha \sum_{i=1}^{c} \rho_i^2 q_i^2 U^*_{(+\alpha;-i,-i)} \\ & + N_\alpha p_\alpha \sum_{(3)} \rho_i\rho_k p_i p_k W^*_{(+\alpha;+i,+k)} + N_\alpha p_\alpha \sum_{(2)} \rho_i\rho_k q_i q_k W^*_{(+\alpha;-i,-k)} \\ & + 2N_\alpha p_\alpha \sum_{(1)} \rho_i\rho_k p_i q_k W^*_{(+\alpha;+i,-k)} \\ & + N_\alpha^2[(L_0^{+(\alpha)})^2 \mu_{2,\alpha} + p_\alpha^2 \sum_{i=1}^{c} \rho_i^2 \mu_{2,i}(L_{1,i}^{+(\alpha)})^2 + 2p_\alpha\rho_\alpha\mu_{2,\alpha} L_0^{+(\alpha)} L_{1,i}^{+(\alpha)}] \end{aligned}$$

where $W^* = U^* + V^*$ with U^* and V^* having the same subscripts as W^*.

(2.29) $\beta^2_{\alpha-}$ = the expression for $\beta^2_{\alpha+}$ with p's and q's interchanged; subscripts of U^* and V^* written with opposite signs; $L_0^{+(\alpha)}$ changed to $-L_0^{-(\alpha)}$; $L_{1,i}^{+(\alpha)}$ changed to $L_{1,i}^{-(\alpha)}$.

(2.30) $d^{xy}_{\alpha,\alpha'} = N_\alpha \rho_\alpha p_\alpha^x p^y_{\alpha'}$ (the expression for $b^{xy}_{\alpha,\alpha'}$ with λ_i, μ_i, U and V changed to $\rho_i p_i$, $\rho_i q_i$, U^* and V^* respectively), where $p_\alpha^x = p_\alpha$ or q_α as x is $+$ or $-$, and $p^y_{\alpha'} = p_{\alpha'}$ or $q_{\alpha'}$ as y is $+$ or $-$.

(2.31) $d_{\alpha+,\alpha-} = N_\alpha \rho_\alpha p_\alpha q_\alpha$ (the expression for $b_{\alpha+,\alpha-}$ with λ_i, μ_i, U and V changed as in (2.30))

(2.32) $d_{\alpha'+,\alpha-} = N_\alpha \rho_\alpha p_{\alpha'} q_\alpha$ (the expression for $b_{\alpha'+,\alpha-}$ with λ_i, μ_i and U and V changed as in (2.30)).

$$\beta_{\alpha+,\alpha-} = d_{\alpha+,\alpha-} + N_\alpha^2 \mu_{2,\alpha} L_0^{+(\alpha)} L_0^{-(\alpha)} + N_\alpha^2 \mu_{2,\alpha} p_\alpha \rho_\alpha L_{1,\alpha}^{+(\alpha)} L_0^{-(\alpha)} - N_\alpha^2 \mu_{2,\alpha} q_\alpha \rho_\alpha L_{1,\alpha}^{-(\alpha)} L_0^{+(\alpha)} - N_\alpha^2 p_\alpha' q_\alpha \sum_{i=1}^c \rho_i^2 \mu_{2,i} L_{1,i}^{+(\alpha)} L_{1,i}^{-(\alpha)}. \tag{2.33}$$

$$\beta_{\alpha+,\alpha'-} = d_{\alpha+,\alpha'-} - N_\alpha N_{\alpha'} \mu_{2,\alpha} q_{\alpha'} \rho_\alpha L_{1,\alpha}^{-(\alpha')} L_0^{+(\alpha)} + N_\alpha N_{\alpha'} \mu_{2,\alpha'} p_\alpha \rho_{\alpha'} L_0^{-(\alpha')} L_{1,\alpha'}^{+(\alpha)} - N_\alpha N_{\alpha'} p_\alpha q_{\alpha'} \sum_{i=1}^c \rho_i^2 \mu_{2,i} L_{1,i}^{+(\alpha)} L_{1,i}^{-(\alpha')}. \tag{2.34}$$

