Mechanics in Differential Geometry

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LEIDEN • BOSTON - 2006

A C.I.P. record for this book is available from the Library of Congress

ISBN-13: 978-90-6764-457-0
ISBN-10: 90-6764-457-9

Printed and bound in The Netherlands.

To Joëlle

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PREFACE

Modern mechanics is undoubtedly a branch of differential geometry and the title '*Mechanics in Differential Geometry*' asserts this obviousness.

In particular the structure *Mechanics-Geometry* presented in this book shows that these disciplines complement one another in a pedagogical manner; mechanics is developed from entities and methods of differential geometry, but mechanics throws light on sometimes arduous geometrical concepts.

Numerous references are made to the classic '*Foundations of Mechanics*' by Abraham and Marsden, and to my previous book entitled '*Differential Geometry with Applications to Mechanics and Physics*'. Some of my published and unpublished research works are also introduced. This '*Mechanics in Differential Geometry*' is also the fruit of my teaching experience, and much work during the 1995-2002 period.

Leading scientists as Newton, Euler, Lagrange, Laplace, Poisson, Jacobi, Hamilton, Liouville and several others have created mechanics from the classical Newton–Leibnitz calculus.

Mathematician and mechanician of genius H. Poincaré viewed mechanics from its global geometric side and considered the phase space as a differentiable manifold notably. So the modern mechanics was born shortly before the 20th century. Moreover, it is a recognized fact that his discoverer spirit has mathematically generated the Einstein's relativistic leap.

The works of M.S. Lie, and later in a deciding manner the intrinsic calculus of E. Cartan elaborated for the first part of the 20th century, have brought mechanics to the field of differential geometry.

Outstanding works of G.D. Birkhoff, C.L. Siegel, A. Kolmogorov, J. Moser, S. Smale, and V. Arnold notably have proven this reality. Published works of mathematicians as S. Marsden, A. Weinstein, etc., and numerous recent papers confirm the position of mechanics in geometry.

This book is mainly a course devoted to the fourth year students in mathematics, physics and certain paths in engineering. These various audiences will discover notions of geometry and mechanics closely connected, they will learn essential mathematical methods for mechanics and physics.

It is also a reference book since it prepares for research. The introduced notions should be known by students when beginning a Ph.D. in theoretical physics, theoretical mechanics, celestial mechanics, stellar dynamics, special relativity, general relativity, cosmology, differential geometry, quantum mechanics, statistical mechanics notably. It could also be used for self-study, given its pedagogical structure and the solved problems which prepare for these disciplines. The reader will quickly discover the importance of successive chapters and, given their controlled and logical progression as well as their role of introducing the above mentioned disciplines, will eagerly take up the challenge.

Since modern mechanics is a branch of differential geometry it was necessary to state the foundations of this 20th century geometry beforehand. So, the fundamental spaces of mechanics are manifolds defined in geometry, the notions of tangent and cotangent bundles are introduced in a practical way which will be particularly appreciate by physicists and engineers. This properly mathematical part of the book largely takes advantage of my previous differential geometry book as proven by the pragmatic development of vector and covector fields, tensor fields, tensor algebra, exterior algebra, Lie derivative, Lie group, Lie algebra, integration of forms and invariants, Riemannian geometry, etc.. Pedagogy, rigor and succinctness highly characterize the first part of this book. These foundations of differential geometry recalled as an unavoidable prerequisite make the work autonomous.

Symplectic geometry in Chapter 2 is the 'interlinking field' of this textbook, because the manifold symplectic structure, canonical forms, brackets, etc. concern modern mechanics. At this stage it is essential to mention that physicists, mechanicians, engineers and mathematicians too will appreciate proofs in local coordinates.

In our modern exposition of mechanics, Lagrangian and Hamiltonian mechanics are first separately elaborated with their own spaces, their own functions and equations. Fundamental principles, forms and invariants, for example, are clearly situated in their respective mechanics formulations.
Secondly, a didactic comparison between these formulations sheds light on notions which could have been tricky without the help of diagrams which, as figures, are numerous throughout this work, for instance, rhombic diagrams in the study of the second-order differential equation, diagrams illustrating relationships between Lagrangian and canonical forms, etc..
Next the Hamilton–Jacobi theory is carefully studied and the powerful integration method of Jacobi is clearly developed in the context of problems with perturbation forces notably.
Finally, a special introduction to perturbations deals with the important problem of stability, notably for infinitesimal canonical transformations, qualitative dynamics, 'third' integral, transverse sections, Poincaré mapping, etc..

In the last chapter, after a succinct recall of equations and integrals of the N-body problem, the mean potential and mean density are properly and explicitly compared with the respective real potential and real density.
The two-body problem in Hamilton–Jacobi theory, Kepler elements, osculating orbits, Lagrange equations in celestial mechanics, perturbated two-body problem are concisely treated as well as statistical mechanics, fluid-dynamical system of the author, etc..
The second part of this chapter presents notions which would deserve further study and could be a matter for research. It particularly illustrates the fact that differential geometry has numerous applications in stellar dynamics, celestial mechanics and statistical mechanics notably.

According to usage the vertical brackets completely enclose the elements of matrices, whereas they partly enclose the normal mathematical expressions.

The important propositions and the formulae to be framed are shown by ☞ and ✍. The glossary of symbols should make the assimilation of notions easier.

All the proofs, examples and the 36 solved exercises are described in detail.

I am preparing special texts about perturbation, stability, chaotic behavior, etc. by referring to galactic dynamics, celestial mechanics, and other areas. These texts should complement this global introduction of mechanics in differential geometry.

This textbook has been essentially written up and typed in 2002-2005.

Yves R. Talpaert

CHAPTER 0

INTRODUCTION TO TENSORS

It is pointless insisting on the considerable importance that tensors have gained through the developments of exact and applied sciences in the 20^{th} century.

Given the purpose of this book and the presentation of various subjects of theoretical mechanics, it was necessary to recall notions of tensor analysis. Definitions and propositions have been introduced for an inventory and the notation, but most of the proofs have been omitted since they are set out in our previous books[1] where in addition numerous exercises are expounded.

1. WELL KNOWN CONCEPTS

1.1 MULTILINEAR FORMS

Let E and F be finite-dimensional real vector spaces.

1.1.1 Linear Mapping

D A mapping

$$\boldsymbol{g} : E \to F : \boldsymbol{x} \mapsto \boldsymbol{g}(\boldsymbol{x})$$

is ***linear*** if $\forall \boldsymbol{x}, \boldsymbol{y} \in E, \forall k \in \boldsymbol{R}$:

$$\boldsymbol{g}(\boldsymbol{x}+\boldsymbol{y}) = \boldsymbol{g}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{y}), \qquad \boldsymbol{g}(k\boldsymbol{x}) = k\,\boldsymbol{g}(\boldsymbol{x}).$$

[1] For instance, in *Mechanics, Tensors & Virtual Works* (2002), and *Tensor Analysis & Continuum Mechanics* (2003) which are mentioned in the bibliography.

Let us denote by $L(E;F)$ the set of (continuous) linear mappings of E to F.

D The ***addition*** in $L(E;F)$ is the mapping

$$L(E;F)\times L(E;F)\to L(E;F):(\boldsymbol{g},\boldsymbol{h})\mapsto \boldsymbol{g}+\boldsymbol{h}$$

such that the ***sum*** $\boldsymbol{g}+\boldsymbol{h}$ is the linear mapping defined by

$$E\to F:\boldsymbol{x}\mapsto(\boldsymbol{g}+\boldsymbol{h})(\boldsymbol{x})=\boldsymbol{g}(\boldsymbol{x})+\boldsymbol{h}(\boldsymbol{x}).$$

D The ***multiplication*** of a linear mapping $\boldsymbol{g}$ of E into F ***by a scalar*** k is the mapping

$$\boldsymbol{R}\times L(E;F)\to L(E;F):(k,\boldsymbol{g})\mapsto k\boldsymbol{g}$$

such that the ***product*** $k\boldsymbol{g}$ (of $\boldsymbol{g}$ by k) is the linear mapping defined by

$$E\to F:(k,\boldsymbol{g})(\boldsymbol{x})=k\boldsymbol{g}(\boldsymbol{x}).$$

We know that $L(E;F)$ provided with the two previous laws of addition and multiplication has the structure of a vector space.

1.1.2 Multilinear Form

In mechanics we particularize F by choosing this vector space to be $\boldsymbol{R}$. So we will consider the vector space $L(E;\boldsymbol{R})$ later on.

D A ***linear form*** on E is a mapping

$$\boldsymbol{f}:E\to\boldsymbol{R}:\boldsymbol{x}\mapsto\boldsymbol{f}(\boldsymbol{x})$$

such that $\forall\boldsymbol{x},\boldsymbol{y}\in E,\ \forall k\in\boldsymbol{R}$:

$$\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{y})=\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{f}(\boldsymbol{y}),\qquad \boldsymbol{f}(k\boldsymbol{x})=k\boldsymbol{f}(\boldsymbol{x}).$$

A linear form on E is also called a ***one-form*** or ***covector***.

Let $E_{(1)},\ldots,E_{(p)}$ be p vector spaces.

D A ***p-linear form*** defined on the Cartesian product of p spaces $E_{(1)}\times\ldots\times E_{(p)}$ is a mapping

$$\boldsymbol{f}:E_{(1)}\times\ldots\times E_{(p)}\to\boldsymbol{R}:(\boldsymbol{x}_{(1)},\ldots,\boldsymbol{x}_{(p)})\mapsto\boldsymbol{f}(\boldsymbol{x}_{(1)},\ldots,\boldsymbol{x}_{(p)})$$

which is linear with respect to each vector, that is,

$\forall\boldsymbol{x}_{(1)},\boldsymbol{y}_{(1)}\in E_{(1)},\ldots,\ \forall\boldsymbol{x}_{(p)},\boldsymbol{y}_{(p)}\in E_{(p)},\ \forall k\in\boldsymbol{R}$:

$$f(\boldsymbol{x}_{(1)}+\boldsymbol{y}_{(1)},\boldsymbol{x}_{(2)},\ldots,\boldsymbol{x}_{(p)})=f(\boldsymbol{x}_{(1)},\boldsymbol{x}_{(2)},\ldots,\boldsymbol{x}_{(p)})+f(\boldsymbol{y}_{(1)},\boldsymbol{x}_{(2)},\ldots,\boldsymbol{x}_{(p)})$$

$$\vdots$$

$$f(\boldsymbol{x}_{(1)},\ldots,k\boldsymbol{x}_{(p)})=k f(\boldsymbol{x}_{(1)},\ldots,\boldsymbol{x}_{(p)}).$$

Provided with laws of addition and multiplication by a scalar defined as before, the space $L_p(E;\boldsymbol{R})$ of p-linear forms on E has the structure of a vector space.

1.2 DUAL SPACE, VECTORS AND COVECTORS

1.2.1 Dual Space

D The vector space of linear forms defined on E is called the ***dual space*** of E.

It is denoted by E^*.

So the dual space is a vector space the elements of which, called *covectors*, are linear functions $E \to \boldsymbol{R}$. It is a space of functions.

1.2.2 Expression of a Covector

Let E be a real vector space of dimension n.

A covector on E is a linear mapping $f: E \to \boldsymbol{R}$ which associates a real $f(\boldsymbol{x})$ to each vector $\boldsymbol{x} \in E$.

We denote by $x^1,\ldots,x^n$ the components of $\boldsymbol{x}$ with respect to a basis $(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_n)$ of E.

The real $f(\boldsymbol{x})$ is written:

$$\begin{aligned} f(\boldsymbol{x}) &= f(x^1\boldsymbol{e}_1+\ldots+x^n\boldsymbol{e}_n) \\ &= x^1 f(\boldsymbol{e}_1)+\ldots+x^n f(\boldsymbol{e}_n). \end{aligned}$$

By letting

$$f_i = f(\boldsymbol{e}_i)$$

we have

$$f(\boldsymbol{x}) = \sum_{i=1}^{n} f_i x^i.$$

We mention that the image of $\boldsymbol{x}$ under $\boldsymbol{f}$ is sometimes called the *value* of the form (for $\boldsymbol{x}$).

We are now going to express the covector $\boldsymbol{f}$ with respect to the dual basis.

