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1.1. BANACH SPACES AND LINEAR OPERATORS

A set $\mathcal{X}$ is called a *linear* or *vector space* over the field of real (complex) numbers if

1) the addition operation is defined: to any elements $x, y \in \mathcal{X}$ there corresponds a definite element $x + y \in \mathcal{X}$ called their sum;
2) $x + y = y + x$;
3) $x + (y + z) = (x + y) + z$;
4) there exists a zero element $0 \in \mathcal{X}$ such that $x + 0 = x$;
5) for any $x \in \mathcal{X}$ there exists $-x \in \mathcal{X}$ such that $x + (-x) = 0$;
6) the operation of multiplication by a number is defined: to any $x \in \mathcal{X}$ and any number $\lambda \in \mathbb{R}$ ($\mathbb{C}$) there corresponds a definite element $\lambda x \in \mathcal{X}$;
7) $\lambda(\mu x) = (\lambda\mu)x$;
8) $1 \cdot x = x$;
9) $\lambda(x + y) = \lambda x + \lambda y$;
10) $(\lambda + \mu)x = \lambda x + \mu x$.

The elements of a linear space will be called *vectors* or *points*.

Remark 1.1.1. It is easy to show that the zero and inverse elements are uniquely defined.

A linear space $\mathcal{X}$ is called *normed* if to any $x \in \mathcal{X}$ a nonnegative number $\|x\|_{\mathcal{X}}$ (the subscript will sometimes be omitted) called the *norm* of x is assigned so that the following axioms are satisfied:

1) $\|x\| \geq 0$;

2) $\|x\| = 0$ if and only if $x = 0$;

3) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for any $\lambda \in \mathbb{R}$ $(\mathbb{C})$;

4) $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in \mathcal{X}$.

To any real linear space $\mathcal{X}$ there corresponds a complex linear space $\tilde{\mathcal{X}}$ consisting of all possible formal sums $z = x + iy$, where $x, y \in \mathcal{X}$ and i is the imaginary unit. Clearly, $\mathcal{X} \subset \tilde{\mathcal{X}}$. Such an inclusion of $\mathcal{X}$ in the space $\tilde{\mathcal{X}}$ is called the *complexification* of the Banach space $\mathcal{X}$.

A sequence of vectors $\{x_n\}_{n=1}^{\infty} \subset \mathcal{X}$ is called *convergent* to a vector $x \in \mathcal{X}$, which is denoted as $x = \lim_{n\to\infty} x_n$, if $\|x_n - x\| \to 0$ as $n \to \infty$.

The set $S_r(x_0) = \{x \in \mathcal{X} \mid \|x - x_0\| < r\}$ is called *open ball* with radius $r > 0$ centered at the point $x_0 \in \mathcal{X}$.

A set $\mathcal{A} \subset \mathcal{X}$ is called *bounded* if $\exists K \in \mathbb{R}_+ \quad \forall x \in \mathcal{A} \ \|x\| \leq K$.

A point $a \in \mathcal{X}$ is called a *limit (accumulation) point* of a set $\mathcal{A} \subset \mathcal{X}$ if $\forall \varepsilon > 0 \ S_\varepsilon(a) \cap \mathcal{A} \neq \emptyset$. In other words, a is a limit point of the set $\mathcal{A}$ if there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{A}$ convergent to a. The union of a set $\mathcal{A}$ and all its limit points is called the *closure* of this set. It is denoted by $\overline{\mathcal{A}}$. A set coinciding with its closure is called *closed*. A set $\mathcal{A}$ is called *open* if the set $\mathcal{X} \setminus \mathcal{A}$ is closed.

A set $\mathcal{A}$ is referred to as *dense* in a space $\mathcal{X}$ if $\overline{\mathcal{A}} = \mathcal{X}$.

A set $\mathcal{L} \subset \mathcal{X}$ is called a *lineal* (or *linear manifold*) if $x + y \in \mathcal{L}$ and $\lambda x \in \mathcal{L}$ for any $x, y \in \mathcal{L}$ and any $\lambda \in \mathbb{R}$ $(\mathbb{C})$. A closed lineal $\mathcal{L} \subset \mathcal{X}$ is called a *linear subspace* of the space $\mathcal{X}$.

Let $\mathcal{L}$ and $\mathcal{M}$ be lineals in a space $\mathcal{X}$ and let $\mathcal{L} \cap \mathcal{M} = \{0\}$. The set of all possible vectors z of the form $x + y$, where $x \in \mathcal{L}$ and $y \in \mathcal{M}$, will be referred to as the *direct sum* of these lineals and will be denoted by $\mathcal{L} \dot{+} \mathcal{M}$. If $\mathcal{L}$ and $\mathcal{M}$ are closed, their direct sum is denoted by $\mathcal{L} \oplus \mathcal{M}$.

A sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{X}$ is called *fundamental* if $\forall \varepsilon > 0 \ \exists N \ \forall n > N \ \forall p \in \mathbb{N} \ \|x_{n+p} - x_n\| < \varepsilon$. A linear space $\mathcal{X}$ is called *complete* if every fundamental sequence in it converges. A complete linear normed space is called a *Banach* space

The symbols I and $\mathbb{O}$ will denote the identity and "zero" operators, respectively, whose domains of definition are clear from the context. In other words, $Ix = x$ and $\mathbb{O}x = 0$.