$$\beta_{\alpha'+,\alpha-} = d_{\alpha'+,\alpha-} - N_\alpha N_{\alpha'} \mu_{2,\alpha'} q_\alpha \rho_{\alpha'} L_{1,\alpha'}^{-(\alpha)} L_0^{+(\alpha')} + N_\alpha N_{\alpha'} \mu_{2,\alpha} p_{\alpha'} \rho_\alpha L_{1,\alpha}^{+(\alpha')} L_0^{-(\alpha)} - N_\alpha N_{\alpha'} p_{\alpha'} q_\alpha \cdot \sum_{i=1}^c \rho_i^2 \mu_{2,i} L_{1,i}^{+(\alpha')} L_{1,i}^{-(\alpha)}. \tag{2.35}$$

$$\beta_{\alpha-,\alpha'-} = d_{\alpha-,\alpha'-} - N_\alpha N_{\alpha'} \mu_{2,\alpha} q_{\alpha'} \rho_\alpha L_{1,\alpha}^{-(\alpha')} L_0^{-(\alpha)} - N_\alpha N_{\alpha'} \mu_{2,\alpha'} q_\alpha \rho_{\alpha'} L_{1,\alpha'}^{-(\alpha)} L_0^{-(\alpha')} + N_\alpha N_{\alpha'} q_\alpha q_{\alpha'} \sum_{i=1}^c \rho_i^2 \mu_{2,i} L_{1,i}^{-(\alpha)} L_{1,i}^{-(\alpha')}. \tag{2.36}$$

$$\beta_{\alpha+,\alpha'+} = d_{\alpha+,\alpha'+} + N_\alpha N_{\alpha'} \mu_{2,\alpha} p_{\alpha'} \rho_\alpha L_{1,\alpha}^{+(\alpha')} L_0^{+(\alpha)} + N_\alpha N_{\alpha'} \mu_{2,\alpha'} p_\alpha \rho_{\alpha'} L_{1,\alpha'}^{+(\alpha)} L_0^{+(\alpha')} + N_\alpha N_{\alpha'} p_\alpha p_{\alpha'} \sum_{i=1}^c \rho_i^2 \mu_{2,i} L_{1,i}^{+(\alpha)} L_{1,i}^{+(\alpha')}. \tag{2.37}$$

The methods used in the proofs for the asymptotic normality of $\tau_N^{(\alpha)}$'s are mainly adaptations of the methods of [18] and [7]. It is assumed that the sample sizes N_α tend to infinity in such a way that $N_\alpha = \rho_\alpha \cdot N$, $N \to \infty$.

III.3. Joint asymptotic normality.

THEOREM 3.1. *If*

(i) $E(\nu_\alpha) = p_\alpha \to p_{\alpha_0}$ *such that* $0 < p_{\alpha_0} < 1$,

(ii) $\mu_{2,\alpha} = E(\nu_\alpha - p_\alpha)^2 = O(1/N)$,

(iii) *for* m_α *such that* $|s_\alpha| \leqq \omega$ *for some fixed* $\omega > 0$,

$$\Pr(\underline{m_\alpha} = m_\alpha) = p(m_\alpha) = (N_\alpha(\mu_{2,\alpha})^{\frac{1}{2}})^{-1}\phi(s_\alpha) + o(1/N^{\frac{1}{2}})$$

where ϕ is the standard normal density function, and $s_\alpha = (\nu_\alpha - p_\alpha)/\mu_{2,\alpha}^{\frac{1}{2}}$ and, if for given $F^{+(\alpha)}(x)$, $F^{-(\alpha)}(x)$; λ_α, μ_α bounded away from zero and one,

(iv) *the conditions Ω_3 of Section 2, Part I, are satisfied then the random vector $({\tau_N}^{(1)} - {d_N}^{(1)}, \cdots, {\tau_N}^{(c)} - {d_N}^{(c)})$ has a limiting normal distribution with zero mean vector and covariance matrix.*

$$
\begin{aligned}
&\operatorname{var}({\tau_N}^{(\alpha)} - {d_N}^{(\alpha)}) \\
&= {\beta_N}^{(\alpha)^2} = \beta^2_{\alpha^+} + \beta^2_{\alpha^-} + 2\, d_{\alpha^+,\alpha^-} + 2{N_\alpha}^2 \mu_{2,\alpha} {L_0}^{+(\alpha)} {L_0}^{-(\alpha)} \\
&\quad - 2{N_\alpha}^2 q_\alpha \mu_{2,\alpha} \rho_\alpha L^{-(\alpha)}_{1,\alpha} {L_0}^{+(\alpha)} - 2{N_\alpha}^2 p_\alpha q_\alpha \textstyle\sum_{i=1}^c {\rho_i}^2 \mu_{2,i} L^{+(\alpha)}_{1,i} L^{-(\alpha)}_{1,i}
\end{aligned} \tag{3.1}
$$

where $\beta^2_{\alpha^+}$, $\beta^2_{\alpha^-}$, d_{α^+,α^-}, $\mu_{2,i}$, ${L_0}^{+(\alpha)}$ and ${L_0}^{-(\alpha)}$, $L^{+(\alpha)}_{1,i}$ and $L^{-(\alpha)}_{1,i}$ are given by (2.28), (2.29), (2.33), (2.6), (2.14), (2.15) and (2.16) respectively.