The dual basis $(\boldsymbol{e}^{*1},\ldots,\boldsymbol{e}^{*n})$ of the basis $(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_n)$ is such that:

☞ $$\boldsymbol{e}^{*i}(\boldsymbol{e}_j)=\delta^i_j\,, \tag{0-1}$$

where δ^i_j is the Kronecker delta[1] and the n linear forms making up the dual basis are

$$\boldsymbol{e}^{*i}:E\to\boldsymbol{R}:\boldsymbol{x}\mapsto\boldsymbol{e}^{*i}(\boldsymbol{x})=x^i\,.$$

Thus $\forall\boldsymbol{x}\in E$:

$$\boldsymbol{f}(\boldsymbol{x})=\sum_{i=1}^{n}f_i\,\boldsymbol{e}^{*i}(\boldsymbol{x})\,,$$

which leads to the expression of the covector

☞ $$\boldsymbol{f}=\sum_{i=1}^{n}f_i\,\boldsymbol{e}^{*i}\,. \tag{0-2}$$

Remark. The reader will compare the previous expression with that of a vector:

$$\boldsymbol{x}=\sum_{i=1}^{n}x^i\boldsymbol{e}_i\,.$$

So according to usage the components of vectors show an upper index and the components of covectors a lower index.

Notation. Generally we will represent the covectors (or linear forms) by Greek characters, and since they are the elements of a vector space, namely E^*, we have decided to write them in bold characters.

1.2.3 Einstein Summation Convention

The Einstein summation convention consists in removing the summation sign Σ, more precisely:

Summation is implied when an index is repeated on upper and lower levels.

---

[1] The Kronecker delta is the symbol

$$\delta^i_j=\delta_{ij}=\delta^{ij}=\begin{cases}1 & if\quad i=j,\\ 0 & if\quad i\neq j.\end{cases}$$

For example, we denote:

$$c_{ijk}x^i y^j z^k = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{ijk}x^i y^j z^k .$$

Remark. By extension, the Einstein convention is sometimes used with indices at the same height in the frame of the usual Euclidean space when considering orthonormal bases.

On the one hand, any repeated index of summation is called a *dummy index* because it does not matter what the letter is; for instance:

$$c_i x^i = c_j x^j = c_1 x^1 + c_2 x^2 + \cdots \quad .$$

On the other hand, there is another type of index. An index which appears once in each expression is called a *free index*.

So, for instance, the equation

$$z^i = c^i_j x^j \qquad\qquad i = 1,2 \,;\; j = 1,2,3,4$$

represents the following system of equations:

$$z^1 = c^1_k x^k ,$$
$$z^2 = c^2_r x^r .$$

In this example the index i is free and the index j is dummy.

1.2.4 Change of Basis and Cobasis

In an n-dimensional space E we say:

PR1 The matrix associated with the expression of *unprimed* components as functions of those *primed* is the transpose of the matrix associated with the expression of *primed* basis vectors as functions of the *unprimed.*

Proof. The matrices $\left(\alpha^j_i\right)$ and $\left(\beta^i_j\right)$ being inverse, we denote

$$\boldsymbol{e}'_j = \alpha^i_j \boldsymbol{e}_i , \tag{0-3}$$

$$\boldsymbol{e}_i = \beta^k_i \boldsymbol{e}'_k . \tag{0-4}$$

The equations

$$\boldsymbol{x} = x^i \boldsymbol{e}_i = x'^k \boldsymbol{e}'_k = x'^k \alpha^i_k \boldsymbol{e}_i$$

imply

$$x^i = \alpha^i_k x'^k . \tag{0-5}$$

By comparing the following explicit expressions

$$x^1 = \alpha_1^1 x'^1 + \alpha_2^1 x'^2 + \,\dots\,, \qquad\qquad e'_1 = \alpha_1^1 e_1 + \alpha_1^2 e_2 + \,\dots$$

and so on, the proposition is thus proved.

PR2 The matrix associated with the expression of *primed* components as a function of the *unprimed* components is the transpose of the matrix associated with the expression of *unprimed* basis vectors as functions of the *primed.*
It is the inverse and transpose of the matrix associated with the expression of *primed* basis vectors as functions of the *unprimed.*

Proof. Since

$$\boldsymbol{e}_i = \beta_i^j \boldsymbol{e}'_j\,,$$

the equalities

$$\boldsymbol{x} = x'^j \boldsymbol{e}'_j = x^i \boldsymbol{e}_i = x^i \beta_i^j \boldsymbol{e}'_j$$

imply

☞ $$x'^j = \beta_i^j x^i\,. \tag{0-6}$$

By comparing the following explicit expressions

$$x'^1 = \beta_1^1 x^1 + \beta_2^1 x^2 + \,\dots\,,$$
$$\boldsymbol{e}_1 = \beta_1^1 \boldsymbol{e}'_1 + \beta_1^2 \boldsymbol{e}'_2 + \,\dots$$
$$\boldsymbol{e}'_1 = \alpha_1^1 \boldsymbol{e}_1 + \alpha_1^2 \boldsymbol{e}_2 + \,\dots$$

and so on, the proposition is thus proved.

We are now going to show the formulae of the change of dual bases (a dual basis is also called a *cobasis*).
From every

$$\boldsymbol{x} = x^i \boldsymbol{e}_i = x'^j \boldsymbol{e}'_j\,,$$

because $\boldsymbol{e}'^{*j}(\boldsymbol{e}_k) = \delta_k^j$ we deduce:

$$\boldsymbol{e}'^{*j}(\boldsymbol{x}) = x'^j = \beta_i^j x^i = \beta_i^j \boldsymbol{e}^{*i}(\boldsymbol{x})$$

which implies

☞ $$\boldsymbol{e}'^{*j} = \beta_i^j \boldsymbol{e}^{*i} \tag{0-7}$$

and

☞ $$\boldsymbol{e}^{*i} = \alpha_j^i \boldsymbol{e}'^{*j}\,. \tag{0-8}$$

The reader will easily say the propositions which refer to (0-7) and (0-8).

1.3 TENSORS AND TENSOR PRODUCT

Let $E_{(1)}, \dots, E_{(p)}, \dots, E_{(p+q)}$ (or simply E) be finite-dimensional vector spaces.

1.3.1 Tensor Product of Multilinear Forms

Let f be a p-linear form defined by

$$E_{(1)} \times \ldots \times E_{(p)} \to \boldsymbol{R} : (\boldsymbol{x}_{(1)}, \ldots, \boldsymbol{x}_{(p)}) \mapsto \boldsymbol{f}(\boldsymbol{x}_{(1)}, \ldots, \boldsymbol{x}_{(p)}),$$

let $\boldsymbol{h}$ be a q-linear form defined by

$$E_{(p+1)} \times \ldots \times E_{(p+q)} \to \boldsymbol{R} : (\boldsymbol{x}_{(p+1)}, \ldots, \boldsymbol{x}_{(p+q)}) \mapsto \boldsymbol{h}(\boldsymbol{x}_{(p+1)}, \ldots, \boldsymbol{x}_{(p+q)}) .$$

D ☞ The ***tensor product*** of a p-linear form f and a q-linear form $\boldsymbol{h}$ is the $(p+q)$-linear form denoted $\boldsymbol{f} \otimes \boldsymbol{h}$:

$$E_{(1)} \times \ldots \times E_{(p+q)} \to \boldsymbol{R} : (\boldsymbol{x}_{(1)}, \ldots, \boldsymbol{x}_{(p+q)}) \mapsto \boldsymbol{f} \otimes \boldsymbol{h}(\boldsymbol{x}_{(1)}, \ldots, \boldsymbol{x}_{(p+q)})$$

such that

$$\boldsymbol{f} \otimes \boldsymbol{h}\ (\boldsymbol{x}_{(1)}, \ldots, \boldsymbol{x}_{(p+q)}) = \boldsymbol{f}(\boldsymbol{x}_{(1)}, \ldots, \boldsymbol{x}_{(p)})\, \boldsymbol{h}(\boldsymbol{x}_{(p+1)}, \ldots, \boldsymbol{x}_{(p+q)}) . \qquad (0\text{-}9)$$

1.3.2 Tensor of Type $\binom{0}{1}$

D A ***tensor of type*** $\binom{0}{1}$ or ***covector*** is a linear form defined on E.

It is an element of the vector space E^*.

According to usage the covectors are generally denoted by Greek letters; for instance:

$$\boldsymbol{\omega} \in E^* .$$

Thus the definition of a covector or tensor of type $\binom{0}{1}$ is expressed as follows:

$$\boldsymbol{\omega} : E \to \boldsymbol{R} : \boldsymbol{x} \mapsto \boldsymbol{\omega}(\boldsymbol{x})$$

with $\forall a, b \in \boldsymbol{R}, \forall \boldsymbol{x}, \boldsymbol{y} \in E$:

$$\boldsymbol{\omega}(a\boldsymbol{x} + b\boldsymbol{y}) = a\boldsymbol{\omega}(\boldsymbol{x}) + b\boldsymbol{\omega}(\boldsymbol{y}) .$$

The covector expressed as (0-2) is written:

$$\boldsymbol{\omega} = \omega_i \boldsymbol{e}^{*i},$$

where

$$\omega_i = \boldsymbol{\omega}(\boldsymbol{e}_i).$$

The image of any vector $\boldsymbol{x}$ under $\boldsymbol{\omega}$ is the real

$$\boldsymbol{\omega}(\boldsymbol{x}) = \omega_i \boldsymbol{e}^{*i}(x^j \boldsymbol{e}_j) = \omega_i x^j \boldsymbol{e}^{*i}(\boldsymbol{e}_j)$$
$$= \omega_i x^i,$$

this value being also denoted by

☞ $$\langle \boldsymbol{\omega}, \boldsymbol{x} \rangle = \boldsymbol{\omega}(\boldsymbol{x}). \tag{0-10}$$

Change of basis

We recall that a linear form $\boldsymbol{\omega}$ behaves towards any vector $\boldsymbol{x}$ in the following way:

$$\boldsymbol{\omega}(\boldsymbol{x}) = \boldsymbol{\omega}(x^i\, \boldsymbol{e}_i) = \boldsymbol{\omega}(x'^j\, \boldsymbol{e}'_j)$$

$$\Leftrightarrow$$

$$\omega_i x^i = \omega'_j x'^j = \langle \boldsymbol{\omega}, \boldsymbol{x} \rangle \tag{0-11}$$

where $\omega'_j = \boldsymbol{\omega}(\boldsymbol{e}'_j)$.

This obvious requirement allows testing of the 'tensor character'. Let us use it in order to obtain the formulae of transformation of components of $\boldsymbol{\omega}$.

By recalling (0-5) and (0-6):

$$x^i = \alpha^i_k x'^k, \quad x'^p = \beta^p_n x^n,$$
$$y^j = \alpha^j_r y'^r, \quad y'^s = \beta^s_m y^m,$$

the condition (0-11) that is

$$\omega_i x^i = \omega_i \alpha^i_k x'^k$$
$$= \omega'_k x'^k$$

implies

$$\omega'_k = \alpha^i_k \omega_i \tag{0-12}$$

and conversely

$$\omega_i = \beta^r_i \omega'_r. \tag{0-13}$$

1.3.3 Tensor of Type $\binom{1}{0}$

D A ***tensor of type*** $\binom{1}{0}$ or ***vector*** is a linear form defined on E^*.

So the definition of a tensor of type $\binom{1}{0}$ or vector $\boldsymbol{x}$ is expressed as follows:

$$\boldsymbol{x} : E^* \to \boldsymbol{R} : \boldsymbol{\omega} \mapsto \boldsymbol{x}(\boldsymbol{\omega})$$

such that $\forall\, a,b \in \boldsymbol{R}, \forall\, \boldsymbol{\omega}, \boldsymbol{\mu} \in E^*$:

$$\boldsymbol{x}(a\boldsymbol{\omega} + b\boldsymbol{\mu}) = a\boldsymbol{x}(\boldsymbol{\omega}) + b\boldsymbol{x}(\boldsymbol{\mu}).$$

A linear form $\boldsymbol{x}$ defining a tensor of type $\binom{1}{0}$ is obviously written with respect to a basis $(\boldsymbol{e}_i)$ of E as follows:

$$\boldsymbol{x} = x^i \boldsymbol{e}_i$$

where

$$x^i = \boldsymbol{x}(\boldsymbol{e}^{*i}).$$

So the image of any covector $\boldsymbol{\omega}$ under the linear form $\boldsymbol{x}$ is the real

$$\boldsymbol{x}(\boldsymbol{\omega}) = x^i \boldsymbol{e}_i(\omega_j \boldsymbol{e}^{*j}) = x^i \omega_j \delta_i^j = x^i \omega_i$$

which is written:

$$\boldsymbol{x}(\boldsymbol{\omega}) = \langle \boldsymbol{x}, \boldsymbol{\omega} \rangle .$$

In conclusion, we have obtained the following equality

$$\langle \boldsymbol{x}, \boldsymbol{\omega} \rangle = \langle \boldsymbol{\omega}, \boldsymbol{x} \rangle \tag{0-14}$$

and this important result expresses the duality between covectors and vectors.

