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The Link Invariants of the Chern-Simons Field Theory

New Developments in Topological Quantum Field Theory

by

Enore Guadagnini

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to my parents
Elena and Eugenio

Preface

For almost two decades, several topological arguments have been used systematically in quantum field theory. These methods have been applied mainly in the study of the semiclassical approximation of the theory and, also, in the description of physical phenomena at low energy. Apart from a few exceptions, like the discovery of the general structure of anomalies, the applicability of these arguments was controlled by the validity of certain limits in some given parameters. So, even if the use of topological arguments has produced several important results, the fundamental structure of the field theory models has never been interpreted in purely topological terms.

In this context, the revolution of the last few years is represented by the quantum Chern-Simons field theory. In the Chern-Simons model, the applicability of topological arguments is not restricted to a region in which the semiclassical approximation is valid. In fact, no kind of approximation is at all necessary. We need not consider either low-energy limit or any other limit. The quantum Chern-Simons theory is a "true" topological field theory; any observable and any result obtained in this model has exclusively a topological origin and a topological meaning. The Chern-Simons theory is exactly soluble at the full quantum level not only in $\mathbb{R}^3$ and in S^3 but also in any closed, connected and orientable three-manifold $\mathcal{M}$.

This series of lecture notes is devoted to the discussion of several aspects of the quantum Chern-Simons field theory. In the first part of these notes, the relevant properties of the model in $\mathbb{R}^3$ are studied by means of the standard methods used in quantum field theory.

In the second part, it is shown how the observables of the Chern-Simons theory, associated with knots and links in $\mathbb{R}^3$, can equivalently be described in more abstract and purely topological terms. The exact solution of the model is obtained by combining the general properties of the observables, which are consequences of the symmetries of the system, with the numerical information derived in the canonical approach. The expectation values of the observables have the form of polynomials and represent the values of an ambient isotopy invariant of framed links in $\mathbb{R}^3$ (or S^3). A constructive method for the computation of the observables is presented and the reconstruction theorems for the non-Abelian $SU(N)$ Chern-Simons theory are proved. The link polynomial defined by the Chern-Simons field theory is characterized by the existing relations between satellites and their companions. The structure of these relations is universal and is described by the representation rings of the Lie algebras associated with compact simple Lie groups. For this reason, the invariant obtained in the Chern-Simons theory is called the universal link polynomial.

In the last part of these notes, the solution of the theory in any closed, connected and orientable three-manifold $\mathcal{M}$ is constructed by means of the operator surgery method. The case in which the gauge group is $SU(2)$ is studied in detail. A certain set of models based on the $U(1)$ gauge group is also considered. The three-manifold invariant, naturally associated with the Chern-Simons theory, is defined and its values are computed for several examples of three-manifolds. The present notes are essentially self-contained and include some introductory reviews of knot theory and surgery on three-manifolds.

The topological and algebraic structures described in these lectures are known in mathematics. The purpose of the present notes is to show how these structures emerge from the physical point of view. Intuition and working hypotheses play a fundamental role in physics; for this reason, the solution of the quantum Chern-Simons field theory is presented on the basis of inductive and constructive methods. For example, new rules for the computation of the universal link polynomial are introduced; these rules are derived from the properties of the field theory and are illustrated by means of several examples. Quite often, physical arguments will be used in the place of more rigorous but technical derivations. Consequently, some of the proofs have been simplified and their completion is left to the reader.

The quantum Chern-Simons field theory provides an intrinsic three-dimensional description of the link invariants. Thus, several features of the link polynomials admit a simple physical interpretation which is a consequence of the symmetry properties of the field theory. The interplay between the symmetries of the field theory and the properties of the link polynomials plays a crucial role in our discussion. Symmetry arguments will be used to derive satellite relations and to construct three-manifold invariants. The rigorous and formal proofs on the subject, which can be found in mathematical literature, are essentially based on the algebraic properties of certain modular Hopf algebras. Our approach, instead, gives prominence to the symmetry principles. On the one hand, this shows that the different constructions of three-manifold invariants, which have been produced in literature, represent slightly different versions of the same general structure. On the other hand, we will see how symmetry arguments can conveniently be used to simplify the explicit computation of these invariants.

The quantum Chern-Simons theory is the first example of a non-trivial gauge field theory whose exact solution can be explicitly produced in any three-manifold $\mathcal{M}$. The algebraic operations, involved in the computation of the observables, are surprisingly simple and elementary. The resulting theory is remarkable; its structure presents several new quantum field theory aspects and has deep connections with different fields of physics and mathematics. It is not clear how useful, in the description of physical phenomena, the possible applications of the Chern-Simons theory will be. Certainly, the Chern-Simons model represents an important starting point for new developments in quantum field theory.

The present book is an extended version of the lectures delivered at LAPP, Laboratoire d'Annecy-le-Vieux de Physique des Particules (France), in April 1991. I wish to thank Paul Sorba for having kindly invited me to give those lectures and I should also like to thank all the members of the Theory Division for their warm hospitality.

Most of the material presented here appears in published articles and deep gratitude is due to R. Cappuccio, P. Cotta-Ramusino, N. Maggiore, M. Martellini, S. Panicucci, S.P. Sorella and F. Volpi for their valuable collaboration. I thank beyond measure Mihail Mintchev; with Mihail I have shared the efforts of several computations and the joys for the first surprising results obtained in perturbative Chern-Simons theory. I also wish to thank Riccardo Benedetti and Corrado De Concini for several illuminating discussions on various topological aspects of three-manifolds and on the algebraic structure of quantum groups.

It is a pleasure to acknowledge the admirable work of Lucy Amasanti who improved the exposition.

These lecture notes would never have been written without Raymond Stora's constant interest in the subject. My debt to Raymond is enormous; many of the achievements obtained in the topological Chern-Simons theory are the results of innumerable discussions characterized by his insistent and tireless desire to get a deeper understanding of quantum field theory.

Pisa 1992 Enore Guadagnini

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Chapter 1

Introduction

Quantum field theories provide a very accurate description of physical phenomena. Local gauge invariance turns out to be one of the basic principles underlying the construction of realistic particle models. On the other hand, the inclusion of quantum gravitational effects within some unified theory of the fundamental processes should be based on the principle of general covariance. In these lectures, we shall consider the case in which both gauge invariance and general covariance are realized at the full quantum level in a particular class of non-trivial field theories in three dimensions.

