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Compact Projective Planes

With an Introduction to Octonion Geometry

by

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Preface

The old and venerable subject of geometry has been changed radically by the famous book of Hilbert [1899, 30] on the foundations of geometry. It was Hilbert's aim to give a simple axiomatic characterization of the real (Euclidean) geometries. He expressed the necessary continuity assumptions in terms of properties of an order. Indeed, the real projective plane is the only desarguesian ordered projective plane where every monotone sequence of points has a limit, see the elegant exposition in Coxeter [61].

However, the stipulation of an order excludes the geometries over the complex numbers (or over the quaternions or octonions) from the discussion. In order to include these geometries, the order properties have been replaced by topological assumptions (like local compactness and connectedness), see Kolmogoroff [32], Köthe [39], Skornjakov [54], Salzmann [55, 57], Freudenthal [57a,b]. This is the historical origin of topological geometry in the sense of this book.

Topological geometry studies incidence geometries endowed with topologies which are compatible with the geometric structure. The prototype of a topological geometry is a topological projective plane, that is, a projective plane such that the two geometric operations of joining distinct points and intersecting distinct lines are continuous (with respect to given topologies on the point set and on the line set). Only few results can be proved about topological planes in general. In order to obtain deeper results, and in order to stay closer to the classical geometries, we concentrate on compact, connected projective planes. Planes of this type exist in abundance; topologically they are very close to one of the four classical planes treated in Chapter 1, but they can deviate considerably from these classical planes in their incidence-geometric structure. Most theorems in this book have a 'homogeneity hypothesis' requiring that the plane in question admits a collineation group which is large in some sense. Of course, there is a multitude of possibilities for the meaning of 'large'. It is a major theme here to consider compact projective planes with collineation groups of large topological dimension. This approach connects group theory and geometry, in the spirit of F. Klein's Erlangen program. We shall indeed use various methods to describe a geometry in group-theoretic terms, see the remarks after (32.20). Usually, the groups appearing in our context turn out to be Lie groups.

*

In this book we consider mainly projective (or affine) planes. This restriction is made for conciseness. Let us comment briefly on some other types of incidence geometries (compare Buekenhout [95] for a panorama of incidence geometry). Topological projective spaces have been considered by Misfeld [68], Kühne–Löwen [92] and others, see also Groh [86a,b]. It is a general phenomenon that spatial geometries are automatically much more homogeneous than plane geometries; we just mention the validity of Desargues' theorem (and its consequences) in each projective space, and the more recent classification of all spherical buildings of rank at least 3 by Tits. This phenomenon effectuates a fundamental dichotomy between plane geometry and space geometry. Stable planes are a natural generalization of topological projective planes, leading to a rich theory, compare (31.26). Typical examples are obtained as open subgeometries of topological projective planes. The reader is referred to Grundhöfer–Löwen [95] and Steinke [95] for surveys on locally compact space geometries (including stable planes) and circle geometries, respectively.

Projective planes are the same thing as generalized triangles, and the generalized polygons are precisely the buildings of rank 2. A theory of topological generalized polygons and of topological buildings is presently developing, see Burns–Spatzier [87], Knarr [90] and Kramer [94] for fundamental results in this direction, compare also Grundhöfer–Löwen [95] Section 6. These geometries are of particular interest in differential geometry, see Thorbergsson [91, 92]. Another connection with differential geometry is provided by the study of symmetric planes, see Löwen [79a,b, 81b], Seidel [90a, 91], H. Löwe [94, 95], Grundhöfer–Löwen [95] 5.27ff.

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Now we give a rough description of the contents of this book (see also the introduction of each chapter).

In Chapter 1, we consider in detail the classical projective planes over the real numbers, over the complex numbers, over Hamilton's quaternions, and over Cayley's octonions; these classical division algebras are denoted by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. The four classical planes are the prime examples (and also the most homogeneous examples, as it turns out) of compact, connected projective planes. We describe the full collineation groups of the classical planes, as well as various interesting subgroups, like motion groups with respect to polarities. In the case of the octonion plane, this comprises a complete and elementary description of some exceptional Lie groups (and of their actions on the octonion plane), including proofs of their simplicity; e.g. the full collineation group has type E_6, and the elliptic motion group is the compact group of type F_4.

Chapter 2 is a brief summary of notions and results concerning projective and affine planes, coordinates and collineations. It is meant as a reference for known facts which entirely belong to incidence geometry.

In Chapter 3 we study planes on the point set $\mathbb{R}^2$, with lines which are homeomorphic to the real line $\mathbb{R}$, so that each line is a curve in $\mathbb{R}^2$. This chapter is the most intuitive part of the book. If the parallel axiom is satisfied, that is, if we have an affine plane, then we can form the usual projective completion, which leads to a (topologically) 2-dimensional compact projective plane. These planes have been studied by Salzmann in 1957–1967 with remarkable success. He proved that the full collineation group Σ of such a plane is a Lie group of dimension at most 8, that the real projective plane $\mathcal{P}_2\mathbb{R}$ is characterized by the condition $\dim \Sigma > 4$, and that the Moulton planes are the only planes of this type with $\dim \Sigma = 4$. Furthermore, he explicitly classified all 2-dimensional compact projective planes with $\dim \Sigma = 3$. All these classification results are proved in Chapter 3.

In Chapter 4 we begin a systematic study of topological projective planes in general. Most results require compactness, and some results (like contractibility properties) are based on connectedness assumptions. Note that, for every prime p, the plane over the p-adic numbers $\mathbb{Q}_p$ provides an example of a compact, totally disconnected plane. We show that the four classical planes studied in Chapter 1 are precisely the compact, connected Moufang planes; Moufang planes are defined by a very strong homogeneity condition, which implies transitivity on triangles (and even on quadrangles). Furthermore, we prove that the group of all continuous collineations of a compact projective plane is always a locally compact group (with respect to the compact-open topology).

Chapter 5 deals with the algebraic topology of compact, connected projective planes of finite topological dimension. As Löwen has shown, the point spaces of these planes have the very same homology invariants as their classical counterparts considered in Chapter 1; moreover, the lines are homotopy equivalent to spheres. We obtain that the topological dimension of a line in such a plane is one of the numbers $1, 2, 4, 8$; the (point sets of the) corresponding planes have topological dimensions $2, 4, 8, 16$. The topological resemblance to classical planes has strong geometric consequences, which are discussed in Section 55 and in Chapters 6–8. In fact, these results determine a subdivision of the whole theory into four cases. In order to understand Chapters 6–8, it suffices to be acquainted with the main *results* of Chapter 5; the *methods* of proof in that chapter are not used in other chapters.

In Chapter 6 we consider compact, connected projective planes which are homogeneous in some sense. As indicated above, the idea of homogeneity plays a central rôle in this book. We prove that a compact, connected projective plane which admits an automorphism group transitive on points is isomorphic to one of the four classical planes treated in Chapter 1. This is a remarkable result; it says that for compact, connected projective planes, the Moufang condition is a consequence of transitivity on points. Furthermore, we consider groups of axial collineations and transitivity conditions for these groups, and we study planes which admit a classical motion group. Often, these homogeneity conditions are strong enough to allow an explicit classification of the planes in question. In Section 65 we employ the topological dimension of the automorphism group as a measure of homogeneity (this idea is fully developed in Chapters 3, 7, 8), and Section 66 is a short report on groups of projectivities in our context.

In Chapters 7 and 8 we determine all compact projective planes of dimension 4, 8 or 16 which admit an automorphism group of sufficiently large topological dimension. This approach leads first to the classical planes over $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, and then the most homogeneous non-classical planes appear in a systematic fashion. In contrast to Chapter 3, deeper methods are required, and proper translation planes arise.

In Chapter 7 we study compact projective planes of topological dimension 4; these planes are the topological relatives of the complex projective plane $\mathcal{P}_2\mathbb{C}$. We prove that the automorphism group Σ of such a plane $\mathcal{P}$ is a (real) Lie group of dimension at most 16, and that the complex projective plane is characterized by the condition $\dim \Sigma > 8$. This result is one of the highlights of the theory of 4-dimensional planes. If $\dim \Sigma \geq 7$, then $\mathcal{P}$ is a translation plane (up to duality) or a shift plane. All translation planes $\mathcal{P}$ with $\dim \Sigma \geq 7$ and all shift planes $\mathcal{P}$ with $\dim \Sigma \geq 6$ have been classified explicitly; Chapter 7 contains a classification of the translation planes with $\dim \Sigma \geq 8$ and of the shift planes with $\dim \Sigma \geq 7$. Finally, we show in Section 75 that only the complex plane admits a complex analytic structure.

The theory of 4-dimensional compact planes is distinguished from the theory of higher-dimensional compact planes, regarding both the phenomena and the methods. For instance, the class of shift planes appears only in low dimensions. In higher dimensions, special tools connected with the recognition and handling of low-dimensional manifolds are not available.

Chapter 8 deals with compact projective planes of topological dimension 8 or 16, that is, with the relatives of the quaternion plane $\mathcal{P}_2\mathbb{H}$ or of the octonion plane $\mathcal{P}_2\mathbb{O}$. In some parts of this chapter, the results are only surveyed, with references to the literature. Again, classification results on planes admitting automorphism groups of large dimension constitute the main theme. It turns out that such planes are often translation planes up to duality. They carry a vector space structure, whence special tools become available. Accordingly, the theory of translation planes and the classification of the most homogeneous ones form a theory on their own. Fundamental results of this theory are developed in Section 81; in Section 82, the classification of all 8-dimensional compact translation planes $\mathcal{P}$ satisfying $\dim \mathrm{Aut}\,\mathcal{P} \geq 17$ and of all 16-dimensional compact translation planes $\mathcal{P}$ satisfying $\dim \mathrm{Aut}\,\mathcal{P} \geq 38$ is presented.

In the following sections, classification results of this kind are extended to 8- and 16-dimensional compact planes in general. For reasons of space, the results often are not proved in their strongest form. Salzmann [81a, 90] proved, on the basis of Hähl [78], that the quaternion plane $\mathcal{P}_2\mathbb{H}$ is the only compact projective plane of dimension 8 such that $\dim \mathrm{Aut}\,\mathcal{P} > 18$. In Section 84, this result is proved under the stronger assumption $\dim \mathrm{Aut}\,\mathcal{P} \geq 23$. Similarly, the octonion plane is known to be the only compact projective plane $\mathcal{P}$ of dimension 16 such that $\dim \mathrm{Aut}\,\mathcal{P} > 40$, see Salzmann [87], Hähl [88]. In Section 85, we characterize the octonion plane by the stronger condition that $\dim \mathrm{Aut}\,\mathcal{P} \geq 57$. The proofs of these characterization results make use of the corresponding characterization results for translation planes.

In Section 86, we construct and characterize the compact Hughes planes of dimensions 8 and 16. They form two one-parameter families of planes with rather singular properties. Section 87 contains basic results indicating a viable route towards an extension of the classification results presented here. This should help the reader to go beyond the limitations of the exposition here; moreover, it may serve as a guide to future research. Among other things, these results explain the special rôle played in the classification by translation planes on the one hand and by Hughes planes on the other hand.

The final Chapter 9 is an appendix. Here we collect a number of results from topology, and we give a systematic outline of Lie theory, as required in this book. In this chapter we usually do not give proofs, but rather refer to the literature (with an attempt to give references also for folklore results). The topics covered in Chapter 9 include the topological characterization of Lie groups (Hilbert's fifth problem), and the structure and the classification of (simple) Lie groups; in fact, we require only results for groups of dimension at most 52. Furthermore we report on real linear representations of almost simple Lie groups, and we list all irreducible representations of these groups on real vector spaces of dimension at most 16. Finally, we deal with various classification results on (not necessarily compact) transformation groups.

*

This book gives a systematic account of many results which are scattered in the literature. Some results are presented in improved form, or with simplified proofs, others only in weakened versions. A few of the more recent results are only mentioned, because their proofs appear to be too complex to be included here. However, we hope to provide a convenient introduction to compact, connected projective planes, as well as a sound foundation for future research in this area.

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Tübingen, September 1994 The authors

Contents

Chapter 1

The classical planes

In this introductory chapter, the classical examples of topological projective planes will be presented and studied. These are the planes over the following coordinate domains: the field $\mathbb{R}$ of real numbers, the field $\mathbb{C}$ of complex numbers, the skew field $\mathbb{H}$ of quaternions, and the alternative field $\mathbb{O}$ of octonions. In a later chapter, see (42.7), it will turn out that these classical planes are the only locally compact, connected topological planes which either satisfy Desargues' law (valid in the planes over the fields $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$), or at least possess the Moufang property (which holds in the plane over $\mathbb{O}$, as well).

A careful elementary study of the latter plane, the plane over the octonions $\mathbb{O}$, is a particular objective of this chapter. This comprises the investigation of the group Σ of its collineations, and of certain subgroups of Σ, including the elliptic motion group Φ. The group Σ is known to be the real exceptional simple Lie group $\mathrm{E}_6(-26)$, and Φ is the compact exceptional simple Lie group of type F_4. One of the aims of our presentation is to give a detailed study of these groups which is mainly based on incidence geometry, and which makes only marginal references to the theory of simple Lie groups (just what we need for the identification of these groups among the simple Lie groups, for instance). In particular, the simplicity of these groups is proved without recourse to Lie group theory. It seems to us that this approach offers a pleasant road to an intimate understanding of $\mathrm{E}_6(-26)$ and its distinguished subgroups.

It should be said here that the material of this chapter is classical and well known. Distinctive features of our presentation are, we believe, the particular blend of arguments and techniques in dealing with the octonion plane, and our systematic use of methods from incidence geometry.

The subject is rooted in incidence geometry on the one hand, and it has aspects which are important for topology and Lie group theory on the other hand. There may be readers whose interest is primarily on one side and who are less familiar with the other side. The presentation in this chapter is therefore intended to be accessible with few prerequisites from either side. In particular, Sections 12 and 13 about the affine planes over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, and about the projective planes over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, may serve at the same time as a concrete introduction to some basic notions about affine and projective planes and their collineations, before one turns to Chapter 2, which is a short and rather abstract summary on projective planes in general. We trust that the more experienced reader will find it easy to skip the extra explanations implied by this approach.

In Section 11, the Cayley–Dickson process is applied to the field $\mathbb{R}$ in order to construct the algebras $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. We present their characteristic properties and study their automorphism groups. Section 12 is concerned with the affine planes over these algebras. For the planes over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, a complete description of their (affine) collineation groups is given. For the plane over $\mathbb{O}$, this is less easy due to non-associativity, and a full treatment is therefore postponed until Sections 15–18. However, first results will be given in the octonion case, as well. They concern special collineations, which are in close connection with the Moufang identities in $\mathbb{O}$, and the description of all collineations fixing the coordinate axes; the latter topic is closely related to the triality principle. It should be pointed out that in our presentation the mentioned algebraic laws, viz. the Moufang identities and the triality principle, are obtained as a by-product of the geometrical reasoning, whereas they are usually proved by algebraic means and then employed for geometrical conclusions, among other things. Section 13 is devoted to a description of the projective planes over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ by homogeneous coordinates (a tool not available for the non-associative algebra $\mathbb{O}$) and to a study of their collineation groups. The fundamental theorem of projective geometry describing all collineations is proved, and the elliptic and hyperbolic motion groups are presented. In Section 14, these planes are studied as topological planes; the topological structure is introduced using homogeneous coordinates.

