Groups – Korea '94

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Editors
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Preface

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A. C. Kim
D. L. Johnson

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On the Rank of a Finite Product of Two p-Groups

*Bernhard Amberg and Lev S. Kazarin**

1. Introduction

If the finite p-group $G = AB$ is the product of two subgroups A and B whose Prüfer ranks are bounded by r, then the Prüfer rank of G is bounded by a polynomial function of r; see [5] and [1]. Although no bound is given there explicitly, the analysis of the proof of this theorem leads to polynomial bounds of relatively high degree (see [1]). In the following we shall give better bounds. A natural way to do this is to obtain first a bound for the normal rank of $G = AB$ which will immediately give a bound for the Prüfer rank of G by Lemma 2.6 below. Our bound for the normal rank of $G = AB$ depending on the Prüfer ranks of A and B is close to being linear. Even if this result may not be best possible, it will be useful in the study of the structure of finite products of groups with low rank.

Recall that a group X has ***Prüfer rank*** $r = r(X)$ if every finitely generated subgroup of X can be generated by r elements and r is the least such integer. The ***normal rank*** $r_n(X)$ of X is the maximum of the minimal number of generators of each normal subgroup of X.

Our main result is the following.

Theorem 1.1. *Let the finite p-group $G = AB$ be the product of two of its subgroups A and B. Let $r_0 = \min\{r(A), r(B)\}$ and $r_1 = r(A) + r(B)$. Then the normal rank $r_n(G)$ satisfies the following inequality:*

$$r_n(G) \leq r_0(\lceil \log_p r_n(G) \rceil + 1 + \lceil \log_p r_0 \rceil \lceil \log_2 2r_0 \rceil + \delta_{2p}) + r_1.$$

The inequality in this theorem may look unusual, but it shows that for any $\epsilon > 0$ and for sufficiently large r_0 we have the following almost linear bound

$$r_n(G)^{1-\epsilon} \leq r_0(3 + \lceil \log_p r_0 \rceil \lceil \log_2 2r_0 \rceil) + r_1.$$

If the two subgroups A and B are abelian, Theorem 1.1 can be improved as follows.

*The authors like to thank the Departments of Mathematics of the Universities of Mainz and Yaroslavl for their excellent hospitality during the preparation of this paper

Theorem 1.2. *Let the finite p-group $G = AB$ be the product of two abelian subgroups A and B. Let $r_0 = \min\{r(A), r(B)\}$ and $r_1 = r(A) + r(B)$. Then the normal rank $r_n(G)$ of G satisfies the inequality*

$$r_n(G) \le r_0 \lceil \log_p r_n(G) \rceil + r_1.$$

If the finite p-group $G = AB$ is the product of two cyclic subgroups A and B, then it follows from Theorem 1.2 that $r_n = r_n(G) \le \lceil \log_p r_n \rceil + 2$. This implies $r_n \le 3$ and even $r_n \le 2$ for $p > 3$. Note however that there exists a finite 2-group of normal rank 3 which is a product of two of its cyclic subgroups. (see [2], Aufgabe 28, p. 341).

The notation is standard and can be found in [2] and [1]. If X is a finite p-group, we note in particular

$$\Omega_i(X) = \text{subgroup generated by all elements } g \text{ in } X \text{ such that } g^{p^i} = 1.$$

$$\mho_i(X) = \text{subgroup generated by all } g^{p^i} \text{ with } g \in X.$$

The ***exponent*** $\exp(X)$ of X is the largest order of its elements. If a minimal generating system of X consists of m elements then we write $d(X) = m$. If α is a real number, then $\lceil \alpha \rceil = m$ is the smallest integer such that $\alpha \le m$. δ_{ij} denotes the Kronecker symbol.

2. Preliminaries

The first lemma is well-known.

Lemma 2.1. *If G is a finite p-group with nilpotency class c, then the derived length of G does not exceed $\lceil \log_2 c \rceil + 1$.*

Proof. See [4], 5.1.12.

Lemma 2.2 (Alperin). *Let G be a finite p-group, n be an integer such that $p^n \ne 2$ and let A be a maximal element in the set of all abelian normal subgroups with exponent $\le p^n$. If $x \in C_G(A)$ and $x^{p^n} = 1$, then $x \in A$.*

Proof. See [2], p. 341.

Lemma 2.3. *Let G be a regular finite p-group with Prüfer (or normal) rank $r > 1$ and exponent p^ν. Then*

$$|G| \le p^{\nu r (\lceil \log_2 (r-1) \rceil + 2)}.$$

Furthermore, if $p = 2$ then $|G| \le 2^{\nu r}$.

Proof. It is obvious that for each i the group $H = \mho_i(G)/\mho_{i+1}(G)$ has exponent less or equal to p. If $A = C_H(A)$ is an abelian self-centralizing normal subgroup of H with rank less or equal to r, then H/A is isomorphic to a subgroup of $\mathrm{Aut}(A) \subseteq GL(r, p)$. If $p = 2$, then G is abelian (see [2], p. 327) and $H = A$. By Theorem 16.3 of [2], p. 382, H/A has nilpotency class at most $r - 1$. By Lemma 2.1 the derived length of H/A does not exceed $\lceil \log_2(r-1) \rceil + 1$. Therefore the derived length of H does not exceed $\lceil \log_2(r-1) \rceil + 2$, since A is abelian. By [2], Theorem on p. 327, we have that $\mho_{\nu+1}(G) = 1$. The lemma is proved.

Lemma 2.4 (Thompson). *Let G be a finite p-group where $p \neq 2$ is a prime. If every abelian normal subgroup of G can be generated by s elements, then a minimal generating system of G contains at most $s(s+1)/2$ elements.*

Proof. See [2], p. 343.

Lemma 2.5. *Let G be a finite p-group. Suppose that if $p > 2$ then every element of order p of G lies in its center, and if $p = 2$ then every element of order ≤ 4 of G lies in its center. Then the following holds:*

(i) $d(G) \leq d(Z(G)) = d(\Omega_1(G))$,

(ii) $|G| \leq p^{\nu r}$ *where r is the Prüfer (or normal) rank of G and* $p^\nu = \exp(G)$.

Proof. We use an idea of Blackburn to prove both statements simultaneously (see [2], p. 342).

Suppose that $A = \Omega_1(G)$ if $p > 2$ and $A = \Omega_2(G)$ if $p = 2$. Let B be an element with maximal order among the normal subgroups X of G containing A with elementary abelian factor group X/A. If $b \in B$ and $g \in G$, then $b^g = bc$ for some $c \in B$. As $b^p \in A \leq Z(G)$, we have $b^p = (b^p)^g = (b^g)^p = (bc)^p = b^p c^p [c, b]^{\binom{p}{2}}$. It follows that if $p > 2$ then $\binom{p}{2} \equiv 0 \pmod p$ and $[c, b]^{\binom{p}{2}} = 1$. In this case we have also $c^p = 1$ and $c \in A$. If $p = 2$ then $c^2[c, b] = 1$. We prove now that $c^4 = 1$. As

$$x^2 y = y x^2 = x y x [y, x] = y x^2 [y, x]^2$$

for each pair $x, y \in B$, we have $B' \subseteq \Omega_1(A)$. Hence it follows from $c^2[c, b] = 1$ that $c^4 = 1$ and so $c \in Z(G)$. Therefore $[B, G] \subseteq \Omega_1(A)$ in each case and $B/\Omega_1(A)$ is an abelian group of exponent p for $p > 2$ and a group of exponent 4 for $p = 2$. Furthermore, this group is not contained in any larger abelian normal subgroup of exponent p (for $p > 2$) or of exponent 4 (for $p = 2$) of the group $G/\Omega_1(A)$. By Lemma 2.2 we have

$$B/\Omega_1(A) = \Omega_1(G/\Omega_1(A)) \subseteq Z(G/\Omega_1(A)) \text{ if } p > 2 \text{ and}$$

$$B/\Omega_1(A) = \Omega_2(G/\Omega_1(A)) \subseteq Z(G/\Omega_1(A)) \text{ if } p = 2.$$

If $p > 2$ then by Theorem 12.2 of [2], p. 342, we have

$$d(B) \leq d(\Omega_1(B)) = d(\Omega_1(A)) = d(\Omega_1(G)).$$

As $\exp(G/\Omega_1(G)) \leq p^{\nu-1}$ it is easy to prove by induction that $|G| \leq p^{\nu r}$

For the remainder of the proof we may suppose now that $p = 2$. If G is abelian, it is easy to see that both statements of Lemma 2.5 hold. Assume now that Lemma 2.5 holds for all groups whose order is less than the order of G. If $B \neq G$ then $d(B) \leq d(\Omega_1(B)) = d(Z(G))$ and $\exp(G/\Omega_1(B)) = \exp(G/\Omega_1(G)) \leq 2^{\nu-1}$ by induction. Since $B/\Omega_1(B) = B/\Omega_1(A)$ plays the same role as $A = \Omega_2(G)$ in G for $G/\Omega_1(A)$, then $|G/\Omega_1(G))| \leq 2^{(\nu-1)r}$ where $r = d(\Omega_1(G)) \geq d(G/\Omega_1(G))$. This proves (ii). If $\Omega_1(B) = \Omega_1(A) \subseteq \Phi(G)$, then $d(G/\Omega_1(G)) = d(G)$ and so (i) is also proved.