The models we shall consider and which are defined by a pure Chern-Simons action are solvable and possess quite remarkable properties. The expectation values of the basic observables of these systems can conveniently be expressed in terms of polynomials of a certain variable which is a function of the coupling constant of the theory. These polynomials, which are associated with knots and links, encode some universal features of physics in two and three dimensions and, at the same time, provide a useful set of invariants of three-manifolds. This introductory chapter contains a preliminary discussion of some of the essential concepts we shall encounter.

1.1 Quantum physics and classical electromagnetism

We shall begin our discussion with an example of quantum field theory computation in which some notions of classical electromagnetism are involved. Before introducing the action of the model in which we are interested, it is useful to recall a simple method of evaluating path-integrals.

There is a particular situation in which one can easily perform path-integral computations; namely, when the system is characterized by a "quadratic" action of the type

$$S_0[\phi] = \int \tfrac{1}{2}\, \phi\, \Delta\, \phi\,. \tag{1.1}$$

In equation (1.1), ϕ denotes the field (or the set of fields) of the model and Δ is some differential operator. For the moment, we do not need to specify the explicit form of Δ nor the number of dimensions of the space-time manifold in which the theory is defined. Let us introduce some external source coupled to the fields; we shall be interested in computing the vacuum expectation value

$$\langle e^{i\int J\phi} \rangle \equiv Z^{-1} \int d\phi \;\; e^{iS_0[\phi]} \;\; e^{i\int J\phi}, \tag{1.2}$$

where

$$Z = \int d\phi \;\; e^{iS_0[\phi]}. \tag{1.3}$$

The simplest way of computing the expression (1.2) is to perform a linear change in the integration variables in the numerator so that the free path-integral factorizes and then cancels out with the denominator. The whole idea is to put

$$\phi = \widetilde{\phi} + \psi, \tag{1.4}$$

where $\widetilde{\phi}$ is a fixed classical configuration. Then, in the path-integral, it is assumed that $d\phi = d\psi$ because this is precisely the fundamental property which should hold in any reasonable definition of integral. Now, the clever choice of $\widetilde{\phi}$ is the one for which ψ and $\widetilde{\phi}$ decouple; this means

$$\frac{\delta}{\delta\phi}\left(S_0[\phi] + \int J\phi\right)\Bigg|_{\phi=\widetilde{\phi}} = \Delta\widetilde{\phi} + J = 0. \tag{1.5}$$

Finally, in terms of the solution $\widetilde{\phi}$ of equation (1.5), the expression (1.2) takes the form

$$\langle e^{i\int J\phi} \rangle = \exp\left(\frac{i}{2}\int J\widetilde{\phi}\right). \tag{1.6}$$

Of course, when no solution of equation (1.5) exists or when several inequivalent solutions (with the same fixed boundary conditions) exist, this simple method of computing the expectation value (1.2) may need some improvements. For the purposes of the present introductory section, however, it is not necessary to discuss in detail these more complicated situations.

Let us now consider the Abelian Chern-Simons action [1,2]

$$S_0 = \frac{k}{8\pi}\int d^3x\, \epsilon^{\mu\nu\rho}\, A_\mu \partial_\nu A_\rho\,. \tag{1.7}$$

In this example, the vector field A_μ is the analogue of ϕ, the differential operator $(k/4\pi)\,\epsilon^{\mu\nu\rho}\,\partial_\nu$ is the analogue of Δ and the source term is chosen to be

$$\int J\,\phi \;\rightarrow\; \int d^3x\; J^\mu\, A_\mu \;=\; e_1 \oint_{C_1} A_\mu dx^\mu + e_2 \oint_{C_2} A_\mu dx^\mu\,. \tag{1.8}$$

In the expression (1.8), the line integrals are performed along the two oriented non-intersecting closed paths C_1 and C_2 in $\mathbb{R}^3$ shown in Fig. 1.1.

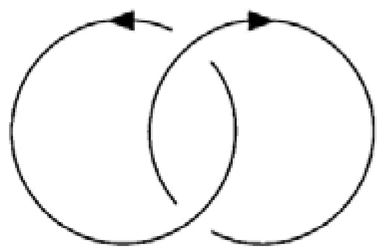


Figure 1.1.

Let us introduce a parametrization $y^\mu(s)$ $(0 \le s \le 1)$ for C_1 and $z^\mu(t)$ $(0 \le t \le 1)$ for C_2. The source J^μ appearing in equation (1.8) can be written as

$$J^\mu(x) \;=\; e_1 \int_0^1 ds\; \dot{y}^\mu(s)\, \delta^3(x - y(s)) + e_2 \int_0^1 dt\; \dot{z}^\mu(t)\, \delta^3(x - z(t))\,. \tag{1.9}$$

In order to compute the expectation value $\langle\, e^{i \int d^3x\, J^\mu A_\mu}\,\rangle$, we shall use the same method as before. The vector field is decomposed into two parts

$$A_\mu(x) \;=\; B_\mu(x) + C_\mu(x)\,, \tag{1.10}$$

where the classical configuration $B_\mu(x)$ should satisfy the analogue of equation (1.5), that is

$$\epsilon^{\mu\nu\rho}\,\partial_\nu\, B_\rho(x) \;=\; -\left(\frac{4\pi}{k}\right) J^\mu(x)\,. \tag{1.11}$$

This equation is well known in classical electromagnetism; it shows the connection between the magnetic field B_μ and the stationary current density which originates the field itself. Finding the solution of equation (1.11) means, in our case, finding the expression of the magnetic field in the presence of two filamentary wires (shown in Fig. 1.1) which carry the currents

$$I_1 \;=\; -e_1 \frac{c}{k} \qquad I_2 \;=\; -e_2 \frac{c}{k}\,. \tag{1.12}$$

The solution of this problem is expressed by the so-called Ampère Law: the total magnetic field is the linear combination

$$B_\mu(x) = B_\mu^{(1)}(x) + B_\mu^{(2)}(x), \tag{1.13}$$

of the two components

$$B_\mu^{(1)}(x) = -\frac{e_1}{k}\int_0^1 ds\,\epsilon_{\mu\nu\rho}\,\dot{y}^\nu(s)\,\frac{(x-y(s))^\rho}{|x-y(s)|^3}, \tag{1.14}$$

and

$$B_\mu^{(2)}(x) = -\frac{e_2}{k}\int_0^1 dt\,\epsilon_{\mu\nu\rho}\,\dot{z}^\nu(t)\,\frac{(x-z(t))^\rho}{|x-z(t)|^3}, \tag{1.15}$$

generated by the wires C_1 and C_2 respectively.