Remarks. (i) In fact, we may identify $E^{**} = L(E^*; \boldsymbol{R})$ with E:

$$E^{**} = E .$$

By referring to the formulae (0-3) and (0-4) for change of basis in E on the one hand, and to their corresponding (0-7) and (0-8) in E^* on the other hand, we immediately see that the relevant vectors $\boldsymbol{e}_i^{**}$ of E^{**} and $\boldsymbol{e}_i$ of E are transformed according to the same rule.

To each vector expressed with respect to a basis $(\boldsymbol{e}_i)$ of E there corresponds a vector with the same components with respect to the corresponding basis $(\boldsymbol{e}_i^{**})$ of E^{**} and conversely, such that to the sum of any two vectors of E corresponds the sum of two corresponding vectors of E^{**}, to the product of a vector of E by a scalar corresponds the product of the corresponding element of E^{**} by this scalar.

Since there is no reason to distinguish the elements of E^{**} from those of E, we have the right to identify these vector spaces.

Algebra courses deal with this question, and the existence of an isomorphism between the finite-dimensional vector spaces E and E^{**} is easily proved.

(ii) Following from the duality expressed by (0-1), we point out that the covector $\boldsymbol{e}^{*i}$ of the dual basis associates with $\boldsymbol{x}$ the ith component x^i:

$$\langle \boldsymbol{e}^{*i}, \boldsymbol{x} \rangle = \boldsymbol{e}^{*i}(\boldsymbol{x}) = \boldsymbol{e}^{*i}(x^j \boldsymbol{e}_j) = x^i .$$

(iii) We note that the law (0-12) of the change of components of any covector is that of change of basis vectors (0-3).
It is not the case for a vector: the matrix is inverse! That is the reason why, initially, every vector (element of E) was called a *contravariant vector* and every covector (element of E^*) was called a *covariant vector.*
This terminology is logically given up because vectors and covectors exist as their own entities regardless of any basis change.
But later it could well be that we say 'indices of contravariance' and 'indices of covariance'.
According to convention the components of vectors show an upper index and the components of covectors a lower index.

1.3.4 Tensor of Type $\binom{0}{2}$

D A ***tensor of type*** $\binom{0}{2}$ is a bilinear form defined on $E \times E$.[1]

D The vector space of bilinear forms defined on $E \times E$ is called the ***tensor product space*** of two spaces E^*.

It is denoted

$$E^* \otimes E^* .$$

So any tensor of type $\binom{0}{2}$ is an element of $E^* \otimes E^*$ and we denote such a tensor as

$$\boldsymbol{t} \in E^* \otimes E^* .$$

Given a basis $(\boldsymbol{e}_i)$ of E we say:

[1] Sometimes called a covariant *tensor of order* 2.

PR3 A tensor of type $\binom{0}{2}$ is expressed as

$$t = t_{ij} e^{*i} \otimes e^{*j}, \tag{0-15}$$

where

$$t_{ij} = t(e_i, e_j)$$

and $(e^{*i} \otimes e^{*j})$ is a basis of $E^* \otimes E^*$.

Remark. As a vector x is sometimes and excessively referred to its components x^i, a tensor t of type $\binom{0}{2}$ can be referred to its components t_{ij} (and so for higher order tensors).

Change of basis

We recall that any tensor is an '*intrinsic mathemathical entity*'; that is, independent of the choice of basis; in other words, each real defined by a bilinear form t is not 'altered' by a change of basis.

Given a change of basis defined by $e'_j = \alpha^i_j e_i$, the components t_{ij} of a tensor $t \in E^* \otimes E^*$ are transformed as follows:

$$\begin{aligned} \forall (x,y),(x',y') \in E \times E: \quad & t(x,y) = t(x',y') \\ \Leftrightarrow \quad & t(x^i e_i, y^j e_j) = t(x'^r e'_r, y'^s e'_s) \\ \Leftrightarrow \quad & t_{ij} x^i y^j = t'_{rs} x'^r y'^s . \end{aligned} \tag{0-16}$$

Such a (general) requirement of tensor theory allows testing the 'tensor character'. So, let us use it in order to obtain the formulae of transformation of components of t.

We recall [see (0-5) and (0-6)]:

$$\begin{aligned} x^i &= \alpha^i_k x'^k, \quad x'^p = \beta^p_n x^n, \\ y^j &= \alpha^j_r y'^r, \quad y'^s = \beta^s_m y^m . \end{aligned}$$

The condition (0-16), namely:

$$\begin{aligned} t_{ij} x^i y^j &= t_{ij} \alpha^i_k \alpha^j_r x'^k y'^r \\ &= t'_{kr} x'^k y'^r \end{aligned}$$

implies

$$t'_{kr} = \alpha^i_k \alpha^j_r t_{ij} . \tag{0-17}$$

Rule. We will notice the presence of elements of two matrices α in (0-17) and of one matrix α in (0-12); that is, to each covariance index corresponds one matrix α. It is the reason why every tensor of type $\binom{0}{2}$ is sometimes called a *second order covariant tensor*.

Conversely we have:

$$t_{ij} = \beta_i^r \beta_j^s t'_{rs}. \qquad (0\text{-}18)$$

1.3.5 Tensor of Type $\binom{2}{0}$

D A **tensor of type** $\binom{2}{0}$ is a bilinear form defined on $E^* \times E^*$.

D The vector space of bilinear forms defined on $E^* \times E^*$ is called the ***tensor product space*** of two spaces E.

This space being denoted $E \otimes E$, any tensor of type $\binom{2}{0}$ is such that

$$\boldsymbol{t} \in E \otimes E.$$

Tensor expression. The reader can transpose the previous developments from tensors of type $\binom{0}{2}$ to tensors of type $\binom{2}{0}$.

He will define n linear forms on E^*:

$$\boldsymbol{e}_i : E^* \to \boldsymbol{R} : \boldsymbol{\omega} \mapsto \boldsymbol{e}_i(\boldsymbol{\omega}) = \omega_i$$

and then will consider the n^2 tensor products

$$\boldsymbol{e}_i \otimes \boldsymbol{e}_j : E^* \times E^* \to \boldsymbol{R} : (\boldsymbol{\omega}, \boldsymbol{\mu}) \mapsto \boldsymbol{e}_i \otimes \boldsymbol{e}_j(\boldsymbol{\omega}, \boldsymbol{\mu})$$

such that

$$\boldsymbol{e}_i \otimes \boldsymbol{e}_j(\boldsymbol{\omega}, \boldsymbol{\mu}) = \boldsymbol{e}_i(\boldsymbol{\omega})\, \boldsymbol{e}_j(\boldsymbol{\mu}) = \omega_i\, \mu_j.$$

Therefore he will be able to state:

PR4 A tensor of type $\binom{2}{0}$ is expressed as

$$\boldsymbol{t} = t^{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad (0\text{-}19)$$

where

$$t^{ij} = \boldsymbol{t}(\boldsymbol{e}^{*i}, \boldsymbol{e}^{*j})$$

and $(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)$ is a basis of $E \otimes E$.

Change of basis

It is easily proved that the components of a tensor $\boldsymbol{t}$ of type $\binom{2}{0}$ are transformed as follows:

$$t'^{rs} = \beta_i^r \beta_j^s t^{ij}. \tag{0-20}$$

Rule. We will notice the presence of elements of two matrices β in (0-20) and of one matrix β in (0-6); that is, to each contravariance index corresponds one matrix β (inverse of the matrix of basis change). It is the reason why every tensor of type $\binom{2}{0}$ is sometimes called a *second order contravariant tensor*.

Conversely we have:

$$t^{ij} = \alpha_p^i \alpha_q^j t'^{pq}. \tag{0-21}$$

Remark. Given two vector spaces E^q and E^r of respective dimensions q and r, the corresponding tensor product space is $E^q \otimes E^r$ of dimension qr. It is the set of tensor products $\boldsymbol{x} \otimes \boldsymbol{y}$ of any $\boldsymbol{x} \in E^q$ and any $\boldsymbol{y} \in E^r$.
Bases $(\boldsymbol{e}_i)$ and $(\boldsymbol{e}'_j)$ of respective spaces E^q and E^r imply that $(\boldsymbol{e}_i \otimes \boldsymbol{e}'_j)$ is a basis of the qr-dimensional space $E^q \otimes E^r$.

1.3.6 Tensor of Type $\binom{1}{1}$

D A ***tensor of type*** $\binom{1}{1}$ is a bilinear form[1] defined either on $E \times E^*$ or on $E^* \times E$.

Thus a tensor of type $\binom{1}{1}$ is either an element of the tensor product space $E^* \otimes E$ or an element of the tensor product space $E \otimes E^*$.

From the covectors of the dual basis:

$$\boldsymbol{e}^{*i} : E \to \boldsymbol{R} : \boldsymbol{x} \mapsto \langle \boldsymbol{e}^{*i}, \boldsymbol{x} \rangle = x^i$$

and the vectors of the basis of E:

$$\boldsymbol{e}_j : E^* \to \boldsymbol{R} : \omega \mapsto \langle \boldsymbol{e}_j, \omega \rangle = \omega_j,$$

the reader will define the n^2 tensor products:

[1] Sometimes called *mixed tensor of order* 2.

$$e^{*i} \otimes e_j : E \times E^* \to \boldsymbol{R} : (x,\omega) \mapsto e^{*i} \otimes e_j(x,\omega)$$

such that

$$e^{*i} \otimes e_j(x,\omega) = x^i \omega_j, \tag{0-22}$$

that is

$$\langle e^{*i}, x\rangle\langle e_j,\omega\rangle = x^j \langle e^{*i}, e_j\rangle\, \omega_i \langle e_j, e^{*i}\rangle = x^i \omega_j.$$

As before, the reader will establish that the various tensors $e^{*i} \otimes e_j$ form a basis of the vector space $E^* \otimes E$ (likewise for $E \otimes E^*$).

From

$$t(\omega,x) = t(\omega_i e^{*i}, x^j e_j) = \omega_i x^j t(e^{*i}, e_j)$$
$$= \omega_i x^j t^i{}_j = t^i{}_j e_i \otimes e^{*j}(\omega,x),$$

it will be deduced:

$$t = t^i{}_j\, e_i \otimes e^{*j} \in E \otimes E^*$$

and also

$$u = u_i{}^j e^{*i} \otimes e_j \in E^* \otimes E.$$

In addition, the reader will easily establish the formulae of transformation of the previous tensors:

$$t'^r{}_s = \beta^r_i \alpha^j_s\, t^i{}_j$$
$$u'_r{}^s = \alpha^i_r \beta^s_j\; u_i{}^j,$$

the rule being: To every covariance index corresponds one matrix α and to every contravariance index one matrix β.

Given $t = t^i{}_j\, e_i \otimes e^{*j} \in E \otimes E^*$ we say:

D The ***transposed tensor*** of t is the tensor of $E^* \otimes E$, denoted ${}^t t$, such that

$\forall e_p \in E$, $\forall e^{*q} \in E^*$:

$${}^t t(e_p, e^{*q}) = t(e^{*q}, e_p). \tag{0-23a}$$

that is:

$$({}^t t)_p{}^q = t^q{}_p. \tag{0-23b}$$

In other words:

$$\forall\; t = t^i{}_j e_i \otimes e^{*j}: \quad {}^t t = ({}^t t)_j{}^i e^{*j} \otimes e_i = t^i{}_j\, e^{*j} \otimes e_i. \tag{0-23c}$$

This last result actually verifies (0-23a).

1.3.7 Tensor of Type $\binom{q}{p}$

D The vector space of *p*-linear forms defined on $E \times ... \times E$ (*p* spaces *E*) is called the ***tensor product space*** of *p* identical vector spaces E^*.

It is denoted

$$\otimes^p E^*$$

and has dimension n^p.

In the same manner we say:

D The vector space of *q*-linear forms defined on $E^* \times \cdots \times E^*$ (*q* spaces E^*) is called the ***tensor product space*** of *q* identical vector spaces *E*.

It is denoted

$$\otimes^q E$$

and has dimension n^q.

D The $\binom{q}{p}$***-tensor space*** associated with *E* is the vector space of (*p*+*q*)-linear forms defined on the Cartesian product $(\times^p E) \times (\times^q E^*)$ of *p* spaces *E* and *q* spaces E^*.

