Rings and Nearrings

Kostia Beidar fishing in Hsiao Liu-chiu

Rings and Nearrings

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Preface

This volume is dedicated to the outstanding ring theorist of our time, Kostia Beidar, a distinguished professor of National Cheng Kung University, one of the leading universities in Asia. It consists of seven papers related to the various kind of research work of Kostia. Written by the leading experts of these areas the papers are not only in the aim of mathematical sense, but also emphasize the versatile applications to other fields of mathematics.

Most papers are based on the talks that were presented in the memorial conference which was held in March, 2005 in NCKU. The speakers were Tomoyuki Arakawa (Japan), Tatiana Bandman (Israel), Matej Brešar (Slovenia), Chen-Lian Chuang (Taiwan), Miguel Ferrero (Brazil), Antonio Giambruno (Italy), Koichiro Harada (USA), Shigeru Kobayashi (Japan), Ching-Hung Lam (Taiwan), Tsiu-Kwen Lee (Taiwan), Ying-Fen Lin (Taiwan), Christian Lomp (Portugal), Leonid Makar-Limanov (USA), Wallace S. Martindale, 3rd (USA), Edmund Puczyłowski (Poland), Peter Šemrl (Slovenia), Lance Small (USA), Richard Wiegandt (Hungary), Robert Wisbauer (Germany), Efim Zelmanov (USA). We would like to use this possibility to thank all the speakers for coming to this very special conference.

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Further, we cannot have this volume come into fruition without the help of many others. Here, acknowledgement should go to those who have encouraged and supported financially and spiritually through years. These include Professor Chiou-Shing Wang (Kao Yuan University), Mr. Yu-Lam Chu (the General Director of Mr. Mathematics Pig Publisher), Dr. Chuang Leo (the President of Cheng-Lin Investment Co., Ltd., and Cheng-Lin Education Co., Ltd.). Finally, special mention must be made to Professor Ka Wai Fong who has been supportive for the past decade.

January 2007

Mikhail Chebotar
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Contents

Functional identities and d-free sets: fundamental contributions of Kostia Beidar

Matej Brešar

Dedicated to the memory of K. I. Beidar

1. Introduction

In a relatively short period 1994–2004 Kostia Beidar published, jointly with different coauthors, a number of papers [1]–[23] that, in one way or another, all base on *functional identities* (FI's): either by creating the general theory of FI's, or by dealing with applications of this theory to other mathematical areas. His impact on FI's was, and still is, unmeasurable. Some of his works, especially [1] (introducing "general" FI's) and two papers with Chebotar [14, 15] (introducing d-free sets), were path-breaking. They gave the foundations of the general theory of FI's.

Roughly speaking, functional identities are identical relations in rings that involve arbitrary ("unknown") functions together with arbitrary elements from a ring; the usual goal when treating an FI is to describe either the form of these functions or (when this is not possible) the structure of the ring admitting the FI in question. This topic was initiated in the early 90's by the present author, studying various special FI's in a series of papers. We refer to survey articles [32, 34] for details concerning this early period in the development of FI's. Let us just mention that many of these results, especially those on commuting maps [28, 29] and (using the present terminology) those on (generalized) functional identities of degree 2 [30, 31], were suggesting that something deeper is hidden behind all these. Some applications of the results on commuting maps, particularly those concerning Lie isomorphisms and derivations [29], made the problem of creating some "general theory" of FI's really attractive. For several years we were searching for a proper setting for such a theory.

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The first important break-through in this direction was made by Misha Chebotar who studied the so-called generalized functional identities and generalized our result from [31] from $n = 2$ to a general n [35] (the reason for the name "generalized functional identities" is that one can view these identities as generalizations of generalized polynomial identities, while functional identities generalize polynomial identities). Chebotar's paper was soon followed by Beidar's fundamental work [1] where in particular our result from [30] was generalized from $n = 2$ to a general n. But more importantly, the right concept was found; in our opinion, in [1] Beidar finally gave the answer to the question what should be the right setting for the general theory. And indeed such a theory was created in this setting soon afterwards – in the papers [14, 15] Beidar and Chebotar introduced and studied the so-called d-free sets which are now considered as the central concept in the theory of FI's. In our opinion these papers are astonishing also from the technical point of view.

The development of FI's has been always closely connected with applications. In particular, almost all abstract theory from [14, 15] was later used in the proofs of complete solutions of long–standing Herstein's conjectures [37] concerning Lie maps of associative rings.