Now suppose that $\Omega_1(B) = \Omega_1(A) = \Omega_1(G) \nsubseteq \Phi(G)$. Then there exists a maximal subgroup M of G such that $M\Omega_1(A) = G$ with $\Omega_1(A) \nsubseteq M$. Obviously we have $M \cap \Omega_1(A) = \{x | x \in M, x^2 = 1\} = \Omega_1(Z(M))$. By induction

$$d(M) \leq d(Z(M)) = d(\Omega_1(Z(M)) = d(M \cap Z) \leq d(Z(G)) - 1.$$

Therefore $d(G) \leq d(M) + 1 \leq d(Z(G))$, and we are done.

Suppose now that $B = G$. As $(xy)^4 = x^4y^4$ for each pair of elements x, y of G, then the map $g \to g^4$ is a homomorphism from G into $\Omega_1(G) = \Omega_1(A)$. It is easy to see that the kernel of this homomorphism is $A = \Omega_2(G)$. By [2], p. 272, we have that $\Omega_1(G) = \Phi(G)$ and if $\Omega_1(G) = A$ then $d(G) = d(G/A) \leq d(\Omega_1(A)) = d(Z(G))$. Hence $\Omega_1(G) \neq A$ and in particular there exists an element of $\Omega_1(A)$ which does not have a root of degree 4 in G.

The set of all elements of $\Omega_1(A)$ having a root of degree 4 in G is a subgroup D of G which has a complement C in $\Omega_1(A)$ such that $A = A_1 \times A_2$ where $\Omega_1(A_1) = D$ and $\Omega_1(A_2) = C = A_2$. If $g^4 \in A_2$ for some $g \in G$ then $g^4 \in C$ and so $g^4 = 1$. Hence $\Omega_2(G/A_2) = A/A_2 \subseteq Z(G/A_2)$. By induction we have

$$d(G/A_2) \leq d(\Omega_1(Z(G/A_2))) = d(A_1).$$

It follows that $d(G) \leq d(A_1) + d(A_2) = d(Z(G))$. This proves Lemma 2.5.

Lemma 2.6. *If every abelian normal subgroup of the finite p-group G can be generated by at most s elements, then the Prüfer rank of G is at most $1/2(s + s^2)$ for $p > 2$ and at most $3/2(s^2 + s)$ for $p = 2$.*

Proof. For $p > 2$ this is a theorem of Thompson (see [2], Satz 12.3, p. 343). If $p = 2$ the proof follows from a slight modification of this theorem. One has only to use Lemma 2.2 for $p^n = 4$ and the arguments of Thompson.

Next we will obtain a bound for the order of a finite p-group in terms of its rank and its exponent.

Lemma 2.7. *Let G be a finite p-group with Prüfer rank r and exponent p^ν. Then the following inequalities hold.*

(i) *If $p > 2$ then $|G| \leq p^{\nu r + r(\lceil \log_p r \rceil)(\lceil \log_2 2r \rceil)}$.*

(ii) *If $p = 2$ then $|G| \leq 2^{\nu r + r + r(\lceil \log_2 r \rceil)(\lceil \log_2 2r \rceil)}$.*

Proof. Let A be the largest abelian normal subgroup of G of exponent p for $p > 2$ and of exponent ≤ 4 for $p = 2$. Suppose first that $\exp(A) = p$. Then clearly the factor group $H = G/C_G(A)$ can be embedded into $\mathrm{Aut}(A) \leq GL(r, p)$. Since G is a p-group it is isomorphic to a subgroup of a Sylow p-subgroup of $GL(r, p)$. By Theorem 16.3 of [2], p.382, the group H has nilpotency class less or equal to $r - 1$ and its derived length is less or equal to $\lceil \log_2(r-1) \rceil + 1$. Since every p-element $a \in GL(r, p)$ has a normal Jordan form with Jordan matrices of size less or equal to r then $a - 1$ is a nilpotent element and $(a-1)^r = 0$. If $\lceil \log_p r \rceil = m$, then $(a-1)^{p^m} = 0$ which implies $a^{p^m} = 1$. Hence the exponent of H does not exceed p^m. Each factor $H^{(i)}/H^{(i+1)}$ of the commutator series of H has order not exceeding p^{mr} and so $|H| \leq p^{mr(\lceil \log_2(r-1) \rceil + 1)}$. Since $C_G(A)$ contains each element of order p in its center by Lemma 2.2, then by Lemma 2.5(i) we have $|C_G(A)| \leq p^{\nu r}$ and so the first assertion is proved.

Suppose now that $\exp(A) = 4$. Then $H = G/C_G(A)$ is isomorphic to a subgroup of a group of invertible (r, r)-matrices with entries in $\mathbb{Z}_4$. Let U be the subgroup of this group consisting of all matrices (a_{ij}) such that $a_{ij} \equiv \delta_{ij} \bmod 2$. It is easy to see that the inverse image V of U in G is normal in G and the group $U = V/C_G(A)$ is abelian. Since $u^2 = 1$ for each matrix $u = (a_{ij})$ in U we have $|U| \leq 2^r$ Now $G/V \subseteq GL(r, 2)$ and we may use the previous arguments. Hence

$$|G| \leq |G/V||U||C_G(A)| \leq 2^{r(\lceil \log_2 r \rceil \lceil \log_2 2r \rceil) + r + \nu r}$$

Corollary 2.8. *Let G be a finite p-group with Prüfer rank r and exponent p^ν. Then*

$$|G| \leq p^{\nu r(2 + \lceil \log_2 r \rceil) + r\delta_{2p}}.$$

This result corresponds to Lemma 2.3 for regular p-groups. Note that there is a similar formula in [5], but its proof is not correct.

3. Some Special Cases

The proofs of our theorems will be reduced to the following special situation of a triply factorized group.

Lemma 3.1. *Let the finite p-group $G = AN = BN = AB$ be the product of two subgroups A and B and an elementary abelian normal subgroup N of G such that $A \cap N = B \cap N = 1$. If the Prüfer rank of one of the subgroups A and B is bounded by r, then*

$$d(N) \leq r(\lceil \log_p d(N) \rceil + 1 + \lceil \log_p r \rceil \lceil \log_2 2r \rceil + \delta_{2p}).$$

Proof. Obviously we have $|G| = |A||B||A \cap B|^{-1} = |A||N| = |B||N|$. Hence $|A| = |B| = |N||A \cap B|$. Let $|N| = p^n = m$ and let the elements of N be $c_i = a_i b_i$

with $a_i \in A$ and $b_i \in B$ where $1 \le i \le m$. We show first that $\{a_i | 1 \le i \le m\}$ (respectively $\{b_i | 1 \le i \le m\}$) is a full system of representatives of A (respectively of B) with respect to the subgroup $H = A \cap B$. Assume that on the contrary for some $1 \le i \ne j \le m$ we have $a_i H = a_j H$ with $i \ne j$. Then $c_i^{-1} c_j = b_i^{-1} a_i^{-1} a_j b_j \in N \cap B = 1$, so that $c_i = c_j$, a contradiction. Similarly, $b_i H \ne b_j H$ if $i \ne j$. As $m = |A \quad H| = |B \quad H|$ the assertion about the representatives is proved.

It is easy to see that for each choice of a system of representatives $a_1, a_2, \dots, a_m$ of A for the subgroup H there is a system of representatives $b_1, b_2, \dots, b_m$ of B such that $N = \{a_i b_i | 1 \le i \le m\}$. The subgroup $D = C_A(N)$ is normal in A, so that DH is a subgroup of A. We may choose a system of representatives $a_1, a_2, \dots, a_m$ in A such that $\bigcup_{i=1}^{k} a_i H = DH$ for some $k \le m$.

Let $c_i = a_i b_i$ and $c_j = a_j b_j$ be elements in N where $1 \le i, j \le k$. It is easy to see that $[a_i, c] = [a_j, c] = 1$ for each $c \in N$. Hence $[a_i, b_i] = [a_j, b_j] = 1$ for $1 \le i, j \le k$. As $c^p = 1$ for each $c \in N$ and $c_i c_j = c_j c_i$ for each pair i, j then we have $a_i^p b_i^p = 1$ and so $a_i b_i a_j b_j = a_j a_i b_i b_j = a_j b_j a_i b_i = a_i a_j b_j b_i$ for each $1 \le i, j \le k$. Therefore $a_i^p \in H$, $[a_j, a_i] = [b_j^{-1}, b_i^{-1}] \in H$ for $1 \le i, j \le k$. By [2], p. 272, it follows that $R = \Phi(D) \subseteq H$. Obviously R is normal in G and $G/R \simeq (AR/R)(BR/R) \simeq (A/R)(NR/R) \simeq (B/R)(NR/R)$, where G/R satisfies the conditions of Lemma 3.1. Without loss of generality we may suppose now that $D = C_A(N)$ is an elementary abelian group of rank at most r, and $\bar{A} = A/D$ is isomorphic to a subgroup of $\mathrm{Aut}(N) \simeq GL(n, p)$. Consider now N as a natural $\bar{A}$-module over $F = GF(p)$. Let $\exp(A) = p^\nu$ Then $p^{\nu-1} \le \exp(\bar{A}) \le p^\nu$ Therefore the minimal polynomial of each $\bar{a} \in \bar{A}$ divides $x^{p^\nu} - 1 = (x-1)^{p^\nu}$ In this case $u = \bar{a} - 1$ is a nilpotent element. It is not difficult to see that $u^d = 0$ for some $d \le \dim_F N = n$. Hence if an integer α satisfies the inequality $p^{\alpha-1} < n \le p^\alpha$ then we have $(1+u)^{p^\alpha} = \bar{a}^{p^\alpha} = 1$. Thus $-1 + \nu \le \alpha$. Since $\alpha = \lceil \log_p n \rceil$ by Lemma 2.7 we have the inequality

$$
\begin{aligned}
n &= \log_p |N| = \log_p |A \quad H| \le \log_p |A| \\
&\le \nu r + r \lceil \log_p r \rceil \lceil \log_2 2r \rceil + r\delta_{2p} \\
&\le \lceil \log_p n \rceil r + r \lceil \log_p r \rceil \lceil \log_2 2r \rceil + r\delta_{2p} + r \\
&= (\lceil \log_p n \rceil + 1 + \lceil \log_p r \rceil \lceil \log_2 2r \rceil + \delta_{2p}) r.
\end{aligned}
$$

The lemma is proved.