This n^{p+q}-dimensional space is denoted[1]

$$T_p^q = (\otimes^p E^*) \otimes (\otimes^q E).$$

D A ***tensor of type*** $\binom{q}{p}$ associated with E is an element[2] of the $\binom{q}{p}$-tensor space T_p^q.

We denote this (*p*+*q*)-linear form by

$$\boldsymbol{t} \in T_p^q.$$

---

[1] To simplify the presentation, we have first chosen *p* spaces *E* and next *q* spaces E^*. The order of successive spaces must be specified.

[2] Also called a *p-order covariant and q-order contravariant mixed tensor.*

PR5☞ A tensor of type $\binom{q}{p}$ is expressed as

$$t = t_{i_1 \dots i_p}{}^{j_1 \dots j_q}\, e^{*i_1} \otimes \dots \otimes e^{*i_p} \otimes e_{j_1} \otimes \dots \otimes e_{j_q} \tag{0-24}$$

where

$$t_{i_1 \dots i_p}{}^{j_1 \dots j_q} = t(e_{i_1}, \dots, e_{i_p}, e^{*j_1}, \dots, e^{*j_q})$$

and where the different $e^{*i_1} \otimes \dots \otimes e^{*i_p} \otimes e_{j_1} \otimes \dots \otimes e_{j_q}$ constitute a basis of $T_p^q = (\otimes^p E^*) \otimes (\otimes^q E)$.

Change of basis

Every transformation of components of a tensor of type $\binom{q}{p}$ associated with a basis change (0-3) is immediately obtained by considering the *rule*:

$$[\exists \text{ covariance index } \Rightarrow \exists \text{ matrix } \alpha]$$

and

$$[\exists \text{ contravariance index } \Rightarrow \exists \text{ matrix } \beta].$$

From this rule it is easy to express any transformation of components.

For example, let

$$t_{ij}{}^{k}{}_{l}\, e^{*i} \otimes e^{*j} \otimes e_k \otimes e^{*l}$$

be a tensor of type $\binom{1}{3}$.

To express the 'primed' components in function of the 'unprimed' we simply proceed as follows.

Having written

$$t'_{pq}{}^{r}{}_{s} = \alpha^{\cdot}_{p} \alpha^{\cdot}_{q} \beta^{r}_{\cdot} \alpha^{\cdot}_{s} t_{ij}{}^{k}{}_{l}$$

where the various α and β follow from the rule, we immediately replace the dots by successive indices of $t_{ij}{}^{k}{}_{l}$, that is:

$$t'_{pq}{}^{r}{}_{s} = \alpha^{i}_{p} \alpha^{j}_{q} \beta^{r}_{k} \alpha^{l}_{s} t_{ij}{}^{k}{}_{l}.$$

Remarks. (i) According to usage, it is necessary and useful to consider tensors of type $\binom{0}{0}$. They are the scalars (independent of basis choice!).

(ii) We recall that the previously introduced Kronecker symbol is only a symbol (and not a tensor).

But we can introduce:

D The ***Kronecker tensor*** δ is a tensor of type $\binom{1}{1}$ whose components are

$$\delta^i_j = \begin{matrix} 1 & \text{if} & i = j, \\ 0 & \text{if} & i \neq j. \end{matrix}$$

It is a very helpful tensor because its components are unaltered under any change of basis:

$$\delta'^{\,j}_i = \alpha^r_i \beta^j_s \delta^s_r = \alpha^r_i \beta^j_r = \delta^j_i .$$

D The ***zero tensor*** is a zero multilinear form denoted by $\boldsymbol{0}$.

1.3.8 Symmetric and Antisymmetric Tensors

Let us consider for instance elements of $E^* \otimes E^*$ [resp. $E \otimes E$].

D A tensor of type $\binom{0}{2}$ [resp. of type $\binom{2}{0}$] is *symmetric* if

$\forall \boldsymbol{x}, \boldsymbol{y} \in E: \quad \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{t}(\boldsymbol{y}, \boldsymbol{x})$

[resp. $\forall \boldsymbol{\omega}, \boldsymbol{\mu} \in E^*: \ \boldsymbol{t}(\boldsymbol{\omega}, \boldsymbol{\mu}) = \boldsymbol{t}(\boldsymbol{\mu}, \boldsymbol{\omega})$].

This definition is equivalent to the following

D A tensor $t_{ij}\boldsymbol{e}^{*i} \otimes \boldsymbol{e}^{*j}$ [resp. $t^{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$] is *symmetric* if

$$t_{ij} = t_{ji} \qquad\qquad [\text{resp. } t^{ij} = t^{ji}].$$

The previous definition may be generalized to higher order tensors.

D A tensor of type $\binom{q}{p}$ is ***partially symmetric*** if it is symmetric with respect to pair(s) of corresponding indices;
in other words:
If there are symmetries following from every transposition of two indices of same variance.

D A tensor of type $\binom{0}{p}$ or type $\binom{q}{0}$ is ***completely symmetric*** if every transposition of indices changes the corresponding component into itself.

Remark. Given an n-dimensional vector space, every symmetric tensor of order 2 has $n(n+1)/2$ independent components.

Now let us consider tensors which play an important role in mathematics and physics: the antisymmetric tensors.

D A tensor $\boldsymbol{t}$ of type $\binom{0}{2}$ [resp. of type $\binom{2}{0}$] is ***antisymmetric*** if

$$\forall \boldsymbol{x}, \boldsymbol{y} \in E: \quad \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{y}) = -\boldsymbol{t}(\boldsymbol{y}, \boldsymbol{x})$$

$$[\text{resp. } \forall \boldsymbol{\omega}, \boldsymbol{\mu} \in E^*: \quad \boldsymbol{t}(\boldsymbol{\omega}, \boldsymbol{\mu}) = -\boldsymbol{t}(\boldsymbol{\mu}, \boldsymbol{\omega})].$$

This definition is equivalent to the following

D A tensor $t_{ij}\boldsymbol{e}^{*i} * \boldsymbol{e}^{*j}$ [resp. $t^{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$] is ***antisymmetric*** if

$$t_{ij} = -t_{ji} \qquad [\text{resp. } t^{ij} = -t^{ji}].$$

We deduce

$$\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{x}) = 0.$$

The previous definition may be generalized to higher order tensors.

D A tensor of type $\binom{q}{p}$ is ***partially antisymmetric*** if it is antisymmetric with respect to pair(s) of corresponding indices;
equivalently,
if there are antisymmetries following from every transposition of two indices of same variance.

D A tensor of type $\binom{0}{p}$ or of type $\binom{q}{0}$ is ***completely antisymmetric*** if every transposition of indices changes the corresponding component into its opposite.

Remark. Given an n-dimensional vector space, every antisymmetric tensor of order 2 has $n(n-1)/2$ independent components.

PR6 Every tensor of type $\binom{0}{2}$ [or of type $\binom{2}{0}$] can always be decomposed, in a unique manner, into the sum of a symmetric tensor $\boldsymbol{t}_S$ and an antisymmetric tensor $\boldsymbol{t}_A$:

$$\boldsymbol{t} = \boldsymbol{t}_S + \boldsymbol{t}_A.$$

2. OPERATIONS ON TENSORS

The various operations on tensors are assumed to be known, but we recall them for the notation essentially.

2.1 TENSOR ALGEBRA

Before defining operations on tensors, we say:

D Tensors are ***equal*** if they are the same element of a same tensor space.

2.1.1 Addition of Tensors

An inner law, namely the addition, can be defined on the set of same type tensors.

D The ***sum*** of two tensors whose n^{p+q} components are respectively $t^{i_1 \dots i_q}_{j_1 \dots j_p}$ and $u^{i_1 \dots i_q}_{j_1 \dots j_p}$ is the tensor of type $\binom{q}{p}$ the components of which are $t^{i_1 \dots i_q}_{j_1 \dots j_p} + u^{i_1 \dots i_q}_{j_1 \dots j_p}$.

The ***addition*** of two tensors of type $\binom{q}{p}$ is

$$T^q_p \times T^q_p \to T^q_p : (\boldsymbol{t}, \boldsymbol{u}) \mapsto \boldsymbol{t} + \boldsymbol{u}$$

where $\boldsymbol{t} + \boldsymbol{u}$ is the tensor sum.

2.1.2 Multiplication of a Tensor by a Scalar

D The ***product of a tensor*** with components $t^{i_1 \dots i_q}_{j_1 \dots j_p}$ ***by a scalar*** k is the tensor whose components are $k\, t^{i_1 \dots i_q}_{j_1 \dots j_p}$.

The ***multiplication of a tensor*** of type $\binom{q}{p}$ ***by a scalar*** k is

$$\boldsymbol{R} \times T^q_p \to T^q_p : (k, \boldsymbol{t}) \mapsto k\,\boldsymbol{t},$$

where $k\,\boldsymbol{t}$ is the product of $\boldsymbol{t}$ by k.

2.1.3 Tensor Multiplication

D The ***tensor multiplication*** of any tensor $\boldsymbol{t}$ of type $\binom{q}{p}$ and any tensor $\boldsymbol{u}$ of type $\binom{s}{r}$ is the mapping

$$\otimes : T_p^q \times T_r^s \to T_{p+r}^{q+s} : (\boldsymbol{t},\boldsymbol{u}) \mapsto \boldsymbol{t}\otimes\boldsymbol{u},$$

the ***tensor product*** $\boldsymbol{t}\otimes\boldsymbol{u}$ being such that

$$\begin{aligned}&\boldsymbol{t}\otimes\boldsymbol{u}(\boldsymbol{\omega}_{(1)},\dots,\boldsymbol{\omega}_{(q)},\boldsymbol{\omega}_{(q+1)},\dots,\boldsymbol{\omega}_{(q+s)},\boldsymbol{x}_{(1)},\dots,\boldsymbol{x}_{(p)},\boldsymbol{x}_{(p+1)},\dots,\boldsymbol{x}_{(p+r)})\\ &= \boldsymbol{t}(\boldsymbol{\omega}_{(1)},\dots,\boldsymbol{\omega}_{(q)},\boldsymbol{x}_{(1)},\dots,\boldsymbol{x}_{(p)})\ \boldsymbol{u}(\boldsymbol{\omega}_{(q+1)},\dots,\boldsymbol{\omega}_{(q+s)},\boldsymbol{x}_{(p+1)},\dots,\boldsymbol{x}_{(p+r)}).\end{aligned} \tag{0-25}$$

This law $\otimes$ verifies the following *properties*:

P1. The tensor multiplication is *bilinear*:

$$\forall \boldsymbol{t}\in T_p^q, \forall \boldsymbol{u}_{(1)},\boldsymbol{u}_{(2)}\in T_r^s : \boldsymbol{t}\otimes(\boldsymbol{u}_{(1)}+\boldsymbol{u}_{(2)}) = \boldsymbol{t}\otimes\boldsymbol{u}_{(1)} + \boldsymbol{t}\otimes\boldsymbol{u}_{(2)},$$
$$\forall \boldsymbol{t}_{(1)},\boldsymbol{t}_{(2)}\in T_p^q, \forall \boldsymbol{u}\in T_r^s : (\boldsymbol{t}_{(1)}+\boldsymbol{t}_{(2)})\otimes\boldsymbol{u} = \boldsymbol{t}_{(1)}\otimes\boldsymbol{u}+\boldsymbol{t}_{(2)}\otimes\boldsymbol{u},$$
$$\forall k\in \boldsymbol{R},\ \forall \boldsymbol{t}\in T_p^q, \forall \boldsymbol{u}\in T_r^s : k(\boldsymbol{t}\otimes\boldsymbol{u}) = k\,\boldsymbol{t}\otimes\boldsymbol{u} = \boldsymbol{t}\otimes k\,\boldsymbol{u}.$$

P2. The tensor multiplication is *associative*:

$$\forall \boldsymbol{t},\boldsymbol{u},\boldsymbol{s}\in T_p^q : (\boldsymbol{t}\otimes\boldsymbol{u})\otimes\boldsymbol{s} = \boldsymbol{t}\otimes(\boldsymbol{u}\otimes\boldsymbol{s}) = \boldsymbol{t}\otimes\boldsymbol{u}\otimes\boldsymbol{s}.$$

P3. The tensor multiplication is *not commutative*.

2.1.4 Tensor Algebra

D ☞ ***The tensor algebra*** is the infinite-dimensional vector space:

$$\mathcal{T} = \boldsymbol{R}\oplus E\oplus E^*\oplus T_2^0\oplus T_0^2\oplus T_1^1\oplus\cdots\oplus T_p^q\oplus\cdots,$$

direct sum of vector spaces the dimensions of which are higher and higher, and where $\boldsymbol{R}$ represents the tensors of type $\binom{0}{0}$ (also called *scalars*).