A mapping (operator) $A : \operatorname{dom} A \to \mathcal{Y}$ of the subset $\operatorname{dom} A$ (the domain of definition of the operator A) of a linear normed space $\mathcal{X}$ to a linear normed space $\mathcal{Y}$ is called *continuous at a point* $x_0 \in \operatorname{dom} A$ if $\lim_{n\to\infty} A(x_n) = A(x_0)$ for any sequence $\{x_n\}_{n=1}^{\infty} \subset \operatorname{dom} A$ converging to x_0. An operator A is called *continuous* if it is continuous at every point $x \in \operatorname{dom} A$.

An operator A is called *bounded on the set* $\operatorname{dom} A$ if $\exists C \in \mathbb{R}_+ \ \forall x \in \operatorname{dom} A \ \|A(x)\|_{\mathcal{Y}} \leq C\|x\|_{\mathcal{X}}$. If $\operatorname{dom} A = \mathcal{X}$, the operator is called *bounded.*

The *image* of an operator A is the set $\operatorname{im} A = \{y \in \mathcal{Y} \mid \exists x \in \operatorname{dom} A \quad y = A(x)\}$, and its *kernel* is the set $\ker A = \{x \in \operatorname{dom} A \mid A(x) = 0\}$.

An operator A is called *linear*, if $\operatorname{dom} A$ is a lineal and $A(\lambda x + \mu y) = \lambda Ax + \mu Ay$ for any $x, y \in \operatorname{dom} A$ and for any $\lambda, \mu \in \mathbb{R}$ ($\mathbb{C}$). (Arguments of linear operators will be written without parentheses).

Theorem 1.1.1. *Let an operator* $A : \mathcal{X} \to \mathcal{Y}$, $\operatorname{dom} A = \mathcal{X}$, *be linear. Then the following statements are equivalent: a) the operator A is continuous at one point; b) the operator A is continuous; c) the operator A is bounded.*

We shall denote by $\mathcal{L}(\mathcal{X}; \mathcal{Y})$ the linear normed space of linear continuous operators A with $\operatorname{dom} A = \mathcal{X}$ if the addition of operators and their multiplication by a number are defined in a natural way: $(A + B)x = Ax + Bx$ and $(\lambda A)x = \lambda Ax$ for all $A, B \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$, $x \in \mathcal{X}$, and $\lambda \in \mathbb{R}$ ($\mathbb{C}$). The norm in $\mathcal{L}(\mathcal{X}; \mathcal{Y})$ is defined as follows:

$$\begin{aligned} \|A\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} &= \sup\{\|Ax\|_{\mathcal{Y}} \mid x \in \mathcal{X}, \|x\|_{\mathcal{X}} \leq 1\} \\ &= \sup\{\|Ax\|_{\mathcal{Y}} \mid x \in \mathcal{X}, \|x\|_{\mathcal{X}} = 1\} = \sup\{\|Ax\|_{\mathcal{Y}}/\|x\|_{\mathcal{X}} \mid x \in \mathcal{X} \setminus \{0\}\} \\ &= \inf\{C \in \mathbb{R}_+ \mid \forall x \in \mathcal{X} \quad \|Ax\|_{\mathcal{Y}} \leq C\|x\|_{\mathcal{X}}\}. \end{aligned}$$

If $\mathcal{Y} = \mathcal{X}$, the notation of the space of linear continuous operators will be abbreviated to $\mathcal{L}(\mathcal{X})$.

A sequence of operators $\{A_n\} \subset \mathcal{L}(\mathcal{X}; \mathcal{Y})$ is called *uniformly convergent* to an operator $A \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$ if $\lim_{n\to\infty} \|A_n - A\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} = 0$. This sequence is *strongly convergent* to A if $\forall x \in \mathcal{X} \ \lim_{n\to\infty} \|A_n x - Ax\|_{\mathcal{Y}} = 0$. Such a convergence is denoted as follows: $A = s\text{-}\lim_{n\to\infty} A_n$.

Theorem 1.1.2. *Let $\mathcal{Y}$ be a Banach space. Then the space $\mathcal{L}(\mathcal{X};\mathcal{Y})$ is a Banach space.*

Theorem 1.1.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, for all $n \in \mathbb{N}$ $A_n \in \mathcal{L}(\mathcal{X};\mathcal{Y})$, and for any $x \in \mathcal{X}$ the sequence $\{\|A_n x\|_{\mathcal{Y}}\}$ be bounded. Then the sequence $\{\|A_n\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})}\}$ is bounded.*

Theorem 1.1.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. A sequence $\{A_n\} \subset \mathcal{L}(\mathcal{X};\mathcal{Y})$ is strongly convergent to an operator $A \in \mathcal{L}(\mathcal{X};\mathcal{Y})$ if and only if the sequence $\{\|A_n\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})}\}$ is bounded and for every x from a lineal dense in $\mathcal{X}$ $\lim_{n\to\infty} A_n x = Ax$.*

Let $\mathcal{X}$ and $\mathcal{Y}$ be linear normed spaces. Let an operator $A : \mathrm{dom}\, A \to \mathcal{Y}$, $\mathrm{dom}\, A \subset \mathcal{X}$, be injective. Then there exists an *inverse* operator A^{-1} : $\mathrm{dom}\, A^{-1} \to \mathcal{X}$, which bijectively maps $\mathrm{dom}\, A^{-1} = \mathrm{im}\, A$ onto $\mathrm{dom}\, A$. The operator A^{-1} is linear. An operator A is called *continuously invertible* if there exists an operator $A^{-1} \in \mathcal{L}(\mathcal{Y};\mathcal{X})$.