Lemma 3.2. *Let the finite p-group $G = AN = BN = AB$ be the product of two subgroups A and B and an elementary abelian normal subgroup N of G. Let the Prüfer rank of A be bounded by r and each element of order p of A lie in its center for $p > 2$ and each element of order ≤ 4 of A lie in its center for $p = 2$. Then $d(N) \le r(\lceil \log_p d(N) \rceil + 1)$.*

For the proof one only has to replace Lemma 2.7 by Lemma 2.5 in the proof of Lemma 3.1.

The following lemma is obvious.

Lemma 3.3. *Let $G = AB$ be the product of two normal p-subgroups A and B. If the Prüfer rank of A is r_1 and the Prüfer rank of B is r_2 then the Prüfer rank of G does not exceed $r_1 + r_2$.*

4. Proof of the Main Results

4.1. Proof of Theorem 1.1

Assume that Theorem 1.1 is false, and let the finite p-group $G = AB$ be a counterexample with minimal order. Let $r_0 = \min\{r(A), r(B)\}$ and $r_1 = r(A) + r(B)$. Let N be a normal subgroup of G with maximal rank. The subgroup $\bar{N} = N/\Phi(N)$ of the factor group $\bar{G} = G/\Phi(N)$ has the same rank as N. Clearly $r(\bar{A}) \leq r(A)$ and $r(\bar{B}) \leq r(B)$ where $\bar{A} = A\Phi(N)/\Phi(N)$ and $\bar{B} = B\Phi(N)/\Phi(N)$. Hence $\min\{r(\bar{A}), r(\bar{B})\} = \bar{r}_0 \leq r_0$ and $\bar{r}_1 = r(\bar{A}) + r(\bar{B}) \leq r_1$. If $\Phi(N) \neq 1$ then $r(\bar{N}) = r(N)$ satisfies the inequality

$$r_n(G) = r(N) \leq \bar{r}_0(\lceil \log_p r(N) \rceil + 1 + \lceil \log_p \bar{r}_0 \rceil \lceil \log_2 2\bar{r}_0 \rceil + \delta_{2p}) + \bar{r}_1.$$

Since $\bar{r}_0 \leq r_0$ and $\bar{r}_1 \leq r_1$ then $r(N) = r_n(G)$ satisfies the required inequality in Theorem 1.1.

Hence we may assume that $\Phi(N) = 1$ and so N is an elementary abelian normal subgroup of G. Now suppose that $AN = H \neq G$ or $BN = H \neq G$. It is clear that $H = (A \cap H)(B \cap H), r(A \cap H) + r(B \cap H) \leq r_1$ and $\min\{r(A \cap H), r(B \cap H)\} \leq r_0$. Thus $r_n(G) = r(N)$ satisfies the conclusion of Theorem 1.1, a contradiction. Therefore we may assume that $AN = BN = AB = G$. Since N is abelian, the subgroups $A \cap N$ and $B \cap N$ are normal in G, so that also $C = (A \cap N)(B \cap N)$ is normal in G. By Lemma 3.3 $r((A \cap N)(B \cap N) \leq r(A) + r(B) = r_1$. If $\bar{G} = G/C$, then $\bar{G} = \bar{A}\bar{B} = \bar{A}\bar{N} = \bar{B}\bar{N}$ where $\bar{A} = AC/C, \bar{B} = BC/C, \bar{N} = NC/C$ and $\bar{A} \cap \bar{N} = 1 = \bar{B} \cap \bar{N}$. In particular $\bar{A} \simeq \bar{B} \simeq \bar{G}/\bar{N}$. By Lemma 3.1 we have

$$d(\bar{N}) \leq r_0(\lceil \log_p d(\bar{N}) \rceil + 1 + \lceil \log_p r_0 \rceil \lceil \log_2 2r_0 \rceil + \delta_{2p}).$$

Furthermore, we have

$$r_n(G) = r(N) = d(N) \leq d(\bar{N}) + r(C) \leq d(\bar{N}) + r_1.$$

Since $\lceil \log_p d(\bar{N}) \rceil \leq \lceil \log_p d(N) \rceil = \lceil \log_p r_n(G) \rceil$ the theorem follows.

4.2. Proof of Theorem 1.2

Assume that Theorem 1.2 is false, and let the finite p-group $G = AB$ be a minimal counterexample where the two subgroups A and B are abelian. Let $r_0 =$

$\min\{r(A), r(B)\}$ and $r_1 = r(A) + r(B)$. As in the proof of Theorem 1.1 it is easy to reduce the proof to the case $G = AN = BN = AB$ where N is an elementary abelian normal subgroup of G with maximal rank. Since $C(N) = N(C(N) \cap A)$ is also abelian then $C(N) \cap A \leq N$ and $C(N) = N$. Hence G/N is isomorphic to a subgroup of $\mathrm{Aut}(N) = GL(n, p)$ where $n = r(N) = d(N)$. Moreover, the subgroups $A \cap N$, $B \cap N$ and $A \cap B$ are central in G so that $D = A \cap B = A \cap N \cap (B \cap N)$. From $|G| = |A||B|/|D| = |A||N|/|A \cap N| = |B||N|/|B \cap N|$ it follows that $|G/N|^2 = (|A|/|A \cap N|)(|B|/|B \cap N|) = |G||D|/(|A \cap N||B \cap N|)$. If $Z = (A \cap N)(B \cap N)$, then $|Z||G/N| = |N|$. Hence if $|Z| = p^x$ then the Jordan form of each element of the group $G/N = A/(A \cap N)$ has at least x Jordan matrices. Now the maximal size of a Jordan matrix is less than $n - x + 2$. It follows from the proof of Lemma 3.1 that $\log_p(\exp(G/N))$ does not exceed $\lceil \log_p(n - x + 1) \rceil$. By Lemma 2.5 this implies $\log_p(|G/N|) \leq r_0 \lceil \log_p(n - x + 1) \rceil$ and $n - x \leq r_0 \lceil \log_p(n - x + 1) \rceil$. We have $n \leq r_0 \lceil \log_p(n - x + 1) \rceil + x$ with $x \leq r_1$. Now the function $f_i = r_0 \lceil \log_p(n - x + 1) \rceil + x$ is increasing in the interval $0 \leq x \leq r_1$. Thus $n \leq \max(f_i) = r_0 \lceil \log_p(n - r_1 + 1) \rceil + r_1$. This proves Theorem 1.2.

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Locally Soluble Products of Two Minimax Subgroups

*Bernhard Amberg and Yaroslav P. Sysak**

1. Introduction

Lennox and Roseblade in [4] and Zaitsev in [13] have shown that a soluble group $G = AB$, which is the product of two polycyclic subgroups A and B, is likewise polycyclic. Moreover, Wilson in [11] and independently Sysak in [9] proved that a soluble product of two minimax subgroups is likewise a minimax group. These authors obtained similar theorems for the finiteness conditions "finite Prüfer rank" and "finite abelian section rank" (see [9] and [12]).

The question arises whether these results can be extended to locally soluble products of two subgroups (see [2], Question 10). Obviously by the theorem of Lennox, Roseblade and Zaitsev also locally soluble products of two polycyclic groups are polycyclic. But even locally finite-soluble products of two subgroups with finite abelian section rank need not have finite abelian section rank (see [8], Theorem 1, p. 4).

In this note we consider locally soluble products of minimax groups. Recall that a group G is a ***minimax group*** if it has a finite series whose factors satisfy the minimum or the maximum condition for subgroups.

Theorem 1.1. *If the locally soluble group $G = AB$ is the product of two minimax subgroups A and B, then G is a soluble minimax group*

The proof of Theorem 1.1 will be reduced to the case when G is hyperabelian by the following result. Recall that a group G if ***residually of bounded finite Prüfer rank*** if there exist normal subgroups N_i of G with $\bigcap N_i = 1$ and a positive integer k such that the Prüfer ranks r_i of the factor groups G/N_i satisfy $r_i \leq k$ for every i in the index set I. Here a group is said to have ***finite Prüfer rank*** r if all its finitely generated subgroups can be generated by r elements and r is the least positive integer with this property.