In terms of the resulting magnetic field (1.13), the Chern-Simons vacuum expectation value of the source term takes the form

$$\langle e^{i\int d^3x\,J^\mu A_\mu}\rangle = \exp\left(i\frac{e_1}{2}\oint_{C_1} B_\mu\,dx^\mu + i\frac{e_2}{2}\oint_{C_2} B_\mu\,dx^\mu\right). \tag{1.16}$$

The expression (1.16), which is the analogue of equation (1.6), presents a problem: each line-integral of the magnetic field along a wire contains a self-interaction part which has certain ambiguities. The solution to this problem will be discussed in detail in Chapter 3. For the moment, one can conclude that, *neglecting self-interactions*, the desired expectation value is given by

$$\langle e^{i\int d^3x\,J^\mu A_\mu}\rangle = \exp\left[-2ie_1e_2\left(\frac{2\pi}{k}\right)\chi(C_1,C_2)\right], \tag{1.17}$$

where

$$\chi(C_1,C_2) = \frac{1}{4\pi}\oint_{C_1} dx^\mu \oint_{C_2} dy^\nu\,\epsilon_{\mu\nu\rho}\,\frac{(x-y)^\rho}{|x-y|^3}. \tag{1.18}$$

Note that, in the Abelian Chern-Simons theory, there are several field configurations satisfying equation (1.11). Indeed, the longitudinal part of B_μ is not constrained by equation (1.11). This means that any B_μ configuration, which differs from the classical expression (1.13) by the gradient of an arbitrary function, still satisfies equation (1.11). Apparently, one has a situation in which the general method leading to equation (1.6) presents some ambiguities. In the Chern-Simons model, however, these ambiguities are completely harmless because of the particular choice of the source, equation (1.9). Since we are considering closed paths, the longitudinal part of B_μ is totally irrelevant (see equation (1.16)) and the result shown in equation (1.17) is unique. In fact, as we shall see, equation (1.17) represents the exact result (neglecting self-interactions, of course).

With the usual (right-handed) convention in the definition of the antisymmetric tensor $\epsilon^{\mu\nu\rho}$ ($\epsilon^{123} = 1$), the value of $\chi(C_1, C_2)$ for the paths shown in Fig. 1.1 is

$$\chi(C_1, C_2) = 1 . \tag{1.19}$$

The result (1.17) has a simple physical interpretation. The exponent in the expression (1.17) contains the circuitation along one closed path, say C_1, of the magnetic field generated by the second wire. This quantity is precisely the energy gain $\mathcal{E}$ of an imaginary magnetic monopole moving along C_1 in the presence of the magnetic field generated by C_2. For each "winding" of the magnetic monopole around C_2, the energy increases by a definite amount which, in our units, is given by $\mathcal{E} = -2e_1e_2(2\pi/k)$. For arbitrary non-intersecting closed paths C_1 and C_2, the value of the expression (1.18) (called the ***Gauss integral***) is an integer representing exactly how many times C_1 "winds" C_2.

One of the most remarkable features of the Abelian Chern-Simons theory is that the expectation value (1.17) is invariant under smooth deformations of the paths C_1 and C_2. We would now like to understand the reasons for this behaviour and construct new models possessing the same property.

1.2 Abelian Chern-Simons action

All the information on the physics of the different systems is encoded in their action; so, let us reconsider the Abelian Chern-Simons (CS) action [1,2]

$$S_0 = \frac{k}{8\pi} \int d^3x \, \epsilon^{\mu\nu\rho} \, A_\mu \partial_\nu A_\rho \, . \tag{1.20}$$

The functional (1.20) is invariant under (Abelian) gauge transformations acting on the vector field $A_\mu(x)$. The quantization of the system defined by the action (1.20) will be discussed in Chapter 3.

The action S_0 is also invariant under general coordinate transformations with $A_\mu(x)$ transforming as a covariant vector. This last property is called ***general covariance***. Note that, in our example, general covariance is realized in quite a peculiar way. In ordinary field theories, the action is not invariant under general coordinate transformations (acting on the fields of the theory) unless the metric also is a dynamical variable. In the Abelian CS model, the metric that one can introduce on the three-manifold is not a variable (or a "field") of the theory. Nevertheless, the action is invariant under general coordinate transformations for the simple reason that S_0 does not depend on the metric at all. In fact, S_0 can be understood as the integral of a three-form on a three-manifold.

Gauge invariance and general covariance are the real reasons for the properties of the expectation value (1.17) that we have observed. Gauge invariance forced us to choose the external source to be expressed in terms of closed paths (conserved external currents), since only gauge-invariant quantities have an intrinsic meaning in gauge theories. Because of general covariance, the final result (1.17) only depends on the topological structure of the closed contours. This is why there is invariance under smooth deformations of the paths in $\mathbb{R}^3$.

In the previous section, the source term was represented by the simple two-component link shown in Fig. 1.1. But one can consider more complicated links, of course; an example is shown in Fig. 1.2.

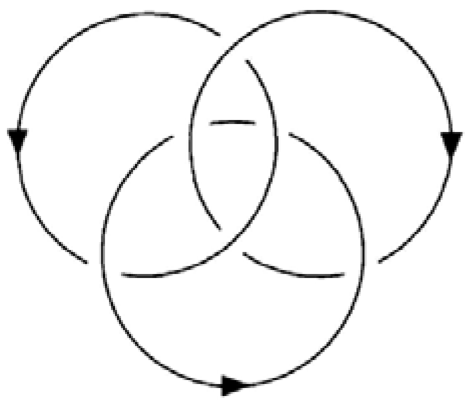


Figure 1.2.

Exercise. Consider the Abelian CS theory with a source term corresponding to the link shown in Fig. 1.2. In this case, what is the expression (neglecting self-interactions) of the vacuum expectation value $\langle e^{i \int d^3x\, J^\mu A_\mu} \rangle$?

1.3 Non-Abelian Chern-Simons action

The action (1.20) can be generalized [1,3,4] to the case in which the gauge group G is a non-Abelian. The corresponding CS action reads

$$S_{CS} = \frac{k}{4\pi} \int d^3x\, \epsilon^{\mu\nu\rho}\ \mathrm{Tr}\left(A_\mu \partial_\nu A_\rho + i\, \tfrac{2}{3}\, A_\mu A_\nu A_\rho \right) , \qquad (1.21)$$

where $A_\mu = A_\mu^a T^a$, $\{T^a\}$ are the Hermitian generators of a compact simple Lie group G in its defining representation and the real parameter k is the coupling constant of the model. The integral appearing in equation (1.21) has to be performed in $\mathbb{R}^3$.