Theorem 1.1.5. *The operator A^{-1} exists and is bounded on $\mathrm{im}\, A$ if and only if there exists $m \in \mathbb{R}_+$ such that for all $x \in \mathrm{dom}\, A$ $\|Ax\|_{\mathcal{Y}} \geq m\|x\|_{\mathcal{X}}$.*

Theorem 1.1.6. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, the operator $A \in \mathcal{L}(\mathcal{X};\mathcal{Y})$, and $\mathrm{im}\, A = \mathcal{Y}$ and A is invertible. Then the operator A is continuously invertible.*

Theorem 1.1.7. *Let $\mathcal{X}$ be a Banach space, $A \in \mathcal{L}(\mathcal{X})$, and $\|A\| < 1$. Then the operator $I - A$ is continuously invertible and*

$$\|(I-A)^{-1}\|_{\mathcal{L}(\mathcal{X})} \leq \frac{1}{1 - \|A\|_{\mathcal{L}(\mathcal{X})}}.$$

A linear operator $A : \mathrm{dom}\, A \to \mathcal{Y}$ is called *closed* if it follows from $\{x_n\} \subset \mathrm{dom}\, A$, $\lim_{n\to\infty} x_n = x$, and $\lim_{n\to\infty} Ax_n = y$ that $x \in \mathrm{dom}\, A$ and $Ax = y$. The set of closed operators $A : \mathrm{dom}\, A \to \mathcal{Y}$ with domains of definition dense in the space $\mathcal{X}$ will be denoted by $\mathcal{C}l(\mathcal{X};\mathcal{Y})$. The set of operators $\mathcal{C}l(\mathcal{X};\mathcal{X})$ will be denoted by $\mathcal{C}l(\mathcal{X})$.

Theorem 1.1.8. *An operator A belongs to the space $\mathcal{L}(\mathcal{X};\mathcal{Y})$ if and only if it is closed and defined on the whole space.*

Theorem 1.1.9. *If an operator A is closed and invertible, then the operator A^{-1} is closed.*

Let us introduce a *graph norm* $\|\cdot\|_g = \|\cdot\|_{\mathcal{X}} + \|A\cdot\|_{\mathcal{X}}$ on the domain of definition dom A of a linear closed operator A.

Theorem 1.1.10. *If an operator $A : \text{dom } A \to \mathcal{X}$, $\text{dom } A \subset \mathcal{X}$, is linear and closed, then the normed space* dom A *is a Banach space with respect to the graph norm, and the operator $A \in \mathcal{L}(\text{dom } A)$.*

Let $\mathcal{X}$ be a complex Banach space and let an operator $A : \text{dom } A \to \mathcal{X}$, $\text{dom } A \subset \mathcal{X}$, be linear. A complex number λ is called a *regular point* of the operator A if the operator $\lambda I - A$ is continuously invertible (there exists the operator $(\lambda I - A)^{-1} \in \mathcal{L}(\mathcal{X})$). The set of all regular points of the operator A is called the *resolvent set* of the operator and is denoted by $\rho(A)$. If $\lambda \in \rho(A)$, then the operator $R_\lambda(A) = (\lambda I - A)^{-1}$ is called the *resolvent* of the operator A. The *spectrum* of the operator A is the set $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

Theorem 1.1.11. *The resolvent set $\rho(A)$ is open, and the spectrum $\sigma(A)$ is closed.*

Theorem 1.1.12. *The spectrum of a continuous operator A lies in the circle $\{\lambda \in \mathbb{C} \mid |\lambda| \le \|A\|_{\mathcal{L}(\mathcal{X})}\}$.*

A complex number λ is called an *eigenvalue* of an operator A if there exists a vector $x \in \text{dom } A \setminus \{0\}$ such that $Ax = \lambda x$. Here, x is called the *eigenvector* of the operator A corresponding to the eigenvalue λ. Every eigenvalue λ of the operator A is a point of its spectrum because the operator $\lambda I - A$ is not invertible in this case.

Theorem 1.1.13. *Let $\mathcal{X}$ be a Banach space and let $A \in \mathcal{L}(\mathcal{X})$. Then there exists a finite limit*

$$r_\sigma(A) = \lim_{n\to\infty} \|A^n\|^{1/n}_{\mathcal{L}(\mathcal{X})} = \inf_{n\in\mathbb{N}} \|A^n\|^{1/n}_{\mathcal{L}(\mathcal{X})}$$

called the spectral radius of the operator A and $r_\sigma(A) \le \|A\|_{\mathcal{L}(\mathcal{X})}$.