A step of major conceptual and technical importance is the study of the geometry of a projective line in Section 15. A projective line is represented as a quadric of index 1 in a certain real vector space. This interpretation is valid over $\mathbb{O}$ as well as over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. It provides a description in algebraic terms of the Möbius geometry ('conformal geometry') on a projective line, which was introduced and used by Tits [53] in his fundamental paper on the octonion plane. The Möbius geometry helps to understand the stabilizer of a line in the collineation group, because the action of that stabilizer on the given line respects this additional geometric structure. This fact is an important tool for the study of the collineation group in the octonion plane.

Sections 16 through 18 are devoted to a close study of the projective plane over $\mathbb{O}$. In Section 16, 'Veronese coordinates' are introduced as a substitute for homogeneous coordinates, which are not available over $\mathbb{O}$. (Veronese coordinates could equally well be used for the planes over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, so that a unified treatment would result.) The topological properties of the octonion plane can now be derived in virtually the same way as for the other classical planes.

In Section 17, the group Σ of all collineations of the octonion plane is determined. We show that Σ is generated by elations, and we deduce that it is a simple group. Incidentally, we obtain that all collineations are induced by appropriate linear transformations acting on Veronese coordinates; this result is an octonion analogue of the fundamental theorem of projective geometry over (not necessarily commutative) fields. Then we study the stabilizer of a triangle. In affine language, such a stabilizer has already been described in Section 12 as the stabilizer of the coordinate axes in the affine collineation group. We now show that its maximal compact, connected subgroup, which is a normal subgroup, is the universal covering group $\mathrm{Spin}_8\mathbb{R}$ of $\mathrm{SO}_8\mathbb{R}$, and we exhibit the triality automorphism of $\mathrm{Spin}_8\mathbb{R}$. In the stabilizer of a degenerate quadrangle, we accordingly find the universal covering group $\mathrm{Spin}_7\mathbb{R}$ of $\mathrm{SO}_7\mathbb{R}$. One thus obtains a concrete geometric understanding of these groups and of the homogeneous space $\mathrm{Spin}_7\mathbb{R}/\mathrm{G}_2 \approx \mathbb{S}_7$.

In Section 18, we study the groups of collineations which commute with the standard elliptic polarity or with the standard hyperbolic polarity of the octonion plane, the so-called elliptic and hyperbolic motion groups. We prove their simplicity by a geometric argument; using their dimensions, they are then easily identified as real simple Lie groups of exceptional type F_4. The elliptic motion group turns out to be a maximal compact subgroup of the full collineation group; this fact finally allows us to recognize the latter among the simple Lie groups as a real form of type E_6. A crucial step in the study of the motion groups is the analysis of the stabilizer of a point. For an arbitrary point in the elliptic case and for an interior point in the hyperbolic case, the stabilizer is isomorphic to the universal covering group $\mathrm{Spin}_9\mathbb{R}$ of $\mathrm{SO}_9\mathbb{R}$; the covering map $\mathrm{Spin}_9\mathbb{R} \to \mathrm{SO}_9\mathbb{R}$ has a very simple geometric description. As by-products of this analysis one obtains that the homogeneous space $\mathrm{F}_4/\mathrm{Spin}_9\mathbb{R}$ is homeomorphic to the point space of the projective octonion plane, and that $\mathrm{Spin}_9\mathbb{R}/\mathrm{Spin}_7\mathbb{R} \approx \mathbb{S}_{15}$. The section closes with a classification of all polarities of the octonion plane, up to equivalence; besides the standard elliptic polarity and the standard hyperbolic polarity, there appears just one further possibility.

We close this summary by a list of references to places in this chapter where descriptions of various classical or exceptional groups or further information about them may be found.

$SO_3\mathbb{R}$	(11.22 through 25), (11.29)	$SU_2\mathbb{C}$	(11.26)
$O_3\mathbb{R}$	(13.13)	$U_2\mathbb{C}$	(13.14)
$SO_4\mathbb{R}$, $O_4\mathbb{R}$	(11.22 and 23)	$SU_3\mathbb{C}$	(11.34 and 35)
$SO_8\mathbb{R}$, $O_8\mathbb{R}$	(11.22 and 23), (12.18)	$PU_3\mathbb{C}$	(13.13 and 15)
$SO_9\mathbb{R}$	(18.8)	$PU_3(\mathbb{C}, 1)$	(13.13 and 17)
$Spin_3\mathbb{R}$	(11.26)	$U_2\mathbb{H}$	(13.14), (18.9)
$Spin_5\mathbb{R}$	(18.9)	$PU_3\mathbb{H}$	(13.13 and 15)
$Spin_7\mathbb{R}$	(17.14 and 15)	$PU_3(\mathbb{H}, 1)$	(13.13 and 17)
$Spin_8\mathbb{R}$	(17.13 and 16)	$PGL_2\mathbb{F}$	(12.12), (15.6)
$Spin_9\mathbb{R}$	(18.8, 13 and 16)	$PGL_3\mathbb{F}$	(13.4) ($\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$)
$O_3(\mathbb{R}, 1)$	(13.13 and 17), (15.6)	$G_2 = \mathrm{Aut}\,\mathbb{O}$	(11.30 through 33), (17.15)
$PSO_4(\mathbb{R}, 1)$	(15.6)	$F_4 = F_4(-52)$	(18.10, 15 and 16)
$PSO_6(\mathbb{R}, 1)$	(15.6)	$F_4(-20)$	(18.23 and 26)
$O'_9(\mathbb{R}, 1)$	(18.22)	$E_6(-26)$	(18.19)
$PSO_{10}(\mathbb{R}, 1)$	(15.6)		

11 The classical division algebras

In this section, we apply the Cayley–Dickson process to the field $\mathbb{R}$ of real numbers in order to construct the field $\mathbb{C}$ of complex numbers, the skew field $\mathbb{H}$ of quaternions and the (non-associative) alternative field $\mathbb{O}$ of octonions, and we derive the characteristic properties of these algebras. They will be called the classical division algebras; notice that, in our terminology, a division algebra is not necessarily associative. The multiplication of these algebras provides useful descriptions of certain orthogonal groups. This will be explained and applied for a study of the automorphism groups $\mathrm{Aut}\,\mathbb{H}$ and $\mathrm{Aut}\,\mathbb{O}$. Little will be said about $\mathrm{Aut}\,\mathbb{C}$, as this is more a matter of field theory.

11.1 The Cayley–Dickson process serves to construct a sequence $\mathbb{F}_m$ of $\mathbb{R}$-algebras, each furnished with an involutory antiautomorphism $a \mapsto \bar{a}$, called *conjugation*. The construction proceeds inductively in the following way: One starts with

$$\mathbb{F}_0 = \mathbb{R}, \quad \bar{a} = a \quad \text{for } a \in \mathbb{R}\,;$$

then, assuming that $\mathbb{F}_{m-1}$ $(m \geq 1)$ with its conjugation has been constructed, one puts

$$\mathbb{F}_m := \mathbb{F}_{m-1} \times \mathbb{F}_{m-1}\,,$$

with addition, multiplication, and conjugation defined by

$$\begin{aligned}(a,b)+(c,d) &:= (a+c,b+d)\\ (a,b)(c,d) &:= (ac-\bar{d}b, da+b\bar{c})\\ \overline{(a,b)} &:= (\bar{a},-b)\end{aligned}$$

(for $a,b,c,d \in \mathbb{F}_{m-1}$). Obviously, the dimension of $\mathbb{F}_m$ over $\mathbb{R}$ is 2^m.

For $m=1$, this is the familiar definition of the field $\mathbb{C}$ of complex numbers, $\mathbb{F}_2 =: \mathbb{H}$ is the algebra of Hamilton's *quaternions*, and $\mathbb{F}_3 =: \mathbb{O}$ is the algebra of *octonions* (or Cayley numbers). The further steps in the ladder will not be of interest to us, because they lead to algebras having zero divisors (11.17).

Via the map $\mathbb{F}_{m-1} \to \mathbb{F}_m : a \mapsto (a,0)$ we may identify $\mathbb{F}_{m-1}$ with a subalgebra of $\mathbb{F}_m$. In this way, $\mathbb{R} = \mathbb{F}_0$ is a central subfield of all the algebras $\mathbb{F}_m$, and we have inclusions

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}\,.$$

It is easily verified by induction that conjugation is indeed an antiautomorphism of $\mathbb{F}_m$, and that its fixed elements are precisely the real numbers:

11.2. $\{\, x \in \mathbb{F}_m \mid \bar{x} = x \,\} = \mathbb{R}$. □

11.3 The norm form. For $x = (a,b) \in \mathbb{F}_m$, $a,b \in \mathbb{F}_{m-1}$, one computes that

$$\begin{aligned}x\bar{x} &= (a,b)(\bar{a},-b) = (a\bar{a}+\bar{b}b, 0)\\ \bar{x}x &= (\bar{a},-b)(a,b) = (\bar{a}a+\bar{b}b,0)\,.\end{aligned}$$

By induction on m, one obtains that $x\bar{x} = \bar{x}x$, and that for $x \neq 0$ this is a positive element of the subfield $\mathbb{R}$. This positive real number will also be written as

$$\|x\|^2 := x\bar{x} = \bar{x}x\,.$$

The map $x \mapsto \|x\|^2$ is a positive definite quadratic form (the so-called *norm form*) on the $\mathbb{R}$-vector space $\mathbb{F}_m$. The associated bilinear form is

$$\langle\, x \mid y \,\rangle := \|x+y\|^2 - \|x\|^2 - \|y\|^2 = \bar{x}y + \bar{y}x\,,$$

as is easily computed. It is a positive definite inner product; note that

$$\langle\, x \mid x \,\rangle = 2\|x\|^2\,.$$

Since conjugation is involutory, one has

$$\begin{aligned}\|\bar{x}\|^2 &= \|x\|^2\\ \langle\, \bar{x} \mid \bar{y} \,\rangle &= \langle\, x \mid y \,\rangle\,.\end{aligned}$$

Our initial calculation shows that the norms in $\mathbb{F}_m$ and in $\mathbb{F}_{m-1}$ are connected as follows:

$$\|(a, b)\|^2 = \|a\|^2 + \|b\|^2$$

for $a, b \in \mathbb{F}_{m-1}$. In particular, the subspaces $\mathbb{F}_{m-1} \times \{0\}$ and $\{0\} \times \mathbb{F}_{m-1}$ of $\mathbb{F}_m = \mathbb{F}_{m-1} \times \mathbb{F}_{m-1}$ are orthogonal with respect to the inner product.

11.4 Inverses. For a nonzero element $x \in \mathbb{F}_m$, one easily verifies that

$$x^{-1} := \|x\|^{-2}\bar{x}$$

is a two-sided multiplicative inverse, and that

$$\|x^{-1}\|^2 = (\|x\|^2)^{-1} .$$

11.5 $\mathbb{F}_m$ as a quadratic algebra. By (11.2), one immediately obtains that for every $x \in \mathbb{F}_m$

$$x + \bar{x} \in \mathbb{R}, \quad \text{so that} \quad \bar{x} \in \mathbb{R} + \mathbb{R}x .$$

Consequently,

$$x^2 = (x + \bar{x})x - \|x\|^2 \in \mathbb{R} + \mathbb{R}x .$$

In particular, every element of $\mathbb{F}_m$ satisfies a quadratic equation with real coefficients.

11.6 Pure elements. The subspace

$$\operatorname{Pu}\mathbb{F}_m := \{\, x \in \mathbb{F}_m \mid \bar{x} = -x \,\} \leq \mathbb{F}_m$$

of *pure* elements is the orthogonal complement of $\mathbb{R} = \mathbb{R} \cdot 1$ with respect to the inner product. One infers directly from the quadratic equation in (11.5) that this subspace can also be described as

$$\operatorname{Pu}\mathbb{F}_m = \left\{\, x \in \mathbb{F}_m \;\middle|\; x^2 \in \mathbb{R}, x^2 \leq 0 \,\right\} .$$

More precisely,

(1) $$x^2 = -\|x\|^2 \quad \text{if, and only if, } x \in \operatorname{Pu}\mathbb{F}_m .$$

For $u, v \in \operatorname{Pu}\mathbb{F}_m$, we have

(2) $$\langle\, u \mid v \,\rangle = 0 \iff uv = -vu \iff uv \in \operatorname{Pu}\mathbb{F}_m ;$$

this is clear from the definitions, since $\overline{uv} = \bar{v}\bar{u} = (-v)(-u) = vu$.

We now deal with the associativity properties of our algebras. $\mathbb{F}_1 = \mathbb{C}$ is associative and commutative, and $\mathbb{F}_2 = \mathbb{H}$ is associative; $\mathbb{F}_3 = \mathbb{O}$ is not associative, but weak forms of associativity survive, which we shall now consider. Our discussion will also cover the facts mentioned about $\mathbb{C}$ and $\mathbb{H}$. For $m \geq 4$, little associativity is left in $\mathbb{F}_m$, see (11.17); the following fact is nevertheless quite general.

11.7 Mono-associativity. *For $x \in \mathbb{F}_m \setminus \mathbb{R}$, the span $\mathbb{R} + \mathbb{R}x$ of 1 and x is an associative and commutative subalgebra of $\mathbb{F}_m$, and this subalgebra is isomorphic to $\mathbb{C}$.*

Proof. The span $A = \mathbb{R} + \mathbb{R}x$ intersects the hyperplane $\operatorname{Pu}\mathbb{F}_m$ in a 1-dimensional subspace $\mathbb{R}u$ with $u \in \operatorname{Pu}\mathbb{F}_m$ and $\|u\|^2 = 1$; then $\{1, u\}$ is a basis of A. By (11.6), we have $u^2 = -\|u\|^2 = -1 \in \mathbb{R}$. One now easily verifies that A is a subalgebra, which is associative and commutative. Clearly, it is isomorphic to $\mathbb{C}$. □

The key to stronger associativity properties of $\mathbb{F}_m$ for $m \leq 3$ is

11.8 Alternativity. *Assume that $\mathbb{F}_{m-1}$ is associative. Then $\mathbb{F}_m$ has the following property:*

(1) $$\bar{x}(xy) = (\bar{x}x)y = \|x\|^2 y$$

for all $x, y \in \mathbb{F}_m$, and hence, upon conjugation,

(2) $$(yx)\bar{x} = y(x\bar{x}) = \|x\|^2 y\,.$$

Equivalently, $\mathbb{F}_m$ is alternative; i.e., the following identities hold:

$$x(xy) = x^2 y \quad \text{and} \quad (yx)x = yx^2.$$

Proof. Let $x = (a, b)$, $y = (c, d)$ for $a, b, c, d \in \mathbb{F}_{m-1}$. By direct computation and the associativity of $\mathbb{F}_{m-1}$ one obtains that $\bar{x}(xy) = (\bar{a}ac - \bar{a}\bar{d}b + \bar{a}\bar{d}b + c\bar{b}b,\ da\bar{a} + b\bar{c}\bar{a} - b\bar{c}\bar{a} + b\bar{b}d)$. Since $a\bar{a} = \bar{a}a = \|a\|^2$ and $b\bar{b} = \bar{b}b = \|b\|^2$ are scalars, and since $\|a\|^2 + \|b\|^2 = \|x\|^2$, we conclude that $\bar{x}(xy) = (\|a\|^2 c + \|b\|^2 c,\ \|a\|^2 d + \|b\|^2 d) = \|x\|^2 (c, d) = \|x\|^2 y$. This proves (1), and (2) follows upon conjugation. The alternative laws are identical to (1) and (2) if x is a pure element, i.e., if $x = -\bar{x}$. The general case is dealt with by decomposing x into a scalar in $\mathbb{R}$ and a pure element. □

From (11.8), we now derive the associativity of $\mathbb{H}$ together with the following weaker associativity property of $\mathbb{O}$, which includes alternativity:

11.9 Biassociativity. *For all $x \in \mathbb{O} \setminus \mathbb{R}$ and $y \in \mathbb{O} \setminus (\mathbb{R} + \mathbb{R}x)$, the span $\mathbb{R} + \mathbb{R}x + \mathbb{R}y + \mathbb{R}xy$ is an associative subalgebra of $\mathbb{O}$ isomorphic to $\mathbb{H}$.*

Remark. Taken together with (11.7), this implies that $\mathbb{O}$ is *biassociative* in the sense that any two elements $x, y \in \mathbb{O}$ belong to an associative subalgebra (containing 1). By (11.4 and 5), then, this subalgebra also contains $\bar{x}$, $\bar{y}$, $x^{-1} = \|x\|^{-2}\bar{x}$, and $y^{-1} = \|y\|^{-2}\bar{y}$. As a consequence, brackets are of no importance in multiple products whose factors are among x, y, $\bar{x}$, $\bar{y}$, x^{-1}, and y^{-1}, or are scalars in $\mathbb{R}$.