The expectation values $\{\langle W(L)\rangle\}$ of Wilson line operators associated with oriented links $\{L\}$ represent a useful set of gauge-invariant observables for the CS theory. Given an oriented knot C and an irreducible representation ρ of G, the associated Wilson line operator is

$$W_\rho(C) = W(C;\rho) = \mathrm{Tr}\,\mathrm{P}\exp\left(i\oint_C A^a_\mu(x)\, T^a_{(\rho)}\, dx^\mu\right), \tag{1.22}$$

where the path ordering is performed along C and $\{T^a_{(\rho)}\}$ are the generators of G in the ρ representation. Consider now an oriented link L with m components $\{C_1, \ldots, C_m\}$ and let ρ_i be the irreducible representation of G associated to the i-th component C_i of L. The vacuum expectation values

$$\langle W(L)\rangle \equiv \frac{\langle 0|\, W_{\rho_1}(C_1)\cdots W_{\rho_m}(C_m)\, |0\rangle}{\langle 0|0\rangle}, \tag{1.23}$$

defined for generic links and arbitrary representations $\{\rho_i\}$, are the gauge-invariant observables in which we are interested.

We will show that the expectation values (1.23) are well defined. The proof is divided into two parts: first, one has to verify that the quantum CS theory is renormalizable (the renormalized correlation functions satisfy an action principle based on the functional (1.21)) and, second, the precise meaning of the composite operator $W(C;\rho)$ at the quantum level must be specified. We will see that, in order to preserve general covariance, the Wilson line operators must be defined on framed knots.

As a consequence of general covariance, $\langle W(L)\rangle$ is invariant under smooth deformations of the framed link L in $\mathbb{R}^3$. Therefore, the set $\{\langle W(L)\rangle\}$ defines a link invariant of the same type considered in knot theory and the main problem is how to compute its values on different framed links in closed form. We will derive the rules which permit the computation of $\langle W(L)\rangle$ with a finite number of operations. It turns out that $\langle W(L)\rangle$ takes the form of a polynomial in a certain complex variable which is a function of the coupling constant k of the theory.

After $\{\langle W(L)\rangle\}$ have been found, the main issue is to identify these polynomials. It turns out that the link invariants obtained in the CS theory are those associated with the braid group representations described by the quasi-tensor category of quasi-triangular quasi-Hopf (QTQH) algebras associated to the quantum deformations of ordinary Lie algebras. In this sense, the obtained link polynomials are not completely unknown. In quite general terms, the link invariants described by the quasi-tensor category of QTQH algebras are called by Drinfeld the universal link (or knot) invariants [5]. For this reason, the polynomial $E(L)$, associated to the expectation value $\langle W(L)\rangle$, is called the ***universal link polynomial***.

The link invariant $E(L)$ is of particular interest for knot theory because it is the natural generalization of the Jones polynomial [6]. The most important aspect of this generalization is based on the existence of a Lie algebra structure underlying the construction of the universal link polynomial. The expectation values $\{ \langle W(L) \rangle \}$, computed in the case in which all the components of the links are associated with the fundamental representation of the gauge group $G = SU(2)$, essentially give the Jones polynomial. The first generalization consists of associating arbitrary representations of $G = SU(2)$ to the different components of the links. In this case, one obtains a new polynomial, which can be called the extended Jones polynomial because the $SU(2)$ Lie algebra remains the same. A further step consists of taking a generic real simple Lie algebra as the Lie algebra of the gauge group G and computing $\{ \langle W(L) \rangle \}$ when the different components of the links are associated with arbitrary representations of G. This is the case considered in these lectures. The general properties of the universal link polynomial associated with an arbitrary simple Lie algebra are obtained and the proof of the complete reconstruction of $E(L)$ in the case of $\{A_n\}$ algebras is presented.

The physics of the model described by the CS action is quite peculiar. In a (2+1) decomposition of $\mathbb{R}^3$, any fixed-time plane may intersect a given link at a certain number of points. These punctures can be interpreted as point-like particles whose world-lines represent the components of the link. As time goes by, these particles may move on the space plane; furthermore, two of them may either annihilate or be produced. A possible world-line configuration is shown in Fig. 1.3.

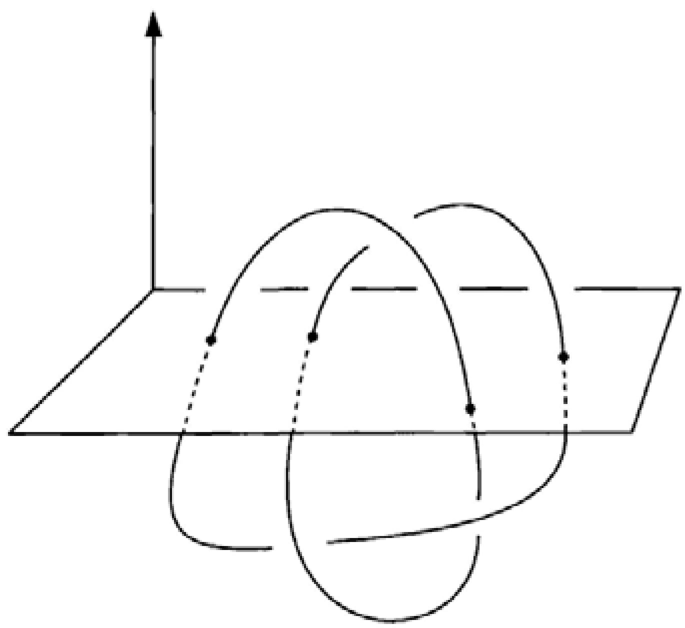


Figure 1.3.

The expectation value $\langle W(L) \rangle$ can be interpreted as the quantum mechanical amplitude associated with the whole process described by the link L. For generic values of k, each particle (puncture) is characterized by a quantum number which labels the inequivalent irreducible representations of the gauge group G. For integer values of k, something special happens: the number of the different kinds of particles is finite. For example, when $G = SU(2)$ and $k > 2$, only $|k| - 2$ different kinds of particles exist. For $|k| = 2$, the state space is one-dimensional and any state vector is proportional to the vacuum vector. When $|k| = 1$, the state space is two-dimensional; in addition to the vacuum state, there is only one nontrivial particle state.