$$
\begin{aligned}
&\operatorname{cov}({\tau_N}^{(\alpha)} - {d_N}^{(\alpha)}, {\tau_N}^{(\alpha')} - {d_N}^{(\alpha')}) = {\beta_N}^{(\alpha,\alpha')} \\
&= d_{\alpha^+,\alpha'^+} + d_{\alpha^+,\alpha'^-} + d_{\alpha'^+,\alpha^-} + d_{\alpha^-,\alpha'^-} \\
&\quad + N_\alpha N_{\alpha'} p_{\alpha'} \rho_\alpha \mu_{2,\alpha} {L_0}^{+(\alpha)} L^{+(\alpha)'}_{1,\alpha} + N_\alpha N_{\alpha'} p_\alpha \rho_{\alpha'} \mu_{2,\alpha'} {L_0}^{+(\alpha')} L^{(+\alpha)}_{1,\alpha'} \\
&\quad + N_\alpha N_{\alpha'} p_\alpha p_{\alpha'} \textstyle\sum_{i=1}^c {\rho_i}^2 \mu_{2,i} L^{+(\alpha)}_{1,i} L^{+(\alpha')}_{1,i} - N_\alpha N_{\alpha'} q_{\alpha'} \rho_\alpha \mu_{2,\alpha} {L_0}^{+(\alpha)} L^{-(\alpha')}_{1,\alpha} \\
&\quad + N_{\alpha'} N_\alpha p_\alpha \rho_{\alpha'} \mu_{2,\alpha'} {L_0}^{-(\alpha')} L^{+(\alpha)}_{1,\alpha'} - N_\alpha N_{\alpha'} p_{\alpha'} q_{\alpha'} \textstyle\sum_{i=1}^c {\rho_i}^2 \mu_{2,i} L^{+(\alpha)}_{1,i} L^{-(\alpha')}_{1,i} \\
&\quad - N_\alpha N_{\alpha'} q_\alpha \rho_{\alpha'} L^{-(\alpha)}_{1,\alpha'} {L_0}^{+(\alpha')} \mu_{2,\alpha'} + N_\alpha N_{\alpha'} p_{\alpha'} \rho_\alpha \mu_{2,\alpha} {L_0}^{-(\alpha)} L^{+(\alpha')}_{1,\alpha} \\
&\quad - N_\alpha N_{\alpha'} p_{\alpha'} q_\alpha \textstyle\sum_{i=1}^c {\rho_i}^2 \mu_{2,i} L^{+(\alpha')}_{1,i} L^{-(\alpha)}_{1,i} \\
&\quad - N_\alpha N_{\alpha'} q_{\alpha'} \rho_\alpha {L_0}^{-(\alpha)} L^{-(\alpha')}_{1,\alpha} \mu_{2,\alpha} - N_\alpha q_\alpha N_{\alpha'} \rho_{\alpha'} \mu_{2,\alpha'} {L_0}^{-(\alpha')} L^{-(\alpha)}_{1,\alpha'} \\
&\quad + N_\alpha N_{\alpha'} q_\alpha q_{\alpha'} \textstyle\sum_{i=1}^c {\rho_i}^2 \mu_{2,i} L^{-(\alpha)}_{1,i} L^{-(\alpha')}_{1,i}.
\end{aligned} \tag{3.2}
$$

REMARKS. (a) The Theorem 3.1 remains valid if the assumption (iii) is replaced by the assumption

(iii)′ $p(m_\alpha) = (1/N_\alpha \mu_{2,\alpha}^{\frac{1}{2}})[\phi(s_\alpha) + h(\phi(s_\alpha))] + o(1/N)$, where ϕ is the standard normal, density, $h(\phi)$ is a polynomial in ϕ whose coefficients involve inverse powers of N_α, and $s_\alpha = (\nu_\alpha - p_\alpha)/\mu_{2,\alpha}^{\frac{1}{2}}$.