This space is provided with a bilinear inner law: the tensor multiplication, and we express:

PR7 The tensor algebra $\mathcal{T}$ is associative, non-commutative, and of infinite dimension.

2.1 CONTRACTION AND TENSOR CRITERIA

We consider tensors of type $\binom{q}{p}$ such that $p,q \geq 1$.

2.2.1 Contraction

D The ***contraction*** of a tensor is the operation which consists in choosing a contravariance index and a covariance index, in equaling these indices, and in summing with respect to the repeated index.

For example, let us consider the tensor

$$t^{mr}{}_{s}\, \mathrm{e}_m \otimes \mathrm{e}_r \otimes \mathrm{e}^{*s} \in T_1^2 .$$

Contracting r and s we obtain a tensor of T_0^1 whose components are

$$u^m = \sum_r t^{mr}{}_r = \delta_r^s t^{mr}{}_s \ .$$

PR8 Every contraction of a tensor removes one contravariance and one covariance.

PR9 After q contractions a tensor of type $\binom{q}{q}$ is reduced to a tensor of type $\binom{0}{0}$ (in principle $q!$ in number).

Example. Given a 2-dimensional space E, two successive contractions of

$$\boldsymbol{t} = t_{ij}{}^{kr}\, \boldsymbol{e}^{*i} \otimes \boldsymbol{e}^{*j} \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_r$$

lead to the following scalars:

$$t_{kr}{}^{kr} = t_{11}{}^{11} + t_{12}{}^{12} + t_{21}{}^{21} + t_{22}{}^{22} ,$$
$$t_{rk}{}^{kr} = t_{11}{}^{11} + t_{12}{}^{21} + t_{21}{}^{12} + t_{22}{}^{22} .$$

D The ***contracted multiplication*** is the tensor multiplication with contraction.

For instance, the contracted multiplication

$$E^* \otimes E \to \boldsymbol{R} : (\omega, \boldsymbol{x}) \mapsto \langle \omega, \boldsymbol{x} \rangle$$

is such that

$$\langle \omega, \boldsymbol{x} \rangle = \omega_i x^i . \qquad (0\text{-}26)$$

Amongst the different contractions of the tensor product $\boldsymbol{t} \otimes \boldsymbol{u}$ of two tensors we emphasize the following:

Notation. The contraction with respect to the *last index* of $\boldsymbol{t}$ and the *first index* of $\boldsymbol{u}$ is denoted by[1]

$$\boldsymbol{t} \cdot \boldsymbol{u}.$$

Amongst the possible *double contractions* of a tensor product of two tensors $\boldsymbol{t} \otimes \boldsymbol{u}$ we emphasize the following:

Notation. The contraction with respect to the *last index* of $\boldsymbol{t}$ and the *first index* of $\boldsymbol{u}$ followed by the contraction with respect to the *penultimate index* of $\boldsymbol{t}$ and the *second index of* $\boldsymbol{u}$ is denoted by

$$\boldsymbol{t} : \boldsymbol{u}.$$

Remark. The dot between two tensors corresponds to the previous type of contraction. If the contraction concerns other indices, then this must be specified by letting both the indices of contraction between brackets.

For instance, given $\boldsymbol{t} = t_{ij}\boldsymbol{e}^{*i} \otimes \boldsymbol{e}^{*j}$ and $\boldsymbol{x} = x^k \boldsymbol{e}_k$, we have the following covector

$$\begin{aligned} \boldsymbol{t} \cdot \boldsymbol{x} &= t_{ik} x^k \boldsymbol{e}^{*i} \\ &= (t_{11}\, x^1 + t_{12}\, x^2 + \ldots)\boldsymbol{e}^{*1} + (t_{21}\, x^1 + t_{22}\, x^2 + \ldots)\boldsymbol{e}^{*2} + \ldots\;, \end{aligned}$$

which is different from the $(1,1)$ contraction:

$$t_{kj} x^k \boldsymbol{e}^{*j} = (t_{11}\, x^1 + t_{21}\, x^2 + \ldots)\boldsymbol{e}^{*1} + (t_{12}\, x^1 + t_{22}\, x^2 + \ldots)\boldsymbol{e}^{*2} + \ldots\;.$$

Following from the properties of the tensor multiplication, we immediately have:

– the associative property:

$$(\boldsymbol{t}_{(1)} \cdot \boldsymbol{t}_{(2)}) \cdot \boldsymbol{t}_{(3)} = \boldsymbol{t}_{(1)} \cdot (\boldsymbol{t}_{(2)} \cdot \boldsymbol{t}_{(3)}) = \boldsymbol{t}_{(1)} \cdot \boldsymbol{t}_{(2)} \cdot \boldsymbol{t}_{(3)}$$

and

$$(\boldsymbol{t}_{(1)} : \boldsymbol{t}_{(2)}) : \boldsymbol{t}_3 = \boldsymbol{t}_{(1)} : (\boldsymbol{t}_{(2)} : \boldsymbol{t}_{(3)}) = \boldsymbol{t}_{(1)} : \boldsymbol{t}_{(2)} : \boldsymbol{t}_{(3)}\,,$$

– the following distributive property:

$$(\boldsymbol{t}_{(1)} + \boldsymbol{t}_{(2)}) \cdot \boldsymbol{t}_{(3)} = \boldsymbol{t}_{(1)} \cdot \boldsymbol{t}_{(3)} + \boldsymbol{t}_{(2)} \cdot \boldsymbol{t}_{(3)}\,.$$

[1] If this conventional notation is allowed.

Examples. **(i)** Given $\boldsymbol{x} = x^i \boldsymbol{e}_i \in E$, $\boldsymbol{y} = y^j \boldsymbol{e}_j \in E$ and $\boldsymbol{t} = t_{pq} \boldsymbol{e}^{*p} \otimes \boldsymbol{e}^{*q} \in E^* \otimes E^*$, we immediately have:

$$\boldsymbol{x} \cdot \boldsymbol{t} \cdot \boldsymbol{y} = t_{ij} x^i y^j$$
$$= \boldsymbol{t}(x,y).$$

(ii) Given $\boldsymbol{x} = x^i \boldsymbol{e}_i \in E$ and $\boldsymbol{t} = t_p{}^q \, \boldsymbol{e}^{*p} \otimes \boldsymbol{e}_q \in E^* \otimes E$, we have:

$$\boldsymbol{x} \cdot \boldsymbol{t} = x^i \, \boldsymbol{e}_i \cdot (t_p{}^q \, \boldsymbol{e}^{*p} \otimes \boldsymbol{e}_q) = x^i t_p{}^q \, \delta_i^p \, \boldsymbol{e}_q$$
$$= x^i t_i{}^q \, \boldsymbol{e}_q \in E$$

and also

$${}^t\boldsymbol{t} \cdot \boldsymbol{x} = ({}^t\boldsymbol{t})^q{}_p \, \boldsymbol{e}_q \otimes \boldsymbol{e}^{*p} \cdot x^i \, \boldsymbol{e}_i = t_p{}^q \, x^i \, \delta_i^p \boldsymbol{e}_q$$
$$= t_i{}^q x^i \, \boldsymbol{e}_q.$$

Thus, we have

$$\boldsymbol{x} \cdot \boldsymbol{t} = {}^t\boldsymbol{t} \cdot \boldsymbol{x}. \tag{0-27}$$

(iii) Given arbitrary second order tensors $\boldsymbol{t}$, $\boldsymbol{u}$ and $\boldsymbol{v}$, we have

$$\boldsymbol{t} : (\boldsymbol{u} \cdot \boldsymbol{v}) = \boldsymbol{u} : (\boldsymbol{v} \cdot \boldsymbol{t}) = \boldsymbol{v} : (\boldsymbol{t} \cdot \boldsymbol{u}). \tag{0-28}$$

Remarks. **(i)** The double contraction of tensors of order 2 decomposed into symmetric and antisymmetric parts is such that:

$$\boldsymbol{t} : \boldsymbol{u} = \boldsymbol{t}_S : \boldsymbol{u}_S + \boldsymbol{t}_A : \boldsymbol{u}_A.$$

(ii) The contraction $\boldsymbol{\delta} \cdot \boldsymbol{t}$ between the Kronecker tensor and $\boldsymbol{t} = t^r{}_k \, \boldsymbol{e}_r \otimes \boldsymbol{e}^{*k}$ is the $\binom{1}{1}$-tensor

$$\delta^i_j \, t^j{}_k \, \boldsymbol{e}_i \otimes \boldsymbol{e}^{*j} \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}^{*k} = t^i{}_k \, \boldsymbol{e}_i \otimes \boldsymbol{e}^{*k}.$$

Thus the double contraction is the following real

$$\boldsymbol{\delta} : \boldsymbol{t} = \delta^k{}_j \, t^j{}_k = t^k{}_k \ , \tag{0-29}$$

with summation over k.

This is the so called *trace* of the second order tensor $\boldsymbol{t}$ and this scalar associated with $\boldsymbol{t}$ is said to be an *invariant* because its value is the same in all coordinate systems

$$t'^p{}_p = \beta^p_i \, \alpha^j_p \, t^i{}_j = \delta^j_i \, t^i{}_j = t^i{}_i.$$

In particular, we mention that the $\binom{1}{1}$-tensor $\boldsymbol{\delta} \cdot \boldsymbol{\delta}$ has the following components

$$\delta_j{}^i \, \delta_i^k = \delta_j^k. \tag{0-30}$$

In a 3-dimensional space the double contraction is the following real value:

$$\delta_j^k \delta_k^r \delta_r^j = \delta_j^r \delta_r^j = \delta_j^j = 3 .$$

2.2.2 Tensor Criteria

Until now we have been able to recognize tensors either from the definition directly, or from the transformation of components through basis changes. Let us recall a very useful criterion ensuring the tensor character of given mathematical entities; it will be based on contractions.

The general tensor criterion is expressed as follows:

If the contracted multiplication of a mathematical entity and an arbitrary tensor is a tensor, then the mathematical entity is a tensor.

Criterion. If the contracted multiplication (k times) of a mathematical entity and a tensor of type $\binom{q}{p}$ leads to a tensor of type $\binom{r}{s}$, then the mathematical entity is a tensor of type $\binom{r+k-q}{s+k-p}$.

For example, the work done by a force $\boldsymbol{f}$ applied to a particle whose infinitesimal displacement is represented by the vector $\boldsymbol{dx}$ is expressed as

$$dW = f_i \, dx^i .$$

The contracted multiplication of the entity of components f_i and the vector of components dx^i leads to a scalar. So the entity of components f_i is a tensor of type

$$\binom{0+1-1}{0+1-0} = \binom{0}{1}.$$

So the force appears to be a covector.

CHAPTER 1

FOUNDATIONS OF DIFFERENTIAL GEOMETRY

Today's *Classical Mechanics* can be considered as a branch of *Differential Geometry*. For example, the fundamental spaces of mechanics are manifolds introduced in this modern geometry and the various well known functions of mechanics are rigorously formulated in this geometric language.

Foundations of differential geometry, necessary and useful to develop mechanics, are recalled. By taking account of this symbiosis *Mechanics-Geometry* the reader can refer, for instance, to *Abraham* and *Marsden* (1980) and *Talpaert* (2000) for proofs and additional material of differential geometry. Unlike these books basic notions of topology and elementary differential calculus in Banach spaces are not here recalled.

1. MANIFOLDS

Let M be a set of elements called points,
F be a finite-dimensional real normed space.

1.1 DIFFERENTIABLE MANIFOLDS

Let us provide M with a topology by considering that every point of M belongs at least to an open U_i of M. We are going to cover M. In other words, let us introduce a *covering* of M by opens.

1.1.1 Chart and Local Coordinates

D A ***(local) chart*** on M is the pair (U_i,φ) made up of:
- an open U_i of M,
- a homeomorphism φ of U_i onto an open subset $\varphi(U_i)$ of F.

The open U_i is called ***domain*** of the chart.

An arbitrary point of M can belong to two distinct opens, for instance U_j and U_k. The corresponding distinct charts are (U_j,φ_j) and (U_k,φ_k).

The homeomorphisms φ_j and φ_k being different, we connect the opens $\varphi_j(U_j)$ and $\varphi_k(U_k)$ of F by introducing the following notion.

Let us denote

$$\varphi^{-1}_{j|U_j \cap U_k}$$

the restriction of φ_j^{-1} to the open $\varphi_j(U_j \cap U_k)$ of F.