Proof of (11.9). Intersecting $\mathbb{R}+\mathbb{R}x$ and $\mathbb{R}+\mathbb{R}x+\mathbb{R}y$ with the hyperplane $\operatorname{Pu}\mathbb{O}$, we obtain elements $u, v \in \operatorname{Pu}\mathbb{O}$ for which $\|u\|^2 = 1 = \|v\|^2$ and $\langle\, u \mid v\,\rangle = 0$, and such that $x = r_0 + r_1 u$, $y = s_0 + s_1 u + s_2 v$, for suitable scalars $r_\nu, s_\nu \in \mathbb{R}$ ($\nu = 0, 1, 2$) with $r_1 \neq 0, s_2 \neq 0$. By (11.6), we have $u^2 = -\|u\|^2 = -1$, and therefore $\mathbb{R}+\mathbb{R}x+\mathbb{R}y+\mathbb{R}xy = \mathbb{R}+\mathbb{R}u+\mathbb{R}v+\mathbb{R}uv$. It thus suffices to prove the following statement, which is formulated so as to give information about $\mathbb{F}_2 = \mathbb{H}$, as well.

11.10 Proposition. *Assume that $2 \leq m \leq 3$, and let $u, v \in \operatorname{Pu}\mathbb{F}_m$ be such that*

(1) $$\|u\|^2 = 1 = \|v\|^2 \quad \textit{and} \quad \langle\, u \mid v\,\rangle = 0\,.$$

Then the product $w := uv$ satisfies

(2) $$w \in \operatorname{Pu}\mathbb{F}_m\,, \quad \langle\, u \mid w\,\rangle = 0 = \langle\, v \mid w\,\rangle\,, \quad \textit{and} \quad \|w\|^2 = 1\,.$$

Moreover, we have the following multiplication table:

	u	v	w
u	-1	w	$-v$
v	$-w$	-1	u
w	v	$-u$	-1

The span $\mathbb{R}+\mathbb{R}u+\mathbb{R}v+\mathbb{R}w$ is an associative subalgebra of $\mathbb{F}_m$ isomorphic to $\mathbb{H}$.

Terminology. A triple u, v, w with these properties is called a *Hamilton triple.*

Proof. At a first stage, we shall assume in addition that $\mathbb{F}_m$, $m = 2, 3$, is already known to be alternative. This assumption will later prove to be innocuous. From (1) and (11.6(2)), we infer $u^2 = -1 = v^2$, and $w \in \operatorname{Pu}\mathbb{F}_m$. Assuming alternativity one obtains

$$uw = u(uv) = u^2 v = -v \quad \text{and} \quad wv = (uv)v = uv^2 = -u\,.$$

By (11.6(2)) again, it now follows that u, v, and w are mutually orthogonal with respect to $\langle\ \mid\ \rangle$ and anticommuting. Using alternativity once more we obtain that $w^2 u = w(wu) = wv = -u$. Since w^2 is a scalar by (11.6(1)), this means that $w^2 = -1$ and $\|w\|^2 = 1$. Thus the multiplication table is as asserted, and it follows that

$$A := \mathbb{R}+\mathbb{R}u+\mathbb{R}v+\mathbb{R}w$$

is a subalgebra. Next we remark that A is *flexible*, which means that

$$a(ba) = (ab)a \tag{3}$$

for all $a, b \in A$. This follows immediately from alternativity by expanding the two sides of the equation $(a+b)((a+b)a) = (a+b)^2 a$.

For the proof of associativity of A, it now suffices to show that triple products composed of u, v and w are associative. If two of the factors of such a triple product coincide, then this is a consequence of alternativity and flexibility. As to triple products with different factors, we have, for instance,

$$(uv)w = w^2 = -1 = u^2 = u(vw) \quad \text{and} \quad (vu)w = -w^2 = 1 = -v^2 = v(uw)\ .$$

All the other triple products with different factors are obtained from these by cyclic permutation of u, v, and w, under which the corresponding equalities remain valid because of the symmetry of our multiplication table. Thus A is associative.

We must now dispose of our supplementary hypothesis of alternativity. In $\mathbb{F}_2$, this is easy. Since $\mathbb{F}_1 = \mathbb{C}$ is associative, $\mathbb{F}_2$ is alternative by (11.8), and so the above arguments are valid in $\mathbb{F}_2$. In this case, $1, u, v, w$ span the 4-dimensional algebra $\mathbb{F}_2 = \mathbb{H}$, since they are linearly independent by (1) and (2); thus $\mathbb{F}_2 = A$. In particular, from the above, $\mathbb{F}_2 = \mathbb{H}$ is now known to be associative.

Using (11.8) again, we infer that also $\mathbb{F}_3 = \mathbb{O}$ is alternative. Thus our previous arguments apply here, as well, with the result that the subalgebra A spanned by 1 and by elements u, v, w of the specified kind is associative. Any two such algebras are isomorphic, since multiplication is entirely determined by the given multiplication table. As $\mathbb{H}$ is also spanned by such a basis, every such subalgebra of $\mathbb{F}_3 = \mathbb{O}$ is isomorphic to $\mathbb{H}$. □

Since, by (11.4), there are multiplicative inverses, associativity of one of our algebras implies that it satisfies all the axioms of a (not necessarily commutative) field. From the preceding discussion, in particular from (11.10), we thus infer the following.

11.11. $\mathbb{C} = \mathbb{F}_1$ *and* $\mathbb{H} = \mathbb{F}_2$ *are associative;* $\mathbb{C}$ *is a commutative field, and* $\mathbb{H}$ *is a skew field.*

(The non-commutativity of $\mathbb{F}_2$ is obvious from the multiplication table in (11.10).) □

It was already noted in the proof of (11.10) that, by (11.8), we now know $\mathbb{F}_3 = \mathbb{O}$ to be alternative. This has the following consequence.

11.12. *For $m \le 3$, the algebra $\mathbb{F}_m$ has no zero divisors, and therefore is a division algebra in the sense that, for $a \neq 0$, the $\mathbb{R}$-linear maps $x \mapsto ax$ and $x \mapsto xa$ are bijective.*

Indeed, if $xy = 0$, then $\|x\|^2 y = \bar{x}(xy) = 0$, so that $x = 0$ or $y = 0$. □

We summarize the properties of $\mathbb{O}$ which we have now obtained (11.8, 9 and 12).

11.13. *$\mathbb{O}$ is an* **alternative field**, *i.e., an alternative division algebra, and it is biassociative.* □

11.14 Multiplicativity of the norm. *If $m \le 3$, i.e., if $\mathbb{F}_m$ is $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$, then*

$$\|xy\|^2 = \|x\|^2 \|y\|^2 \tag{1}$$

for $x, y \in \mathbb{F}_m$. Moreover, for each $a \in \mathbb{F}_m$,

$$\langle\, x \mid \bar{a}y \,\rangle = \langle\, ax \mid y \,\rangle \quad \textit{and} \quad \langle\, x \mid y\bar{a} \,\rangle = \langle\, xa \mid y \,\rangle . \tag{2}$$

In particular, for $p \in \operatorname{Pu}\mathbb{F}_m$, we have

$$\langle\, px \mid x \,\rangle = 0\,. \tag{3}$$

Remark. For $m \ge 4$, these equalities are not true, see (11.17).

Proof. By (11.9) and (11.11), all the algebras in question are at least biassociative. Therefore, $\|xy\|^2 = \overline{(xy)}(xy) = \bar{y}(\bar{x}x)y = \|x\|^2\bar{y}y = \|x\|^2\|y\|^2$, proving (1). For (2), we may assume $a \neq 0$. Put $z = a^{-1}y$, so that $az = a(a^{-1}y) = y$. Using (1), we obtain $\langle\, ax \mid y \,\rangle = \langle\, ax \mid az \,\rangle = \|ax + az\|^2 - \|ax\|^2 - \|az\|^2 = \|a\|^2\big(\|x+z\|^2 - \|x\|^2 - \|z\|^2\big) = \|a\|^2\langle\, x \mid z \,\rangle = \langle\, x \mid \|a\|^2 z \,\rangle = \langle\, x \mid (\bar{a}a)z \,\rangle = \langle\, x \mid \bar{a}(az) \,\rangle = \langle\, x \mid \bar{a}y \,\rangle$. The first equality of (2) is thus proved. The second is obtained by applying conjugation, which preserves the inner product (11.3). Identity (3) follows from (2), since $\bar{p} = -p$. □

The following result is a motivation ex post of the Cayley–Dickson process (11.1).

11.15 Proposition: Constructing $\mathbb{O}$ from quaternion subfields. *Let H be a subalgebra of $\mathbb{O}$ isomorphic to $\mathbb{H}$, and let $z \in \operatorname{Pu}\mathbb{O}$ be of unit length $\|z\|^2 = 1$ and orthogonal to H. Then the 4-dimensional $\mathbb{R}$-linear subspace Hz is orthogonal to H, so that the $\mathbb{R}$-vector space $\mathbb{O}$ decomposes into the direct sum*

$$\mathbb{O} = H \oplus Hz\,.$$

For $a, b, c, d \in H$, we have

$$(a + bz)(c + dz) = (ac - \bar{d}b) + (da + b\bar{c})z\,.$$

Proof. Clearly, H is invariant under conjugation (11.5). By (11.14(2)), we have $\langle a \mid bz \rangle = \langle \bar{b}a \mid z \rangle = 0$, so that H and Hz are orthogonal. Using alternativity (11.8) we obtain, for $x, y, w \in \mathbb{O}$,

$$\|x+y\|^2 \cdot w = (x+y)((\bar{x}+\bar{y})w) = (\|x\|^2 + \|y\|^2)w + x(\bar{y}w) + y(\bar{x}w) .$$

Now $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ whenever $\langle x \mid y \rangle = 0$, so that

(*) if $\langle x \mid y \rangle = 0$, then $x(\bar{y}w) = -y(\bar{x}w)$.

In particular, for $w = 1$,

(**) if $y \in \operatorname{Pu}\mathbb{O}$ and $\langle x \mid y \rangle = 0$, then $xy = y\bar{x}$.

Using these pieces of information together with alternativity, we may now evaluate the terms obtained from expanding the left hand side of the asserted product formula as follows.

$$\begin{aligned}
(bz)c &= \bar{c}(bz) = \bar{c}(z\bar{b}) = -\bar{z}(c\bar{b}) = -(b\bar{c})\bar{z} = (b\bar{c})z ,\\
a(dz) &= a(z\bar{d}) = -\bar{z}(\bar{a}\bar{d}) = z(\overline{da}) = (da)z ,\\
(bz)(dz) &= -\bar{d}((\bar{z}\bar{b})z) = -\bar{d}((b\bar{z})z) = -\bar{d}(b\|z\|^2) = -\bar{d}b .
\end{aligned}$$ □

11.16 Cayley triples are triples $u, v, z \in \operatorname{Pu}\mathbb{O}$, $\|u\|^2 = \|v\|^2 = \|z\|^2 = 1$ such that u and v are mutually orthogonal, and such that z is orthogonal to u, v and uv.

This notion is symmetric in the sense that a permutation of a Cayley triple is another Cayley triple. For example, by (11.15 and 10), we obtain the following identities.

(1) $$u(vz) = (vu)z = -(uv)z ,$$

(2) $$(vz)u = -(vu)z .$$

Thus, in particular, $u(vz) = -(vz)u$. From (11.6(2)) one now infers that the pure element vz is orthogonal to u. In passing, we note that (1) shows $\mathbb{O}$ to be non-associative.

By (11.10 and 9), for a given Cayley triple u, v, z, the pure octonions u, v and $w := uv$ form a Hamilton triple, and together with 1 they span a subalgebra H isomorphic to $\mathbb{H}$. Furthermore, according to (11.15), the elements 1, u, v, w, z, uz, vz, wz constitute a basis of $\mathbb{O}$ as a vector space over $\mathbb{R}$. From (11.10 and 15), it is easy to compute a multiplication table for this basis; even without explicit computation, the following assertion is obvious.

For all Cayley triples u, v, z one obtains the same multiplication table with respect to the basis 1, u, v, w, z, uz, vz, $(uv)z$.

In particular, this shows that, for any two Cayley triples, the $\mathbb{R}$-linear transformation of $\mathbb{O}$ mapping the basis obtained in this way from the first Cayley triple onto the basis corresponding to the second Cayley triple is an automorphism of $\mathbb{O}$. We thus have proved the following.

For any two Cayley triples of $\mathbb{O}$, there is a unique automorphism of $\mathbb{O}$ mapping the first Cayley triple onto the second.

This observation may be used to reduce the effort in computing the multiplication table mentioned above.

The *standard Cayley triple* and the associated basis of $\mathbb{O}$ are obtained as follows. Let $i \in \mathbb{C}$ be an 'imaginary unit', that is, $i \in \operatorname{Pu}\mathbb{C}$ with $i^2 = -1$. In $\mathbb{H} = \mathbb{C} \times \mathbb{C}$, one usually considers the basis

$$1 = (1, 0)\,, \quad i = (i, 0)\,, \quad j := (0, 1)\,, \quad k := ij = (i, 0)(0, 1) = (0, i)\,.$$

The triple i, j, k is clearly a Hamilton triple as in (11.10). In $\mathbb{O} = \mathbb{H} \times \mathbb{H}$, by identifying $\mathbb{H}$ with the subalgebra $\mathbb{H} \times \{0\}$, we find these elements again, namely

$$i \mathrel{\widehat{=}} (i, 0)\,, \quad j \mathrel{\widehat{=}} (j, 0)\,, \quad k \mathrel{\widehat{=}} (k, 0)\,.$$

They are elements of $\operatorname{Pu}\mathbb{O}$. Putting

$$l = (0, 1) \in \operatorname{Pu}\mathbb{O}\,,$$

we note that i, j, l is a Cayley triple, and we compute the following products.

$$\begin{aligned} il &= (i, 0)(0, 1) &&= (0, i)\,, \\ jl &= (j, 0)(0, 1) &&= (0, j)\,, \\ kl &= (k, 0)(0, 1) &&= (0, k)\,. \end{aligned}$$

A discussion and presentation of the complete multiplication table for the basis $1, i, j, k, l, il, jl, kl$ of $\mathbb{O}$ may be found for instance in Porteous [81] Chap. 14, p. 277 ff. Following an idea of Freudenthal [85] 1.5.13, p. 19, one may represent this multiplication table graphically as shown in Figure 11a, using the projective plane with 7 points.