A further generalization of the CS theory consists of considering the model, described by the action (1.21), in a generic three-manifold $\mathcal{M}$ which is closed, connected and orientable. In this case, the computation of the expectation values, denoted by $\{ \langle W(L) \rangle \big|_{\mathcal{M}} \}$, can be performed by means of an operator realization of surgery and the expectation values in $\mathcal{M}$ can be expressed as appropriate linear combinations of the expectation values in $\mathbb{R}^3$ which must be evaluated for integer values of the coupling constant k. We shall give the details of the surgery construction in the case in which $G = SU(2)$. The three-manifold invariant defined by the CS action is also constructed and the values taken by this invariant in some examples of manifolds are given.

The importance of the action (1.21) for knot theory and the study of three-manifolds has been pointed out by Schwarz [1] and Atiyah [3]. Further developments on the subject, together with a large number of possible connections of the CS theory with different areas of mathematics and physics, have been discussed by Witten in Ref.[4], in which several conjectures have been formulated. The first rigorous mathematical construction of the three-manifold invariants, whose existence was conjectured in [4], has been produced by Reshetikhin and Turaev [7,8]. These authors have given a precise definition of these invariants by means of certain modular Hopf algebras. We shall use a different approach based on the Feynman path-integral. But our final results perfectly agree with the Reshetikhin-Turaev invariants. Only recently has the exact solution of the quantum CS theory been produced [9,10,11]. Its possible realistic applications in physics have not been completely explored.

Chapter 2

Basic notions of knot theory

The full program of classifying and studying the properties of knots and links that one can construct in $\mathbb{R}^3$ was formulated on the basis of physical motivations in the second half of the nineteenth century. The interest in this subject mainly originated from the vortex-atoms model proposed by J.C. Maxwell, P.G. Tait and W. Thomson around 1867. Several excellent results have been obtained in knot theory and important developments, connecting different fields of mathematics, have taken place. In this chapter, a few definitions and results are briefly recalled; a more detailed and complete exposition can be found, for instance, in [12,13] and in the references quoted there. We shall begin by considering links in $\mathbb{R}^3$ or, equivalently, in the three-sphere S^3. The more general case of an arbitrary closed, connected and orientable three-manifold $\mathcal{M}$ will be discussed in Chapter 16.

2.1 Ambient and regular isotopy

A smooth non-intersecting closed path C in $\mathbb{R}^3$ is called a ***knot***. Since the definition of the holonomy requires an orientation for the path, we will always consider oriented knots. An oriented ***link*** L with m components is the union of m oriented non-intersecting closed paths. The m components of L will be denoted by $\{C_1, C_2, \ldots, C_m\}$. Smooth deformations in the ambient space do not modify the "topological" properties of links. Two links L_1 and L_2 in $\mathbb{R}^3$ are called ***ambient isotopic*** if L_1 can be smoothly connected with L_2 in $\mathbb{R}^3$. If one is interested in the topological properties of links, only the equivalence classes of ambient isotopic links are relevant, of course.

A convenient description of links is given in terms of ***diagrams*** obtained by projecting the links on a plane. In order to avoid all ambiguities, one usually considers link diagrams containing only simple crossing points; at each crossing point the choice of over/under crossing is specified, as shown for example in Fig. 1.1 and Fig. 1.2. Given two link diagrams D_1 and D_2, the associated links L_1 and L_2 are ambient isotopic if and only if a finite sequence of ***Reidemeister moves*** (shown in Fig. 2.1) which transforms D_1 into D_2 exists.

Reidemeister moves (RM) are very important in knot theory because they encode the symmetry structure which is relevant for the link classification problem. Indeed,

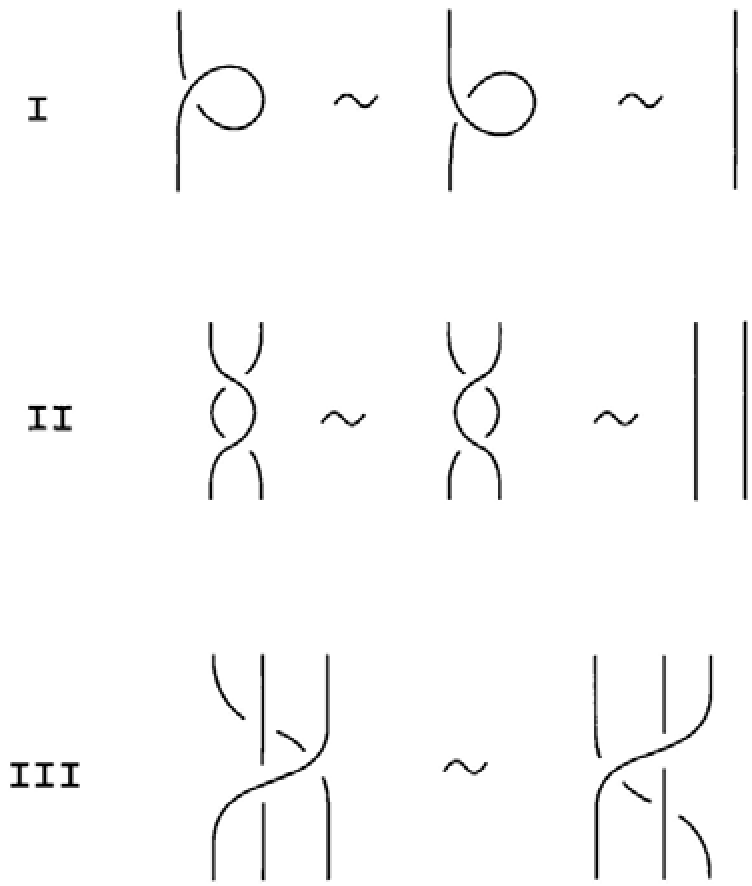


Figure 2.1.

constructing link invariants of ambient isotopy means finding invariants of the symmetry group generated by the RM. Several different methods of constructing link invariants have been discovered and some of them will be mentioned here. Before describing some explicit examples, let us analyse the RM a little more carefully. Reidemeister moves of type I are very special. In fact, one can eliminate them from the list of admissible moves. In this way, one can define an interesting structure which plays an important role in the construction of the universal link polynomial.

Two link diagrams D_1 and D_2 related by RM of types II and III only are called ***regular isotopic***. Consider now the equivalence classes of regular isotopic link diagrams. A useful invariant of regular isotopy is the ***writhe*** number $w(D_L)$ which is defined for any link diagram D_L by

$$w(D_L) = \sum_p \epsilon\,(p)\,. \tag{2.1}$$

The sum in equation (2.1) is performed over all the crossing points of the link diagram D_L and

$$\epsilon(L_{\pm}) = \pm 1 , \tag{2.2}$$

where $L_{\pm}$ are shown in Fig. 2.2. The configurations L_+ and L_- will be called overcrossing and undercrossing respectively.