Our aim in this paper is to present a survey of those results that are particularly important in the general theory of FI's. In Section 2 we will consider the theory of d-free sets, basically just surveying the papers [14, 15]. Section 3 is devoted to Beidar's fundamental theorem from [1] (this is also the only result in this paper which will be proved) and to related subsequent results establishing d-freeness of some concrete classes of sets.

We shall omit several important topics, including all applications of FI's. Although this makes this paper a kind of a torso, we believe that on the other hand it would be difficult to include everything relevant into one article and so it is better to select a particular area. We have tried to choose those results that we find beautiful and that in our opinion adequately represent the mathematical legacy of Kostia Beidar.

2. Beidar-Chebotar theory of d-free sets

As mentioned, in this section we will survey two papers [14, 15] by Beidar and Chebotar on d-free sets. In order to make the paper easily accessible we shall formulate only simplified versions of some results, and omit presenting the most involved results. Moreover, we shall not give any proofs. An interested reader should therefore consult the original papers.

The concept that we are about to introduce concerns arbitrary subsets of arbitrary rings, and is an elementary one in a sense that only the very basic knowledge of algebra is necessary to understand it (at least technically). However, the definition is somewhat complicated and before stating it we have to introduce various notations.

Let $\mathcal{Q}$ be a ring with unity, and let $\mathcal{R}$ be a nonempty subset of $\mathcal{Q}$. In the first concrete situation that was studied (and the one that appears in many applications), $\mathcal{R}$ was a ring and $\mathcal{Q}$ was its maximal ring of quotients (what explains the background for the choice of symbols $\mathcal{R}$ and $\mathcal{Q}$), but in the general theory that we shall now outline, $\mathcal{Q}$ can be just any ring with 1 and $\mathcal{R}$ can be its arbitrary nonempty subset. By $\mathcal{C}$ we denote the center of $\mathcal{Q}$.

Let m be a positive integer. For elements $x_i \in \mathcal{R}$, $i = 1, 2, \dots, m$, we set

$$\begin{aligned} \overline{x}_m &= (x_1, \dots, x_m) \in \mathcal{R}^m, \\ \overline{x}_m^i &= (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) \in \mathcal{R}^{m-1}, \\ \overline{x}_m^{ij} = \overline{x}_m^{ji} &= (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1} \dots, x_m) \in \mathcal{R}^{m-2}; \end{aligned}$$

here $\mathcal{R}^k$ denotes the Cartesian product of k copies of $\mathcal{R}$. Let $\mathcal{I}, \mathcal{J}$ be subsets of $\{1, 2, \dots, m\}$, and for each $i \in \mathcal{I}$ and $j \in \mathcal{J}$ let

$$E_i : \mathcal{R}^{m-1} \to \mathcal{Q} \quad \text{and} \quad F_j : \mathcal{R}^{m-1} \to \mathcal{Q}$$

be arbitrary functions. If $m = 1$, then we can regard E_i's and F_j's as elements in $\mathcal{Q}$.

The basic *functional identities*, that for the first time appeared in this generality in Beidar's paper [1] (the case $m = 2$ was considered earlier in [30]), are:

$$\sum_{i\in\mathcal{I}} E_i(\overline{x}_m^i)x_i + \sum_{j\in\mathcal{J}} x_j F_j(\overline{x}_m^j) = 0 \quad \text{for all } \overline{x}_m \in \mathcal{R}^m, \tag{1}$$

$$\sum_{i\in\mathcal{I}} E_i(\overline{x}_m^i)x_i + \sum_{j\in\mathcal{J}} x_j F_j(\overline{x}_m^j) \in \mathcal{C} \quad \text{for all } \overline{x}_m \in \mathcal{R}^m. \tag{2}$$

For example, if $m = 3, \mathcal{I} = \{1, 2\}$, and $\mathcal{J} = \{2, 3\}$, (1) can be rewritten as

$$E_1(x_2, x_3)x_1 + E_2(x_1, x_3)x_2 + x_2F_2(x_1, x_3) + x_3F_3(x_1, x_2) = 0 \tag{3}$$

for all $x_i \in \mathcal{R}$.

We remark that (1) trivially implies (2), so one should not understand that (1) and (2) are satisfied simultaneously by the same maps E_i and F_j; each of the two identities should be treated separately.