Figure 11a is to be interpreted as follows: If basis elements a, b, c are on a line of this projective plane, then $ab = \pm c$, the sign depending on whether or not the cyclic order of (a, b, c) matches with the orientation indicated in the diagram. For example, $ij = k$ but $ji = -k$. (When comparing this to Freudenthal, loc. cit., one should note that Freudenthal uses a different basis, so that the orientations implicit in his diagram do not completely agree with ours. Also, there is a misprint; e_4e_7 should be e_3.)

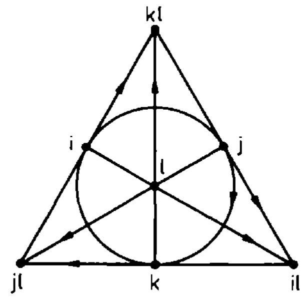

Figure 11a

11.17 Warning. If $m \geq 4$, then $\mathbb{F}_m$ has zero divisors. Consequently, $\mathbb{F}_m$ is not alternative, and the norm in $\mathbb{F}_m$ is not multiplicative.

It suffices to show this for $\mathbb{F}_4 \subseteq \mathbb{F}_m$. Let u, v, z be a Cayley triple of $\mathbb{O}$ as discussed in (11.16). In $\mathbb{F}_4 = \mathbb{O} \times \mathbb{O}$, we then compute $(z, -u)(vz, uv) = \big(z(vz) - (uv)u, (uv)z + u(vz)\big) = \big(-z(zv) + (vu)u, 0\big) = \big((-z^2 + u^2)v, 0\big) = (0, 0)$; here, we have applied (11.10) and the alternativity of $\mathbb{O}$. Thus, there are zero divisors; hence, clearly, the norm cannot be multiplicative. If $\mathbb{F}_m$ were alternative, there could be no zero divisors by the argument of (11.12).

11.18 Notes. Biassociativity as a consequence of alternativity can be derived much more generally for alternative rings. This result is commonly ascribed to Artin, see Zorn [31].

There are further weak associativity properties of $\mathbb{O}$, which involve quadruple products of three elements, the so-called Moufang identities. Usually they are derived from alternativity by algebraic means. We shall obtain them as a by-product from geometric considerations, see (12.14 and 15) and the references given there.

We have seen that among the algebras $\mathbb{F}_m$ only the first few have satisfactory associativity properties, are division algebras (without zero divisors), and have multiplicative norms. These facts are special cases of much more general results: A theorem of Frobenius [1878] says that $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ are the only fields which contain $\mathbb{R}$ as central subfield and have finite dimension over $\mathbb{R}$. By a theorem of Zorn [33], the octonions form the only non-associative alternative field of finite dimension over $\mathbb{R}$.

For modern proofs of these results see Herstein [64] Chap. 7 Sect. 3, p. 326 ff, Palais [75], Ebbinghaus et al. [90, 92] Chap. 8 §2 and Chap. 9 §3. Much more generally, there is a complete structure theory for non-associative alternative fields, culminating in the theorem of Bruck–Kleinfeld–Skornyakov, which is presented, for instance, in the following books: Kleinfeld [63], Pickert [75] Chap. 6, Theorem 13, p. 175 and Theorem 15, p. 177, Schafer [66] Theorem 3.17, p. 56, Zhevlakov et al. [82] 7.3 Corollary 2, p. 152; for further references, in particular to the original papers, one may consult Grundhöfer–Salzmann [90] XI.7.8, p. 322.

The algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ can also be characterized as being the only finite-dimensional composition algebras over $\mathbb{R}$, i.e., finite-dimensional algebras admitting a positive definite quadratic form which is multiplicative; this is a theorem of Hurwitz [1898], see also Freudenthal [85] 1.5.14 p. 19, Harvey–Lawson [82] Appendix IVA, p. 140 ff and Theorem A.12, p. 143, Ebbinghaus et al. [90, 92] Chap. 10 §1, Curtis [90] VD, p. 156 ff. For generalizations see Zhevlakov et al. [82] Chap. 2, p. 22 ff.

These facts are of an algebraic nature. Adams, Atiyah, Bott, Hirzebruch, Kervaire, and Milnor proved the following (much broader) result by using methods of algebraic topology: Finite-dimensional (not necessarily associative) division algebras over $\mathbb{R}$ can only exist in dimensions 1, 2, 4, and 8, that is, in the dimensions of the classical examples $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. See e.g. Bott–Milnor [58], Milnor [58], Kervaire [58]. An even more general, purely topological result determining the possible dimensions of spheres which admit a multiplication with (homotopy) unit will play an important rôle in Chap. 5, cf. (52.5 and 8). A survey by Hirzebruch of these topics may be found in Ebbinghaus et al. [90, 92] Chap. 11. For a recent proof using mainly analytic means see Gilkey [87].

There are variations of the Cayley–Dickson process. When these are carried out over other scalar fields than $\mathbb{R}$, higher stages of the process may remain free of zero divisors, thus producing (non-alternative) division algebras of dimension at least 16, in contrast with the above and (11.17). In this way, one obtains, for instance, a 16-dimensional division algebra over $\mathbb{Q}$. See Schafer [45] and R.B. Brown [67].

The deviation in $\mathbb{F}_m$ from commutavity and associativity is measured by the following results.

11.19 The center of $\mathbb{F}_m$. *For $m > 1$, the only elements of $\mathbb{F}_m$ commuting with all elements are the scalars in $\mathbb{R} \subset \mathbb{F}_m$; i.e., the center of $\mathbb{F}_m$ is $\mathbb{R}$.*

Proof. Let $c, d \in \mathbb{F}_{m-1}$ be such that $(c, d) \in \mathbb{F}_m = \mathbb{F}_{m-1} \times \mathbb{F}_{m-1}$ commutes with each element of $\mathbb{F}_m$. Then, in particular, $(-\bar{d}, \bar{c}) = (0, 1)(c, d) = (c, d)(0, 1) = (-d, c)$, so that $c = \bar{c} \in \mathbb{R}$, $d = \bar{d} \in \mathbb{R}$ (11.2). Moreover, for arbitrary $z \in \mathbb{F}_{m-1}$, we have $(zc, dz) = (z, 0)(c, d) = (c, d)(z, 0) = (cz, d\bar{z})$, so that $d(z - \bar{z}) = 0$. Since d is a real scalar, it must be zero, whence $(c, d) = (c, 0) \in \mathbb{R}$. □

11.20 The kernel of $\mathbb{F}_m$ is defined as

$$\operatorname{Ker} \mathbb{F}_m = \{\, a \in \mathbb{F}_m \mid x(ya) = (xy)a \text{ for all } x, y \in \mathbb{F}_m \,\} .$$

The fields $\mathbb{F}_m$, $m \leq 2$ coincide with their kernels. We shall show that

$$\operatorname{Ker} \mathbb{F}_m = \mathbb{R} \quad \textit{for} \ \ m \geq 3 ,$$

so that, in particular,

$$\operatorname{Ker} \mathbb{O} = \mathbb{R} .$$

Remarks. 1) We are particularly interested in the case of $\mathbb{F}_3 = \mathbb{O}$. The geometric significance of the result will be discussed in (12.13), see also (23.11) and (25.4).

2) The same result can be obtained more generally for arbitrary 8-dimensional real division algebras by means of algebraic topology, see Buchanan–Hähl [77].

Proof of (11.20). Let $m \geq 3$. It is clear that $\operatorname{Ker}\mathbb{F}_m$ contains the scalar field $\mathbb{R}$. Conversely, let $a \in \operatorname{Ker}\mathbb{F}_m$, and write $a = (c, d) \in \mathbb{F}_{m-1} \times \mathbb{F}_{m-1} = \mathbb{F}_m$. For $u, v \in \mathbb{F}_{m-1}$ and for

$$x = (u, 0), \ \ y = (v, 1) \in \mathbb{F}_{m-1} \times \mathbb{F}_{m-1} = \mathbb{F}_m ,$$

by definition of the multiplication in $\mathbb{F}_m$ via the Cayley–Dickson process, one has

$$\begin{aligned} (xy)a &= (uv, u)(c, d) &&= \big((uv)c - \bar{d}u, d(uv) + u\bar{c}\big) \\ x(ya) &= (u, 0)(vc - \bar{d}, dv + \bar{c}) &&= \big(u(vc) - u\bar{d}, (dv)u + \bar{c}u\big) . \end{aligned} \tag{1}$$

Because of $a \in \operatorname{Ker}\mathbb{F}_m$, these two products coincide. With $v = 0$ and u arbitrary, we obtain that $\bar{c}$ and $\bar{d}$ belong to the center of $\mathbb{F}_{m-1}$, which is $\mathbb{R}$ by (11.19), so that $c, d \in \mathbb{R}$. Now, with arbitrary u and v, a comparison of the second components in (1) yields

$$d(uv) = (dv)u = d(vu) . \tag{2}$$

Since $\mathbb{H} = \mathbb{F}_2 \subseteq \mathbb{F}_{m-1}$ is not commutative, we may choose u and v in such a way that $uv \neq vu$; it then follows from (2) that the scalar d must be 0, so that $a = (c, d) = (c, 0) \in \mathbb{R}$. □

Orthogonal groups

In the sequel, let $\mathbb{F}$ be one of the algebras $\mathbb{F}_m$ for $0 \leq m \leq 3$, i.e., one of the algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$. Using the multiplication in $\mathbb{F}$, one may give convenient descriptions of certain groups of orthogonal transformations acting on the $\mathbb{R}$-vector space $\mathbb{F} = \mathbb{R}^n$, where

$$n = \dim_{\mathbb{R}} \mathbb{F} = 2^m \in \{1, 2, 4, 8\} .$$

11.21. The group of $\mathbb{R}$-linear transformations of $\mathbb{F}$ which preserve the norm form $\| \ \|^2$, or, equivalently, which are orthogonal with respect to the inner product $\langle \ \mid \ \rangle$ on $\mathbb{F}$ (11.3), is denoted by

$$\mathrm{O}_n\mathbb{R} = \big\{ C : \mathbb{F} \to \mathbb{F} \ \big| \ C \text{ is } \mathbb{R}\text{-linear}, \forall x \in \mathbb{F} : \|Cx\|^2 = \|x\|^2 \ \big\} .$$

The normal subgroup

$$\mathrm{SO}_n\mathbb{R} = \{ C \in \mathrm{O}_n\mathbb{R} \mid \det C = 1 \}$$

has index 2. Together with the scalar homotheties, these groups generate the group of 'similitudes'

$$\mathrm{GO}_n\mathbb{R} := \{\, \mathbb{F} \to \mathbb{F} : x \mapsto r \cdot Cx \mid r \in \mathbb{R} \setminus \{0\},\ C \in \mathrm{O}_n\mathbb{R} \,\}\ ,$$

and the group of 'direct similitudes'

$$\mathrm{GO}_n^+\mathbb{R} := \{\, \mathbb{F} \to \mathbb{F} : x \mapsto r \cdot Cx \mid r \in \mathbb{R} \setminus \{0\},\ C \in \mathrm{SO}_n\mathbb{R} \,\}\ .$$

Note that $\mathrm{GO}_1\mathbb{R} = \mathrm{GO}_1^+\mathbb{R} = \{\, x \mapsto rx \mid r \in \mathbb{R} \setminus \{0\} \,\}$.

11.22 Lemma.

(a) *For each* $a \in \mathbb{F} \setminus \{0\}$, *the transformations* $x \mapsto ax$ *and* $x \mapsto xa$ *of* $\mathbb{F} = \mathbb{R}^n$ *belong to* $\mathrm{GO}_n^+\mathbb{R}$. *If* $\|a\|^2 = 1$, *they belong to* $\mathrm{O}_n\mathbb{R}$, *and even to* $\mathrm{SO}_n\mathbb{R}$ *for* $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$.

(b) *The group* $\mathrm{SO}_n\mathbb{R}$ *is generated by the transformations*

$$\mathbb{F} \to \mathbb{F} : x \mapsto axa\ , \quad \textit{where } a \in \mathbb{F},\ \|a\|^2 = 1\ .$$

(c) *The group* $\mathrm{GO}_n^+\mathbb{R}$ *is generated by the transformations*

$$\mathbb{F} \to \mathbb{F} : x \mapsto raxa\ , \quad \textit{where } a \in \mathbb{F},\ \|a\|^2 = 1 \textit{ and } r \in \mathbb{R} \setminus \{0\}\ .$$

Note that in the non-commutative case the transformations in (b) and in (c) do not constitute a group by themselves. For more precise information in the associative case, see (11.23).

Proof. Since the norm is multiplicative (11.14), the $\mathbb{R}$-linear transformations in (a) change the square of the norm by a factor $\|a\|^2$; hence, upon multiplying by $\|a\|^{-1}$, one obtains an element of $\mathrm{O}_n\mathbb{R}$. If $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$, every element belongs to a subfield isomorphic to $\mathbb{C}$, see (11.7), and therefore is a square; by alternativity (11.8), it follows that the given transformations are squares of transformations of the same kind, and thus have positive determinant over $\mathbb{R}$. For $\mathbb{F} = \mathbb{R}$ use the fact that $\mathrm{GO}_1\mathbb{R} = \mathrm{GO}_1^+\mathbb{R}$.

Assertion (b) is trivial for $\mathbb{F} = \mathbb{R}$, since $\mathrm{SO}_1\mathbb{R} = \{\mathrm{id}\}$. In general, the transformations considered in (b) belong to $\mathrm{SO}_n\mathbb{R}$ by (a). In order to show that they generate $\mathrm{SO}_n\mathbb{R}$, we use reflections in hyperplanes. (These reflections do *not* belong to $\mathrm{SO}_n\mathbb{R}$!) For any $a \in \mathbb{F}$ with $\|a\|^2 = 1$, the reflection in the hyperplane $a^\perp$ orthogonal to a is the mapping

$$\varrho_a : x \mapsto x - 2a \cdot \frac{\langle\, x \mid a \,\rangle}{\langle\, a \mid a \,\rangle} = x - 2a \cdot \frac{(\bar{x}a + \bar{a}x)}{2\|a\|^2} = x - a(\bar{x}a + \bar{a}x) = -a\bar{x}a\ ;$$

note that $a(\bar{a}x) = \|a\|^2 x = x$ by alternativity (11.8). As is well known, every element of $\mathrm{SO}_n\mathbb{R}$ is the product of an even number of such hyperplane reflections, that is, the product of transformations having the form $\varrho_a\varrho_b$ for $a, b \in \mathbb{F}$ with $\|a\|^2 = 1 = \|b\|^2$, see e.g. Porteous [81] p. 159–160. Now $x^{\varrho_a\varrho_b} = -b(\overline{-a\bar{x}a})b = b(\bar{a}x\bar{a})b$, so that $\varrho_a\varrho_b$ can also be written as the composition of the transformations $x \mapsto \bar{a}x\bar{a}$ and $x \mapsto bxb$. □

In contrast to the preceding result, the following does not hold in the case $\mathbb{F} = \mathbb{O}$ for lack of associativity.

11.23 Corollary. *For $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$ and $n = \dim_{\mathbb{R}} \mathbb{F}$, the group $\mathrm{GO}_n^+\mathbb{R}$ consists precisely of the transformations*

$$\mathbb{F} \to \mathbb{F} : x \mapsto b^{-1}xa\,,$$

where $a, b \in \mathbb{F} \setminus \{0\}$, and the group $\mathrm{SO}_n\mathbb{R}$ consists of these transformations for $\|a\|^2 = 1 = \|b\|^2$.