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Theorem 1.2. *If the locally soluble group G is residually of bounded finite Prüfer rank, then G is hyperabelian.*

The proof of Theorem 1.2 depends on the following proposition about the endomorphism ring of an abelian group of finite Prüfer rank, which is of independent interest.

Proposition 1.3. *Let M be an abelian group of finite Prüfer rank r. Then the endomorphism ring* End M *satisfies the standard polynomial of degree* $2r$

The results of this note have earlier been published as Preprint No. 2 (November 1993) of the Preprint-Reihe des Fachbereichs Mathematik der Johannes Gutenberg-Universität Mainz. The notation is standard and can be found in [2], [5], [7] and [6]. In particular the Prüfer rank of the group G will be denoted by $r(G)$.

2. Proof of Proposition 1.3

Recall that the standard polynomial of degree n is the polynomial

$$S_n(x_1, \dots, x_n) = \sum_{\pi \in \mathrm{Sym}(n)} (\operatorname{sgn} \pi) x_{\pi 1} \cdots x_{\pi n}.$$

The ring R satisfies the standard polynomial of degree n if $S_n(r_1, \dots, r_n) = 0$ for all elements $r_1, r_2, \dots, r_n$ of R. It is easy to see that the property that a ring satisfies the standard polynomial for some degree n is inherited by subrings, factor rings and cartesian products. The theorem of Amitsur and Levitzki says that the ring $M_n(R)$ of $n \times n$-matrices with coefficients in the commutative ring R satisfies the standard polynomial of degree $2n$ (see [6], Theorem 1.4.1).

Proof of Proposition 1.3. Assume first that the abelian group of finite Prüfer rank M is radicable. If M is a p-group or torsion-free, then End M is isomorphic to the ring of matrices $M_r(K)$ over the field K of p-adic numbers or of rational numbers, respectively. Hence End M satisfies the standard polynomial of degree $2r$ by the theorem of Amitsur and Levitzki. Clearly if M is periodic, then End M also satisfies this polynomial identity. Therefore we may suppose that the maximal periodic subgroup T of M satisfies $1 \subset T \subset M$.

The endomorphism rings End T and End M/T satisfy the standard polynomials of degree m and n respectively, where $m = 2r(T)$ and $n = 2r(M/T)$. We will show that End M satisfies the standard polynomial of degree $n + m$. The restriction of an endomorphism α of M onto T is an endomorphism of T, the subring $\mathrm{Hom}(M, T)$ is an ideal of End M and the factor ring End $M/\mathrm{Hom}(M, T)$ is isomorphic with End M/T Therefore if $\alpha_1, \dots, \alpha_m, \alpha_{m+1}, \dots, \alpha_{m+n}$ are arbitrary endomorphisms of M and t is an element in T, it follows that $S_m(\alpha_1, \dots, \alpha_m)(t) = 0$ and $S_n(\alpha_{m+1}, \dots, \alpha_{m+n})$ belongs to $\mathrm{Hom}(M, T)$. Hence for every element a in M we have

$$S_m(\alpha_1, \dots, \alpha_m)(S_n(\alpha_{m+1}, \dots, \alpha_{m+n})(a)) = 0.$$

This implies $S_{m+n}(\alpha_1, \dots, \alpha_m, \alpha_{m+1}, \dots, \alpha_{m+n}) = 0$, since

$$S_{m+n}(x_1, \dots, x_{m+n}) = \sum_{\pi \in S}(\operatorname{sgn} \pi) S_m(x_{\pi 1}, \dots, x_{\pi m}) S_n(x_{\pi(m+1)}, \dots, x_{\pi(m+n)}),$$

where S is the set of all permutations of the symmetric group $\operatorname{Sym}(m+n)$ such that for every subset of $N = \{1, \dots, m+n\}$ with m elements $i_1, \dots, i_m$ there is a permutation π in S with $\pi 1 = i_1, \dots, \pi m = i_m$ and $\pi(m+i)$ is the minimal number in the complement set of $\{\pi 1, \dots, \pi m, \pi(m+1), \dots, \pi(m+i-1)\}$ in N. Since $r(M) = r(T) + r(M/T)$, it follows that the ring $\operatorname{End} M$ satisfies the standard polynomial of degree $m+n = 2r(T) + 2r(M/T) = 2(r(T) + r(M/T)) = 2r$. This concludes the proof of Proposition 1.3 for radicable groups.

Suppose now that M is arbitrary and let $\bar{M}$ be the radicable hull of M. Then $r(\bar{M}) = r(M)$. By a theorem of Dlab (see [3], Satz 5)

$$\operatorname{End} M \simeq \operatorname{End}_M(\bar{M}, M)/\operatorname{End}_0(\bar{M}, M),$$

where $\operatorname{End}_M(\bar{M}, M)$ is the subring of all endomorphisms of $\bar{M}$ which map M into itself and $\operatorname{End}_0(\bar{M}, M)$ is the ideal of $\operatorname{End}_M(\bar{M}, M)$ consisting of all endomorphisms of $\bar{M}$ which map M onto 0. We have shown above that the ring $\operatorname{End} \bar{M}$ satisfies the standard polynomial of degree $2r$. In particular the subring $\operatorname{End}_M(\bar{M}, M)$ and so also its factor ring $\operatorname{End}_M(\bar{M}, M)/\operatorname{End}_0(\bar{M}, M) \simeq \operatorname{End} M$ satisfy the standard polynomial of degree $2r$. This proves Proposition 1.3.

3. Proof of Theorem 1.2

For the proof of Theorem 1.2 we need the following lemmas.

Lemma 3.1. *Let the group G be the cartesian product of hyperabelian groups of bounded Prüfer rank. Then there exists a normal subgroup M of G which is nilpotent of class at most 2 such that the factor group G/M is embedded in the multiplicative group of a ring with the standard polynomial identity.*

Proof. Let H be a hyperabelian group and let N be a maximal normal subgroup of G with class at most 2. Then $Z(N) = C_H(N)$ (see for example [7], Chapter 2, proof of Proposition 3). The intersection $C_H(Z(N)) \cap C_H(N/Z(N))$ is a nilpotent normal subgroup of G with class at most 2 by a lemma of Kaluznin (see [7], Chapter 1, Proposition 10). Since $N = C_H(N/Z(N)) \cap C_H(Z(N))$, the factor group H/N is a subgroup of the direct product of the groups $H/C_H(N/Z(N))$ and $H/C_H(Z(N))$. Now these groups are embedded in $\operatorname{End} N/Z(N)$ and $\operatorname{End} Z(N)$, respectively. If $r(H) \leq r$, then the rank of the abelian groups $N/Z(N)$ and $Z(N)$ is likewise at most r. By Proposition 1.3 the rings $\operatorname{End} N/Z(N)$ and $\operatorname{End} Z(N)$ and therefore also their direct product satisfy the standard polynomial of degree $2r$. Thus the factor group H/N is embedded in the multiplicative group of a ring which satisfies the standard polynomial of degree $2r$.

Now let $G = \mathrm{Cr}_{i\in J} H_i$ be the cartesian product of hyperabelian groups H_i with $r(H_i) \leq r$, and let M_i be a maximal nilpotent normal subgroup of H_i with class at most 2. Then $M = \mathrm{Cr}_{i\in J} M_i$ is a nilpotent normal subgroup of G of class at most 2 such that the factor group G/M has the desired property.

Lemma 3.2. *A locally soluble subgroup of the multiplicative group of a ring with polynomial identity is hyperabelian.*

Proof. Let R be a ring with polynomial identity. By Proposition 1.6.25 of [6] the nil radical $N = N(R)$ of R contains a non-zero nilpotent ideal of R or $N = 0$. If I is a nilpotent ideal of R, then $1 + I$ is a nilpotent normal subgroup of the multiplicative group R^* of R (see [7], Chapter 1, Proposition 9). Hence $1 + N$ is a normal subgroup of R^* which has an ascending invariant series of R^* with abelian factors. By Theorem 1.6.27 of [6] the factor ring R/N is embedded in the ring of matrices $M_r(\mathbb{Z}[x])$ for some degree r, as it has trivial nil radical. Since the factor group $R^*/(1+N)$ is a linear group of degree r over the noetherian commutative ring $\mathbb{Z}[x]$, every locally soluble subgroup of $R^*/(1 + N)$ is soluble (see [10], 13.12). It follows that every locally soluble subgroup of R^* is hyperabelian.

Proof of Theorem 1.2. Let the locally soluble group G be residually of bounded finite Prüfer rank. Every locally soluble group with finite Prüfer rank is hyperabelian (see [5], Vol. 2, p. 179). Therefore the group G is isomorphic with a subgroup H of a cartesian product C of hyperabelian groups with bounded finite Prüfer rank. It follows from Lemma 3.1 that C contains a nilpotent normal subgroup M such that C/M is embedded in the multiplicative group of a ring with a polynomial identity. By Lemma 3.2 the locally soluble subgroup $HM/M \simeq H/(H \cap M)$ of C/M is hyperabelian. Since $H \cap M$ is nilpotent, also H and its isomorphic copy G are hyperabelian. This proves Theorem 1.2.