The usual goal when facing a certain FI is to describe the maps appearing in it, that is, we consider an FI as an equation with maps as unknowns. Let us present a natural possibility when (1), and hence automatically also (2), is fulfilled. Suppose there exist maps

$$\begin{aligned} &p_{ij} : \mathcal{R}^{m-2} \to \mathcal{Q}, \;\; i \in \mathcal{I}, \; j \in \mathcal{J}, \; i \neq j, \\ &\lambda_k : \mathcal{R}^{m-1} \to \mathcal{C}, \;\; k \in \mathcal{I} \cup \mathcal{J}, \end{aligned}$$

(for $m = 1$ one should understand this as that $p_{ij} = 0$ and λ_k is an element in $\mathcal{C}$) such that

$$\begin{aligned} E_i(\overline{x}_m^i) &= \sum_{\substack{j\in\mathcal{J},\\ j\neq i}} x_j p_{ij}(\overline{x}_m^{ij}) + \lambda_i(\overline{x}_m^i), \quad i \in \mathcal{I}, \\ F_j(\overline{x}_m^j) &= -\sum_{\substack{i\in\mathcal{I},\\ i\neq j}} p_{ij}(\overline{x}_m^{ij}) x_i - \lambda_j(\overline{x}_m^j), \quad j \in \mathcal{J}, \\ \lambda_k &= 0 \quad \text{if} \quad k \notin \mathcal{I} \cap \mathcal{J}. \end{aligned} \tag{4}$$

A straightforward computation indeed shows that (4) implies (1), that is, (4) is a solution of the equation (1). We call (4) a *standard solution* of (1) (as well as of (2)). For example, a standard solution of (3) is

$$\begin{aligned} E_1(x_2, x_3) &= x_2 p_{12}(x_3) + x_3 p_{13}(x_2), \\ E_2(x_1, x_3) &= x_3 p_{23}(x_1) + \lambda_2(x_1, x_3), \\ F_2(x_1, x_3) &= -p_{12}(x_3) x_1 - \lambda_2(x_1, x_3), \\ F_3(x_1, x_2) &= -p_{13}(x_2) x_1 - p_{23}(x_1) x_2; \end{aligned}$$

here, $p_{12}, p_{13}, p_{23} : \mathcal{R} \to \mathcal{Q}$ and $\lambda_2 : \mathcal{R}^2 \to \mathcal{C}$ are arbitrary maps.

The cases when one of the sets $\mathcal{I}$ and $\mathcal{J}$ is empty are not excluded. We shall follow the convention that the sum over $\emptyset$ is 0. Thus, if $\mathcal{J} = \emptyset$ (resp. $\mathcal{I} = \emptyset$), (1) reads as

$$\sum_{i\in\mathcal{I}} E_i(\overline{x}_m^i) x_i = 0 \quad \text{for all } \overline{x}_m \in \mathcal{R}^m, \tag{5}$$

$$\sum_{j\in\mathcal{J}} x_j F_j(\overline{x}_m^j) = 0 \quad \text{for all } \overline{x}_m \in \mathcal{R}^m. \tag{6}$$

Note that the standard solution of (5) is $E_i = 0$ for each i, and the standard solution of (6) is $F_j = 0$ for each j.

We are now in a position to introduce the basic notion of the theory of FI's.

Definition 2.1. A set $\mathcal{R}$ is said to be a *d-free subset of* $\mathcal{Q}$, where d is a positive integer, if the following two conditions hold for all $m \geq 1$ and all $\mathcal{I}, \mathcal{J} \subseteq \{1, 2, \ldots, m\}$:

(a) If $\max\{|\mathcal{I}|, |\mathcal{J}|\} \leq d$, then (1) implies (4).
(b) If $\max\{|\mathcal{I}|, |\mathcal{J}|\} \leq d - 1$, then (2) implies (4).

So, roughly speaking, on d-free sets the FI's (1) and (2) have only standard solutions provided that $|\mathcal{I}|$ and $|\mathcal{J}|$ are small enough, i.e. they do not exceed d (resp. $d - 1$). It follows easily from the definition that these standard solutions are *unique*.

Note that, in view of (a), (b) can be replaced by (b') If $\max\{|\mathcal{I}|, |\mathcal{J}|\} \leq d - 1$, then (2) implies (1). It turns out that the conditions (a) and (b) are often equivalent, but not in general [33].