(b) The assumptions (ii) and (iii) of Theorem 3.1 are satisfied if the random variable m_α has a binomial distribution with parameters N_α and p_α such that $p_\alpha \to p_{\alpha_0}$, $0 < p_{\alpha_0} < 1$.

(c) The assumptions (ii) and (iii) of Theorem 6.1 are also satisfied if m_α has a hypergeometric distribution, and the size of the population ${N_\alpha}^*$ and the size of the sample N_α, are such that $N^* = O(N^{k+\delta})$ for $k \geqq 2$ and some $\delta > 0$, for then (cf. [7]),

$$p(m_\alpha) = \binom{N_\alpha}{m_\alpha} p_\alpha^{m_\alpha} q_\alpha^{N_\alpha - m_\alpha} + o(1/{N_\alpha}^{k-2}).$$

To prove this theorem, we first consider the case when the sample sizes m_α, n_α; $\alpha = 1, \cdots, c$, are non-random instead of random. In such a case the random variables $(X^+_{\alpha 1}, \cdots, X^+_{\alpha m_\alpha})$ and $(X^-_{\alpha 1}, \cdots, X^-_{\alpha n_\alpha})$ can be regarded as constituting $2c$ independent samples from the distribution functions $F^{+(\alpha)}(x)$ and

$F^{-(\alpha)}(x)$ respectively, $\alpha = 1, \cdots, c$; and we have the following specializations of the conditional analogues of Theorem 3.1, the proofs of which follow by proceeding exactly as in Theorem 6.1 of Puri (1964), and are therefore omitted.

(3A) Non-Random Case.

Lemma 3A.1. *If assumption* (iv) *of Theorem* 3.1 *is satisfied, then the random vector* $N^{\frac{1}{2}}(T_1^+ - a_1^+, \cdots, T_c^+ - a_c^+)$ *where* T^+*'s and* a^+*'s are defined by* (2.9) *and* (2.12) *respectively, has a limiting normal distribution with zero mean vector and variance-covariances given by* $Nb_{\alpha^+}^2$ *and* Nb_{α^+,α'^+} *where* $b_{\alpha^+}^2$ *and* b_{α^+,α'^+} *are defined in* (2.21) *and* (2.23) *respectively.*

Lemma 3A.2. *If assumption* (iv) *of Theorem* 3.1 *is satisfied, then the random vector* $N^{\frac{1}{2}}(T_1^- - a_1^-, \cdots, T_c^- - a_c^-)$ *where* T^-*'s and* a^-*'s are defined by* (2.10) *and* (2.12) *respectively, has a limiting normal distribution with zero mean vector and variance-covariances given by* $Nb_{\alpha^-}^2$ *and* Nb_{α^-,α'^-} *where* $b_{\alpha^-}^2$ *and* b_{α^-,α'^-} *are defined in* (2.22) *and* (2 .23) *respectively.*

Theorem 3A.2. *Under the assumptions of Lemma* 3A.1, *the random vector* $W = (W^{(1)}, \cdots, W^{(c)})$ *where*

$$W^{(\alpha)} = N^{-\frac{1}{2}}(m_\alpha T_\alpha^+ + n_\alpha T_\alpha^- - m_\alpha a_\alpha^+ - n_\alpha a_\alpha^-) \tag{3.3}$$

has a limiting normal distribution with zero mean vector and variance-covariances given by $N^{-1}b_N{}^{(\alpha)2}$ *and* $N^{-1}b_N{}^{(\alpha,\alpha')}$ *where* $b_N{}^{(\alpha)2}$ *and* $b_N{}^{(\alpha,\alpha')}$ *are defined in* (2.26) *and* (2.27) *respectively.*

We have thus established the joint asymptotic normality of the random variables $\tau_N{}^{(\alpha)}$'s when the sample sizes m_α, n_α ($\alpha = 1, \cdots, c$) are non-random. We now drop the assumption that m_α and n_α are non-random. We assume that m_α, n_α are random variables which satisfy the assumptions (i) to (iii) of Theorem 3.1.