4. Proof of Theorem 1.1

A group G satisfies the ***weak minimum condition for subgroups*** if every descending chain of subgroups S_i has only finitely many infinite indices $|S_{i+1} \quad S_i|$. The weak minimum condition for normal subgroups is defined accordingly.

Lemma 4.1 (Amberg [1], Theorem 2.5). *If the group $G = AB$ is the product of two subgroups A and B with weak minimum condition for subgroups, then G satisfies the weak minimum condition for normal subgroups.*

Proof. Let U and V be normal subgroups of G such that $U \subset V$ and the indices $|AV : AU|$ and $|(A \cap V) : (A \cap U)|$ are finite. Then the following indices are finite:

$$|V : U(A \cap V)| = |V : (V \cap AU)| = |AV : AU|$$

and

$$|U(A \cap V) : U| = |(A \cap V) : (A \cap U)|.$$

Therefore the following index is also finite as a product of two finite indices:

$$|V : U| = |V : U(A \cap V)||U(A \cap V) : U| = |AV : AU||(A \cap V) : (A \cap U)|.$$

It is now easy to derive the lemma from this fact.

Proof of Theorem 1.1. It suffices to show that the locally soluble group G is hyperabelian, since then G is a minimax group by [9], Corollary A. Since every epimorphic image of G is likewise a locally soluble product of two minimax subgroups we only need to show that the group $G \neq 1$ has a non-trivial abelian normal subgroup. By Lemma 4.1 the group G satisfies the weak minimum condition for normal subgroups. Therefore there exists a normal subgroup N of G such that either

(1) N is a minimal normal subgroup of G, or

(2) for all normal subgroups M of G contained in N the factor group N/M is finite and the intersection of all these normal subgroups M is trivial.

In case (1) the minimal normal subgroup N of the locally soluble group G is abelian (see [5], Vol. 1, Corollary 1 to Theorem 5.27). Therefore we may suppose that N satisfies condition (2). The factorizer of N has the triple factorization

$$X = X(N) = NA_1 = NB_1 = A_1B_1$$

where $A_1 = A \cap BN$ and $B_1 = B \cap AN$ are minimax groups. Let M be a normal subgroup of G such that $M \subseteq N$ and N/M is finite. Then

$$\bar{X} = X/M = \bar{N}\bar{A}_1 = \bar{N}\bar{B}_1 = \bar{A}_1\bar{B}_1$$

where $\bar{N} = N/M$, $\bar{A}_1 = A_1M/M$ and $\bar{B}_1 = B_1M/M$. Here $\bar{N}$ is a finite normal subgroup of $\bar{X}$. Since $\bar{A}_1$ and $\bar{B}_1$ are soluble minimax groups and in particular have finite Prüfer ranks, also $\bar{X}$ has finite Prüfer rank bounded by a function of the Prüfer ranks of A and B; see [2], Theorem 4.3.5. Therefore every such factor group X/M has bounded Prüfer rank. Hence X is residually of bounded finite Prüfer rank. By Theorem 1.2 the group X is hyperabelian and so by the Theorem of Sysak and Wilson it is a soluble minimax group; see [9], Corollary A. This implies that N contains a non-trivial abelian normal subgroup of G. Theorem 1.1 is proved.

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A Group-Theoretic Reduction of J. H. C. Whitehead's Asphericity Question

W. A. Bogley and M. N. Dyer

Abstract. J. H. C. Whitehead asked in 1941 whether subcomplexes of aspherical two-complexes are aspherical. The question remains unanswered as of this writing. In this note we use a theorem of J. Howie to show that Whitehead's question can be reduced to two problems in combinatorial group theory. Some partial results are surveyed.

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1. Introduction

This article is concerned with group-theoretic aspects of the following topological question, which was posed by J. H. C. Whitehead in 1941 [W41]: "Is any subcomplex of an aspherical, 2-dimensional complex itself aspherical?" A 2-dimensional complex is a CW complex in which each cell has dimension at most two; in short, what we will call a ***two-complex***. A connected space is ***aspherical*** if its universal covering is contractible. For a connected two-complex X, this is equivalent to saying that the second homotopy group $\pi_2 X$ is trivial.

A survey of the extensive work that has been done on Whitehead's question appears in [B93]. The purpose of this article is to publicize the fact that Whitehead's question can be reduced to a pair of problems in combinatorial group theory. It is hoped that the group-theoretic formulations that are presented here will stimulate further work on the problem.

Interest in Whitehead's question can be motivated by the fact that the complement of any tame knot in the three-sphere has the homotopy type of a two-complex that can be embedded in a finite contractible two-complex. A positive solution to Whitehead's question therefore holds the promise of a (new) proof of the asphericity of knot complements. A footnote included in the midst of Whitehead's original question [W41, Footnote 30] suggests that this prospect may have been uppermost in Whitehead's mind at the time.

Our group-theoretic reduction of Whitehead's question is based on a topological reduction of the problem that appears in the following theorem due to J. Howie.

Theorem 1 ([H83]). *If the answer to Whitehead's question is NO, then there exists a connected two-complex L such that either*

1. *L is finite and contractible and $L - e$ is not aspherical for some open two-cell e of L, or*
2. *L is the union of an infinite ascending chain of finite connected nonaspherical subcomplexes $K_0 \subset K_1 \subset \quad$ where each inclusion $K_{i-1} \subset K_i$ is nullhomotopic.*

□

The situation in 1.1 will be referred to as the ***finite case***; 1.2 will be called the ***infinite case***. Of course, there is a converse to Howie's theorem in the sense that if there is a two-complex L with the properties described in either the finite or the infinite case, then the answer to Whitehead's question is NO. In addition, it has been shown by E. Luft [L94] that if there is a two-complex L of the sort described in the finite case, then there is also an example of the sort described in the infinite case. Thus, Whitehead's question actually reduces to the infinite case. This does not detract from the finite case however, which is still very interesting.

We will show that each of the two cases in Theorem 1 can be reduced to a problem in combinatorial group theory. The finite case leads to a problem (Theorem 3) concerning intersections of normal subgroups in finitely generated free groups. A partial result (Theorem 4) essentially solves the problem ***modulo the central series***, and leads to a question about residual nilpotence of certain groups. In the infinite case, we reduce Whitehead's question to one that concerns the existence of groups admitting certain ascending chains of normal subgroups. The particulars are given in Theorem 2.

Following this introductory section, the infinite case is discussed in Section 2. Section 3 treats the finite case. All spaces in this paper will be connected two-complexes. Basepoints for homotopy groups will be suppressed from the notation, but will always be taken to be a fixed zero-cell. If A and B are subgroups of a group G, then $[A, B]$ denotes the subgroup of G that is generated by all commutators $[a, b]$ $(a \in A,\ b \in B)$, where $[a, b] = aba^{-1}b^{-1}$ If A and B are normal in G, then so is $[A, B]$, and in this case we also have $[A, B] \subseteq A \cap B$. The lower central series is defined inductively by $G_1 = G$ and $G_{n+1} = [G, G_n]$. All homology groups will be computed with integer coefficients.

2. The Infinite Case

The possibility of constructing an example as in the infinite case has been considered by M. Dyer [D92]. Suppose that a connected two-complex L is given as a union $K_0 \subset K_1 \subset \quad \subset \bigcup_i K_i = L$ as in 1.2. Replacing each K_i by $K_i \cup L^{(1)}$, where $L^{(1)}$ denotes the one-skeleton of L, we have that for each $i \geq 1$, K_i is obtained from K_{i-1} by attaching two-cells (so that the inclusion-induced homomorphism $\pi_1 K_{i-1} \to \pi_1 K_i$ is surjective) and the inclusion-induced map $\pi_2 K_{i-1} \to \pi_2 K_i$ is trivial.

For an inclusion of two-complexes, the triviality of the induced map on second homotopy modules can be formulated in terms of the subgroup structure of the fundamental group of the subcomplex. Following [BD81], let X be a connected two-complex and let $N \leq \pi_1 X$. The two-complex X is ***N-Cockcroft*** if the lifted Hurewicz map $\pi_2 X \to H_2 X_N$ is trivial, where $X_N \to X$ is the covering corresponding to N. This property derives its name from its earliest consideration by W. H. Cockcroft in his work on Whitehead's question [C51]. Note that if X is N-Cockcroft and $N' \leq \pi_1 X$ contains some $\pi_1 X$-conjugate of N, then X is N'-Cockcroft. Also, X is Cockcroft $\Leftrightarrow$ X is $\pi_1 X$-Cockcroft, while X is aspherical $\Leftrightarrow$ X is $\{1\}$-Cockcroft. Our interest in the Cockcroft properties comes from the following elementary observation.