Proof. By (11.22a), these transformations belong to $\mathrm{SO}_n\mathbb{R}$ and to $\mathrm{GO}_n^+\mathbb{R}$, respectively, and they form subgroups of $\mathrm{SO}_n\mathbb{R}$ and $\mathrm{GO}_n^+\mathbb{R}$. (Here, the associativity of $\mathbb{F}$ is required.) On the other hand, these subgroups contain the transformations of (11.22b) (put $b = a^{-1}$), and of (11.22c) (put $b = (ra)^{-1}$), which generate $\mathrm{SO}_n\mathbb{R}$ and $\mathrm{GO}_n^+\mathbb{R}$, respectively. □

11.24. Using $\mathbb{F} = \mathbb{H}$, we may also obtain a description of $\mathrm{SO}_3\mathbb{R}$ from (11.23). The subspace $\mathrm{Pu}\,\mathbb{H} \cong \mathbb{R}^3$ of the $\mathbb{R}$-vector space $\mathbb{H}$ is the orthogonal complement of $\mathbb{R} = \mathbb{R} \cdot 1$ in $\mathbb{H}$, see (11.6). The group of $\mathbb{R}$-linear transformations of $\mathbb{H}$ which are orthogonal with respect to the inner product and which fix 1 may therefore be identified with the group of orthogonal transformations of $\mathrm{Pu}\,\mathbb{H} \cong \mathbb{R}^3$. Abusing notation somewhat, we denote both groups by $\mathrm{O}_3\mathbb{R}$. The normal subgroup $\mathrm{SO}_3\mathbb{R}$ of elements of determinant 1 has index 2. Now the transformation $\mathbb{H} \to \mathbb{H} : x \mapsto b^{-1}xa$ in (11.23) fixes 1 if, and only if $b = a$, whence the following result.

Corollary. $\mathrm{SO}_3\mathbb{R}$ *consists precisely of the transformations*

$$\mathrm{int}(a) : \mathbb{H} \to \mathbb{H} : x \mapsto a^{-1}xa\,,$$

where $a \in \mathbb{H}$ satisfies $\|a\|^2 = 1$. □

The notation $\mathrm{int}(a)$ reflects the fact that this transformation is an inner automorphism of the skew field $\mathbb{H}$.

The preceding result shows that $\mathrm{SO}_3\mathbb{R}$ is an epimorphic image of the unit sphere

$$\mathbb{S}_3 = \left\{\, a \in \mathbb{H} \;\middle|\; \|a\|^2 = 1 \,\right\}$$

of $\mathbb{H} = \mathbb{R}^4$, which, by multiplicativity of the norm (11.14), is a subgroup of the multiplicative group $\mathbb{H}^\times$.

11.25 Corollary. *The kernel of the epimorphism* $\mathrm{int} : \mathbb{S}_3 \to \mathrm{SO}_3\mathbb{R} : a \mapsto \mathrm{int}(a)$ *is* $\{1, -1\}$.

Proof. An element $a \in \mathbb{S}_3$ belongs to the kernel if, and only if $a^{-1}xa = x$ for all $x \in \mathbb{H}$, or, equivalently, if a belongs to the center of $\mathbb{H}$, which is $\mathbb{R}$, cf. (11.19); now $\mathbb{R} \cap \mathbb{S}_3 = \{1, -1\}$. □

Remark. In other words, $\mathbb{S}_3$ is a two-fold covering group of $\mathrm{SO}_3\mathbb{R}$. Since the sphere $\mathbb{S}_3$ is simply connected, this shows that $\mathbb{S}_3$ is (isomorphic to) the *universal covering group* of $\mathrm{SO}_3\mathbb{R}$, cf. (94.2), which is denoted systematically by $\mathrm{Spin}_3\mathbb{R}$. This group shall now be interpreted as a $\mathbb{C}$-linear group.

11.26 An isomorphism $\mathrm{Spin}_3\mathbb{R} \cong \mathrm{SU}_2\mathbb{C}$. We consider $\mathbb{H} = \mathbb{C} \times \mathbb{C}$ as a right vector space over the subfield $\mathbb{C} \mathrel{\hat{=}} \mathbb{C} \times \{0\}$. Thus, scalar multiplication of $(a, b) \in \mathbb{C} \times \mathbb{C} = \mathbb{H}$ by the scalar $c \in \mathbb{C}$ is given by

$$(a, b)c = (a, b)(c, 0) = (ac, b\bar{c}) .$$

The elements $1 \mathrel{\hat{=}} (1, 0)$ and $j = (0, 1)$ of $\mathbb{H} = \mathbb{C} \times \mathbb{C}$ form a $\mathbb{C}$-basis. Under the standard Hermitian form with respect to this basis, the inner product of $x = x_1 + jx_2$ and $y = y_1 + jy_2$, for $x_1, x_2, y_1, y_2 \in \mathbb{C}$, is, by definition, $x_1\overline{y_1} + x_2\overline{y_2}$. Now note that, for $c \in \mathbb{C}$, we have $cj = j\bar{c}$ as $ij = -ji$, see also (11.15(∗)). Therefore, $\|x\|^2$ may be expressed in the following way.

$$(*)\qquad \begin{aligned} \|x\|^2 &= (x_1 + jx_2)(\overline{x_1} - jx_2) = x_1\overline{x_1} + jx_2\overline{x_1} - j\,\overline{x_1}x_2 - j^2\,\overline{x_2}x_2 \\ &= x_1\overline{x_1} + \overline{x_2}x_2 \,; \end{aligned}$$

this is the inner product of x by itself under the standard Hermitian form. For $a \in \mathbb{H}$, the transformation

$$\lambda_a : \mathbb{H} \to \mathbb{H} : x \mapsto ax$$

is $\mathbb{C}$-linear by associativity. Moreover, if $a \in \mathbb{S}_3$, the norm is preserved, see (11.14), so that λ_a is unitary with respect to the standard Hermitian form.

By $\mathrm{U}_2\mathbb{C}$ we mean the group of all unitary transformations of the 2-dimensional $\mathbb{C}$-vector space $\mathbb{H}$, and $\mathrm{SU}_2\mathbb{C}$ is the normal subgroup consisting of the unitary transformations having (complex) determinant 1.

Proposition. *The group* $\{\lambda_a \mid a \in \mathbb{S}_3\}$, *which clearly is isomorphic to* $\mathbb{S}_3 = \mathrm{Spin}_3\mathbb{R}$, *coincides with* $\mathrm{SU}_2\mathbb{C}$.

Proof. For $a_1, a_2 \in \mathbb{C}$ and $a = a_1 + ja_2 = a_1 + \overline{a_2}j$, we have $aj = a_1 j + \overline{a_2}j^2 = -\overline{a_2} + j\,\overline{a_1}$, so that the matrix of λ_a is

$$\begin{pmatrix} a_1 & -\overline{a_2} \\ a_2 & \overline{a_1} \end{pmatrix} .$$

It has determinant 1 and is unitary if, and only if, $a_1\overline{a_1} + a_2\overline{a_2} = 1$. By $(*)$, this is equivalent to $a \in \mathbb{S}_3$. Hence, $\{ \lambda_a \mid a \in \mathbb{S}_3 \}$ is a subgroup of $\mathrm{SU}_2\mathbb{C}$, and the two groups coincide since both are sharply transitive on $\mathbb{S}_3$. □

Automorphisms

11.27. We now study the group $\operatorname{Aut}\mathbb{F}_m$ of automorphisms of the ring $\mathbb{F}_m$. That is, we consider automorphisms with respect to addition and multiplication in $\mathbb{F}_m$; linearity over $\mathbb{R}$ is not presupposed. As is well known, the only automorphism of $\mathbb{F}_0 = \mathbb{R}$ is the identity. The only *continuous* automorphisms of the field $\mathbb{F}_1 = \mathbb{C}$ are the identity and conjugation, because every continuous automorphism fixes the elements of $\mathbb{R}$ (the closure of the prime field $\mathbb{Q}$) and maps i onto i or $-i$. However, $\mathbb{C}$ has a multitude of other automorphisms, see the references in (44.11). Concerning information about their topological (mis)behaviour, cf. (55.22a) and the references given there.

For $m \geq 2$, the picture becomes simpler again:

11.28 Proposition. *Let $m \geq 2$. Then an automorphism $\alpha \in \operatorname{Aut}\mathbb{F}_m$ fixes every element of the center $\mathbb{R}$ of $\mathbb{F}_m$, and so is $\mathbb{R}$-linear. Moreover, α leaves $\operatorname{Pu}\mathbb{F}_m$ invariant, commutes with conjugation, and is orthogonal with respect to the inner product of $\mathbb{F}_m$.*

Proof. The center $\mathbb{R}$ of $\mathbb{F}_m$ (11.19) is invariant under α, and $\alpha|_{\mathbb{R}} = \mathrm{id}$ since $\mathbb{R}$ has no other automorphism. This is equivalent to $\mathbb{R}$-linearity. The invariance of $\operatorname{Pu}\mathbb{F}_m = \left\{ x \in \mathbb{F}_m \;\middle|\; x^2 \leq 0 \right\}$, see (11.6), is now immediate. Therefore conjugation, which on $\operatorname{Pu}\mathbb{F}_m$ induces $-\mathrm{id}$, commutes with α. It follows that $\|x^\alpha\|^2 = x^\alpha\overline{x^\alpha} = x^\alpha\bar{x}^\alpha = (x\bar{x})^\alpha = (\|x\|^2)^\alpha = \|x\|^2$, whence α is orthogonal. □

We now determine $\operatorname{Aut}\mathbb{H}$.

11.29. Because of associativity of $\mathbb{H}$, conjugation by an element $a \in \mathbb{H} \setminus \{0\}$ is an automorphism of the skew field $\mathbb{H}$, the *inner automorphism* $\operatorname{int}(a) : \mathbb{H} \to \mathbb{H} : x \mapsto a^{-1}xa$. Let $\operatorname{Int}\mathbb{H}$ be the group of all these inner automorphisms. Since $\|a\|$ belongs to the center $\mathbb{R}$ of $\mathbb{H}$, we have $\operatorname{int}\left(\frac{1}{\|a\|}a\right) = \operatorname{int}(a)$. Now $\frac{1}{\|a\|}a$ has norm 1, so that

$$\operatorname{Int}\mathbb{H} = \left\{ \operatorname{int}(a) \;\middle|\; a \in \mathbb{H},\ \|a\|^2 = 1 \right\} .$$

According to (11.28 and 24), we already know that $\mathrm{Aut}\,\mathbb{H}$ is contained in the group $\mathrm{O}_3\mathbb{R}$ of orthogonal transformations of $\mathbb{H} = \mathbb{R}^4$ fixing 1, and that $\mathrm{Int}\,\mathbb{H} = \mathrm{SO}_3\mathbb{R}$.

Proposition. $\mathrm{Aut}\,\mathbb{H} = \mathrm{Int}\,\mathbb{H} = \mathrm{SO}_3\mathbb{R}$.

Proof. One merely has to show that $\mathrm{Aut}\,\mathbb{H} \subseteq \mathrm{O}_3\mathbb{R}$ cannot be bigger than $\mathrm{SO}_3\mathbb{R} = \mathrm{Int}\,\mathbb{H}$. Now, $\mathrm{SO}_3\mathbb{R}$ has index 2 in $\mathrm{O}_3\mathbb{R}$, whence, if $\mathrm{Aut}\,\mathbb{H}$ were bigger than $\mathrm{SO}_3\mathbb{R}$, we would have $\mathrm{Aut}\,\mathbb{H} = \mathrm{O}_3\mathbb{R}$. In particular, conjugation (which on $\mathrm{Pu}\,\mathbb{H}$ induces $-\mathrm{id} \in \mathrm{O}_3\mathbb{R}$) would have to be an element of $\mathrm{Aut}\,\mathbb{H}$, but conjugation is an *anti-*automorphism and not an automorphism, as $\mathbb{H}$ is not commutative. □

Finally, we study $\mathrm{Aut}\,\mathbb{O}$. We begin by stating a few transitivity properties which follow from the sharp transitivity of $\mathrm{Aut}\,\mathbb{O}$ on the set of Cayley triples, compare (11.16).

11.30 Lemma.

(a) $\mathrm{Aut}\,\mathbb{O}$ *acts transitively on* $\{\, (u, v) \mid u, v \in \mathrm{Pu}\,\mathbb{O},\ u \perp v,\ \|u\|^2 = 1 = \|v\|^2 \,\}$. *In other words,* $\mathrm{Aut}\,\mathbb{O}$ *is transitive on the* 6*-sphere* $\{\, u \in \mathrm{Pu}\,\mathbb{O} \mid \|u\|^2 = 1 \,\}$, *and the stabilizer* $(\mathrm{Aut}\,\mathbb{O})_i$ *is transitive on the unit sphere of the orthogonal space of i in* $\mathrm{Pu}\,\mathbb{O}$, *that is, on the* 5*-sphere* $\{\, u \in \mathrm{Pu}\,\mathbb{O} \mid \|u\|^2 = 1,\ u \perp i \,\}$.

(b) *The stabilizer* $(\mathrm{Aut}\,\mathbb{O})_{i,j}$ *is sharply transitive on the unit sphere of* $\{0\} \times \mathbb{H} \subseteq \mathbb{H} \times \mathbb{H} = \mathbb{O}$, *that is, on the* 3*-sphere* $\{\, (0, b) \in \mathbb{H} \times \mathbb{H} \mid \|b\|^2 = 1 \,\}$.

Proof. Concerning (a), note that, for any two pure orthogonal elements u, v of unit length, there is a Cayley triple having u, v as the first two elements. The 3-sphere in (b) consists precisely of the elements $z \in \mathrm{Pu}\,\mathbb{O}$ with $\|z\|^2 = 1$ and such that i, j, z is a Cayley triple; this is because i, j, ij span $\mathrm{Pu}\,\mathbb{H} \times \{0\}$, and $\{0\} \times \mathbb{H}$ is the orthogonal space of $\mathrm{Pu}\,\mathbb{H} \times \{0\}$ in $\mathrm{Pu}\,\mathbb{O}$, cf. (11.3 and 6). □

11.31 Lemma.

(a) *The automorphism group* $\Lambda = \mathrm{Aut}\,\mathbb{O}$ *acts transitively on the set* $\mathcal{H}$ *of subalgebras* H *of* $\mathbb{O}$ *with* $H \cong \mathbb{H}$, *so that the stabilizers* Λ_H *of such subalgebras are conjugate. These stabilizers cover* Λ.

(b) *The stabilizer of* $\mathbb{H} = \mathbb{H} \times \{0\} \subseteq \mathbb{H} \times \mathbb{H} = \mathbb{O}$ *is*

$$\Lambda_{\mathbb{H}} = \{\, (x, y) \mapsto (a^{-1}xa, b^{-1}ya) \mid a, b \in \mathbb{H},\ \|a\|^2 = 1 = \|b\|^2 \,\},$$

where $(x, y) \in \mathbb{O} = \mathbb{H} \times \mathbb{H}$. *It is isomorphic to* $\mathrm{SO}_4\mathbb{R}$.

(c) *The stabilizer of i and j is*

$$\Lambda_{i,j} = \{\, (x, y) \mapsto (x, b^{-1}y) \mid b \in \mathbb{H},\ \|b\|^2 = 1 \,\}.$$

(d) *All involutions of* Λ *are conjugate, and* Λ *is generated by them.*

(e) $\mathrm{Aut}\,\mathbb{O} \subseteq \mathrm{SO}_8\mathbb{R}$.