Theorem 1.1.14. *Let $\mathcal{X}$ be a Banach space, an operator $A \in \mathcal{L}(\mathcal{X})$, and $|\lambda| > r_\sigma(A)$. Then $\lambda \in \rho(A)$.*

An operator function $A(\lambda) : \mathbb{C} \to \mathcal{L}(\mathcal{X})$ is called *analytic* at a point λ_0 if it is expanded in some neighbourhood of the point λ_0 into a power series

$A(\lambda) = \sum_{k=0}^{\infty} A_k(\lambda - \lambda_0)^k$ convergent in this neighbourhood. Note that the notions of analyticity in the sense of the uniform and strong convergence of the series are equivalent.

Theorem 1.1.15. *$R_\lambda(A)$ is an analytic function of λ at any point $\lambda \in \rho(A)$.*

Remark 1.1.2. Let an operator $A \in \mathcal{L}(\mathcal{X})$ and $|\lambda| > r_\sigma(A)$. Then, based on Theorem 1.1.7, the following expansion can be readily obtained:

$$R_\lambda(A) = -\sum_{k=0}^{\infty} \frac{A^k}{\lambda^{k+1}}.$$

Theorem 1.1.16. *Let $\mathcal{X}$ be a Banach space and an operator $A \in \mathcal{L}(\mathcal{X})$. Then $\sigma(A) \neq \emptyset$.*

An operator A is called *idempotent* if $A^2 = A$. *Projector* is an idempotent operator $A \in \mathcal{L}(\mathcal{X})$. On a space $\mathcal{X}$ there exists a projector A if and only if $\mathcal{X} = \mathcal{X}^0 \oplus \mathcal{X}^1$, where $A|_{\mathcal{X}^0} = \mathbb{O}$, $A|_{\mathcal{X}^1} = I$.

The space $\mathcal{L}(\mathcal{X};\mathbb{R}(\mathbb{C}))$ is called *adjoint* to $\mathcal{X}$ and is denoted by $\mathcal{X}'$. Its elements are called *functionals.* If to every element $x \in \mathcal{X}$ an element $\tilde{x} \in \mathcal{X}''$ is assigned by the rule $\tilde{x}(f) = f(x)$ $\forall f \in \mathcal{X}'$, then it is clear that $\mathcal{X} \subset \mathcal{X}''$. A space $\mathcal{X}$ such that $\mathcal{X} = \mathcal{X}''$ is called *reflexive.*

A sequence $\{x_n\} \subset \mathcal{X}$ is called *weakly convergent* to $x \in \mathcal{X}$ and this fact is denoted as $x = w\text{-}\lim_{n\to\infty} x_n$ if $\lim_{n\to\infty} f(x_n) = f(x)$ for any functional $f \in \mathcal{X}'$.

1.2. THEOREMS ON INFINITESIMAL GENERATORS

Let $\mathcal{X}$ be a Banach space and let an operator $A : \operatorname{dom} A \to \mathcal{X}$, $\operatorname{dom} A \subset \mathcal{X}$, be linear and closed. Consider the Cauchy problem

$$x(0) = x_0, \quad x_0 \in \operatorname{dom} A$$

for an operator-differential equation

$$\dot{x} = Ax.$$

The question of solvability of this problem on the semiaxis $\overline{\mathbb{R}}_+ = \{0\} \cup \mathbb{R}_+$ (on the axis $\mathbb{R}$) is equivalent to the problem of finding a semigroup (group) of operators generated by the operator A.

Definition 1.2.1. A *semigroup* of linear continuous operators is a mapping $X^{\cdot} : \overline{\mathbb{R}}_+ \to \mathcal{L}(\mathcal{X})$ such that $X^s X^t = X^{s+t}$ for all $s, t \in \overline{\mathbb{R}}_+$.

A semigroup will be identified with the set $\{X^t \mid t \in \overline{\mathbb{R}}_+\}$.

A semigroup $\{X^t \mid t \in \overline{\mathbb{R}}_+\}$ is called *nondegenerate* if $X^0 = I$ and *strongly continuous* if for any $t \in \mathbb{R}_+$ $\lim_{s\to t} X^s = X^t$, $\lim_{s\to 0+} X^s = X^0$. A nondegenerate strongly continuous semigroup is called *strongly continuous* (C_0)*-semigroup* (or a (C_0)*-continuous semigroup*).

Definition 1.2.2. *An infinitesimal generator* of a nondegenerate semigroup of operators $\{X^t \mid t \in \overline{\mathbb{R}}_+\}$ is the operator

$$Ax = \lim_{t\to 0+} \frac{X^t x - x}{t},$$

defined only on these vectors x for which the above limit exists. In this case the operator A is said to generate the semigroup $\{X^t \mid t \in \overline{\mathbb{R}}_+\}$.

Let us introduce the denotation $\mathbb{R}_{a,+} = \{c \in \mathbb{R} \mid c > a\}$.

Definition 1.2.3. An operator $A \in Cl(\mathcal{V})$ satisfying the conditions

$$\exists a \in \mathbb{R} \quad \forall \mu \in \mathbb{R}_{a,+} \quad \mu \in \rho(A),$$

$$\exists K \in \mathbb{R}_+ \quad \forall \mu \in \mathbb{R}_{a,+} \quad \forall n \in \mathbb{N} \quad \|(R_\mu(A))^n\|_{\mathcal{L}(\mathcal{V})} \leq K/(\mu - a)^n$$

will be called *radial.*

Theorem 1.2.1 [Hille—Yosida]. *An operator A is radial if and only if it generates a strongly continuous (C_0)-semigroup.*

Definition 1.2.4. A *group* of linear continuous operators is a mapping $X^{\cdot} : \mathbb{R} \to \mathcal{L}(\mathcal{X})$ such that $X^s X^t = X^{s+t}$ for all $s, t \in \mathbb{R}$.