Lemma 1. *Suppose that X is a subcomplex of a connected two-complex Y. The inclusion-induced map $\pi_2 X \to \pi_2 Y$ is trivial if and only if X is* $\ker i_\#$*-Cockcroft, where $i_\#$ $\pi_1 X \to \pi_1 Y$ is the inclusion-induced homomorphism of fundamental groups.*

Proof. Let p $\tilde{Y} \to Y$ be the universal covering and let $\bar{X}$ be a connected component of $p^{-1}(X)$; the restriction of p then determines the covering $\bar{X} \to X$ corresponding to $\ker i_\# \leq \pi_1 X$. Since $\pi_2 Y \to H_2 \tilde{Y}$ and $H_2 \bar{X} \to H_2 \tilde{Y}$ are both injective, it readily follows that $\pi_2 X \xrightarrow{0} \pi_2 Y \Leftrightarrow \pi_2 X \xrightarrow{0} H_2 \bar{X}$. □

Quite a lot of work has been done on Cockcroft properties in recent years. Of particular group-theoretic interest is the fact, due independently to J. Harlander [H94] and to N. Gilbert and J. Howie [GH94], that for any two-complex X, there is a minimal subgroup H of $\pi_1 X$ such that X is H-Cockcroft. Such minimal subgroups are referred to as Cockcroft ***thresholds*** for X. Informally, it is appropriate to say that if X has a "small" Cockcroft threshold, then X is "nearly" aspherical.

If X is any topological space, it is obvious that a spherical map $S^2 \to X$ can be rendered nullhomotopic by attaching a three-cell to X: One simply uses the spherical map to attach the three-cell! Somewhat less obvious is the fact that essential spherical maps into two-complexes can be rendered nullhomotopic simply by adding ***two-cells***. The following sort of example is fairly well known. Let X be the real projective plane, modeled on the presentation $(a \quad a^2)$ for the cyclic group of order two. Thus, X is constructed by attaching a disc to a circle S^1_a by a two-fold wrap of the boundary circle of the disc onto S^1_a. One has that $H_2 X = 0$ and that $\pi_2 X$ is infinite cyclic, since X is covered by the two-sphere. Let Y be the two-complex modeled on the presentation $(a : a^2, a)$ for the trivial group. Thus, Y is obtained from X by attaching another disc to X, this time using a homeomorphism of the boundary circle in the disc with S^1_a. Now Y is simply connected (in fact Y has the homotopy type of the two-sphere), and so the Hurewicz homomorphism $\pi_2 Y \to H_2 Y$ is an isomorphism (of infinite cyclic groups). It follows that the inclusion-induced map $\pi_2 X \to \pi_2 Y$ factors through $H_2 X = 0$, and so this map is trivial. (Thus, X is $\pi_1 X$-Cockcroft.) However, this process can not be repeated in any fashion, for if Y is a subcomplex of any two-complex Z, then $\pi_2 Y \cong H_2 Y \neq 0$ embeds in $H_2 Z$, and so $\pi_2 Y \to \pi_2 Z$ is nontrivial. In other words, Y is not $\pi_1 Y$-Cockcroft.

Coupled with this example, Lemma 1 reveals the main difficulty in attempting to construct a two-complex $L = \bigcup_{i \geq 0} K_i$ of the sort described in the infinite case. Having constructed K_{i-1}, one must add (two-)cells in such a way that the resulting adjunction space K_i has a suitable Cockcroft property. We examine the requirements from a group-theoretic perspective.

Suppose that X is a connected two-complex and that

$$Y = X \cup \bigcup_{\alpha \in \mathcal{A}} c_\alpha^2 \text{ and } Z = Y \cup \bigcup_{\beta \in \mathcal{B}} d_\beta^2$$

are obtained from X by attaching two-cells. Set $G = \pi_1 X$. For each $\alpha \in \mathcal{A}$, let $a_\alpha \in G$ denote the (based) homotopy class of an attaching map for the two-cell c_α^2. The element a_α is well-defined up to conjugacy in G. In the same way, let $b_\beta \in G$ be the based homotopy class for an attaching map of d_β^2. We set

$$A = \ker(\pi_1 X \to \pi_1 Y) \text{ and } B = \ker(\pi_1 X \to \pi_1 Z)$$

so that $A \leq B \leq G$ where A and B are normal subgroups of G. Note that A is normally generated in G by $\{a_\alpha \quad \alpha \in \mathcal{A}\}$ and B is normally generated in G by $\{a_\alpha \quad \alpha \in \mathcal{A}\} \cup \{b_\beta \quad \beta \in \mathcal{B}\}$. We have that $\pi_1 Y = G/A$ and $\pi_1 Z = G/B$. The abelianized group $H_1 A = A/[A, A]$ is a (left) $\mathbb{Z}G/A$-module under conjugation in G:

$$g \cdot a[A, A] = gag^{-1}[A, A]$$

for all $g \in G$ and for all $a \in A$. This module is $\mathbb{Z}G/A$-generated by $\{a_\alpha[A, A] : \alpha \in \mathcal{A}\}$. Killing the action of the subgroup B/A of G/A, the group

$$A/[A, B] \cong \mathbb{Z} \otimes_{B/A} H_1 A$$

is a $\mathbb{Z}G/B$-module with generators $\{a_\alpha[A, B] : \alpha \in \mathcal{A}\}$.

Lemma 2. *If $\pi_2 X \xrightarrow{0} \pi_2 Y$, then $\pi_2 Y \xrightarrow{0} \pi_2 Z$ if and only if both of the following conditions are satisfied.*

1. *$A/[A, B]$ is a free $\mathbb{Z}G/B$-module with indexed basis $\{a_\alpha[A, B] : \alpha \in \mathcal{A}\}$.*
2. *$H_2 B \to H_2 B/A$ is injective.*

Proof. Let $\widetilde{X}$, $\widetilde{Y}$, and $\widetilde{Z}$ denote the universal covering complexes for X, Y, and Z, respectively. As shown in the following diagram of inclusions and covering projections, let $\overline{X}$ and $\widehat{X}$ denote the preimages of X in $\widetilde{Y}$ and $\widetilde{Z}$, respectively, and let $\overline{Y}$ denote the preimage of Y in $\widetilde{Z}$. All of these spaces are connected. The covering complexes $\overline{X}$ and $\widehat{X}$ of X are those corresponding to the subgroups A and B of $G = \pi_1 X$, respectively. The covering complex $\overline{Y}$ of Y is that corresponding to the subgroup B/A of $G/A = \pi_1 Y$. Assuming the $\pi_2 X \to \pi_2 Y$ is trivial, it follows that $\pi_2 X \to H_2 \overline{X}$ is trivial by Lemma 1. This implies that $\pi_2 X \to H_2\widehat{X}$ is the zero map, and it follows that the natural surjection $H_2\widehat{X} \to H_2\pi_1\widehat{X} = H_2 B$ is an isomorphism.

$$\begin{array}{ccccc}
\widetilde{X} & & & & \\
\downarrow & & & & \\
\overline{X} & \subseteq & \widetilde{Y} & & \\
\downarrow & & \downarrow & & \\
\widehat{X} & \subseteq & \overline{Y} & \subseteq & \widetilde{Z} \\
\downarrow & & \downarrow & & \downarrow \\
X & \subseteq & Y & \subseteq & Z
\end{array}$$

By Lemma 1, the map $\pi_2 Y \to \pi_2 Z$ is trivial if and only if $\pi_2 Y \to H_2\overline{Y}$ is the zero map. Using the fact that $H_2\widehat{X} \to H_2\overline{Y}$ is injective and $H_2\widehat{X} \to H_2 B$ is an isomorphism, a chase in the following commutative diagram (which has exact rows and columns) shows that $\pi_2 Y \to \pi_2 Z$ is trivial if and only if $H_2 B \to H_2 B/A$ is injective and the composite $(\pi_2 Y \to H_2\overline{Y} \to H_2(\overline{Y}, \widehat{X}))$ is the zero map.

$$\begin{array}{ccccccc}
 & & \pi_2 Y & \longrightarrow & H_2(\widetilde{Y}, \overline{X}) & \longrightarrow & H_1\overline{X} \\
 & & \downarrow & & \downarrow & & \downarrow \\
H_2\widehat{X} & \longrightarrow & H_2\overline{Y} & \longrightarrow & H_2(\overline{Y}, \widehat{X}) & \longrightarrow & H_1\widehat{X} \\
\downarrow & & \downarrow & & & & \\
H_2 B & \longrightarrow & H_2 B/A & & & &
\end{array}$$

It remains to show that the latter condition is equivalent to the condition 1 of the lemma. To see this, note that the boundary map $H_2(\widetilde{Y}, \overline{X}) \to H_1\overline{X}$ can be identified with the $\mathbb{Z}G/A$-module epimorphism

$$\bigoplus_{\alpha \in \mathcal{A}} (\mathbb{Z}G/A)\, t_\alpha \to A/[A, A] \to 0$$

that carries the basis element t_α to $a_\alpha[A, A]$. This follows by excision since Y is obtained from X by attaching one two-cell for each element $\alpha \in \mathcal{A}$ and the covering $\widetilde{Y} \to Y$ has automorphism group G/A. In addition, the map $H_2(\widetilde{Y}, \overline{X}) \to H_2(\overline{Y}, \widehat{X})$ can be identified with the map

$$1 \otimes - \quad \bigoplus_{\alpha \in \mathcal{A}} (\mathbb{Z}G/A)\, t_\alpha \to \mathbb{Z} \otimes_{B/A} \bigoplus_{\alpha \in \mathcal{A}} (\mathbb{Z}G/A)\, t_\alpha \cong \bigoplus_{\alpha \in \mathcal{A}} (\mathbb{Z}G/B)\, t_\alpha.$$

This is because the covering $\widetilde{Y} \to \overline{Y}$ has automorphism group B/A.