A trivial remark: If $\mathcal{R}$ is d-free, then it is also d'-free for every $d' < d$. Usually we are interested in the largest d such that $\mathcal{R}$ is d-free. In the ideal situation $\mathcal{R}$ is d-free for every positive integer d.

A reader facing the definition of a d-free set for the first time might wonder whether d-free sets actually exist, and of course why to consider these complicated identities in the first place. One cannot answer the latter question immediately; let us just assure the reader that the notion of a d-free set has proved to be extremely useful. The first question has a clear answer: yes, d-free sets do exist, in fact there are plenty of them provided that $\mathcal{Q}$ satisfies certain conditions. All proofs of their existence that are known to us are based on Beidar's method that will be presented in the next section.

Of course such a definition does not come out of nothing. When Beidar and Chebotar introduced d-free sets in their paper [14] published in 2000, they knew that the concept of a d-free set is not an empty one, that is, they were aware of various concrete examples as well as of the applicability of the identities (1) and (2). But nevertheless the idea of studying *abstract* d-free sets appeared rather surprising at the time; later development has showed that it was a brilliant idea.

Concrete examples of d-free sets will be presented in the next section. Let us for a start point out some limitations with regard to d-freeness. Suppose that $\mathcal{R}$ is a nonzero commutative subset of $\mathcal{Q}$. Then $x_2x_1 - x_1x_2 = 0$ for all $x_1, x_2 \in \mathcal{R}$.

We can interpret this as that the FI

$$E_1(x_2)x_1 + E_2(x_1)x_2 = 0$$

has a nonzero, and hence a nonstandard solution, namely,

$$E_1(x_2) = x_2 \quad \text{and} \quad E_2(x_1) = -x_1.$$

Therefore $\mathcal{R}$ cannot be 2-free, and so also not d-free for every $d \geq 2$ (it may be 1-free, but this is not so interesting; the case $d = 2$ is the first nontrivial one). Accordingly, a commutative ring $\mathcal{Q}$ cannot contain 2-free subsets. Similarly one can show that if $\mathcal{Q}$ satisfies a polynomial identity of degree d, then $\mathcal{Q}$ does not contain d-free subsets. The question, however, whether $\mathcal{Q}$ contains $(d-1)$-free subsets, in particular whether $\mathcal{Q}$ itself is a $(d-1)$-free subset of itself, may be interesting.

One can view (multilinear) polynomial identities as very special examples of FI's of the type (1). Indeed, (1) reduces to a polynomial identity if all E_i's and F_j's are polynomials. But the theory of polynomial identities has rather different goals than the theory of functional identities, and as observations from the previous paragraph suggest, in PI rings one cannot hope for handling FI's easily. One could say, especially from the point of view of applications, that the two theories are complementary to each other, rather than that of FI's generalizes that of PI's.

Let us now present the basic results of the general theory of d-free sets. They can be, roughly speaking, divided into two groups: the results that yield new d-free sets from a given d-free set, and the results showing that on d-free sets one can handle more general FI's than those from the definition, i.e. (1) and (2).

We shall state just two sample results from the first group, both of them very useful and important. The first one has also a very simple statement. The proof, however, is nontrivial and is based on a generalization of the notion of a d-free set, which we shall avoid introducing in this expository article.

Theorem 2.2. *Let $\mathcal{P} \subseteq \mathcal{R} \subseteq \mathcal{Q}$ be nonempty sets. If $\mathcal{P}$ is a d-free subset of $\mathcal{Q}$, then $\mathcal{R}$ is also d-free.*

The next theorem also involves a concept more general than d-freeness. Although this concept is basically a technical one, perhaps not of great interest in its own right, we can not avoid it. It is too vital for the theory of FI's. This is the concept of a $(t; d)$-*free subset*. As it will be indicated in the next section, in order to show that a certain set is d-free, one is often forced to show more, namely that this set is $(t; d)$-free for some t. Let us introduce this concept.