Note. More precisely, it can be proved that every element in $\operatorname{Aut}\mathbb{O}$ is a product of at most two involutions, see Wonenburger [69].

Proof. (a) A subalgebra $H \cong \mathbb{H}$ intersects $\operatorname{Pu}\mathbb{O}$ in a 3-dimensional subspace, and therefore contains pure elements u, v of unit length which are orthogonal. By (11.10), H is the span of 1, u, v, and uv. It follows from (11.30a) that $\Lambda = \operatorname{Aut}\mathbb{O}$ is transitive on the set $\mathcal{H}$ of such subalgebras. In particular, the stabilizers Λ_H for $H \in \mathcal{H}$ are conjugate, cf. (91.1a).

We must show in addition that every element $\lambda \in \Lambda$ leaves some subalgebra $H \in \mathcal{H}$ invariant. Under the orthogonal action of λ, the 7-dimensional invariant subspace $\operatorname{Pu}\mathbb{O}$ (11.28) decomposes into 1- and 2-dimensional invariant subspaces. In particular, there is an invariant 2-dimensional subspace spanned by orthogonal pure vectors u, v of norm 1. The subalgebra H spanned by 1, u, v, uv is isomorphic to $\mathbb{H}$, see (11.10), and clearly is invariant under λ.

(b and c) By definition of the multiplication in $\mathbb{O} = \mathbb{H} \times \mathbb{H}$, one readily verifies that the right-hand side in assertion (b) is a subgroup M of $\operatorname{Aut}\mathbb{O}$. The stabilizer $\mathsf{M}_{i,j}$ fixes every element of the subalgebra $\mathbb{H} \times \{0\}$, which is spanned by 1, i, j, and $k = ij$. Thus, $\mathsf{M}_{i,j}$ consists of the transformations described in assertion (b) with $a = \pm 1$, cf. (11.25), and is the group on the right-hand side of assertion (c).

In particular, $\mathsf{M}_{i,j}$ acts transitively on the unit sphere of $\{0\} \times \mathbb{H}$. Now, by (11.30), the larger group $\Lambda_{i,j} \supseteq \mathsf{M}_{i,j}$ is sharply transitive on this unit sphere. So, we have

$$(1) \qquad \Lambda_{i,j} = \mathsf{M}_{i,j}\,.$$

Obviously, M leaves $\mathbb{H} \mathrel{\hat{=}} \mathbb{H} \times \{0\}$ invariant, and induces the full automorphism group $\operatorname{Int}\mathbb{H} = \mathrm{SO}_3\mathbb{R}$ (11.29). In particular, M is transitive on the set of pairs of pure orthogonal elements of length 1 contained in $\mathbb{H} \mathrel{\hat{=}} \mathbb{H} \times \{0\} \subseteq \mathbb{H} \times \mathbb{H} = \mathbb{O}$, and this set is invariant under $\Lambda_{\mathbb{H}}$ by (11.28). From (1), we therefore infer that $\Lambda_{\mathbb{H}} = \mathsf{M}$, using the general principle (91.3).

Restriction to $\{0\} \times \mathbb{H}$ provides an epimorphism of $\Lambda_{\mathbb{H}}$ onto $\mathrm{SO}_4\mathbb{R}$ according to the description of $\mathrm{SO}_4\mathbb{R}$ given in (11.23) (with $\mathbb{F} = \mathbb{H}$). We even obtain an isomorphism; indeed, if $b^{-1}ya = y$ for all $y \in \mathbb{H}$, then $b = a$.

(d) Let $\iota \in \Lambda$ be an involution. Since ι is orthogonal (11.28), the $\mathbb{R}$-vector space $\mathbb{O}$ is the orthogonal sum of the eigenspaces F_+ and F_- of ι corresponding to the eigenvalues 1 and -1, and ι is uniquely determined by its fixed space F_+, which is a subalgebra of $\mathbb{O}$.

Now, multiplication by an element $a \in F_- \setminus \{0\}$ is a vector space isomorphism between F_+ and F_-, so that F_+ has dimension 4 over $\mathbb{R}$. By (11.9 and 10), the subalgebra F_+ is isomorphic to $\mathbb{H}$. The fact that Λ is transitive on the set

$\mathcal{H}$ of such subalgebras according to (a) now implies that all involutions are conjugate.

It is well known that $\Lambda_{\mathbb{H}} \cong \mathrm{SO}_4\mathbb{R}$ is generated by its involutions, see e.g. Dieudonné [71] Chap. II §6 no. 1), p. 51. The same then holds for the conjugates Λ_H, $H \in \mathcal{H}$, and for their union Λ, see (a).

(e) From (11.28), we know that $\mathrm{Aut}\,\mathbb{O} \subseteq \mathrm{O}_8\mathbb{R}$, and (a) and (b) imply that every automorphism has determinant 1. □

11.32 Theorem. $\mathrm{Aut}\,\mathbb{O}$ *is a simple group.*

Proof. Let $\mathsf{N} \neq \{\mathrm{id}\}$ be a normal subgroup of $\Lambda = \mathrm{Aut}\,\mathbb{O}$. We must show that $\mathsf{N} = \mathrm{Aut}\,\mathbb{O}$. By (11.31d), it suffices to prove that N contains an involution. From (11.31a) we infer that N intersects some and therefore each of the conjugate subgroups Λ_H, $H \in \mathcal{H}$ non-trivially. Now every non-trivial normal subgroup of $\mathrm{SO}_4\mathbb{R} \cong \Lambda_{\mathbb{H}}$ contains an involution, as is well known, and may be seen as follows.

According to (11.23), write

$$\mathrm{SO}_4\mathbb{R} = \left\{ \mathbb{H} \to \mathbb{H} : y \mapsto b^{-1}ya \;\middle|\; a, b \in \mathbb{H},\ \|a\|^2 = 1 = \|b\|^2 \right\} .$$

The two subgroups

$$\mathsf{A} = \left\{ \mathbb{H} \to \mathbb{H} : y \mapsto ya \;\middle|\; a \in \mathbb{H}, \|a\|^2 = 1 \right\}$$

and

$$\mathsf{B} = \left\{ \mathbb{H} \to \mathbb{H} : y \mapsto b^{-1}y \;\middle|\; b \in \mathbb{H}, \|b\|^2 = 1 \right\}$$

are normal subgroups, and $\mathsf{AB} = \mathrm{SO}_4\mathbb{R}$. The centralizer $\mathrm{Cs}\,\mathsf{B}$ of B is A (and vice versa). If N is a normal subgroup of $\mathrm{SO}_4\mathbb{R}$ such that $\mathsf{N} \cap \mathsf{B} = \{\mathrm{id}\}$, then $\mathsf{N} \subseteq \mathrm{Cs}\,\mathsf{B} \subseteq \mathsf{A}$. As A and B are isomorphic to $\mathbb{S}_3 = \left\{ a \in \mathbb{H} \;\middle|\; \|a\|^2 = 1 \right\}$, it now suffices to ascertain that every nontrivial normal subgroup H of $\mathbb{S}_3$ contains an involution. For this, we consider the epimorphism $\mathrm{int} : \mathbb{S}_3 \to \mathrm{SO}_3\mathbb{R}$ with kernel $\{1, -1\}$ of (11.25). If H contains the involution -1, the proof is finished. If not, then H is mapped isomorphically onto a non-trivial normal subgroup of $\mathrm{SO}_3\mathbb{R}$ by int. As $\mathrm{SO}_3\mathbb{R}$ is simple (see e.g. Artin [57] p. 178), we then have $\mathrm{int}(\mathsf{H}) = \mathrm{SO}_3\mathbb{R}$, so that H must contain involutions since $\mathrm{SO}_3\mathbb{R}$ does. (In fact, the latter case does not occur, but that does not affect the argument.) □

The following result connects our discussion with the theory of simple Lie groups; it will not be used in this chapter in an essential way.

11.33 Theorem. $\mathrm{Aut}\,\mathbb{O}$ *is a compact, connected simple Lie group of dimension* 14. *By the classification of simple Lie groups, it is therefore isomorphic to the exceptional compact Lie group* $\mathrm{G}_2 = \mathrm{G}_2(-14)$.

Proof. By (11.28), $\operatorname{Aut}\mathbb{O}$ is a subgroup of $\mathrm{O}_8\mathbb{R}$. It is closed in the usual topology of $\mathrm{O}_8\mathbb{R} \subseteq \mathrm{GL}_8\mathbb{R}$, as multiplication in $\mathbb{O}$ is continuous. Hence $\operatorname{Aut}\mathbb{O}$ is compact since $\mathrm{O}_8\mathbb{R}$ is (see for instance Porteous [81] Prop. 17.8, p. 337). As a closed linear group, $\operatorname{Aut}\mathbb{O}$ is a Lie group, cf. (94.3).

Simplicity has been proved in (11.32). It follows that $\operatorname{Aut}\mathbb{O}$ is connected, because the connected component of the identity of any topological group is a normal subgroup, and because $\operatorname{Aut}\mathbb{O}$ contains non-trivial connected subsets, see (11.31b and c).

The dimension of $\operatorname{Aut}\mathbb{O}$ may be computed by applying the dimension formula for stabilizers (96.10) to the transitive actions described in (11.30), as follows. By (11.31c or 30b), we have $\dim(\operatorname{Aut}\mathbb{O})_{i,j} = \dim\mathbb{S}_3 = 3$. As $(\operatorname{Aut}\mathbb{O})_{i,j}$ is a stabilizer of the transitive action of $(\operatorname{Aut}\mathbb{O})_i$ on the 5-sphere of pure unit octonions orthogonal to i, it follows that $\dim(\operatorname{Aut}\mathbb{O})_i = \dim(\operatorname{Aut}\mathbb{O})_{i,j} + 5 = 8$. By transitivity of $\operatorname{Aut}\mathbb{O}$ on the 6-dimensional unit sphere of $\operatorname{Pu}\mathbb{O}$, we conclude that $\dim\operatorname{Aut}\mathbb{O} = \dim(\operatorname{Aut}\mathbb{O})_i + 6 = 14$.

Now, according to the classification of almost simple Lie groups, cf. (94.32 and 33), there is just one compact almost simple Lie group of dimension 14, viz. the exceptional Lie group $\mathrm{G}_2 = \mathrm{G}_2(-14)$. □

Remarks. The classification of almost simple Lie groups also yields that $\operatorname{Aut}\mathbb{O}$ is simply connected; we shall prove this independently in (17.15c).

In the above proof, the only information needed about the stabilizer $(\operatorname{Aut}\mathbb{O})_i$ was its dimension. In fact, it is not difficult to determine this stabilizer completely, as we shall see now.

11.34 Automorphisms of $\mathbb{O}$ which are $\mathbb{C}$-linear. Biassociativity (11.13) of $\mathbb{O}$ implies that $\mathbb{O}$ is a left vector space over the subfield $\mathbb{C}$ spanned by 1 and i, scalar multiplication being multiplication within $\mathbb{O}$. Explicitly, scalar multiplication of $(x, y) \in \mathbb{H} \times \mathbb{H} = \mathbb{O}$ by the scalar $c \in \mathbb{C} \subseteq \mathbb{H} = \mathbb{H} \times \{0\}$ is given by

(1) $$c(x, y) := (c, 0)(x, y) = (cx, yc) .$$

An automorphism in $(\operatorname{Aut}\mathbb{O})_i$ has the following properties: it fixes 1, is $\mathbb{C}$-linear, and is orthogonal with respect to the standard inner product of $\mathbb{O} = \mathbb{R}^8$; for the latter property, see (11.28). We shall show that $(\operatorname{Aut}\mathbb{O})_i$ consists precisely of the transformations which have these properties and in addition have *complex* determinant 1.

For convenience of notation, we first identify $\mathbb{O}$ and $\mathbb{C}^4$ in a suitable way. It is easy to see that the vectors 1, j, $l = (0, 1)$, and $jl = (0, j)$ belonging to the standard basis of $\mathbb{O} = \mathbb{H} \times \mathbb{H}$ form a $\mathbb{C}$-basis of $\mathbb{O}$. We note that these vectors are orthonormal with respect to the standard inner product of $\mathbb{O} = \mathbb{R}^8$. If we identify $\mathbb{O}$ and $\mathbb{C}^4$ via this basis, then these vectors are also orthonormal with respect to the standard Hermitian form on $\mathbb{C}^4 = \mathbb{O}$. We note that with these identifications we obtain that $O_8\mathbb{R} \cap GL_4\mathbb{C} = U_4\mathbb{C}$, where $U_4\mathbb{C}$ is the group of unitary transformations of $\mathbb{O} \mathrel{\hat=} \mathbb{C}^4$ with respect to the standard Hermitian form.

The properties of $(\operatorname{Aut}\mathbb{O})_i$ stated above now can be expressed by saying that $(\operatorname{Aut}\mathbb{O})_i$ is contained in the stabilizer of $1 \in \mathbb{O}$ in $U_4\mathbb{C}$. With a slight abuse of notation, this stabilizer will be denoted by $U_3\mathbb{C}$, as it induces the identity on $\mathbb{C} = \mathbb{R} + \mathbb{R}i \mathrel{\hat=} \mathbb{C} \times \{0\}$, and acts in the usual way on the orthogonal space $\mathbb{C}^\perp \mathrel{\hat=} \{0\} \times \mathbb{C}^3$. The normal subgroup consisting of the elements of $U_3\mathbb{C}$ with *complex* determinant 1 will be denoted by $SU_3\mathbb{C}$.

Proposition. $(\operatorname{Aut}\mathbb{O})_i = SU_3\mathbb{C}$.

Proof. The group $SU_3\mathbb{C}$ is transitive on the unit sphere of $\mathbb{C}^\perp$, and so is $\Lambda_i := (\operatorname{Aut}\mathbb{O})_i$ by (11.30a). It therefore suffices to show that $\Lambda_i \subseteq SU_3\mathbb{C}$ and that

$$\Lambda_{i,j} = (SU_3\mathbb{C})_j \,, \tag{2}$$

see (91.3). By (11.31c and 26), the stabilizer $\Lambda_{i,j}$ acts trivially on $\mathbb{H} \times \{0\} \subseteq \mathbb{H} \times \mathbb{H} = \mathbb{O}$ and induces the group $SU_2\mathbb{C}$ on $\{0\} \times \mathbb{H} \mathrel{\hat=} \{0\} \times \mathbb{C}^2$, so that (2) is clear. It remains to prove that $\Lambda_i \subseteq SU_3\mathbb{C}$. Every $\mathbb{C}$-linear map has an eigenvector, hence Λ_i is the union of all stabilizers $\Lambda_{i,\mathbb{C}x}$ with $0 \neq x \in \mathbb{C}^\perp$. The action of Λ_i on the set of such subspaces $\mathbb{C}x$ is transitive by (11.30a); according to the principle of conjugate stabilizers (91.1a), this implies that Λ_i acts transitively on the set of the corresponding stabilizers $\Lambda_{i,\mathbb{C}x}$ by conjugation. Thus, because of $\Lambda_i \subseteq U_3\mathbb{C}$, the proof is finished if we show that $\Lambda_{i,\mathbb{C}j} \subseteq SU_3\mathbb{C}$.