With these identifications, consider the effect of applying the right exact functor $\mathbb{Z} \otimes_{B/A} -$ to the exact sequence

$$\pi_2 Y \to H_2(\widetilde{Y}, \overline{X}) \to H_1 \overline{X} \to 0.$$

We find that the composite $(\pi_2 Y \to H_2 \overline{Y} \to H_2(\overline{Y}, \widehat{X}))$ is the zero map if and only if the map

$$\mathbb{Z} \otimes_{B/A} H_2(\widetilde{Y}, \overline{X}) \to \mathbb{Z} \otimes_{B/A} H_1 \bar{X} \cong A/[A, B]$$

is an isomorphism, as in the condition 1 of Lemma 2. This completes the proof of the lemma. □

Theorem 2. *If there is a two-complex L as described in Theorem 1.2, then there exists a finite connected nonaspherical two-complex K and an infinite ascending chain $\{1\} = N_0 < N_1 < N_2 < \cdots < \pi_1 K$ of normal subgroups of $\pi_1 K$ such that the following two properties hold.*

1. *K is N_1-Cockcroft.*
2. *There are subsets $\mathbf{r}_i \subseteq \pi_1 K$ $(i \geq 1)$ such that $\{r_i N_{i-1} \mid r_i \in \mathbf{r}_i\}$ normally generates N_i/N_{i-1} in $\pi_1 K/N_{i-1}$ and such that the following two conditions are satisfied for each positive integer i.*
   (a) *$N_i/N_{i-1}[N_i, N_{i+1}]$ is a free $\mathbb{Z}\pi_1 K/N_{i+1}$-module with indexed basis $\{r_i N_{i-1}[N_{i+1}, N_{i-1}] \mid r_i \in \mathbf{r}_i\}$.*
   (b) *$H_2(N_{i+1}/N_{i-1}) \to H_2(N_{i+1}/N_i)$ is injective.*

Conversely, if such a two-complex K exists, then the answer to Whitehead's question is NO.

Proof. Suppose that we are given $K_0 \subseteq K_1 \subseteq \cdots \subseteq \bigcup_{i \geq 0} K_i = L$ as in Theorem 1.2. Replace each of the subcomplexes K_i by the union $K_i \cup L^1$, where L^1 denotes the one-skeleton of L. Let $K = K_0$; this two-complex is not aspherical. Each K_i is obtained from K_{i-1} by attaching two-cells and the inclusion-induced map $\pi_2 K_{i-1} \to \pi_2 K_i$ is trivial. For each positive integer i, let N_i be the kernel of the inclusion-induced epimorphism $\pi_1 K \to \pi_1 K_i$ and let $\mathbf{r}_i$ be a subset of $\pi_1 K$ consisting of one based homotopy class of an attaching map for each two-cell of $K_i - K_{i-1}$. Then $\{1\} = N_0 < N_1 < N_2 < \cdots < \pi_1 K$ is an ascending chain of normal subgroups of $\pi_1 K$ and $\{r_i N_{i-1} : r_i \in \mathbf{r}_i\}$ normally generates N_i/N_{i-1} in $\pi_1 K_{i-1} = \pi_1 K/N_{i-1}$.

Now $K = K_0$ is N_1-Cockcroft by Lemma 1, since $\pi_2 K_0 \to \pi_2 K_1$ is the zero map. Fixing a positive integer i, consider the triple

$$K_{i-1} \subseteq K_i \subseteq K_{i+1}$$

and set $G = \pi_1 K_{i-1} = \pi_1 K/N_{i-1}$, $A = N_i/N_{i-1}$, and $B = N_{i+1}/N_{i-1}$. Note that $A/[A, B] = N_i/N_{i-1}[N_i, N_{i+1}]$ is a module over the integral group ring of $G/B = \pi_1 K/N_{i+1}$ and that $B/A = N_{i+1}/N_i$. Since both of these inclusions induce the trivial map in second homotopy, Lemma 2 implies that the conditions 2(a) and

2(b) of Theorem 2 are satisfied.

For the converse, suppose that we are given K, N_i and $\mathbf{r}_i$ that satisfy the conditions 1 and 2 in Theorem 2. The two-complex K is not aspherical. We will show that the answer to Whitehead's question is NO by embedding K in an aspherical two-complex L. Let $K_0 = K$. For each $i \geq 1$ let K_i be obtained from K_{i-1} by attaching two-cells along based loops representing the elements $r_i \in \mathbf{r}_i \subseteq \pi_1 K_0$. By Lemma 1, condition 1 of Theorem 2 implies that the inclusion-induced map $\pi_2 K_0 \to \pi_2 K_1$ is trivial. Arguing inductively, given that $\pi_2 K_{i-1} \to \pi_2 K_i$ is the zero map, the conditions 2(a) and 2(b) of Theorem 2 imply that $\pi_2 K_i \to \pi_2 K_{i+1}$ is trivial by Lemma 2. We set $L = \bigcup_{i \geq 0} K_i$, where L is given the weak topology with respect to the closed subspaces K_i. The nonaspherical two-complex K is thus a subcomplex of the two-complex L, and L is aspherical by compact supports. For each spherical map $S^2 \to L$ has its image in a finite subcomplex of L, and hence in one of the subcomplexes K_i. This spherical map is then nullhomotopic in K_{i+1}, and hence in L. This shows that the answer to Whitehead's question is NO, and so completes the proof of the theorem. □

We remark that in light of the result of E. Luft that was mentioned in the Introduction [L94], the answer to Whitehead's question is NO if and only if there is a two-complex K of the sort described in Theorem 2. If one is trying to construct such an example, then one seeks a group that contains an infinite ascending chain of normal subgroups with certain properties. Theorem 2 indicates that one need only look among finitely presented groups; this is because the two-complex K is finite. However, if one could find a two-complex K, not necessarily finite but which otherwise satisfies the conditions of the theorem, then the proof of the theorem shows how to embed K in an aspherical two-complex.

Before moving to the finite case, we may as well admit that the statement of Theorem 2 is not purely group-theoretic. This can be remedied artificially by defining a group G to be N-Cockcroft (where N is a subgroup of G) if there is a two-complex with fundamental group isomorphic to G and which is N-Cockcroft. In search of a nonaspherical subcomplex of an aspherical two-complex via the infinite route, we would then be asking for a (finitely presented) group G with an ascending chain $\{1\} = N_0 < N_1 < N_2 < \cdots$ of normal subgroups satisfying the conditions 2 and 2 of Theorem 2, and where G is N_1-Cockcroft but is not $\{1\}$-Cockcroft. Less formally, one seeks a group G that is not "aspherical" (i.e. is not $\{1\}$-Cockcroft), but which is "very nearly aspherical" in the sense that it contains a Cockcroft threshold that is small enough to sit underneath an ascending chain of a certain restricted type.

3. The Finite Case

If L is a finite contractible two-complex and $K = L - e$ is obtained by removing an open two-cell e from L, then the fundamental group $G = \pi_1 K$ is a group of deficiency one that has weight one. In other words, G has a finite presentation with one fewer relator than generators, and G is normally generated in itself by a single element. Moreover, it is not difficult to show that all such groups arise in this way. It can be shown that the finite case of Whitehead's question is equivalent to the question of whether finitely presented groups having deficiency and weight one must have cohomological dimension at most two. See [BDS83] for further discussion.

Another algebraic approach to the finite case of Whitehead's question leads to the study of groups of the form $F/[R, S]$ where R and S are normal subgroups of a finitely generated free group F. In order to state this reduction of the problem, we need some terminology.

Let F be a finitely generated free group with basis $\mathbf{x}$ and let $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{t}$ be finite subsets of F. Let R, S, and T be their normal closures in F, respectively. Let K_r, K_s, and K_t be the two-complexes modeled on the group presentations $(\mathbf{x} : \mathbf{r})$, $(\mathbf{x} : \mathbf{s})$, and $(\mathbf{x} : \mathbf{t})$, respectively. Let $K = K_r \cup K_s$ and $L = K \cup K_t$. These are all finite connected two-complexes. Now L is simply connected if and only if $F = RST$ Further, L is contractible if and only if $F = RST$ and $|\mathbf{r}| + |\mathbf{s}| + |\mathbf{t}| = \operatorname{rank} F$. In this case we say that $F = RST$ is an ***efficient factorization*** of F. The following result is proved in [B91] and in [B93].

Theorem 3 ([B91]). *The following two statements are logically equivalent.*

1. *Connected subcomplexes of finite contractible two-complexes are aspherical.*
2. *If R and S are distinct factors from an efficient factorization of a finitely generated free group, then $R \cap S \subseteq [R, S]$.*

□

We shall not reprove this result here, but it is worth mentioning the main ingredient. If $\mathbf{x}$, $\mathbf{r}$, and $\mathbf{s}$ are arbitrarily given as above and if corresponding two-complexes are constructed in the manner indicated, then by [GR81, Theorem 1] there is an exact sequence

$$\pi_2 K_r \oplus \pi_2 K_s \to \pi_2 K \to \frac{R \cap S}{[R, S]} \to 0.$$

At a fundamental level, this is the result that explains our interest in the group $Q = F/[R, S]$. Note that the subgroup $\Theta = (R \cap S)/[R, S]$ is naturally a module over the integral group ring of F/RS, via conjugation in F. This module has been studied in [B84] and in [HK91], to name two sources.