(3B) Random Case. We shall need the following lemmas:

Lemma 3B.1. *Under the assumptions* (ii) *and* (iii) *of Theorem* 3.1

$$\tilde{\mu}_{1,\alpha} = E\{(\nu_\alpha - p_\alpha) \mid |s_\alpha| \leqq \omega\} = o(N^{-\frac{1}{2}}) \tag{3.4}$$

$$|\mu_{2,\alpha} - \tilde{\mu}_{2,\alpha}| = O(\omega e^{-\omega^2/2}/N) + o(N^{-1}), \tag{3.5}$$

where $\tilde{\mu}_{2,\alpha} = E\{(\nu_\alpha - p_\alpha)^2 \mid |s_\alpha| \leqq \omega\}$.

The proof of this lemma is the same as in ([7], p. 37) and is therefore omitted

Lemma 3B.2. *Let* $\{X_N\}$ *be a sequence of random variables and* $\{r_N\}$ *a sequence of numbers. If* $X_N = r_N + O_p(t_N)$ *where* $t_N \to 0$ *and* $r_N \to r$ *as* $N \to \infty$, *and* $h(x)$ *is a function admitting continuous* $(j + 1)$st *derivative in some interval containing* r, *then*

$$\begin{aligned} h(X_N) &= h(r_N) + \textstyle\sum_{i=1}^{j} h^{(i)}(r_N)(X_N - r_N)^i/i! \\ &\quad + [(X_N - r_N)^{j+1}/(j+1)!]\, h^{(j+1)}(eX_n + (1-e)r_N), \qquad 0 < e < 1, \end{aligned} \tag{3.6}$$

$$h(X_n) = h(r_N) + \textstyle\sum_{i=1}^{j} h^{(i)}(r_N)(X_N - r_N)^i/i! + o_p(t_N{}^j). \tag{3.7}$$

Proof. (3.6) is just the Taylor expansion of $h(X_N)$ and (3.7) follows as a special case of the Corollary 3 of Mann and Wald [15].

LEMMA 3B.3. *Under the assumption* (ii) *of Theorem* 3.1

$$(3.8)\qquad J(H) = J(H^*) + J'(H^*) \textstyle\sum_{i=1}^{c} \rho_i \Delta_i(x)(\nu_i - p_i) + o_p(N^{-\frac{1}{2}}),$$

$$(3.9)\qquad J'(H(x))J'(H(y)) = J'(H^*(x)J'(H^*(y)) + o_p(1).$$

PROOF. The proof follows by noting that $H(x) = H^*(x) + O_p(N^{-\frac{1}{2}})$, and applying Lemma 3B.2.

LEMMA 3B.4. *If the assumptions* (ii), (iii) *and* (iv) *of Theorem* 3.1 *are satisfied, then for large N*

$$(3.10)\qquad a_\alpha{}^*X = N_\alpha\, d_\alpha L_0^{X(\alpha)} + N_\alpha(\nu_\alpha - p_\alpha)L_0^{X(\alpha)} + N_\alpha e_\alpha \textstyle\sum_{i=1}^{c} \rho_i(\nu_i - p_i)L_{1,i}^{X(\alpha)} + O(N^{\frac{1}{2}}),$$

$$(3.11)\qquad b_\alpha{}^*X = \beta_\alpha{}^2X - N_\alpha{}^2[(L_0{}^{X(\alpha)})^2\mu_{2,\alpha} + d_\alpha{}^2 \textstyle\sum_{i=1}^{c} \rho_i{}^2\mu_{2,i}(L_{1,i}^{X(\alpha)})^2 + 2\rho_\alpha e_\alpha \mu_{2,\alpha} L_0{}^{X(\alpha)} L_{1,i}^{X(\alpha)}] + O(N)$$

where

$$a_\alpha{}^*X = m_\alpha a_{\alpha^+}, \quad b_\alpha{}^*X = m_\alpha^2 b_{\alpha^+}^2, \quad d_\alpha = e_\alpha = p_\alpha \quad \textit{if} \quad X \textit{ is } +;$$

$$a_\alpha{}^*X = n_\alpha a_\alpha{}^-, \quad b_\alpha{}^*X = n_\alpha^2\text{-}b_{\alpha}^2\text{-}, \quad d_\alpha = e_\alpha = q_\alpha \quad \textit{if} \quad X \textit{ is } -.$$

(3.12) $m_\alpha m_{\alpha'} b_{\alpha^+,\alpha'^+} = N_\alpha \rho_{\alpha'} p_\alpha p_{\alpha'}$ [*the expression for* Nb_{α^+,α'^+} (cf. (2.23) *with* λ_i, μ_i, U *and* V *changed to* $\rho_i p_i$, $\rho_i q_i$, U^* *and* V^* *respectively*] $+ o(N)$;