Now the subalgebra of $\mathbb{O}$ generated by i and $\mathbb{C}j$ is $\mathbb{H} \mathrel{\hat=} \mathbb{H} \times \{0\}$, so that $\Lambda_{i,\mathbb{C}j} = \Lambda_{\mathbb{H},i}$. The explicit description (11.31b) of $\Lambda_\mathbb{H}$ shows that $\Lambda_{i,\mathbb{C}j}$ is the product of $\Lambda_{i,j} = \left\{ (x,y) \mapsto (x, b^{-1}y) \;\middle|\; b \in \mathbb{H},\, \|b\|^2 = 1 \right\}$ by a subgroup of $\Lambda_{i,l}$, namely $\left\{ (x,y) \mapsto (a^{-1}xa, a^{-1}ya) \;\middle|\; a \in \mathbb{C},\, \|a\|^2 = 1 \right\}$. From (11.30a), we know that the stabilizer $\Lambda_{i,l}$ is conjugate in $\Lambda_i \subseteq U_3\mathbb{C}$ to $\Lambda_{i,j} = (SU_3\mathbb{C})_j$, see (2). We conclude that $\Lambda_{i,l} \subseteq SU_3\mathbb{C}$ and that $\Lambda_{i,\mathbb{C}j} \subseteq \Lambda_{i,j}\Lambda_{i,l} \subseteq SU_3\mathbb{C}$. □

The following converse of the preceding result will only be used in later chapters.

11.35 Proposition. *Every closed, connected subgroup of* $\operatorname{Aut}\mathbb{O}$ *which is locally isomorphic to* $SU_3\mathbb{C}$ *is conjugate to* $(\operatorname{Aut}\mathbb{O})_i$.

Proof. Let Δ be a subgroup of this kind; it is a Lie group by (94.3). As $SU_3\mathbb{C}$ is connected and simply connected (see e.g. Porteous [81] Prop. 17.22, p. 340 and Husemoller [66] 12.3, p. 93), there is a surjective covering homomorphism $SU_3\mathbb{C} \to \Delta$, which may be viewed as a representation of $SU_3\mathbb{C}$ on $\mathbb{O} = \mathbb{R}^8$. (For the notion of covering homomorphism, see (94.2).) According to (95.3), $\mathbb{R}^8$

is the direct sum of the subspace F of fixed vectors and of irreducible subspaces. Now, $1 \in F$, so that $\dim F \geq 1$, and, up to equivalence, the only irreducible representation of $\mathrm{SU}_3\mathbb{C}$ of dimension ≤ 7 is the usual representation of $\mathrm{SU}_3\mathbb{C}$ on $\mathbb{C}^3 = \mathbb{R}^6$, see (95.10). Thus F is 2-dimensional, $F = \mathbb{R} + \mathbb{R}u$ for a suitable $u \in \mathrm{Pu}\,\mathbb{O}$ with $\|u\|^2 = 1$. Up to conjugation with an automorphism of $\mathbb{O}$ mapping i to u (11.30a), we may assume that $F = \mathbb{R} + \mathbb{R}i = \mathbb{C}$; then $\Delta \subseteq (\mathrm{Aut}\,\mathbb{O})_i$. By assumption, $\dim \Delta = 8 = \dim(\mathrm{Aut}\,\mathbb{O})_i$, cf. (11.34) or the proof of (11.33). We conclude that $\Delta = (\mathrm{Aut}\,\mathbb{O})_i$ by connectedness, cf. (93.12). □

12 The classical affine planes

This section is concerned with the classical planes over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ from an affine point of view. We introduce the affine planes over these division algebras; the corresponding projective planes can then be obtained by projective completion, i.e., by adjunction of a line at infinity. Our main objects of study are the affine collineations, which may also be viewed as collineations of the projective completion fixing the line at infinity. A special rôle is played by collineations having an axis and a center; these notions are introduced and illustrated by a concrete description of all such collineations for particular axes and centers. The group of collineations of the affine plane over a division algebra $\mathbb{F}$ can be easily determined if $\mathbb{F}$ is a field like $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ in our context here. Then, the affine collineations are just the semilinear affine transformations of the vector space $\mathbb{F} \times \mathbb{F}$; this is the so-called fundamental theorem of affine geometry. On the line at infinity, the affine collineations induce the fractional semilinear transformations.

The plane over $\mathbb{O}$ has no comparable vector space structure, as $\mathbb{O}$ is not associative. Consequently, for the octonion plane, the description of the collineation group is much more involved. A full discussion will therefore take place separately in Sections 16–18. Here, we restrict ourselves to first results. We construct special affine collineations of particular interest; they will be obtained as products of certain easily accessible central collineations (shears). The geometric fact that they are collineations readily translates into well-known algebraic properties of $\mathbb{O}$, the Moufang identities. These collineations fix the coordinate axes, and the group of all collineations doing so will be determined completely, starting from these special collineations. The description obtained for this group is in terms of orthogonal transformations of the coordinate axes and of the line at infinity. As a direct algebraic interpretation of these geometric results, we obtain the so-called triality principle for the group $\mathrm{SO}_8\mathbb{R}$.

12.0 General assumption. Throughout this section, $\mathbb{F}$ shall denote one of the Cayley–Dickson division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.

12.1 The affine plane $\mathscr{A}_2\mathbb{F}$ over $\mathbb{F}$ is constructed as follows. The set of points is $\mathbb{F} \times \mathbb{F}$, and the following subsets of $\mathbb{F} \times \mathbb{F}$ are called lines:

$$\begin{aligned} [s,t] &= \{\, (x, sx+t) \mid x \in \mathbb{F} \,\} \quad \text{for } s,t \in \mathbb{F}\,, \\ [c] &= \{c\} \times \mathbb{F} \quad \text{for } c \in \mathbb{F}\,. \end{aligned}$$

We say that the line $[s,t]$ has *slope* s, and the line $[c]$ has *slope* ∞.

Because $\mathbb{F}$ is a division algebra, it is easily verified that this structure has the following properties:

1) For two points $(x_1, y_1) \neq (x_2, y_2)$ there is a unique line joining them (i.e., containing both of them), namely the line $[s, y_1 - sx_1]$, where s is uniquely determined by the equation

$$(1) \qquad s(x_2 - x_1) = y_2 - y_1$$

if $x_1 \neq x_2$, and the line $[x_1]$ if $x_1 = x_2$.

2) Two lines of different slopes have a unique point of intersection. In fact, we have $[c] \cap [s,t] = \{\, (c, sc+t) \,\}$, and for $s_1 \neq s_2$ we have $[s_1, t_1] \cap [s_2, t_2] = \{\, (x, s_1x + t_1) \,\}$, where x is uniquely determined by the equation

$$(2) \qquad (s_1 - s_2)x = t_2 - t_1\,.$$

3) Two different lines are disjoint if they have the same slope, and by property 2) this condition is also necessary. Two lines will be called *parallel* if they have the same slope, i.e., if they are disjoint or equal.

4) The so-called *parallel axiom* holds: For each line L and each point (x, y), there is a unique line which passes through (x, y) and is parallel to L. Depending on whether the slope of L is ∞ or $s \in \mathbb{F}$, this parallel is given by $[x]$ or by $[s, y - sx]$, respectively.

Properties 1) – 4) say that $\mathscr{A}_2\mathbb{F}$ satisfies the axioms of an *affine plane*, see (21.8).

Since $\mathbb{F}$ is biassociative (11.11 and 13), the solutions of equations (1) and (2) may be written down explicitly as

$$\begin{aligned} (1') \qquad s &= (y_2 - y_1)(x_2 - x_1)^{-1} \\ (2') \qquad x &= (s_1 - s_2)^{-1}(t_2 - t_1)\,. \end{aligned}$$

12.2 The projective completion $\overline{\mathscr{A}_2\mathbb{F}}$ of the affine plane $\mathscr{A}_2\mathbb{F}$ is obtained by adjoining 'points at infinity', one for each parallel class. All the lines of the parallel class are thought to intersect in the respective point at infinity. Furthermore, one adds a 'line at infinity' passing through just these points at infinity.

For $s \in \mathbb{F} \cup \{\infty\}$, the point at infinity on the lines of slope s (which form a parallel class) will be denoted by (s), and the line at infinity is

$$[\infty] := \{ (s) \mid s \in \mathbb{F} \cup \{\infty\} \} .$$

By this construction, we obtain a geometry in which any two lines $L_1 \neq L_2$ always have a unique point of intersection $L_1 \wedge L_2$. (If L_1, L_2 are lines of the affine plane $\mathscr{A}_2\mathbb{F}$, their intersection point in the projective completion $\overline{\mathscr{A}_2\mathbb{F}}$ is a point at infinity if, and only if, in $\mathscr{A}_2\mathbb{F}$ the lines L_1, L_2 are parallel.) The property that for any two points $p_1 \neq p_2$ there is a unique line $\overline{p_1 p_2}$ joining them is preserved in this extension process. Thus, the projective completion $\overline{\mathscr{A}_2\mathbb{F}}$ is a *projective* plane in the sense of definition (21.1).

Of course, this construction applies quite generally to any affine plane, see Hughes–Piper [73] Theorem 3.10, p. 83.

For certain points and lines of the projective completion $\overline{\mathscr{A}_2\mathbb{F}}$, we shall systematically use special names as indicated in the diagram of Figure 12a.

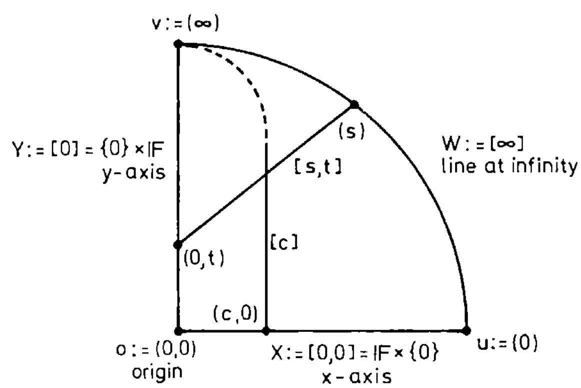

Figure 12a

12.3 Collineations. A *collineation* of an affine or projective plane is a bijection of the set of points onto itself mapping lines onto lines.

As for the latter condition, it suffices to postulate that lines are mapped *into* lines, see (23.2).

A collineation of an affine plane maps every parallel class of lines to a parallel class of lines; one therefore obtains a unique extension to a collineation of the projective completion by permuting the points at infinity accordingly. Conversely, if a collineation of the projective completion leaves the line at infinity invariant, then it arises in this way from an affine collineation. Henceforth, we shall not distinguish between an affine collineation and its projective extension. We remark that, besides these collineations, the projective completion may have collineations which move the line at infinity, cf. (13.5) and (17.6).

It is straightforward that for a collineation φ, for points $p_1 \neq p_2$ and for lines $L_1 \neq L_2$ one has

$$(p_1 p_2)^\varphi = p_1^\varphi p_2^\varphi \quad \text{and} \quad (L_1 \wedge L_2)^\varphi = L_1^\varphi \wedge L_2^\varphi .$$

The collineations of a given plane form a group under composition. For a subgroup Δ of this group, for a point p and a line L, the stabilizers Δ_p and Δ_L are the subgroups consisting of all collineations in Δ which fix p or leave L invariant, respectively.

12.4 Axial collineations. We now present a notion which is basic for the study of collineation groups of projective planes, together with some standard results, cf. also (23.7 ff).

(a) Consider a point p and a line L. We say that a collineation φ has *axis* L if φ fixes every point on L; dually, φ has *center* p if φ leaves every line through p invariant. A collineation of a *projective* plane has an axis if, and only if, it has a center (see Hughes–Piper [73] Theorem 4.9, p. 94; in the present chapter, we shall not use this fact). The center may be on the axis or not.

(b) For a group Δ of collineations, $\Delta_{[p,L]}$ denotes the subset of all collineations in Δ with center p and axis L; it is obviously a subgroup. For $\gamma \in \Delta$ one easily verifies the following

Conjugation formula: $\qquad \gamma^{-1} \Delta_{[p,L]} \gamma = \Delta_{[p^\gamma, L^\gamma]}$,

cf. also (91.1a).

(c) *Uniqueness properties.* A collineation $\delta \in \Delta_{[p,L]}$ having a fixed point q outside $L \cup \{p\}$ must be the identity. Indeed, by joining q to the points of L, one sees that q is a center of δ, too; therefore, every point not on pq is a fixed point, being the intersection of two different lines through the centers p and q, and the same argument applied to such a fixed point instead of q finally shows that $\delta = \mathrm{id}$.

Dually, the only collineation in $\Delta_{[p,L]}$ leaving a line $M \neq L$ with $p \notin M$ invariant is the identity, because then M is an axis as well, its points being the points of intersection of the invariant line M and the invariant lines through the center p.

In particular, if a non-identical collineation has a center and an axis, these are uniquely determined.

We now give concrete descriptions of collineations of $\overline{\mathscr{A}_2}\mathbb{F}$ having a center and an axis. We first deal with such collineations for which center and axis are incident, so-called *elations*. The other case, in which center and axis are not incident, will be taken up later (12.13).

12.5 Proposition: Elations of $\overline{\mathscr{A}_2\mathbb{F}}$.

(a) *The collineations of $\overline{\mathscr{A}_2\mathbb{F}}$ having the line at infinity $W = [\infty]$ as axis and a center on W are precisely the translations*

$$\tau_{a,b} : (x, y) \mapsto (x + a, y + b);\ \ \tau_{a,b}|_W = \mathrm{id}$$

with $a, b \in \mathbb{F}$. The translation group

$$\mathsf{T} = \left\{ \tau_{a,b} \mid a, b \in \mathbb{F} \right\}$$

is commutative and sharply transitive on the affine point set $\mathbb{F} \times \mathbb{F}$.

(b) *The collineations of $\overline{\mathscr{A}_2\mathbb{F}}$ with axis $Y = [0]$ and center $v = (\infty)$ are precisely the shears*

$$\sigma_a : (x, y) \mapsto (x, y + ax),\ \ (s) \mapsto (s + a),\ \ (\infty) \mapsto (\infty)$$

for $a \in \mathbb{F}$; they form a commutative group which is sharply transitive on $W \setminus \{v\}$.

(c) *The collineations of $\overline{\mathscr{A}_2\mathbb{F}}$ with axis $X = [0, 0]$ and center $u = (0)$ are precisely the collineations*

$$\begin{aligned}\sigma'_a : (x, y) \mapsto (x + ay, y),\ (s) \mapsto ((s^{-1} + a)^{-1}) \quad &\textit{for } s \neq 0, -a^{-1},\\ (0) \mapsto (0),\ \ (-a^{-1}) \mapsto (\infty),\ \ &(\infty) \mapsto (a^{-1})\end{aligned}$$

for $a \in \mathbb{F} \setminus \{0\}$ (together with $\sigma'_0 = \mathrm{id}$).

Note that (c) is obtained from (b) by conjugation with the following collineation, see the conjugation formula in (12.4b).

12.6. *The reflection*

$$\begin{aligned}(x, y) \mapsto (y, x),\ (s) \mapsto (s^{-1}) \quad &\textit{for } s \neq 0,\\ (0) \mapsto (\infty),\ \ &(\infty) \mapsto (0)\end{aligned}$$

is a collineation of $\overline{\mathscr{A}_2\mathbb{F}}$ with axis $[1, 0]$ and center (-1).