A group will be identified with the set $\{X^t \mid t \in \mathbb{R}\}$.

A group $\{X^t \mid t \in \mathbb{R}\}$ is called *nondegenerate* if $X^0 = I$ and *strongly continuous* if for any $t \in \mathbb{R}$ $\lim_{s\to t} X^s = X^t$. A nondegenerate strongly continuous group is called a *strongly continuous* (C_0)*-group* (or (C_0)*-continuous group*).

Definition 1.2.5. The *infinitesimal generator* of a nondegenerate group of operators $\{X^t \mid t \in \mathbb{R}\}$ is the operator

$$Ax = \lim_{t\to 0} \frac{X^t x - x}{t}$$

defined on those vectors x for which the above limit exists. In this case it is said that the operator A generates the semigroup $\{X^t \mid t \in \mathbb{R}\}$.

Let $\mathbb{R}_a = \{c \in \mathbb{R} \mid |c| > a\}$.

Definition 1.2.6. If an operator $A \in Cl(\mathcal{V})$ satisfies the conditions

$$\exists a \in \mathbb{R} \quad \forall \mu \in \mathbb{R}_a \quad \mu \in \rho(A),$$

$$\exists K \in \mathbb{R}_+ \quad \forall \mu \in \mathbb{R}_a \quad \forall n \in \mathbb{N} \quad \|(R_\mu(A))^n\|_{\mathcal{L}(\mathcal{V})} \leq K/(|\mu| - a)^n,$$

it will be called *biradial.*

Theorem 1.2.2. *An operator A generates a strongly continuous group if and only if it is biradial.*

A group $\{X^t \mid t \in \mathbb{R}\}$ of operators is called *analytic* if it can be analytically continued to the whole complex plane in the variable t with retaining its group property.

Theorem 1.2.3. *The semigroup generated by an operator $A \in \mathcal{L}(\mathcal{X})$ can be continued to an analytic group. Conversely, the generator of an analytic group is a bounded operator $A \in \mathcal{L}(\mathcal{X})$.*

A semigroup $\{X^t \mid t \in \mathbb{R}\}$ of operators is called *analytic* if it can be analytically continued to a certain sector containing $\mathbb{R}_+$ in the variable t with retaining its semigroup property.

Definition 1.2.7. An operator $A \in Cl(\mathcal{V})$ is called *sectorial* if it satisfies the conditions

$$\begin{gathered} \exists a \in \mathbb{R} \quad \exists \theta \in (\pi/2, \pi) \\ S_{a,\theta}(A) = \{\mu \in \mathbb{C} \mid |\arg(\mu - a)| < \theta, \; \mu \neq a\} \subset \rho(A), \\ \exists K \in \mathbb{R}_+ \quad \forall \mu \in S_{a,\theta}(A) \quad \|R_\mu(A)\|_{\mathcal{L}(\mathcal{V})} \leq K/|\mu - a|. \end{gathered}$$

Remark 1.2.1. The term "sectorial operator" is taken from Clement, Heijmans, Angenent, van Duijn, de Pagter (1987), Henry (1981).

Theorem 1.2.4 [Solomyak—Yosida]. *An operator A is the generator of an analytic semigroup if and only if it is sectorial.*

1.3. FUNCTIONAL SPACES AND DIFFERENTIAL OPERATORS

A bounded domain $\Omega \subset \mathbb{R}^n$ will be referred to as a *domain of class C^k*, $k = 0, 1, 2, \ldots, \infty$, if

(i) the boundary $\partial\Omega$ of the domain Ω is a compact C^k-manifold without border;

(ii) there exist numbers $\alpha, \beta \in \mathbb{R}_+$ and an atlas $\{a_i \mid i = 1, 2, \ldots, m\}$, where every map corresponds to a local coordinate system $\{O_i \mid a_i, x_2^i, \ldots, x_n^i\}$, such that the domain boundary

$$\partial\Omega \subset \bigcap_{i=1}^{m} \{(x_1^i, \bar{x}^i) \mid x_1^i = a_i(\bar{x}^i),\ |\bar{x}^i| < \alpha\}$$

and

$$\{(x_1^i, \bar{x}^i) \mid a_i(\bar{x}^i) < x_1^i < a_i(\bar{x}^i) + \beta,\ |\bar{x}^i| < \alpha\} \subset \mathbb{R}^n \setminus \overline{\Omega};$$
$$\{(x_1^i, \bar{x}^i) \mid a_i(\bar{x}^i) - \beta < x_1^i < a_i(\bar{x}^i),\ |\bar{x}^i| < \alpha\} \subset \Omega.$$

Here $\bar{x}^i = (x_2^i, x_3^i, \ldots, x_n^i)$.

Remark 1.3.1. Condition (ii) formalizes vague statements like "a domain Ω locally lies on one side of its boundary".