(3.13) $n_\alpha n_{\alpha'} b_{\alpha^-,\alpha'^-} = N_\alpha \rho_{\alpha'} q_\alpha q_{\alpha'}$ [*the expression for* Nb_{α^-,α'^-} (cf. (2.23)) *with* λ_i, μ_i, U *and* V *changed as in* (3.12)] $+ o(N)$;

(3.14) $m_{\alpha'} n_\alpha b_{\alpha'^+,\alpha^-} = N_\alpha \rho_{\alpha'} p_{\alpha'} q_\alpha$ [*the expression for* $Nb_{\alpha'^+,\alpha^-}$ *with* λ_i, μ_i, U *and* V *changed as in* (3.12)] $+ o(N)$;

(3.15) $m_\alpha n_{\alpha'} b_{\alpha^+,\alpha'^-} = N_\alpha \rho_{\alpha'} p_\alpha q_{\alpha'}$ [*the expression for* Nb_{α^+,α'^-} *with* λ_i, μ_i, U *and* V *changed as in* (3.12)] $+ o(N)$;

(3.16) $m_\alpha n_\alpha b_{\alpha^+,\alpha^-} = N_\alpha \rho_\alpha p_\alpha q_\alpha$ [*the expression for* Nb_{α^+,α^-} *with* λ_i, μ_i, U *and* V *changed as in* (3.12)] $+ o(N)$.

PROOF. Apply Lemma 3B.3 and make use of the facts that $v_\alpha{}^2\nu_i = p_\alpha{}^2 p_i + o(1)$; $v_\alpha{}^2(1 - \nu_i) = p_\alpha{}^2 q_i + o(1)$ and similar expressions for $\nu_\alpha \nu_i^2$, $\nu_\alpha(1 - \nu_i)^2$, $\nu_\alpha \nu_i \nu_k, \nu_\alpha(1 - \nu_i)(1 - \nu_k)$ and $\nu_\alpha \nu_i(1 - \nu_k)$.

LEMMA 3B.5. *If the hypothesis of Lemma* 3B.4 *hold, then for large* N_α, $\alpha = 1, \cdots, c$,

$$(3.17)\qquad (d_\alpha - m_\alpha a_{\alpha^+})/m_\alpha b_{\alpha^+} = -\textstyle\sum_{i=1}^{c} s_i v_i / I_1 + o(1);$$

$$(3.18)\qquad \beta_{\alpha^+}/m_\alpha b_{\alpha^+} = I_2/I_1 + o(1);$$

where

$$N_\alpha \mu_{2,\alpha} = p_\alpha q_\alpha c_\alpha^{\ 2}, \qquad c_\alpha = O(1);$$

$$v_i = p_\alpha(\rho_i \rho_\alpha p_i q_i)^{\frac{1}{2}} c_i L_{1,i}^{+(\alpha)}), \qquad i = 1, \cdots, c; \quad i \neq \alpha;$$

$$v_\alpha = (p_\alpha q_\alpha)^{\frac{1}{2}}(c_\alpha L_0^{\ +(\alpha)} + p_\alpha \rho_\alpha c_\alpha L_{1,\alpha}^{+(\alpha)};$$

$$I_1^{\ 2} = 2\rho_\alpha p_\alpha^{\ 2} \sum_{i=1, i\neq\alpha}^{c} \rho_i p_i U^*_{(+i;+\alpha,+\alpha)} + 2\rho_\alpha p_\alpha^{\ 2} \sum_{i=1}^{c} \rho_i q_i U^*_{(-i;+\alpha,+\alpha)}$$
$$+ 2p_\alpha \sum_{i=1, i\neq\alpha}^{c} \rho_i^2 p_i^{\ 2} U^*_{(+\alpha;+i,+i)} + 2p_\alpha \sum_{i=1}^{c} \rho_i^2 q_i^{\ 2} U^*_{(+\alpha;-i,-i)}$$
$$+ p_\alpha \sum\nolimits_{(3)} \rho_i \rho_k p_i p_k W^*_{(+\alpha;+i,-k)} + p_\alpha \sum\nolimits_{(2)} \rho_i \rho_k q_i q_k W^*_{(+\alpha;-i,-k)}$$
$$+ 2p_\alpha \sum\nolimits_{(1)} \rho_i \rho_k p_i q_k W^*_{(+\alpha;+i,-k)}$$

where W^* *is defined in* (2.28). *and* $I_2^{\ 2} = I_1^{\ 2} + \sum_{i=1}^{c} v_i^{\ 2}$.