L_+ L_-

Figure 2.2.

The proof that $w(D_L)$ is a regular isotopy invariant is very simple; as a matter of fact, it is immediately verified that $w(D_L)$ is invariant under RM of types II and III.

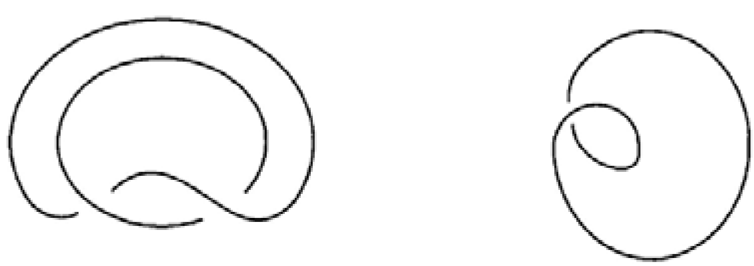
The concept of regular isotopy is useful because, by eliminating the RM of type I, one does not lose any information concerning the topology of links; on the contrary, one gains a free variable for each component of the link. In fact, each equivalence class of ambient isotopy contains, by definition, all the equivalence classes of regular isotopy which are connected by RM of type I. Each RM of type I acts only on a single line of the diagrams; therefore, it can modify only the writhe number of a single component of the link diagrams. This being the case, each equivalence class of ambient isotopy corresponding to a link in $\mathbb{R}^3$ contains an infinite number of equivalence classes of regular isotopy, which are labelled by the writhe numbers $\{w(C_i)\}$ of the different components $\{C_i\}$ of the link. The crucial point is that one can give [13] the following interpretation of the above conclusion. The equivalence classes of regular isotopic link diagrams describe ambient isotopy classes of links in $\mathbb{R}^3$ in which each component is characterized by an integer number which, in turn, is an ambient isotopy invariant.

Now, suppose that one replaces links made of, say, strings with links made of oriented ***bands***. The topology of the links is not modified; the only change is that for each component C_i of the link we now have an extra variable $T(C_i)$ telling us how many times the oriented band is twisted. The ***twist*** T of the band is an ambient isotopy invariant and therefore we are precisely in the same conditions as above. In conclusion, one can represent the equivalence classes of ambient isotopic links made of bands with the equivalence classes of regular isotopic link diagrams. The only thing which remains to be fixed is the connection between the writhe number $w(C_i)$ and the the value of the twist $T(C_i)$ of each component C_i of the links. The simplest choice is

$$T(C_i) = w(C_i) . \tag{2.3}$$

With the convention (2.3), the band shown in Fig. 2.3(a), for example, is represented by the link diagram shown in Fig. 2.3(b).

The importance of regular isotopy for the CS theory is due to the fact that, in studying the properties of the expectation values $\{\langle W(L)\rangle\}$, one has to consider ***framed links***. This means that for each component C of the links one has to introduce another closed and oriented path C_f called the framing of C. We will discuss this point in detail in Chapter 3. For the moment, imagine that C_f lies within an infinitesimal neighbourhood of C, with the condition that C and C_f never intersect. Moreover, C_f is always oriented in the same direction as C (i.e., C and C_f coincide in the limit in which the thickness of the neighbourhood vanishes). An example of a framed knot is shown in Fig. 2.4.



(a) (b)

Figure 2.3.

Since C and C_f can be considered to be the two components of the boundary of an oriented band, framed links can be interpreted as links made of bands. Consequently, we will represent oriented framed links in $\mathbb{R}^3$ with the regular isotopy classes of link diagrams with the identification (2.3); this representation will be called ***vertical framing***. In order to simplify the notations, framed links in $\mathbb{R}^3$ and their corresponding link diagrams in vertical framing will be indicated by the same symbol.

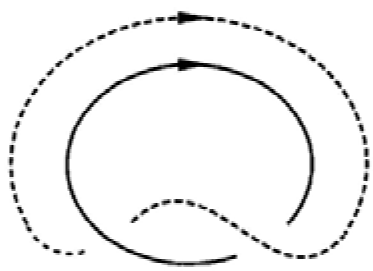


Figure 2.4. The unknot (solid line) with a particular framing (dashed line).

RM of type I are very peculiar for another reason too; as we shall see, they are the only moves in which the partial closure of braids is involved. As far as RM of types II and III are concerned, one should note that they essentially determine the algebraic structure of the Artin braid group B_n. For open braids, the invariance under RM of type II is quite trivial because this invariance is automatically satisfied in terms of the B_n generators. However, in considering the closure of braids, RM of type II have the important effect of associating the conjugacy classes of B_n to the links. Finally, let us consider the RM of type III; they represent the main feature of the braid group and enter directly the defining relations satisfied by the B_n generators. Finding a complete classification of the inequivalent realizations of the RM of type III is still an open problem. In considering matrix representations of B_n and with an appropriate choice of the form of the generators, RM of type III give origin to the famous quantum Yang-Baxter equation. Some relevant properties of the braid group will be considered in Chapter 8.

2.2 Link invariants

In this section some examples of link invariants are reported. Consider a two-component oriented link L in $\mathbb{R}^3$ with components C_1 and C_2 and let D_L, D_1 and D_2 be the associated diagrams. As we have stated above, $w(D_L)$ is a regular isotopy invariant; it is easy to see that also $w(D_1)$ and $w(D_2)$ are separately regular isotopy invariants. Let us now try to construct an invariant of ambient isotopy by combining these three writhe numbers. Under a RM of type I, $w(D_L)$ and $w(D_1) + w(D_2)$ transform as

$$\Delta w(D_L) = \Delta [w(D_1) + w(D_2)] = \pm 1 . \tag{2.4}$$

Therefore, the combination

$$\chi(C_1, C_2) = \frac{1}{2}\left[\, w(D_L) - w(D_1) - w(D_2) \,\right] \tag{2.5}$$

is an ambient isotopy invariant. The invariant $\chi(C_1, C_2)$ is called the ***linking number*** of C_1 and C_2. Roughly speaking, the value of $\chi(C_1, C_2)$ tells us how many times C_2 winds around C_1. This quantity can also be expressed in terms of the Gauss integral

$$\chi(C_1, C_2) = \frac{1}{4\pi} \oint_{C_1} dx^\mu \oint_{C_2} dy^\nu \, \epsilon_{\mu\nu\rho} \, \frac{(x-y)^\rho}{|x-y|^3}\,, \tag{2.6}$$

where the distance $|x - y|$ is computed by means of the ordinary flat metric in $\mathbb{R}^3$.