Returning to Whitehead's question, in the finite case there is a partial result on the group-theoretic problem.

Theorem 4 ([B91]). *If R and S are distinct factors from an efficient factorization of a finitely generated free group F, then*

$$R \cap S \subseteq \bigcap_{n \geq 1} [R, S] F_n.$$

It follows easily that the quotient $(R \cap S)/[R, S]$ embeds naturally in $Q_\omega = \bigcap_{n \geq 1} Q_n$. The proof of Theorem 4 essentially amounts to a determination of the structure of the Lie algebra that is built out of the lower central series of Q [B91, Theorem 2]. In particular it is shown that Q_n/Q_{n+1} is finitely generated and free abelian for all $n \geq 1$. Little seems to be known about Q_ω however. With Theorem 4, one might be led to ask for conditions under which the group Q is residually nilpotent (i.e. $Q_\omega = 1$).

There are many test cases to consider in the finite case. The model of any finite balanced presentation for the trivial group is a finite contractible two-complex. A large and interesting class of examples arises as follows. Let Γ be a finite tree with vertices $V\Gamma$ and (geometric) edges $E\Gamma$ Assume that each edge of Γ is oriented and is labeled by a vertex of Γ (Thus Γ is a ***labeled oriented tree*** or LOT.) Associated to Γ is a group presentation

$$\mathcal{P}(\Gamma) = (V\Gamma \quad i(e)\lambda(e)t(e)^{-1}\lambda(e)^{-1}(e \in E\Gamma)).$$

Here, $i(e)$, $\lambda(e)$ and $t(e)$ denote the initial vertex, label and terminal vertex of the edge $e \in E\Gamma$, respectively. Let $K(\Gamma)$ denote the two-complex modeled on $\mathcal{P}(\Gamma)$.

It is not difficult to show that upon adding a single relation of the form $v = 1$, ($v \in V\Gamma$), there results a finite balanced presentation for the trivial group. If we denote the cellular model of the presentation $\mathcal{P}(\Gamma)$ by $K(\Gamma)$ (so that $K(\Gamma)$ is an ***LOT complex***), then $K(\Gamma)$ is a connected subcomplex of a finite contractible two-complex. If one wishes to prove that there are no examples of the sort described in the finite case, one must therefore prove that LOT complexes are aspherical. J. Howie has proved some partial results in this area [H85]. Notably, if an LOT Γ has diameter less than four, then $K(\Gamma)$ is aspherical. A crucial element in Howie's proof is the fact that a tree of diameter three has at most two nonextremal vertices. The structure of larger trees can be far more varied. The complexes $K(\Gamma)$ therefore provide a wide open playing field.

This area includes the connection to knots that was mentioned in the introduction, for the complement of any tame knot in the three-sphere has the homotopy type of an LOT complex. See [H83, H85] for references and further discussion. It is an open question whether each proper subcomplex of a finite contractible two-complex has the homotopy type of a subcomplex of an LOT complex.

We close with the following problem, which is seen to contain a large and interesting portion of Whitehead's question. Let Γ be a labeled oriented tree. Let F be the free group on the set of vertices of Γ and let $\mathbf{r} \cup \mathbf{s}$ be a nontrivial partition of the set of edges of Γ. Let R (resp. S) be the normal closure of the set of all element of the form $i(e)\lambda(e)t(e)^{-1}\lambda(e)^{-1}$, where $e \in \mathbf{r}$ (resp. $e \in \mathbf{s}$). Is $R \cap S = [R, S]$? Equivalently, does $Q = F/[R, S]$ embed in $F/R \times F/S$? If not, then the answer to Whitehead's question is NO.

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Efficiency and Direct Products of Groups

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Abstract. We extend techniques introduced in [3] to obtain efficient presentations for certain direct products and give some general results.

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1. Introduction

Let $\mathcal{P}$ be the finite presentation $\langle X \mid R\rangle$. The ***deficiency*** of $\mathcal{P}$ is $|R| - |X|$ and, if $\mathcal{P}$ defines a finite group, the deficiency of $\mathcal{P}$ is non-negative. The ***deficiency of a group** G*, def G, is the minimum of the deficiencies of all finite presentations of G. It is well known that def $G \geq \mathrm{rk}(M(G))$ where $M(G)$ is the Schur multiplier of G. A group G is said to be ***efficient*** if def $G = \mathrm{rk}(M(G))$.

The efficiency of finite groups has been studied over many years; see for example [5], [11]. In particular the efficiency of direct products of groups, stimulated by questions asked by Wiegold in [11], has been studied by several authors; see for example [5], [7], and [8]. Recently a new approach to finding efficient presentations for certain direct products was suggested by Izumi Miyamoto and is used in [3] to show that, for p a prime, $PSL(2, p) \times SL(2, p)$ and $PSL(2, p) \times PSL(2, p) \times PSL(2, p)$ are efficient.

In this paper we extend Miyamoto's method and prove a more general theorem. We then apply the theorem to obtain other classes of efficient direct products of groups. The key idea in Miyamoto's method is contained in the following easily proved result:

Lemma 1.1 (Lemma 2.1 of [3]). *Let G be a group with $a, b \in G$ satisfying $a^{\epsilon} = (a^m b^{\delta})^n$ where $\epsilon, \delta = \pm 1$, m and n integers. Then $\langle a, b\rangle$ is a cyclic subgroup of G and $a^{\epsilon - mn} = b^{\delta n}$.*

2. Some Direct Products of Groups Having Trivial Multiplier with Those Having Cyclic Multiplier

Our first result on efficient direct products generalises the efficiency of the group $PSL(2, p) \times SL(2, p)$.

Theorem 2.1. *Let G_1, G_2 be finite perfect groups with trivial centres, G_1 having multiplier C_2 and G_2 having trivial multiplier. Let G_1, G_2 have presentations of the form*

$$\langle a, b \mid a^{\alpha_1} = b^{\alpha_2} = (ab)^{\alpha_3} = w(a, b) = 1\rangle$$
$$\langle x, y \mid x^{\beta_1} = y^{\beta_2} = (xy)^{\beta_3} = v(x, y) = 1\rangle$$

with the α_i, β_i satisfying the congruences $\beta_3 \equiv \pm 1 \pmod{\alpha_1}$, $\beta_2 \equiv \pm 1 \pmod{\alpha_3}$, $\alpha_2 \equiv \pm 1 \pmod{\beta_1}$, $\alpha_3 \equiv \pm 1 \pmod{\beta_1}$. Let the group G, given by the presentation below, be perfect:

$$\langle a, b, x, y \mid (xy)^{\pm 1}((xy)^{(\beta_3 \mp 1)/\alpha_1} a)^{\alpha_1} = y^{\pm 1}(y^{(\beta_2 \mp 1)/\alpha_3} ab)^{\alpha_3} =$$
$$(ab)^{\pm 1}((ab)^{(\alpha_3 \mp 1)/\beta_1} x)^{\beta_1} = b^{\pm 1}(b^{(\alpha_2 \mp 1)/\beta_1} x)^{\beta_1} = a^{-\alpha_1} w(a, b) v(x, y) = 1\rangle.$$

(The four congruences decide the choice of $\pm$ and $\mp$ in each of the four relations so that all of the powers within the relations are integer powers. In each relation one is chosen to be $+$ and the other $-$.)

Let G_1 be such that in the group presented by

$$\langle a, b \mid a^{\alpha_1} = s,\ b^{\alpha_2} = (ab)^{\alpha_3} = u,\ w(a, b) = t;\ s,\ u,\ t \text{ central involutions}\rangle$$

we have $s = t$ and let G_2 be such that

$$\langle x, y \mid x^{\beta_1} = y^{\beta_2} = u,\ (xy)^{\beta_3} = s,\ v(x, y) = 1;\ u,\ s \text{ central involutions}\rangle$$

is perfect. Then $G \cong G_1 \times G_2$ and so this direct product is efficient.

Proof. By Lemma 1.1, the following relations hold in G:

$$[ab, y] = [ab, x] = [b, x] = [a, xy] = 1,$$
$$b^{\alpha_2} = (ab)^{\alpha_3} = x^{-\beta_1} = y^{-\beta_2},\ a^{\alpha_1} = (xy)^{-\beta_3}.$$

Let $H = \langle a, b\rangle$, $K = \langle x, y\rangle$. From the relations $[ab, x] = 1$ and $[b, x] = 1$ we have $[a, x] = 1$. Similarly we have $[a, y] = 1$ and $[b, y] = 1$. Hence $[H, K] = 1$. Let $D = \langle x^{\beta_1}, (xy)^{\beta_3}, v(x, y)\rangle$. Clearly $D \leq H \cap K \leq Z(G)$ and, since $G/D \cong G_1 \times G_2$ which has trivial centre, we must have that $D = H \cap K = Z(G)$. G is perfect with D central and $G/D \cong G_1 \times G_2$ so, by Lemma 4.1 of [12], D must be an epimorphic image of $M(G_1 \times G_2)$. Therefore D is either trivial or cyclic of order two. We also have that $G/H \cong K/D \cong G_2$ and $G/K \cong H/D \cong G_1$. Now, in H the following