The proof of this lemma involves straightforward algebraic computations and is therefore omitted.

LEMMA 3B.6. *If*

(i) $0 < \lambda_0 \leqq \lambda_1, \cdots, \lambda_c \leqq 1 - \lambda_0 < 1$ *for some* $\lambda_0 \leqq 1/2c$,
$0 < \mu_0 \leqq \mu_1, \cdots, \mu_c \leqq 1 - \mu_0 < 1$ *for some* $\mu_0 \leqq 1/2c$,

(ii) *the assumptions* (ii), (iii), *and* (iv) *of Theorem* 3.1 *hold,*

(iii) $E(T_{\alpha^+} \mid \lambda_1, \cdots, \lambda_c)$, $E(T_{\alpha^-} \mid \lambda_1, \cdots, \lambda_c)$, $\operatorname{var}(T_{\alpha^+} \mid \lambda_1, \cdots, \lambda_c)$, $\operatorname{var}(T_{\alpha^-} \mid \lambda_1, \cdots, \lambda_c)$ *exist, then for large* N_α *such that* $\omega(\mu_{2,\alpha})^{\frac{1}{2}} < p_\alpha q_\alpha$,

$$\tilde{E}(\tau_{\alpha\mp}) = d_{\alpha\mp} + o(N^{\frac{1}{2}}); \tag{3.19}$$

$$\tilde{E}(\tau_N^{\ (\alpha)}) = d_N^{\ (\alpha)} + o(N^{\frac{1}{2}}); \tag{3.20}$$

$$\tilde{\operatorname{var}}(\tau_{\alpha\mp}) = B^2_{\alpha\mp} + O(N\omega e^{-\omega^2/2}) + o(N); \tag{3.21}$$

$$\tilde{\operatorname{cov}}(\tau_{\alpha^+}, \tau_{\alpha^-}) = \beta_{\alpha^+,\alpha^-} + O(N\omega e^{-\omega^2/2}) + o(N); \tag{3.22}$$

$$\tilde{\operatorname{var}}(\tau_N^{\ (\alpha)}) = \beta^2_{\alpha^+} + \beta^2_{\alpha^-} + 2\beta_{\alpha^+,\alpha^-} + O(N\omega e^{-\omega^2/2}) + o(N); \tag{3.23}$$

$$\tilde{\operatorname{cov}}(\tau_N^{\ (\alpha)}, \tau_N^{\ (\alpha')}) = \beta_{\alpha^+,\alpha'^+} + \beta_{\alpha^+,\alpha'^-} + \beta_{\alpha'^+,\alpha^-} + \beta_{\alpha^-,\alpha'^-} + O(N\omega e^{-\omega^2/2}) + o(N). \tag{3.24}$$

NOTE. The quantities d_{α^+}, d_{α^-}, $d_N^{\ (\alpha)}$, $\beta^2_{\alpha^+}$, $\beta^2_{\alpha^-}$, $\beta_{\alpha^+,\alpha^-}$, $\beta_{\alpha^+,\alpha'^+}$, $\beta_{\alpha^+,\alpha'^-}$, $\beta_{\alpha'^+,\alpha^-}$, $\beta_{\alpha^-,\alpha'^-}$ are all defined in Section 2.

The proof of the lemma follows by straightforward computations.

LEMMA 3B.7. *Under the assumptions of Theorem* 3.1, *the random vector* $(\tau_1^{\ +} - d_1^{\ +}, \cdots, \tau_c^{\ +} - d_c^{\ +})$ *has a limiting normal distribution with zero mean vector and covariance matrix*

$$\operatorname{var}(\tau_{\alpha^+} - d_{\alpha^+}) = \beta^2_{\alpha^+}, \qquad \operatorname{cov}(\tau_{\alpha^+} - d_{\alpha^+}, \tau_{\alpha'^+} - d_{\alpha'^+}) = \beta_{\alpha^+,\alpha'^+} \tag{3.25}$$

where $\beta^2_{\alpha^+}$ *and* $\beta_{\alpha^+,\alpha'^+}$ *are given by* (2.28) and (2.39) *respectively.*

The proof of this lemma follows from Theorem 3.1 of [7] as does Lemma 3A.1 (or Theorem 6.1 of [18]) from Theorem 1 of [3].