We continue to assume that $\mathcal{R}$ is a nonempty subset of $\mathcal{Q}$, m is a positive integer, and $\mathcal{I}, \mathcal{J}$ are subsets of $\{1, 2, \ldots, m\}$. Now everything shall center round a fixed element $t \in \mathcal{Q}$. Let a, b be nonnegative integers, and $E_{iu} : \mathcal{R}^{m-1} \to \mathcal{Q}$, $i \in \mathcal{I}$, $0 \le u \le a$, and $F_{jv} : \mathcal{R}^{m-1} \to \mathcal{Q}$, $j \in \mathcal{J}$, $0 \le v \le b$, be arbitrary functions. The two identities that we shall now consider are clearly generalizations of (1) and (2):

$$\sum_{i\in\mathcal{I}} \sum_{u=0}^{a} E_{iu}(\overline{x}_m^i) x_i t^u + \sum_{j\in\mathcal{J}} \sum_{v=0}^{b} t^v x_j F_{jv}(\overline{x}_m^j) = 0 \quad \text{for all } \overline{x}_m \in \mathcal{R}^m, \tag{7}$$

and

$$\sum_{i\in\mathcal{I}} \sum_{u=0}^{a} E_{iu}(\overline{x}_m^i) x_i t^u + \sum_{j\in\mathcal{J}} \sum_{v=0}^{b} t^v x_j F_{jv}(\overline{x}_m^j) \in \mathcal{C} \quad \text{for all } \overline{x}_m \in \mathcal{R}^m. \tag{8}$$

Indeed, if $a = b = 0$ then we get (1) and (2). We define a *standard solution* of (7) and (8) as follows: there exist maps

$$\begin{aligned} p_{iujv} &: \mathcal{R}^{m-2} \to \mathcal{Q}, \ \ i \in \mathcal{I},\ j \in \mathcal{J},\ i \ne j,\ 0 \le u \le a,\ 0 \le v \le b, \\ \lambda_{kuv} &: \mathcal{R}^{m-1} \to \mathcal{C}, \ \ k \in \mathcal{I} \cup \mathcal{J}, 0 \le u \le a,\ 0 \le v \le b, \end{aligned}$$

such that

$$\begin{aligned} E_{iu}(\overline{x}_m^i) &= \sum_{\substack{j\in\mathcal{J},\\ j\ne i}} \sum_{v=0}^{b} t^v x_j p_{iujv}(\overline{x}_m^{ij}) + \sum_{v=0}^{b} \lambda_{iuv}(\overline{x}_m^i) t^v, \\ F_{jv}(\overline{x}_m^j) &= -\sum_{\substack{i\in\mathcal{I},\\ i\ne j}} \sum_{u=0}^{a} p_{iujv}(\overline{x}_m^{ij}) x_i t^u - \sum_{u=0}^{a} \lambda_{juv}(\overline{x}_m^j) t^u, \\ \lambda_{kuv} &= 0 \quad \text{if} \quad k \notin \mathcal{I} \cap \mathcal{J} \end{aligned} \tag{9}$$

for all $\overline{x}_m \in \mathcal{R}^m$, $i \in \mathcal{I}$, $j \in \mathcal{J}$, $0 \le u \le a$, $0 \le v \le b$.

Definition 2.3. A set $\mathcal{R}$ is said to be a $(t; d)$*-free subset of* $\mathcal{Q}$, where d is a positive integer and $t \in \mathcal{Q}$, if the following two conditions hold for all $m \ge 1$, all $\mathcal{I}, \mathcal{J} \subseteq \{1, 2, \ldots, m\}$, and all $a, b \ge 0$:

(a) If $\max\{|\mathcal{I}| + a, |\mathcal{J}| + b\} \le d$, then (7) implies (9).
(b) If $\max\{|\mathcal{I}| + a, |\mathcal{J}| + b\} \le d - 1$, then (8) implies (9).

Considering the case where $a = b = 0$ we see that if $\mathcal{R}$ is $(t; d)$-free for some $t \in \mathcal{Q}$, then it is also d-free. Although trivial, this observation is essential for us.

We say that $t \in \mathcal{Q}$ is algebraic over $\mathcal{C}$ of degree $\leq n$ if there exist $c_0, c_1, \ldots, c_n \in \mathcal{C}$, not all zero, such that $c_0 + c_1 t + \ldots + c_n t^n = 0$. Now we are ready to state the second theorem.

Theorem 2.4. *Let $\mathcal{P}$ be a $(t; d{+}1)$-free subset of $\mathcal{Q}$. Suppose that t is not algebraic over $\mathcal{C}$ of degree ≤ 2. Let $\epsilon \in \{1, -1\}$, and suppose that a set $\mathcal{R}$ is such that $tx + \epsilon xt \in \mathcal{R}$ for all $x \in \mathcal{P}$. Then $\mathcal{R}$ is a d-free subset of $\mathcal{Q}$.*

This theorem is useful when one wants to establish the d-freeness of Lie or Jordan ideals of (Lie or Jordan) subrings of $\mathcal{Q}$. Its proof is very complicated and rather long.