Let $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

Let us introduce the *Sobolev spaces*

$$W_p^l = \{\partial^\alpha u \in L_p(\Omega) \mid \forall \alpha \in \mathbb{N}_0^n,\ |\alpha| < l\},$$

where $l \in \mathbb{N}_0$, $1 \le p < \infty$.

The space W_p^l is a Banach space with the norm

$$\|u\|_{l,p} = \Big(\sum_{|\alpha| \le l} \int_\Omega |\partial^\alpha u|^p \, dx \Big)^{1/p}.$$

The space W_p^l is reflexive for $1 < p < \infty$. For $l > k$ the bounded set $\{u \in W_p^l \mid \|u\|_{l,p} \le \text{const}\}$ is compact in W_p^k. If for $p = 2$ the space W_p^l is equipped with a scalar product

$$(u,v)_l = \int_\Omega \sum_{|\alpha|\le l} \partial^\alpha u\, \partial^\alpha v\, dx,$$

it will be a Hilbert space. Let us denote $H^l = W_2^l$. In addition to these spaces, we will also need the *Hölder spaces*

$$C^{l+\lambda} = C^{l+\lambda}(\Omega) = \Big\{ u \in C^l(\overline{\Omega}) \mid \\ \|u\|_{l+\lambda} = \|u\|_l + \sum_{|\alpha|=l} \sup_{x,y\in\Omega,\ x\ne y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\lambda} < \infty \Big\},$$

where $l \in \mathbb{N}_0$, $0 < \lambda < 1$, and $\|\cdot\|_l$ designates the uniform norm in $C^l(\overline{\Omega})$

$$\|u\|_l = \sum_{|\alpha|\le l} \max_{x\in\overline{\Omega}} |\partial^\alpha u(x)|.$$

The spaces $C^{l+\lambda}$ are Banach spaces with a norm $\|\cdot\|_{l+\lambda}$. When $l+\lambda > k+\mu$, the bounded set $\{u \in C^{l+\lambda} \mid \|u\|_{l+\lambda} \le \text{const}\}$ is compact in $C^{k+\mu}$.

The connection between Sobolev and Holder spaces is established by the *Sobolev embedding theorems.*

(TS1) If an integer k, $0 \le k < l$, is such that $0 \le 1/q = 1/p - (l-k)/n \le 1$, then the embedding of W_p^l in W_q^k is continuous. If in addition $q' < q$, then the embedding of W_p^l in $W_{q'}^k$ is compact.

(TS2) If an integer k, $0 \le k < l$, is such that $0 < \lambda = l - n/p - k < 1$, then the embedding of W_p^l in $C^{k+\lambda}$ is compact.

Let us now consider differential operators. Henceforth we will consider that the domain $\Omega \subset \mathbb{R}^n$ is of the class C^∞. A set $\{B_j \mid j = 0,1,\dots,k\}$ of differential operators defined on $\partial\Omega$,

$$B_j = \sum_{|\alpha|\le m_j} b_\alpha^j(x)\partial^\alpha, \qquad b_\alpha^j \in C^\infty(\partial\Omega), \tag{1.3.1}$$

is called a *normal system* if $0 \le m_0 < m_1 < \cdots < m_k$ and for every vector ν_x, $x \in \partial\Omega$, normal to $\partial\Omega$ the following condition is satisfied

$$\sum_{|\alpha|=m_j} b_\alpha^j(x)\nu_x^\alpha \ne 0, \qquad j = 0,1,\dots,k.$$

Let $\{B_j \mid j = 0, 1, \dots, k\}$ be a normal system and $m_k < l$. Let us introduce the spaces

$$W^l_{p,\{B_j\}} = \{u \in W^l_p \mid B_j u = 0 \quad \text{on } \partial\Omega,\ j = 0, 1, \dots, k\},$$
$$C^{l+\lambda}_{\{B_j\}} = \{u \in C^{l+\lambda} \mid B_j u = 0 \quad \text{on } \partial\Omega,\ j = 0, 1, \dots, k\};$$

$W^l_{p,\{B_j\}}$ and $C^{l+\lambda}_{\{B_j\}}$ are Banach subspaces of the spaces W^l_p and $C^{l+\lambda}$, respectively.

The differential operator

$$A = \sum_{|\alpha| \le 2m} a_\alpha(x)\partial^\alpha, \qquad a_\alpha \in C^\infty(\overline{\Omega}),$$

satisfies the *Petrovskii ellipticity condition* if

$$\sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \neq 0 \ \ \forall \xi \in \mathbb{R}^n \setminus \{0\} \quad \forall x \in \overline{\Omega}.$$

A set $\{B_j \mid j = 0, 1, \dots, m-1\}$ of differential operators on $\partial\Omega$ of the form (1.3.1) satisfies the *complementary condition* (*Shapiro—Lopatinskii condition*) with respect to an operator A if for any normal vector ν_x and any tangent vector ξ_x at any point $x \in \partial\Omega$ the polynomials

$$b_j(x, \xi_x + \tau\nu_x) = \sum_{|\alpha|=m_j} b^j_\alpha(x)(\xi_x + \tau\nu_x)^\alpha, \qquad j = 0, 1, \dots, m-1,$$

in a variable τ are linearly independent modulo of the polynomial $\prod_{k=0}^{m-1}(\tau - \tau_k^+)$, where τ_k^+ are the roots with positive imaginary part of the polynomial

$$\sum_{|\alpha|=2m} a_\alpha(x)(\xi + \tau\eta)^\alpha$$

in the variable τ. Here the vectors $\xi, \eta \in \mathbb{R}^n$ are linearly independent.