This example shows that the same link invariant can be constructed by using very different methods. In equation (2.5), χ is obtained through computations performed by looking at the link diagrams, whereas equation (2.6) provides a more direct geometrical interpretation of χ. The expression (2.5) has to be computed on a specific link diagram but the result, being ambient isotopic, does not depend on the particular choice of this diagram. Similarly, in the expression (2.6) the integrand depends on the metric $\delta_{\mu\nu}$ of $\mathbb{R}^3$, but the result of the integral is metric-independent: it depends exclusively on the topology. It is obvious that, for explicit computations, equation (2.5) is more practical to use than the expression (2.6). Usually, the invariants constructed by operating on the link diagrams are easier to compute than those constructed by means of geometrically more intrinsic methods.

Equations (2.5) and (2.6) are useful for illustrating the strategy pursued in solving the CS theory. The contributions to $\langle\, W(L)\,\rangle$ obtained, for instance, by computing the Feynman diagrams are the analogue of the expression (2.6) and are in general quite difficult to evaluate. The idea is to find the analogue of equation (2.5); that is, all the contributions of the Feynman diagrams will be expressed in terms of simple algebraic operations based on the structure of the link diagrams.

As we have already said, a framed oriented knot C with framing C_f can be interpreted as a knot made of a band; the twist T of this band is simply given by

$$T = \chi(C, C_f)\,. \tag{2.7}$$

Among the several link invariants that have been discovered, there are the so-called ***link polynomials***. Some of them are strictly connected with the universal link polynomial described by the CS theory and their construction is simply formulated in terms of the concept of the ***skein relation***. Three oriented links L_+, L_- and L_0 are skein related if they have diagrams which are identical except for a small part

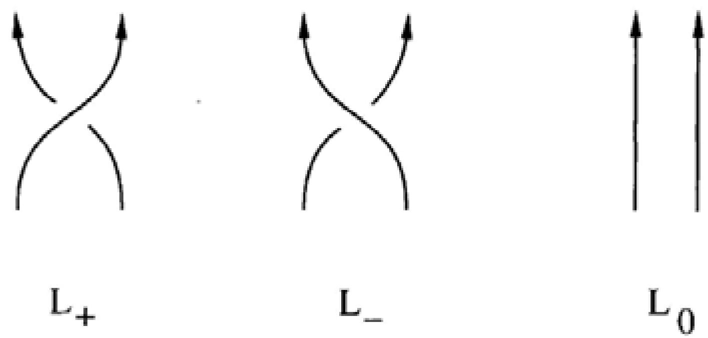


Figure 2.5.

contained inside a fixed open disc of the plane of the pictures. Moreover, inside this disc the three diagrams look as shown in Fig. 2.5.

The Alexander-Conway polynomial $\nabla(L; z)$, associated to the link L, is defined by [14]

$$\begin{aligned} &\text{(i)} \quad \text{ambient isotopy invariance,} \\ &\text{(ii)} \quad \nabla(U; z) = 1 \quad , \\ &\text{(iii)} \; \nabla(L_+; z) - \nabla(L_-; z) = z\,\nabla(L_0; z)\,, \end{aligned} \tag{2.8}$$

where U is the unknotted knot and condition (ii) fixes the normalization. With our notations, $\nabla(L; z)$ is also called the Conway potential function and it is a (finite) Laurent polynomial in the variable z with integer coefficients. Of course, if $\nabla(L_1; z) \neq \nabla(L_2; z)$, then the two links L_1 and L_2 are not ambient isotopic. However, it is easy to find examples of links which are not ambient isotopic but have the same Alexander-Conway polynomial.

The Jones polynomial $V(L; q^{1/2}) \in Z[q^{\pm 1/2}]$ is defined by [6]

$$\begin{aligned} &\text{(i)} \quad \text{ambient isotopy invariance,} \\ &\text{(ii)} \quad V(U; q^{1/2}) \; = \; 1 \quad , \\ &\text{(iii)} \; qV(L_+) - q^{-1}V(L_-) \; = \; (q^{1/2} - q^{-1/2})\, V(L_0)\,. \end{aligned} \tag{2.9}$$

The $V(L)$ polynomial is more selective than $\nabla(L)$; however, there are still ambient non-isotopic links with the same Jones polynomial. Our convention on the form of the exchange relation (2.9) should be noted; the dependence of equation (2.9) on the variable q is not standard. In these notations, the field theory results take a simple form. In Chapter 12, we shall compare our definition of the Jones polynomial with the standard convention used in mathematics.

The two-variable HOMFLY polynomial $P(L; t, z)$ is defined by [15]

$$
\begin{array}{ll}
\text{(i)} & \text{ambient isotopy invariance,} \\
\text{(ii)} & P(U;t,z) = 1 \quad , \\
\text{(iii)} & t\,P(L_+) - t^{-1}P(L_-) = z\,P(L_0)\,,
\end{array}
\tag{2.10}
$$

and it represents essentially the most general polynomial [16] constructed by means of the skein relation involving the configurations shown in Fig. 2.5. In fact, $P(L;t,z)$ reduces to the Alexander-Conway and Jones polynomials with the obvious choices for the values of the variables t and z. The HOMFLY polynomial also does not provide a complete classification of knots or links.

A common feature of all these polynomials is that, by means of the skein relation, they can be easily constructed by analysing the link diagrams. By using the conditions (i) and (iii) recursively, the polynomial of whatever link can be written in terms of the polynomial of the unknotted knot U, which is conventionally taken to be the identity. At this stage, it is not completely obvious that the construction based on the recursive use of the skein relation is well defined. Several different proofs of the internal consistency of the defining conditions (i), (ii) and (iii) have been produced in literature, see for instance [6,15,17].

By looking at the defining conditions (2.8)-(2.10), one notes that the progress in the construction of the "classical" link polynomials has been made by modifying the skein (or exchange) relation. However, along this line it is hard to imagine how to improve the HOMFLY polynomial significantly. In fact, the strategy leading to the universal link polynomial is to come back to the Jones polynomial and provide it with a Lie algebra interpretation.