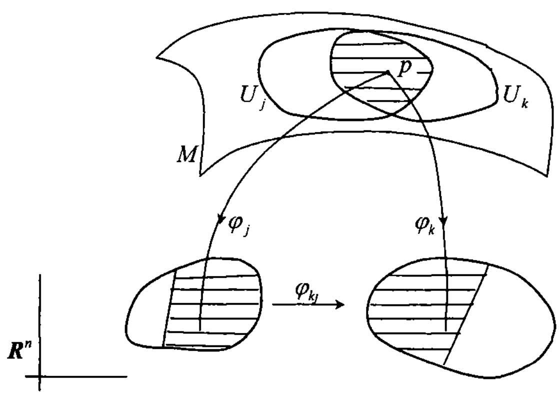

Fig. 1

Henceforward the space F will be only $\boldsymbol{R}^n$, then to each point M is associated a chart (U,φ) such that $\varphi(U)$ is an open of $\boldsymbol{R}^n$.

D Two charts (U_j,φ_j) and (U_k,φ_k) on M, such that $U_j \cap U_k \neq \varnothing$, are called ***$C^q$-compatible*** $(q \geq 1)$ if the overlap mapping

$$\varphi_{kj} = \varphi_k \circ \varphi^{-1}_{j|U_j \cap U_k}$$

is a C^q diffeomorphism[1] between the opens $\varphi_j(U_j \cap U_k)$ and $\varphi_k(U_j \cap U_k)$ of $\boldsymbol{R}^n$.

---

[1] A diffeomorphism of class C^q is called a C^q *diffeomorphism.*

This diffeomorphism of class C^q is the mapping φ_{kj} between the two hatched domains as illustrated in Fig.1.

D The ***local coordinates*** x^i of a point p belonging to the domain U of a chart (U,φ) of M are the coordinates of point $\varphi(p)$ of $\boldsymbol{R}^n$.

Let us denote by

$$(x^1,...,x^n)$$

the ordered n-tuple of real numbers linked to point p.

Thus the bijection

$$\varphi : U \to \boldsymbol{R}^n : p \mapsto \varphi(p) = (x^1,...,x^n)$$

assigns to any point p of $U \subset M$ the n-tuple of reals $(x^1,...,x^n)$.

Conversely φ^{-1} assigns to every ordered n-tuple of real numbers a point of U.

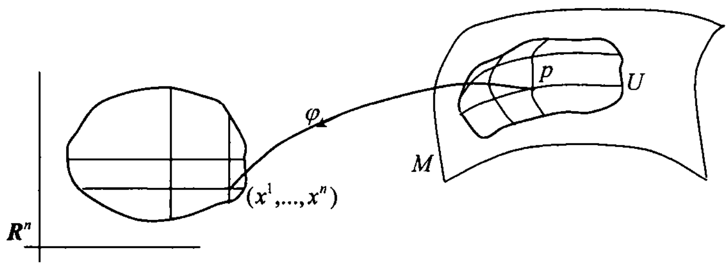

Fig. 2

So, to coordinate lines of $\boldsymbol{R}^n$ are associated coordinate lines on M and a chart defines a *local coordinate system* on M.

The reader will notice that the reference made to $\boldsymbol{R}^n$ in which differential calculus is well known will later be profitable for mechanics.

1.1.2 Atlas

Of course before defining the rules of differential calculus on M we must properly define a set of charts on the whole of M.

D ☞ An ***atlas*** of class C^q on M is a family A of charts (U_i,φ_i) such that:

(*i*) the domains U_i of charts make up a covering of M:

$$\bigcup_{i\in I} U_i \ ,$$

(*ii*) any two charts (U_i,φ_i), (U_j,φ_j) of A, with $U_i \cap U_j \neq \varnothing$, are C^q-compatible.

Let us observe that an atlas of class C^q generates an atlas of class C^p such that $p \leq q$.

Classic examples. Atlases of S^2 and S^n.

Let the unit 2-sphere be:

$$S^2 = \left\{(x^1,x^2,x^3) \in \boldsymbol{R}^3 : \sum_{i=1}^{3}(x^i)^2 = 1\right\}.$$

Consider the mapping φ_1, stereographic projection from the north pole n onto the plane $\left\{q \in \boldsymbol{R}^3 : x^3(q) = 0\right\}$. It is a bijection between $S^2 - \{n\}$ and this plane locally. More details about stereographic projections are given in *Talpaert* (2000) for instance.

Similarly, the stereographic projection φ from the south pole s onto the previous plane is a bijection between $S^2 - \{s\}$ and the plane locally.

Because of poles the sphere cannot be covered by only one chart: no single homeomorphism φ can be used between S^2 and the plane.

On S^2, with the topology induced by the one of $\boldsymbol{R}^3$, we know that two arbitrary charts (U_1,φ_1) and (U_2,φ_2) are compatible. The 2-sphere atlas is composed of (at least) two charts.

The reader will immediately generalize to the n-sphere:

$$S^n = \left\{(x^1,\ldots,x^{n+1}) \in \boldsymbol{R}^{n+1} : \sum_{i=1}^{n+1}(x^i)^2 = 1\right\}.$$

He will consider two homeomorphisms: the stereographic projections from respectively the north pole and the south pole, i.e. two mappings of $S^n - \{n\}$ (resp. $S^n - \{s\}$) onto the hypersurface of equation $x^{n+1} = 0$. With the usual topology on S^n as a subset of $\boldsymbol{R}^{n+1}$ then an atlas with two charts will be defined.

D A chart (U,φ) is ***compatible*** with the atlas $\{(U_i,\varphi_i)_{i\in I}\}$ or is ***admissible*** if the union $\{(U,\varphi)\} \cup \{(U_i,\varphi_i)_{i\in I}\}$ is again an atlas; in other words if it is a chart of the atlas.

D Two atlases of class C^q are ***equivalent*** or ***compatible*** if their union is still an atlas.

1.1.3 Differentiable Manifold Structure

To avoid that different atlases lead to the same calculus on M we are going to consider the two following notions.

D The ***maximal atlas*** $\overline{A}$, associated with an atlas A is the atlas being composed of all (equivalent) charts compatible with A.

D A maximal atlas on M provides M with a ***differentiable manifold structure.***

In practice, a differentiable manifold structure is defined from an atlas representative of its equivalence class (all the equivalent atlases defining the same differentiable manifold structure).

Let us emphasize that the definition of a differentiable manifold structure has two requirements:

(i) The opens of local charts cover M.
(ii) Two any charts (U_j,φ_j), (U_k,φ_k) such that $U_j \cap U_k \neq \varnothing$ are C^q-compatible.

Let us make more explicit the second requirement with the aid of the notion of change of charts (or change of local coordinates).

Let p be a point belonging to the intersection $U_j \cap U_k$ of domains of distinct charts (U_j,φ_j) and (U_k,φ_k).

The reader will easily sketch the following situation by referring to Fig.1 and where the C^q diffeomorphism φ_{kj} is the C^q diffeomorphism $\varphi_j^{-1} \circ \varphi_k$ between the hatched opens of $\boldsymbol{R}^n$ included in $\varphi_k(U_k)$ and $\varphi_j(U_j)$ respectively.

Thus let us consider two local coordinate systems.

The definition of an atlas of class C^q means the coordinates $x'^1,...,x'^n$ of p with respect to a local coordinate system are functions of class C^q of coordinates $x^1,...,x^n$ of p with respect to the other system of local coordinates. Thus we express:

D The change of charts (U_j,φ_j) and (U_k,φ_k) or local coordinate transformation of point p is ***admissible*** if there is a C^q diffeomorphism between opens of $\boldsymbol{R}^n$:

$$\varphi_k \circ \varphi_j^{-1} : \boldsymbol{R}^n \to \boldsymbol{R}^n : (x^1,...,x^n) \mapsto (x'^1,...,x'^n) \ ,$$

that is:

if the functions f^i defining the coordinate transformation

$$x'^1 = f^1(x^1,...,x^n), \ . \ . \ . \ , \ x'^n = f^n(x^1,...,x^n)$$

have continuous qth-order partial derivatives with respect to variables x^i.

1.1.4 Differentiable Manifolds

D A ***differentiable manifold*** of class C^q is a pair consisting of a topological space and a maximal atlas.

It is denoted by

$$(M,\overline{A}).$$

Henceforward we will assume that the basis for the topology defined by chart domains is countable. We recall that a topological space is said to be a space with countable basis if there is (at least) one basis consisting of a countable number of elements, countable meaning finite or denumerable. Since every topological space with countable basis is separable, that is containing a (everywhere) dense countable set, then every manifold is assumed *separable*.

In addition every manifold will be assumed of *Haussdorff* type. We recall that a topological space is a Haussdorff space if for any two distinct points of this space there are neighborhoods of these points that do not overlap.

In practice, we define a differentiable manifold from an atlas on M. Unless otherwise specified, the differentiable manifolds will be of class C^∞ and we express:

D ☞ *A* ***differentiable manifold*** is a pair consisting of a Haussdorff space with countable basis and an atlas.

A differentiable manifold is an ***n-manifold*** if for every point x of space there exists an admissible local chart (U,φ) with $x \in U$ and $\varphi(U) \subset \boldsymbol{R}^n$.

Notation. Its dimension being n, the manifold (M_n, A) will be denoted M_n or simply M.

Examples. (i) *Space* $\boldsymbol{R}^n$ is a manifold such that the atlas $\{(\boldsymbol{R}^n, id)\}$ is made up of only one chart $(\boldsymbol{R}^n, id)$.

(ii) By considering

$$S^1 = \left\{(x^1,x^2) \in \boldsymbol{R}^2 : (x^1)^2 + (x^2)^2 = 1\right\},$$

then circle S^1 is provided with the induced topology.

The circle is evidently not homeomorphic to $\boldsymbol{R}$ and thus S^1 cannot be covered by only one chart.

Let us check an atlas of S^1 defined by two charts (U_1,φ_1) and (U_2,φ_2) which are

$$U_1 = \{(x^1, x^2) \in S^1 : x^1 < 1\}$$
$$\varphi_1 : U_1(\subset S^1) \to]0,2\pi[: (x^1 = \cos\theta, x^2 = \sin\theta) \mapsto \theta$$

and

$$U_2 = \{(x^1, x^2) \in S^1 : x^1 > -1\}$$
$$\varphi_2 : U_2(\subset S^1) \to]-\pi,\pi[: (x^1 = \cos\theta, x^2 = \sin\theta) \mapsto \theta.$$

These charts evidently cover S^1.

Now from

$$U_1 \cap U_2 = S^1 - \{(1,0),(-1,0)\}$$

let us check that the mapping $\varphi_1 \circ \varphi_2^{-1}$ is a diffeomorphism between the opens $\varphi_2(U_1 \cap U_2)$ and $\varphi_1(U_1 \cap U_2)$ of $\boldsymbol{R}$.

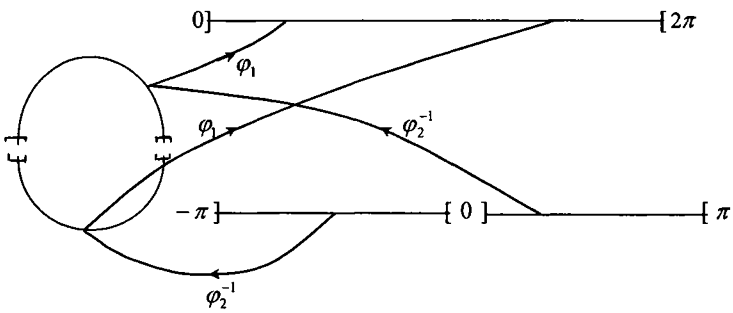

Fig. 3

On the one hand, for $\theta \in]0,\pi[$ we have:

$$\varphi_1(\varphi_2^{-1}\theta) = \varphi_1(\cos\theta, \sin\theta) = \theta.$$

The example corresponding to $\theta = \pi/4$ is sketched.

On the other hand, for $\theta \in]-\pi, 0[$ we have:

$$\varphi_1(\varphi_2^{-1}\theta) = \varphi_1(\cos\theta, \sin\theta) = \theta + 2\pi.$$

The case of the value $\theta = -\dfrac{\pi}{2}$ is illustrated, that is:

$$\varphi_1(0,-1) = \frac{3\pi}{2} \in]0,2\pi[.$$

In conclusion, $\varphi_1 \circ \varphi_2^{-1}$ is a diffeomorphism between the above mentioned opens of $\boldsymbol{R}$.