We now turn to the other group of results, showing that on d-free sets one can also settle other, more general and more complicated FI's than (1) and (2). Let us emphasize that the study of these identities was motivated by concrete problems like characterizing Lie homomorphisms etc.; most of the abstract theory of FI's, which we are now outlining, was developed because of applications that the authors had in their minds.

Consider the following situation. Let $\mathcal{S}$ be a nonempty set and let $\alpha : \mathcal{S} \to \mathcal{Q}$ be an arbitrary (but fixed!) map. We shall write x^α for $\alpha(x)$. Let $m, \mathcal{I}, \mathcal{J}$ have the usual meaning, while the functions E_i and F_j now map from $\mathcal{S}^{m-1}$ into $\mathcal{Q}$. We are now interested in FI's

$$\sum_{i\in\mathcal{I}} E_i(\overline{x}_m^i)x_i^\alpha + \sum_{j\in\mathcal{J}} x_j^\alpha F_j(\overline{x}_m^j) = 0 \quad \text{for all } \overline{x}_m \in \mathcal{S}^m, \tag{10}$$

$$\sum_{i\in\mathcal{I}} E_i(\overline{x}_m^i)x_i^\alpha + \sum_{j\in\mathcal{J}} x_j^\alpha F_j(\overline{x}_m^j) \in \mathcal{C} \quad \text{for all } \overline{x}_m \in \mathcal{S}^m. \tag{11}$$

If $\mathcal{S} = \mathcal{R}$ and α is the identity map, then (10) and (11) coincide with (1) and (2), respectively. Now one can already guess how to define a *standard solution* of (10) and (11): there exist maps

$$p_{ij} : \mathcal{S}^{m-2} \to \mathcal{Q}, \;\; i \in \mathcal{I}, \; j \in \mathcal{J}, \; i \neq j,$$
$$\lambda_k : \mathcal{S}^{m-1} \to \mathcal{C}, \;\; k \in \mathcal{I} \cup \mathcal{J}$$

such that

$$\begin{aligned} E_i(\overline{x}_m^i) &= \sum_{\substack{j\in\mathcal{J},\\ j\neq i}} x_j^\alpha p_{ij}(\overline{x}_m^{ij}) + \lambda_i(\overline{x}_m^i), \quad i \in \mathcal{I}, \\ F_j(\overline{x}_m^j) &= -\sum_{\substack{i\in\mathcal{I},\\ i\neq j}} p_{ij}(\overline{x}_m^{ij}) x_i^\alpha - \lambda_j(\overline{x}_m^j), \quad j \in \mathcal{J}, \\ \lambda_k &= 0 \quad \text{if} \quad k \notin \mathcal{I}\cap\mathcal{J}. \end{aligned} \tag{12}$$

When do (10) and (11) have only standard solutions? The next theorem tells us that this depends *only* on the range of α.

Theorem 2.5. *Suppose that $\mathcal{S}^\alpha$ is a d-free subset of $\mathcal{Q}$.*

(a) *If* $\max\{|\mathcal{I}|, |\mathcal{J}|\} \leq d$, *then* (10) *implies* (12).
(b) *If* $\max\{|\mathcal{I}|, |\mathcal{J}|\} \leq d-1$, *then* (11) *implies* (12).

Of course, if $\mathcal{S} = \mathcal{R}$ and α is the identity map, Theorem 2.5 is nothing but the restatement of the definition of a d-free set; it seems rather surprising that the two conditions (a) and (b) remain true for any map α.

The last topic we wish to discuss in this section is the theory of *quasi-polynomials*. We shall state only a sample result, and moreover only a special case of a more general theorem. There are two reasons for choosing this particular result: the first one is that it has turned out to be important because of applications, and the second one is that it has a simple and clear statement, which makes it possible for us to avoid introducing a complicated notation necessary for discussing more advanced results on quasi-polynomials.