Let $k \in \mathbb{N}_0$ and let $\{B_j \mid j = 0, 1, \dots, m-1\}$ be a normal system of differential operators, complementary with respect to a operator A satisfying the ellipticity condition. The operator A defined on the spaces $W^{2m+k}_{p,\{B_j\}}$ or $C^{2m+k+\lambda}_{\{B_j\}}$ will be called an *elliptic differential operator.*

Let us summarize the main results on elliptic differential operators which will be useful for further treatment.

(1) An elliptic differential operator

$$A : W^{2m+k}_{p,\{B_j\}} \to W^k_p, \quad A : C^{2m+k+\lambda}_{\{B_j\}} \to C^{k+\lambda}$$

is a Noether operator and its index $\operatorname{ind} A$ depends neither on $k \in \mathbb{N}_0$ nor on p $(1 < p < \infty)$ and λ $(0 < \lambda < 1)$.

Remark 1.3.2. Henceforth, among the Noether operators we will single out the *Fredholm* operators, i. e., the Noether operators with zero index.

(2) The kernel $\ker A$ depends neither on k, p nor on λ and the following embedding takes place:

$$\ker A \subset \{u \in C^{l+\lambda} \mid B_j u = 0 \quad \text{on } \partial\Omega, \ j = 0, 1, \ldots, m-1\}.$$

(3) The cokernel $\operatorname{coker} A$ can be selected so that the embedding $\operatorname{coker} A \subset C^\infty(\overline{\Omega})$ takes place and in this case the cokernel depends neither on p, k nor on λ.

(4) Either the resolvent set of an operator A is empty or the spectrum $\sigma(A)$ consists of isolated points that are eigenvalues of finite multiplicity and is condensed only at infinity.

(5) The set of eigenfunctions and adjoint functions of an operator A $\{\varphi_k \mid k \in \mathbb{N}\} \subset C^\infty(\overline{\Omega})$ forms a basis of the spaces $W^{l+2m}_{p,\{B_j\}}$, W^l_p, $C^{l+2m+\lambda}_{\{B_j\}}$, and $C^{l+\lambda}$ irrespective of $l = 0, 1, \ldots$ and of p and λ.

Finally, let us define spaces of functions with values in a Banach space.

Let $\mathcal{X}$ be a Banach space. Consider the space $\tilde{W}^l_q([a,b];\mathcal{X})$, $l = 0, 1, 2, \ldots$, $1 < q < \infty$, consisting of functions $u : [a,b] \to \mathcal{X}$ continuously differentiable l times with the norm

$$\|u\|_{\tilde{W}^l_q[a,b]} = \|u\|_{l,q} = \Big(\sum_{k=0}^{l} \int_a^b \|u^{(k)}(t)\|^q_{\mathcal{X}}\, dt\Big)^{1/q}.$$

By definition, the space $W^l_q(a,b;\mathcal{X})$ is the completion of $\tilde{W}^l_q([a,b];\mathcal{X})$ in this norm and is called the *Sobolev—Bochner space.* The space $W^0_q(a,b;\mathcal{X})$ is denoted by $L_q(a,b;\mathcal{X})$ and is called the *Lebesgue—Bochner space.*

Chapter 2.

Relatively p-radial operators and degenerate strongly continuous semigroups of operators

2.0. INTRODUCTION

Let $\mathcal{U}$ and $\mathcal{F}$ be Banach spaces; operators $L \in \mathcal{L}(\mathcal{U};\mathcal{F})$ and $M \in Cl(\mathcal{U};\mathcal{F})$. Let us consider a Cauchy problem

$$u(0) = u_0 \tag{2.0.1}$$

for a linear Sobolev-type operator

$$L\dot{u} = Mu \tag{2.0.2}$$

Suppose that there exists an operator $L^{-1} \in \mathcal{L}(\mathcal{F};\mathcal{U})$, then problem (2.0.1), (2.0.2) is reduced to a couple of problems equivalent to it

$$\dot{u} = Su, \qquad u(0) = u_0; \tag{2.0.3}$$

$$\dot{f} = Tf, \qquad f(0) = f_0, \tag{2.0.4}$$

where operators $S = L^{-1}M \in Cl(\mathcal{U})$, $T = ML^{-1} \in Cl(\mathcal{F})$; vectors $f = Lu$, $f_0 = Lu_0$.

Problems (2.0.3), (2.0.4) with an accuracy of notation coincide with the problem

$$\dot{v} = Av, \qquad v(0) = v_0, \tag{2.0.5}$$

where $A \in \mathcal{C}l(\mathcal{V})$, $\mathcal{V}$ is a Banach space and $v_0 \in \operatorname{dom} A$. If an operator A is radial, then, as follows from the Hille—Yosida theorem (Hille and Phillips, 1957; Yosida, 1965), the unique solution of problem (2.0.5) is $v(t) = V^t v_0$, where $\{V^t \mid t \in \overline{\mathbb{R}}_+\}$ is a strongly continuous (C_0)-semigroup of solving operators of equation (2.0.5).