Note that the only one-dimensional connected manifolds are $\boldsymbol{R}$ and S^1.

PR1 A manifold M is such as the topological space is locally compact and locally connected.

Proof. First, let us show that every point x of M has a compact neighborhood.
Let (U,φ) be a chart containing x where φ is a homeomorphism $U \to \varphi(U)$ such that $\varphi(U)$ is a neighborhood of $\varphi(x)$ in $\boldsymbol{R}^n$ locally compact. Thus there is a compact K of $\boldsymbol{R}^n$ such that $\varphi(x) \in K \subset \varphi(U)$.
But, φ^{-1} being continuous and M being Haussdorff, we can conclude $\varphi^{-1}(K)$ is a compact neighborhood containing x since any continuous mapping f of a compact space S into a Haussdorff space T implies the subset $f(S)$ of T is compact.

Next, to prove that a manifold M is a locally connected topological space we proceed as previously so that $\varphi(U)$ is a neighborhood of $\varphi(x)$ in $\boldsymbol{R}^n$ containing a connected neighborhood C of $\varphi(x)$ and then $\varphi^{-1}(C)$ is a connected neighborhood in M containing x.

Let us now introduce the notion of *product manifold.*

Let M be a manifold of class C^q defined by an atlas

$$A = \{ (U_i, \varphi_i) \mid i \in I \}.$$

Let N be a manifold of class C^q defined by an atlas

$$\widetilde{A} = \{ (V_j, \psi_j) \mid j \in J \}.$$

D The ***product atlas*** $A \times \widetilde{A}$ is

$$\{ (U_i \times V_j, \varphi_i \times \psi_j) \mid (i,j) \in I \times J \}$$

where

$$U_i \times V_j = \{ (x_i, y_j) \mid x_i \in U_i,\ y_j \in V_j \}$$

$$\varphi_i \times \psi_j : U_i \times V_j \to \boldsymbol{P}^n \times \boldsymbol{P}^m : (x,y) \mapsto (\varphi_i(x), \psi_j(y)).$$

PR2 The structure of differentiable manifold $M \times N$ follows from the structures of differentiable manifolds M and N.

Proof. If we denote

$$W_{ij} = U_i \times V_j, \qquad W'_{ij} = U'_i \times V'_j,$$

then the following set

$$W_{ij} \cap W'_{ij} = (U_i \cap U'_i) \times (V_j \cap V'_j)$$

is an open of $M \times N$.

If we define

$$\varphi_{ij}(x,y) = (\varphi_i(x), \psi_j(y))$$

and analogically for φ'_{ij}, then the C^∞diffeomorphism between opens of $\boldsymbol{R}^{n+m}$ is:

$$\varphi'_{ij} \circ \varphi_{ij}^{-1} = (\varphi'_i \circ \varphi_i^{-1}) \times (\psi'_j \circ \psi_j^{-1}) .$$

Indeed

$$\begin{aligned}(\varphi'_{ij} \circ \varphi_{ij}^{-1})(x,y) &= \varphi'_{ij}(\ \varphi_i^{-1}(x), \psi_j^{-1}(y)\ ) \\ &= (\ \varphi'_i(\varphi_i^{-1}(x)), \psi'_j(\psi_j^{-1}(y))\ ) \\ &= (\ (\varphi'_i \circ \varphi_i^{-1})(x), (\psi'_j \circ \psi_j^{-1})(y)\ ) .\end{aligned}$$

Thus we naturally express:

D The ***product manifold*** $M \times N$ of two manifolds is the manifold defined from the product atlas of M and N.

Its dimension is the sum of dimensions of each manifold.

Examples. (i) The 2-torus $T^2 = S^1 \times S^1$ is the product of two manifolds S^1 which are two circles respectively around the inner tube and the cross section.

More generally the *n*-dimensional torus is the product of *n* circles.

(ii) The cylinder $S^1 \times \boldsymbol{R}$, provided with the product manifold structure is a two dimensional manifold.

More generally the $(n+1)$ - dimensional cylinder is the product manifold $S^n \times \boldsymbol{R}$.

Let us conclude this section with the notion of *orientable manifolds.*

Let (x^i) and (y^j) be two coordinate systems of an open U of M.

D A differentiable manifold is ***orientable*** if there is one atlas $\{(U_i, \varphi_i)\}_{i \in I}$ such that in the common domain of any two charts the orientations are the *same*;
in other words

if $$\frac{D(x^i)}{D(y^j)} > 0 .$$

Let us note that the orientations associated with each coordinate system (in the common domain) are *opposite* if, at every point of the domain:

$$\frac{D(x^i)}{D(y^j)} < 0 .$$

D ☞ A differentiable manifold is ***orientable*** if there is one atlas $(U_i,\varphi_i)_{i\in I}$ such as the Jacobian of every coordinate transformation $\varphi_i \circ \varphi_j^{-1}$ is positive at every point.

We immediately deduce from the definition:

PR3 The product manifold of orientable manifolds is orientable.

PR4 Any open of an orientable manifold is an orientable manifold.

Examples. (i) The manifold S^n ($n \geq 1$) is orientable.

It is readily proven that S^n has a differentiable manifold structure. See for instance *Talpaert* (2000), but the atlas which has permitted defining the differentiable manifold structure of S^n does not allow using the definition of orientation; thus we are going to choose another atlas.

Let us consider the opens

$$U_1 = \left\{(x^1,\ldots,x^{n+1}) \mid x^{n+1} < 1\right\}, \quad U_2 = \left\{(x^1,\ldots,x^{n+1}) \mid x^{n+1} > -1\right\},$$

and the poles $N = (0,\ldots,0,1)$ and $S = (0,\ldots,0,-1)$.

Let φ_1 be the stereographic projection from the north pole N onto the plane of equation $x^{n+1} = 0$.
Let φ_2 be the stereographic projection from the south pole S onto the previous plane.
Let ψ be the symmetry with respect to the plane of equation $x^1 = 0$.

The atlas consisting of charts (U_1,φ_1) and $(U_2,\psi \circ \varphi_2)$ is in accordance with the definition. Indeed the coordinate transform

$$(\psi \circ \varphi_2) \circ \varphi_1^{-1} = \psi \circ (\varphi_2 \circ \varphi_1^{-1})$$

is the composition of an inversion (mapping presenting an always negative Jacobian) with a symmetry with respect to a plane. Thus it is a diffeomorphism with positive Jacobian. Therefore the sphere S^n is orientable.

(ii) Any torus or cylinder is orientable.

This is an obvious consequence of PR3 and of the fact that S^n and $\boldsymbol{R}^n$ are orientable.

To prove a manifold is not orientable it is easy to consider the next proposition.

PR5 In order that a differentiable manifold M be orientable it is necessary, for any pair of connected charts (U,φ) and (V,ψ), the Jacobian of $\psi \circ \varphi^{-1}$ to have a constant sign on $\varphi(U \cap V)$.

Example. The Möbius strip is not an orientable manifold [see e.g. *Talpaert* (2000)].

1.2 DIFFERENTIABLE MAPPINGS

Let us introduce the notion of mapping of class C^q between differentiable manifolds.

Let M_n, N_m be manifolds of class C^p,
f be a continuous mapping of M_n into N_m,
x be a point of M_n.

1.2.1 Differentiable Mapping between Manifolds

D A mapping f of M_n into N_m is of ***class*** $\boldsymbol{C^q}$ $(q \leq p)$ at point x of M_n if, for each chart (U,φ) such as $x \in U$ and each chart (V,ψ) such as $y = f(x) \in V$, the mapping called *'local representative'* of f

$$f_{\varphi\psi} \equiv \psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \subset \boldsymbol{R}^n \to \boldsymbol{R}^m$$

is of class C^q.

Let us express this definition in coordinate systems.

Let $x^1,\ldots,x^n$ be the local coordinates of x in (U,φ),
$y^1,\ldots,y^n$ be the local coordinates of $y = f(x)$ in (V,ψ).

D A mapping f of M_n into N_m is *of* ***class*** $\boldsymbol{C^q}$ at point x of M_n if the m local coordinates y^i of point $y = f(x)$ are, in the neighborhood of x, the m functions of class C^q

$$y^i = f^i(x^j) \qquad j = 1,\ldots,n$$

of n coordinates x^j of x.

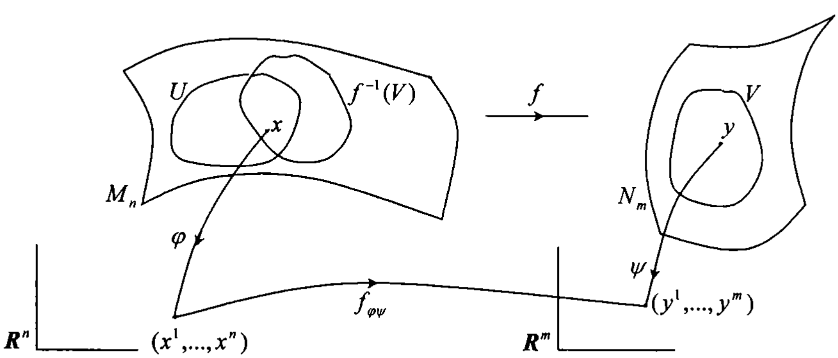

Fig. 4

D A mapping f of M_n into N_m is a mapping ***of class C^q*** of M_n into N_m if, for every x in M_n, to any (admissible) chart (V,ψ) on N_m is associated a chart (U,φ) on M_n such as $x\in U$, $f(x)\in f(U)\subset V$ and also

$$f_{\varphi\psi} \equiv \psi \circ f \circ \varphi^{-1} : \varphi(U) \subset \boldsymbol{R}^n \to \boldsymbol{R}^m$$

is of class C^q.

Notation.
$C^q(M_n;N_m)$ denotes the set of mappings of class C^q of M_n into N_m;
$C^\infty(M_n;N_m)$ denotes the set of differentiable mappings (class C^∞) of M_n into N_m.

It can be proved:

PR6 A mapping f of M_n into N_m is a mapping of class C^q *iff* for each x of M_n there exists *one* chart (U,φ) with $x\in U$ and *one* chart (V,ψ) with $f(x)\in V$ such that $f(U)\subset V$ and $f_{\varphi\psi} \in C^q(\varphi(U);\boldsymbol{R}^m)$.

Now, we consider the notion of *canonical projection.*

PR7 The canonical projections are differentiable mappings of the product differentiable manifold $M_n \times N_m$ into the respective manifolds M_n and N_m.

Proof. Let us consider the canonical projection

$$p : M_n \times N_m \to M_n .$$

It is sufficient to prove that there are a chart $(U\times V, \varphi\times\psi)$ on $M_n \times N_m$ at each $(x,y)\in M_n\times N_m$ and a chart (U',φ') on M_n at $x = p(x,y)$ such as $p(U\times V)\subset U'$ and $\varphi'\circ p\circ(\varphi\times\psi)^{-1}$ is a mapping of class C^∞ of $(\varphi\times\psi)(U\times V)$ into $\boldsymbol{R}^n$.

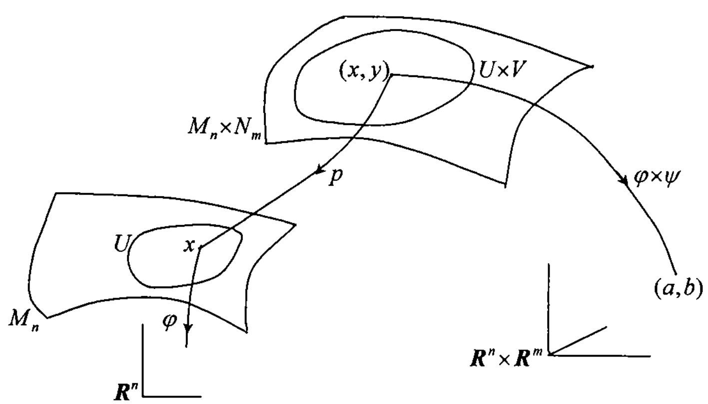

Fig. **5**

Let $(U \times V, \varphi \times \psi)$ be a chart at (x,y) on the product manifold, and (U,φ) be the corresponding chart on M_n at x.

We have:

$$p(U \times V) = U,$$

and the following mapping

$$\varphi \circ p \circ (\varphi \times \psi)^{-1} : (\varphi \times \psi)(U \times V) \to \boldsymbol{R}^n :$$

$$(a,b) \xrightarrow[(\varphi\times\psi)^{-1}]{} (\varphi^{-1}(a), \psi^{-1}(b)) \xrightarrow[p]{} \varphi^{-1}(a) \xrightarrow[\varphi]{} a$$

is of class C^∞.