We conclude this section by considering a link polynomial of regular isotopy which is of particular interest for the CS theory. The $S(L;\alpha,\beta,z)$ polynomial [18] is defined by

$$
\begin{array}{ll}
\text{(i)} & \text{regular isotopy invariance,} \\
\text{(ii)} & S(U_0) = 1\,, \\
\text{(iii)} & S(L^{(+)}) = \alpha\,S(L^{(0)}), \qquad S(L^{(-)}) = \alpha^{-1}\,S(L^{(0)})\,, \\
\text{(iv)} & \beta\,S(L_+) - \beta^{-1}S(L_-) = z\,S(L_0)\,,
\end{array}
\tag{2.11}
$$

where $L^{(\pm)}$ and $L^{(0)}$ are shown in Fig. 2.6 and U_0 is the unknotted knot with zero writhe. The use of a simplified notation should cause no problems here. From the definition (2.11) it is clear that $S(L)$ is defined on the equivalence classes of regular isotopic link diagrams or, equivalently, on the set of framed links.

The polynomial $S(L)$ was introduced in Ref.[18] to describe the behaviour of the Wilson line operators when the link components are associated with certain representations of the gauge group. $S(L)$ also can be constructed by using the skein relation recursively and is related [18] to the HOMFLY polynomial by

$$
P(L;t=\alpha\beta,z) \;=\; \alpha^{-w(L)}\,S(L;\alpha,\beta,z)\,. \tag{2.12}
$$

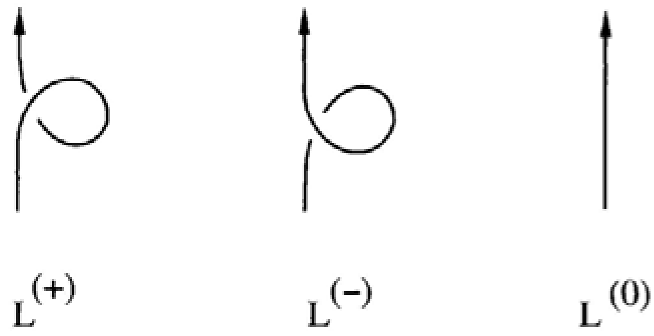


Figure 2.6.

The factor multiplying $S(L)$ in equation (2.12) just compensates the covariant variation of $S(L)$ under RM of type I. Because of the identity (2.12), the information concerning the link classification problem contained in $S(L)$ and $P(L)$ is essentially the same. However, the $S(L)$ polynomial is particularly significant; as we shall see in Chapter 8, $S(L)$ is in a way the ancestor of all the polynomials which are obtained in Hecke algebra representations of the braid group. Moreover, the structure of the relations (2.11) naturally extends to the general case described by $E(L)$.

Clearly, all the link polynomials defined by a skein relation represent invariants of links in $\mathbb{R}^3$ and of links in S^3 as well. Indeed, in the construction of the invariant what is important is the fact that any link is contained in the interior of a three-ball where the ordinary skein relation holds. We shall return to this important point later.

2.3 Framing and satellites

This section completes our rapid review of knot theory and introduces several definitions and concepts which are useful for our discussion. Consider a knot $C \subset \mathbb{R}^3$ or in S^3 and the two-dimensional disc D^2 (i.e. the unit ball in $\mathbb{R}^2$ centered at the origin). If we represent D^2 in the complex plane, its points have coordinates $(r\, e^{i\theta})$ with $0 \leq r \leq 1$. An embedding $f \,:\, C \times D^2 \to \mathbb{R}^3$, such that $f(x, 0) = x$ for $x \in C$, defines a ***tubular neighbourhood*** of C. In other words, a tubular neighbourhood of C is just a solid torus whose core is C, as shown in Fig. 2.7.

A ***solid torus*** V is a three-dimensional space homeomorphic with $S^1 \times D^2$; a specified homeomorphism $h \;:\; S^1 \times D^2 \;\to\; V$ is called a ***framing*** of V. In Chapter 16, we shall analyse in detail the main properties of solid tori; for the moment, only some useful definitions are recalled. A ***meridian*** of V is a simple closed curve on the boundary ∂V of the solid torus which is essential

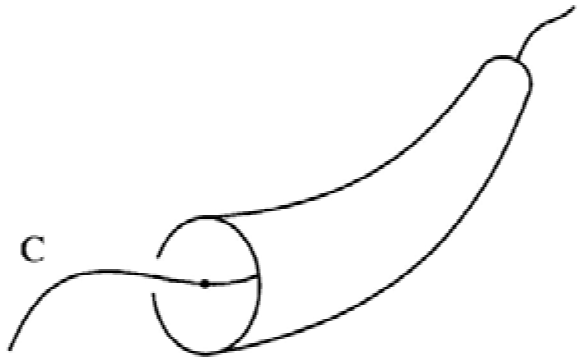


Figure 2.7. Tubular neighbourhood of the knot C

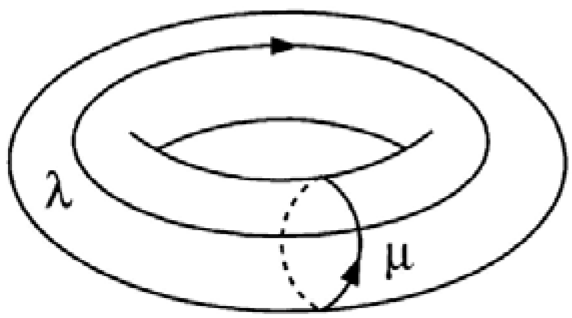


Figure 2.8.

in ∂V and is homotopically trivial in V. The curve μ, shown in Fig. 2.8, is a meridian. A ***longitude*** of V is a closed curve in ∂V intersecting some meridian of V (transversally) in a single point. The curve λ, shown in Fig. 2.8, is a longitude. For any given framing h of V, $h(S^1 \times 1)$ is a longitude. It should be noted that any two meridians are ambient isotopic in V; in this sense, the meridian is an intrinsic part of a solid torus. On the other hand, there is an infinite number of ambient isotopy classes of longitudes. These classes can be classified by the linking number of the longitude and the core of the solid torus.

Given a tubular neighbourhood N of the knot C and a specific framing h of N, the longitude $h(S^1 \times 1)$ defines a framing C_f of C. One can always assume that the image of the core of $S^1 \times D^2$, namely $h(S^1 \times 0)$, coincides with the knot C. Therefore, a given orientation for the knot C defines an orientation for S^1 and, consequently, $C_f = h(S^1 \times 1)$ also is oriented. In conclusion, a framing h of a tubular neighbourhood of a knot C defines a framing C_f of C. Viceversa, a given framing C_f of C determines, up to ambient isotopy, a unique framing h for a