It is easy to see that the operator S of problem (2.0.3) is radial exactly when the operator T is radial. Therefore, in this case a pair of operators (L, M) generates a pair $(\{U^t\}, \{F^t\})$ of strongly continuous (C_0)-semigroups defined on the spaces $\mathcal{U}$ and $\mathcal{F}$ respectively. At the same time, the semigroup $\{U^t \mid t \in \overline{\mathbb{R}}_+\}$ consists of solving operators of equation (2.0.2), i. e. a unique solution $u = u(t)$ of problem (2.0.1), (2.0.2) for every $u_0 \in \operatorname{dom} M$ has the form $u(t) = U^t u_0$.

The Hille—Yosida theorem establishes bijection between a set of radial operators and a set of strongly continuous (C_0)-semigroups. In this case, however, operators S and T are similar (i. e. $T = LSL^{-1}$), consequently semigroups $\{U^t \mid t \in \overline{\mathbb{R}}_+\}$ and $\{F^t \mid t \in \overline{\mathbb{R}}_+\}$ are also similar (that is $F^t = LU^tL^{-1}$ for every $t \in \overline{\mathbb{R}}_+$), therefore, there is no longer any bijection between the set of operator pairs (L, M) and the set of pairs of random strongly continuous (C_0)-semigroups. The only bijection now is that between a set of pairs of similar operators (S, T) and a set of pairs of similar strongly continuous (C_0)-semigroups $(\{U^t\}, \{F^t\})$.

The situation becomes more complicated when an operator L is non-invertible, in particular, when its kernel $\ker L \neq \{0\}$. Sviridyuk (1995) was one of the first to consider this case from the viewpoint of the theory of radial operators and strongly continuous semigroups. His results were later developed by Fedorov (1996, 2001). The results presented in this chapter were mostly obtained by Fedorov.

In Section 2.1, we introduce a L-resolvent set and a L-spectrum of an operator M generalising the concepts of a resolvent set and of a spectrum of the operator S (or T) when an operator L is invertible; and study the properties of L-resolvents of the operator M coinciding with the resolvents of the operators S and T for the case $L^{-1} \in \mathcal{L}(\mathcal{U}; \mathcal{F})$. In addition, this section contains a detailed study of M- adjoint vectors of the operator L introduced by Vainberg and Trenogin (1969).

In Section 2.2 relatively p-radial operators are introduced and studied, in particular, if an operator L is continuously invertible, an operator S (or T)

is radial then an operator M is (L,p)-radial. For $p = 0$ the reverse is also true.

In Section 2.3 the existence of degenerate strongly continuous semigroups of operators is proved generated by an (L,p)-radial operator M. The proof is based on the approximations of the Yosida-type. In Section 2.4 the same fact is proved by approximations of the Hille—Widder—Post-type. The result generalises the direct statement of the Hille—Yosida theorem.

In Section 2.5 conditions are discussed sufficient for splitting the space $\mathcal{U} = \mathcal{U}^0 \oplus \mathcal{U}^1$, $\mathcal{F} = \mathcal{F}^0 \oplus \mathcal{F}^1$ and for splitting the actions of operators $L : \mathcal{U}^k \to \mathcal{F}^k$, M : dom $\cap \mathcal{U}^k \to \mathcal{F}^k$, $k = 0, 1$. In addition, here the conditions of existence of an operator $L_1^{-1} \in \mathcal{L}(\mathcal{F}^1; \mathcal{U}^1)$ are considered.

In Section 2.6, infinitesimal generators of restrictions of degenerate strongly continuous semigroups and phase spaces of equation (2.0.2) are studied. The results of this section are used in Section 2.7 to prove the generalisation of the invertible statement of the Hille—Yosida theorem. In Section 2.8, all the obtained results are employed for studying degenerate strongly continuous groups generated by operators L and M.

2.1. RELATIVE RESOLVENTS

Let $\mathcal{U}$ and $\mathcal{F}$ be Banach spaces, operator $L \in \mathcal{L}(\mathcal{U}; \mathcal{F})$, and operator M : dom $M \subset \mathcal{U} \to \mathcal{F}$ be linear and closed.

Definition 2.1.1. Set

$$\rho^L(M) = \{\mu \in \mathbb{C} \mid (\mu L - M)^{-1} \in \mathcal{L}(\mathcal{F}; \mathcal{U})\}$$

is called a *resolvent set* of an operator M *with respect to* an operator L (or, briefly, *L-resolvent set* of an operator M). The set $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ is called *spectrum* of an operator M *with respect to* an operator L (or, briefly, *L-spectrum* of an operator M).

Remark 2.1.1. When there exists an operator $L^{-1} \in \mathcal{L}(\mathcal{F}; \mathcal{U})$ L-resolvent set and L-spectrum of the operator M coincide with the resolvent set and the spectrum of the operator $L^{-1}M$ (or the operator ML^{-1}).

Remark 2.1.2. The L-resolvent set of the operator M is always open, and, consequently, the L-spectrum of the operator M is always closed.