The following propositions are proved, for instance, in *Talpaert* (2000).

PR8 The composition of mappings of class C^q between manifolds is a mapping of class C^q.

PR9 Let M_n, N_m, P_r be differentiable manifolds,
p_1 be the canonical projection of $M_n \times N_m$ onto M_n,
p_2 be the canonical projection of $M_n \times N_m$ onto N_m.
The mapping

$$f : P_r \to M_n \times N_m$$

is of class C^q *iff* the coordinate functions

$$p_1 \circ f : P_r \to M_n, \qquad p_2 \circ f : P_r \to N_m$$

are of class C^q.

PR10 A mapping $f : M_n \to N_m$ is of class C^q *iff* there is an (open) covering $(U_i)_{i\in I}$ of M_n such that $f\big|_{U_i}$ is of class C^q for every $i \in I$.

1.2.2 Diffeomorphism, Immersion, Submersion, and Embedding

First, let M_n and N_n be differentiable manifolds of *same* dimension.

D A mapping f of M_n onto N_n is a C^q ***diffeomorphism*** of M_n onto N_n if f is a bijection of $C^q(M_n; N_m)$ and $f^{-1} \in C^q(N_n; M_n)$.

Notation.

Let $Diff^q(M_n;N_n)$ denote the set of C^q diffeomorphisms of M_n onto N_n,
$Diff(M_n;N_n)$ denote the set of (C^∞) diffeomorphisms of M_n onto N_n.

By referring to PR8 it can be easily proven the following.

PR11 If M_n is a differentiable manifold then $Diff(M_n;N_n)$ is a group with respect to the composition of mappings.

D A differentiable mapping between manifolds (of same dimension) $f: M_n \to N_n$ is ***a local diffeomorphism at*** a point x of M_n if the rank of f at x is n.

It is ***a local diffeomorphism on*** M_n if it is a local diffeomorphism at every point of M_n.

PR12 A mapping of class C^q of M_n onto N_n is a C^q diffeomorphism *iff* it is bijective and is a local diffeomorphism on M_n.

Let us emphasize the importance of the bijective assumption.

PR13 A bijection f of M_n onto N_n is a diffeomorphism of M_n onto N_n *iff*, in local coordinates x^i, the n differentiable functions

$$f^j(x^i) \qquad i = 1,...,n,$$

that define f, show a nonzero Jacobian.

Secondly, let M_n, N_m be differentiable manifolds.

D A differentiable mapping $f: M_n \to N_m$ is an ***immersion at*** point x of M_n if the rank of f is equal to the dimension of M_n.

It is called ***immersion*** of M_n into N_m if it is an immersion at every point of M_n.

It is necessary that $n \le m$.

D A differentiable mapping $f: M_n \to N_m$ is a ***submersion at*** point x of M_n if the rank of f is equal to the dimension of N_m.

It is called ***submersion*** of M_n into N_m if it is a submersion at every point of M_n.

It is necessary that $n \ge m$.

D A differentiable mapping $f: M_n \to N_m$ is an ***embedding*** of M_n into N_m if f is an injective immersion and a homeomorphism of N_m onto $f(M_n)$ (for the induced topology).

The reader will prove the following propositions by having in mind the constant rank theorem.

PR14 If $f: M_n \to N_m$ is an immersion (resp. submersion) at point x of M_n, a chart (U,φ) on M_n containing x and a chart (V,ψ) on N_m exist such that

$$f(U) \subset V$$

and

$$f_{\varphi\psi} \equiv \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \boldsymbol{R}^m : (x^1,\dots,x^n) \mapsto (x^1,\dots,x^n,0,\dots,0)$$

$$[\text{resp. } (x^1,\dots,x^n) \mapsto (\mathrm{x}^1,\dots,\mathrm{x}^m),\, n \geq m].$$

PR15 If $f: M_n \to N_m$ is a mapping of class C^q of constant rank r on M_n then, for every x of M_n, a chart (U,φ) on M_n containing x and a chart (V,ψ) on N_m exist such that:

$$f(U) \subset V$$

and

$$f_{\varphi\psi} \equiv \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \boldsymbol{R}^m : (x^1,\dots,x^n) \mapsto (x^1,\dots,x^r,0,\dots,0).$$

PR16 If f is an embedding of M_n into N_m then the set $f(M_n)$ is provided with a differentiable manifold structure (induced by the embedding).

Proof. If $\{(U_i,\varphi_i)_{i\in I}\}$ is an atlas of M_n, let us prove that $\{(f(U_i),\varphi_i \circ f^{-1})_{i\in I}\}$ is an atlas of $f(M_n)$.

Every point $f(x)$ of $f(U_i)$ corresponding to $x \in U_i$ has a neighborhood which is homeomorphic to an open of $\boldsymbol{R}^n$; and the opens $f(U_i)_{i\in I}$ cover $f(M_n)$; moreover the image of any $x \in U_i \cap U_j$ is a point $f(x) \in f(U_i) \cap f(U_j)$.

The mapping

$$(\varphi_i \circ f^{-1}) \circ (\varphi_j \circ f^{-1})^{-1}$$

is a diffeomorphism between the opens (of $\boldsymbol{R}^n$) $\varphi_j(U_j)$ and $\varphi_i(U_i)$ because:

$$(\varphi_i \circ f^{-1}) \circ (\varphi_j \circ f^{-1})^{-1} = (\varphi_i \circ f^{-1}) \circ (f \circ \varphi_j^{-1}) = \varphi_i \circ \varphi_j^{-1}$$

and that (U_i,φ_i) and (U_j,φ_j) are charts of atlas on M_n.

1.3 SUBMANIFOLDS

1.3.1 Submanifolds of R^n

D A subset V of $\boldsymbol{R}^n$ is a ***submanifold*** of $\boldsymbol{R}^n$, of dimension m ($\leq n$) and of class C^q if, for every $x \in V$, there exists an open U_x of $\boldsymbol{R}^n$ containing x and a C^q diffeomorphism g of U_x onto the open $g(U_x)$ of $\boldsymbol{R}^n$ such that

$$g(U_x \cap V) = g(U_x) \cap \boldsymbol{R}^m .$$

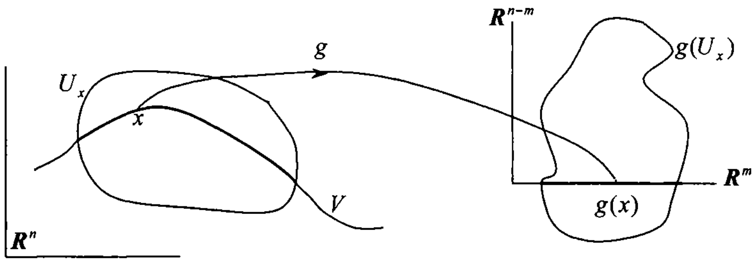

Fig. **6**

The following propositions which allow to find submanifolds of $\boldsymbol{R}^n$ are illustrated and proved in *Talpaert* (2000).

Let U be an open of $\boldsymbol{R}^n$.

PR17 Let $f: U(\subset \boldsymbol{R}^n) \to \boldsymbol{R}^m$ be a mapping of class C^q,
y be a point of $\boldsymbol{R}^m$,
$V = f^{-1}(y)$.
If f is a submersion at every point of V, then V is an $(n-m)$-dimensional submanifold of $\boldsymbol{R}^n$.

From this proposition we can immediately deduce another one very useful in practice.

PR18 Let $f^i : U(\subset \boldsymbol{R}^n) \to \boldsymbol{R}$ be m functions of class C^q,
$V = \left\{ x = (x^1,\dots,x^n) \in \boldsymbol{R}^n : f^i(x^1,\dots,x^n) = 0, \forall i \in \{1,\dots,m\} \right\}.$
If for every x of V, the rank of the Jacobian matrix

$$\left(\frac{\partial f^i}{\partial x^j} \right) \qquad 1 \leq i \leq m, \; 1 \leq j \leq n$$

is m ($\leq n$), then V is an $(n-m)$-dimensional submanifold of $\boldsymbol{R}^n$.

A special case of the previous proposition is the following.

PR19 Let $f : U(\subset \boldsymbol{R}^n) \to \boldsymbol{R}$ be a function of class C^q,

$$V = \{ x \in \boldsymbol{R}^n \mid f(x) = 0 \}.$$

If for every x of V one of partial derivatives of f is nonzero (nonzero gradient of f), then V is an $(n-1)$-dimensional submanifold of $\boldsymbol{R}^n$.

Another interesting proposition lets conclude to the existence of submanifolds of $\boldsymbol{R}^n$:

PR20 If $f : U(\subset \boldsymbol{R}^m) \to \boldsymbol{R}^n$ is an injective immersion $(m \le n)$,
if $f^{-1} : V = f(U) \to U$ is a continuous mapping,
then V is an m-dimensional submanifold of $\boldsymbol{R}^n$.

1.3.2 Submanifold of Manifold

D A subset W of a manifold M_n is an m-dimensional ***submanifold*** of M_n $(m \le n)$ if for each $x \in W$ there is a chart (U,φ) in M_n containing x such that:

$$\varphi(U \cap W) = \varphi(U) \cap \boldsymbol{R}^m.$$

A chart (U,φ) such that $\varphi(U \cap W)$ is the set of points $(x^1,\dots,x^n)$ of $\varphi(U)$ fulfilling $x^{n+1} = \dots = x^n = 0$ is said to be *'adapted'* to W.

Example. Let us consider the classic case $n = 3$, $m = 2$.

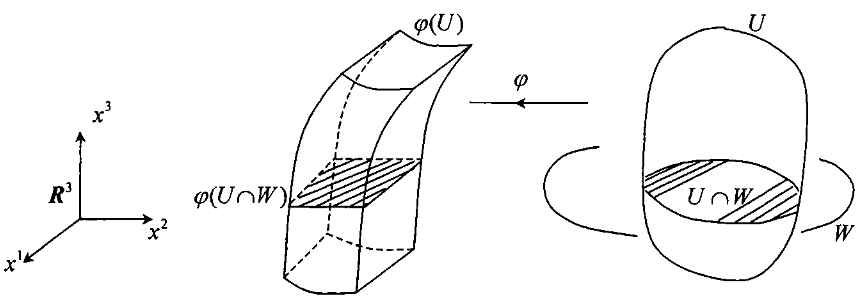

Fig. 7

In this example we have:

$$\begin{aligned}\varphi(U \cap W) &= \varphi(U) \cap \boldsymbol{R}^2 \\ &= \{ (x^1,x^2,x^3) \in \varphi(U) \mid x^3 = 0 \}.\end{aligned}$$

PR21 Given two differentiable manifolds M_n and N_m if $f: M_n \to N_m$ is of class C^q and of (constant) rank r on M_n then, for each $y \in f(M_n)$, $f^{-1}(y)$ is an $(n-r)$-dimensional submanifold of M_n.

PR22 If $f: M_n \to N_m$ is a mapping of class C^q between differentiable manifolds, if y is a point of $f(M_n)$ and if f is a submersion at each point of $f^{-1}(y)$, then $f^{-1}(y)$ is an $(n-m)$-dimensional submanifold of M_n.

We express in another manner:

PR23 Given m differentiable functions $f^i : M_n \to \boldsymbol{R} : x \mapsto f^i(x)$ on M_n defining a differentiable mapping $f: M_n \to \boldsymbol{R}^m : x \mapsto (f^1(x),\ldots,f^m(x))$, then a subset W of M_n, defined by m equations $f^i(x) = 0$ and f having rank p at each point of W, is an $(n-m)$-dimensional differentiable submanifold of M_n.

2. TANGENT VECTOR SPACE

We are going to associate an n-dimensional vector space at any point x of a differentiable manifold M which will be called the *tangent space* to M at x. A decisive progress in differential geometry occurred when tangent space was defined given a manifold without reference to $\boldsymbol{R}^n$. Different techniques can be used, for example the algebraic approach using the notion of ideal, but we have chosen the method which is the most used by engineers and physical scientists.

2.1 TANGENT VECTOR

Let M be a differentiable manifold,
x_0 be a point of M,
I be an open interval in $\boldsymbol{R}$ containing 0,
(U,φ) be an admissible chart of M.

2.1.1 Tangent Curves

D A *(**differentiable**) **curve***[1], passing through x_0, in M is a differentiable mapping

$$c: I \to M : t \mapsto c(t)$$

such that

$$c(0) = x_0 .$$

[1] Strictly speaking it is a matter of an arc.