Let us define a quasi-polynomial in an informal manner. As above, let $\alpha : \mathcal{S} \to \mathcal{Q}$ be a fixed map. We say that a map $P_1 : \mathcal{S} \to \mathcal{Q}$ is a quasi-polynomial of degree 1 if there exist $\lambda \in \mathcal{C}$ and $\mu : \mathcal{S} \to \mathcal{C}$, at least one of them nonzero, such that

$$P_1(x) = \lambda x^\alpha + \mu(x)$$

for all $x \in \mathcal{S}$. A quasi-polynomial of degree 2 is a map $P_2 : \mathcal{S}^2 \to \mathcal{Q}$ of the form

$$P_2(x_1, x_2) = \lambda_1 x_1^\alpha x_2^\alpha + \lambda_2 x_2^\alpha x_1^\alpha + \mu_1(x_2) x_1^\alpha + \mu_2(x_1) x_2^\alpha + \nu(x_1, x_2)$$

where $\lambda_1, \lambda_2 \in \mathcal{C}$, $\mu_1, \mu_2 : \mathcal{S} \to \mathcal{C}$, $\nu : \mathcal{S}^2 \to \mathcal{C}$, and at least one of them is nonzero. Now, a quasi-polynomial of degree 3 consists of summands such as

$$\lambda_1 x_1^\alpha x_2^\alpha x_3^\alpha,\ \mu_1(x_1) x_2^\alpha x_3^\alpha,\ \nu_1(x_1, x_2) x_3^\alpha, \quad \text{etc.}$$

with at least one "coefficient" (i.e. λ_1, μ_1, ν_1 etc.) nonzero. It should now be clear what we mean by a quasi-polynomial of degree n.

Theorem 2.6. *Let $P : \mathcal{S}^{m-1} \to \mathcal{Q}$ be a map. Let $\lambda_i, \mu_i \in \mathcal{C}$ be such that at least one of them is invertible, and define $R : \mathcal{S}^m \to \mathcal{Q}$ by*

$$R(\overline{x}_m) = \sum_{i=1}^{m} \lambda_i P(\overline{x}_m^i)x_i^\alpha + \mu_i x_i^\alpha P(\overline{x}_m^i).$$

Suppose that $\mathcal{S}^\alpha$ is an $(m+1)$-free subset of $\mathcal{Q}$. If R is a quasi-polynomial, then P is a quasi-polynomial too.

Of course P is of degree $m-1$, unless all its coefficients are 0. We remark that the latter is equivalent to the condition that $P(\overline{x}_{m-1}) = 0$ for all $x_i \in \mathcal{S}$. This is a corollary to Theorem 2.5; in fact, to establish this it suffices to assume that $\mathcal{S}^\alpha$ is m-free (instead of $(m+1)$-free).

More general results in this area consider FI's involving summands such as

$$x_{i_1}^\alpha \ldots x_{i_p}^\alpha P(x_{j_1}, \ldots, x_{j_q}) x_{k_1}^\alpha \ldots x_{k_r}^\alpha$$

where P is an arbitrary (unknown) function. We refer to [15] for these results, their proofs and all details.

3. Beidar's fundamental theorem and related results

Let us first fix the notation. By $\mathcal{A}$ we denote a prime ring, and by $\mathcal{Q}_{ml}$ we denote its maximal left ring of quotients (incidentally, "left" is chosen by chance, we could also work with the maximal right ring of quotients). The center $\mathcal{C}$ of $\mathcal{Q}_{ml}$ is a field called the extended centroid of $\mathcal{A}$. These notions are studied in detail in the book [24] by Beidar, Martindale and Mikhalev, to which we shall occasionally refer in the sequel.

Given $t \in \mathcal{Q}$, we denote by $\deg(t)$ the degree of algebraicity of t over $\mathcal{C}$ (if t is algebraic over $\mathcal{C}$) or ∞ (if it is not algebraic). We set $\deg(\mathcal{A}) = \sup\{\deg(t) \mid t \in \mathcal{A}\}$. It is known that the condition $\deg(\mathcal{A}) \leq n < \infty$ is equivalent to the condition that $\mathcal{A}$ satisfies the standard polynomial identity of degree $2n$, and in this case $\mathcal{A}$ can be embedded into a ring of $n \times n$ matrices over a field.

Theorem 3.1 (Beidar's fundamental theorem). *Let $\mathcal{A}$ be a prime ring and let d be a positive integer. If $\deg(\mathcal{A}) \geq d$, then $\mathcal{A}$ is a d-free subset of $\mathcal{Q}_{ml}$.*

This result was published in Beidar's seminal paper [1] in 1998. The original formulation was of course different because the notion of d-freeness was introduced only later. Also, in [1] the additional assumption that all maps E_i and F_j from (1) and (2) are multiadditive was used, but in the subsequent paper [22] it was noticed that this assumption is redundant.