Proof of (12.5 and 6). 1) We first consider the restrictions of the maps in (12.5a and b) and in (12.6) to the affine plane with point set $\mathbb{F} \times \mathbb{F}$. One may easily verify by direct computation that these restrictions transform affine lines into lines and therefore are collineations of the affine plane $\mathscr{A}_2\mathbb{F}$. For (12.5a and b), this only requires the distributive laws; for (12.6), one also uses biassociativity (11.9). The necessary computations show that lines are mapped as follows:

(1) $$[s, t]^{\tau_{a,b}} = [s, t + b - sa];\quad [c]^{\tau_{a,b}} = [c + a]$$

(2) $$[s, t]^{\sigma_a} = [s + a, t];\quad [c]^{\sigma_a} = [c]\,,$$

and that the reflection of (12.6) interchanges

$$(3)\qquad \begin{array}{lll} [s,t] & \text{with} & [s^{-1}, -s^{-1}t] \quad \text{for } s \neq 0\,, \\ [0,t] & \text{with} & [t]\,. \end{array}$$

The effect of these collineations on the slopes of lines shows that their projective extensions act on the line at infinity W in the specified way.

From (3), one infers that the collineation defined in (12.6) leaves all the lines of slope -1 invariant, so that it has center (-1). It is clear that this collineation has axis $[1,0] = \{\, (x,x) \mid x \in \mathbb{F} \,\}$.

Thus, (12.6) is proved. Concerning (12.5), it now suffices to prove (a) and (b), because (c) then follows from (b), as was noted above.

2) The collineation $\tau_{a,b}$ has axis W. If $a \neq 0$ and if $s \in \mathbb{F}$ is the solution of $sa = b$, then by (1) all the lines $[s,t]$, $t \in \mathbb{F}$, are invariant under $\tau_{a,b}$. Now these are precisely the affine lines through the point (s), so that $\tau_{a,b}$ has center (s). For $a = 0$, the center is (∞), since then all the lines $[c]$ are invariant according to equation (1).

Commutativity of T and transitivity on $\mathbb{F} \times \mathbb{F}$ are obvious. We use a standard transitivity argument in order to show that T contains every collineation with axis W and center p on W. For fixed $p \in W$, the group $\mathsf{A}_{[p,W]}$ of all these collineations contains $\mathsf{T}_{[p,W]}$. Now, for any line $M \neq W$ through p, the latter group is transitive on $M \setminus \{p\}$; indeed, for two different points $q_1, q_2 \in M \setminus \{p\}$, the translation $\tau \in \mathsf{T}$ mapping q_1 to q_2 leaves $M = pq_1 = pq_2$ invariant, so that, by the uniqueness properties (12.4c), the center of τ belongs to M and therefore equals $p = M \wedge W$. Again by the uniqueness properties (12.4c), the group $\mathsf{A}_{[p,W]}$ is now seen to be sharply transitive on $M \setminus \{p\}$, so that $\mathsf{A}_{[p,W]} = \mathsf{T}_{[p,W]}$; see also (23.9). This completes the proof of (12.5a).

3) It is obvious that σ_a has axis $Y = \{0\} \times \mathbb{F}$, and from (2) one sees that all the lines $[c]$ of slope ∞, in other words the lines through $v = (\infty)$, are invariant, so that σ_a has center v. Transitivity of the group $\{\, \sigma_a \mid a \in \mathbb{F} \,\}$ on $W \setminus \{v\}$ is obvious. By the same argument as above, it follows that this group coincides with the group $\mathsf{A}_{[v,Y]}$ of all collineations with axis Y and center v, since the latter group is seen to be sharply transitive on $W \setminus \{v\}$ by virtue of the uniqueness properties (12.4c). Thus, (12.5b) is verified. □

Next, we prove some general statements about collineations having non-collinear fixed points.

12.7 Lemma: Stabilizer of a triangle. *The collineations of $\overline{\mathscr{A}_2\mathbb{F}}$ fixing the points $o = (0,0)$, $u = (0)$, and $v = (\infty)$ are precisely the transformations of the form*

$$(x,y) \mapsto (x^\alpha, y^\beta),\ \ (s) \mapsto (s^\gamma),\ \ (\infty) \mapsto (\infty)$$

with automorphisms α, β, γ *of the additive group of* $\mathbb{F}$ *satisfying*

$$(*) \qquad (sx)^\beta = s^\gamma x^\alpha$$

for all $s, x \in \mathbb{F}$. *In particular, these collineations are completely determined by their actions on the coordinate axes* $\mathbb{F} \times \{0\}$ *and* $\{0\} \times \mathbb{F}$.

Proof. A collineation of this kind leaves the coordinate axes $X = ou$ and $Y = ov$ invariant and maps parallels of X and Y to parallels of X and Y, respectively; this means that in the image of (x, y) the first coordinate is independent of y and the second one is independent of x, so that the collineation is of the stated form with bijections α, β, γ of $\mathbb{F}$ fixing 0. We now find the algebraic conditions for such a transformation to be a collineation. Consider the point $(x, sx+t)$ on the line $[s, t]$ joining $(0, t)$ and (s). The corresponding condition for the image points, requiring that the point $(x^\alpha, (sx+t)^\beta)$ lies on the line joining $(0, t^\beta)$ and (s^γ), that is, on the line $[s^\gamma, t^\beta]$, is expressed algebraically by

$$(1) \qquad (sx+t)^\beta = s^\gamma x^\alpha + t^\beta.$$

The special case $t = 0$, which is just $(*)$, allows us to write (1) as $(sx+t)^\beta = (sx)^\beta + t^\beta$, thus exhibiting the additivity of β, which in turn, once more via $(*)$, implies the additivity of α and γ.

Conversely, if $(*)$ holds and if β is additive, then (1) is valid, and we have a collineation. □

12.8 Corollary: Stabilizer of a quadrangle. *The collineations of* $\overline{\mathscr{A}_2}\mathbb{F}$ *fixing the points* $o = (0, 0)$, $u = (0)$, $v = (\infty)$, *and* $e = (1, 1)$ *are precisely the transformations*

$$(x, y) \mapsto (x^\alpha, y^\alpha), \;\; (s) \mapsto (s^\alpha), \;\; (\infty) \mapsto (\infty) \quad \textit{with } \alpha \in \operatorname{Aut}\mathbb{F}.$$

Proof. These collineations are the transformations given in (12.7) which, in addition, satisfy $1^\alpha = 1 = 1^\beta$. From (12.7$(*)$) it then follows that also $1^\gamma = 1$, that $\alpha = \beta = \gamma$, and that, consequently, α is an automorphism of $\mathbb{F}$. □

12.9 Remark. *The collineations in* (12.8) *coincide with the affine collineations fixing the points* o, $e_1 = (1, 0)$, *and* $e_2 = (0, 1)$.

Indeed, if we consider them as collineations of the projective completion $\overline{\mathscr{A}_2}\mathbb{F}$, the line at infinity W remains invariant; now $u = (oe_1) \wedge W$, $v = (oe_2) \wedge W$, $e = (e_1 v) \wedge (e_2 u)$, and conversely $e_1 = (ve) \wedge (ou)$, $e_2 = (ue) \wedge (ov)$. □

In the field case $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, the affine collineation group can now be determined completely.

12.10 Fundamental theorem of affine geometry. *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. The collineations of the affine plane $\mathscr{A}_2\mathbb{F}$ are precisely the mappings*

$$(x, y) \mapsto \varphi(x^\alpha, y^\alpha) + (a, b)$$

for $a, b \in \mathbb{F}$, $\alpha \in \operatorname{Aut}\mathbb{F}$, and $\varphi \in \mathrm{GL}_2\mathbb{F}$ (the group of linear transformations of $\mathbb{F} \times \mathbb{F}$ considered as a right *$\mathbb{F}$-vector space). They form the group $\mathsf{A} = \mathrm{A\Gamma L}_2\mathbb{F}$, which is the semidirect product $\mathsf{A} = \mathsf{T} \rtimes \mathsf{A}_o$ of the translation group T described in (12.5) by the stabilizer of the origin*

$$\mathsf{A}_o = \Gamma\mathrm{L}_2\mathbb{F} = \operatorname{Aut}\mathbb{F} \ltimes \mathrm{GL}_2\mathbb{F}\,.$$

Remark. This theorem and the proof given below are in fact valid over any (not necessarily commutative) field $\mathbb{F}$, cf. (23.6) and the references given there.

Proof of (12.10). Let A denote the group of all affine collineations. As T is the group of all collineations with axis W and center on W, see (12.5), the conjugation formula (12.4b) implies that T is a normal subgroup of A . By transitivity of T on the affine point set, the Frattini argument (91.2a) shows that $\mathsf{A} = \mathsf{A}_o \cdot \mathsf{T} = \mathsf{T} \cdot \mathsf{A}_o$.

The group $\mathrm{GL}_2\mathbb{F}$ consists of collineations because in the field case the lines are just the one-dimensional affine subspaces of the *right* $\mathbb{F}$-vector space $\mathbb{F} \times \mathbb{F}$, as is immediate from their definition (12.1), so that linear transformations map lines to lines. Now $\mathrm{GL}_2\mathbb{F}$ is sharply transitive on the set of bases of $\mathbb{F} \times \mathbb{F}$, whence $\mathsf{A}_o = \mathsf{A}_{o,(1,0),(0,1)} \cdot \mathrm{GL}_2\mathbb{F}$ by the Frattini argument. Finally, the stabilizer $\mathsf{A}_{o,(1,0),(0,1)}$ is described by $\operatorname{Aut}\mathbb{F}$ according to (12.9 and 8). □

We now determine how these affine collineations act on the line at infinity W. Since $\mathsf{A} = \mathsf{T} \cdot \mathsf{A}_o$, and since the translation group T acts trivially on W, only the collineations fixing o need to be considered.

12.11. *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. The stabilizer A_o consists of all the mappings*

$$\gamma_{\alpha,\varphi} : (x, y) \mapsto \varphi(x^\alpha, y^\alpha)$$

for $\alpha \in \operatorname{Aut}\mathbb{F}$ and $\varphi \in \mathrm{GL}_2\mathbb{F}$. If φ is described by a matrix as

$$\varphi : \mathbb{F}^2 \to \mathbb{F}^2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

then the projective extension of the collineation $\gamma_{\alpha,\varphi}$ acts on the line at infinity as the fractional semilinear transformation

$$\gamma_{\alpha,\varphi}|_W : W \to W : (s) \mapsto \left((c + d \cdot s^\alpha)(a + b \cdot s^\alpha)^{-1}\right)\,.$$

This should be understood also for $s = 0$ and $s = \infty$, with the usual conventions about the rôle of 0 and ∞ in such expressions, e.g. with $0^{-1} = \infty$, $\infty^{-1} = 0$.

In order to obtain the asserted description of $\gamma_{\alpha,\varphi}|_W$, we decompose the given collineation as $\gamma_{\alpha,\varphi} = \gamma_{\alpha,\mathrm{id}}\gamma_{\mathrm{id},\varphi}$. The projective extension of $\gamma_{\alpha,\mathrm{id}}$ is the collineation described in (12.8) mapping (s) to (s^α). The projective extension of $\gamma_{\mathrm{id},\varphi}$ maps (s) to (s'), where s' is the slope of the image of the line $[s,0] = \{ (x,sx) \mid x \in \mathbb{F} \} = (1,s)\mathbb{F}$. The image line is $((a+bs),(c+ds))\mathbb{F} = (1,s')\mathbb{F}$, so that $s' = (c+ds)(a+bs)^{-1}$. The description of $\gamma_{\alpha,\varphi}|_W$ now follows by composition. □

We thus have obtained the following result.

Fundamental theorem for the projective line. *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. The group $\mathsf{A}|_W$ of transformations induced on the line at infinity W by the group A of collineations of $\mathscr{A}_2\mathbb{F}$ (via projective extension) is the product*

$$\mathsf{A}|_W = \operatorname{Aut}\mathbb{F} \cdot \mathrm{PGL}_2\mathbb{F}$$

of the group

$$\mathrm{PGL}_2\mathbb{F} = \left\{ (s) \mapsto \left((c+ds)(a+bs)^{-1}\right) \;\middle|\; \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2\mathbb{F} \right\}$$

of fractional linear transformations by the group

$$\{ (s) \mapsto (s^\alpha),\ (\infty) \mapsto (\infty) \mid \alpha \in \operatorname{Aut}\mathbb{F} \} \cong \operatorname{Aut}\mathbb{F}\,.$$

Note. In (15.6), we shall present a version of the preceding theorem which is valid for $\mathbb{F} = \mathbb{O}$, as well.

Addenda.

(a) *The last-mentioned group is the stabilizer of* (0), (1), *and* (∞) *in* $\mathsf{A}|_W$.

(b) *The stabilizer of* (0) *and* (∞) *in* $\mathrm{PGL}_2\mathbb{F}$ *consists of the transformations* $(s) \mapsto (dsa^{-1})$ *for* $a, d \in \mathbb{F}^\times$. *These are precisely the maps* $(s) \mapsto (Bs)$ *where* B *belongs to the group* $\mathrm{GO}_n^+\mathbb{R}$ *defined in* (11.21).

(c) *The fractional linear group* $\mathrm{PGL}_2\mathbb{F}$ *is triply transitive on* W. *It is sharply triply transitive if, and only if* $\mathbb{F}$ *is commutative.*

(d) $\operatorname{Aut}\mathbb{F}$ *is trivial for* $\mathbb{F} = \mathbb{R}$. *In the case* $\mathbb{F} = \mathbb{H}$, *the transformation* $(s) \mapsto (s^\alpha)$ *for* $\alpha \in \operatorname{Aut}\mathbb{F}$ *belongs to* $\mathrm{PGL}_2\mathbb{H}$. *Thus,*

$$\mathsf{A}|_W = \mathrm{PGL}_2\mathbb{F} \quad \textit{for } \mathbb{F} = \mathbb{R}, \mathbb{H}\,.$$

Proof of Addenda. We begin with assertion (b). The first part of the assertion is immediate from the explicit description of $\mathrm{PGL}_2\mathbb{F}$. The second part has been proved in (11.23).

We now turn to (c). The stabilizer of (∞) in $\mathrm{PGL}_2\mathbb{F}$ contains the transformations $(s) \mapsto (s+c)$ for $c \in \mathbb{F}$ and hence is transitive on $W \setminus (\infty)$. Under $(s) \mapsto (s^{-1})$, the points (0) and (∞) are interchanged. Hence, the group $\mathrm{PGL}_2\mathbb{F}$ is doubly transitive on W, and (b) shows that it is even triply transitive. Also from (b), we infer that the stabilizer Λ of the triple (0), (1), (∞) in $\mathrm{PGL}_2\mathbb{F}$ is induced by the inner automorphisms of $\mathbb{F}$, so that this stabilizer is trivial precisely if $\mathbb{F}$ is commutative. Thus, (c) is proved. At the same time, we have obtained that this stabilizer is contained in the subgroup of $\mathsf{A}|_W$ corresponding to $\mathrm{Aut}\,\mathbb{F}$, which proves (a). Moreover, the latter subgroup reduces to $\Lambda \subseteq \mathrm{PGL}_2\mathbb{F}$ whenever all automorphisms of $\mathbb{F}$ are inner, so that then $\mathsf{A}|_W = \mathrm{PGL}_2\mathbb{F}$. This is the case for $\mathbb{F} = \mathbb{H}$ by (11.29), whence (d). $\square$

From now on, we include the case $\mathbb{F} = \mathbb{O}$ in our discussion again. Our next topic is a concrete description of collineations with *non*-incident center and axis. Such collineations are called *homologies.*

12.13 Proposition: Homologies of $\overline{\mathscr{A}_2}\mathbb{F}$.

(a) *The collineations of $\overline{\mathscr{A}_2}\mathbb{F}$ having the line at infinity $W = [\infty]$ as axis and fixing the origin $o = (0, 0)$ are precisely the mappings*

$$\mu_a : (x, y) \mapsto (xa, ya); \ \ \mu_a|_W = \mathrm{id}$$

$$\textit{for } 0 \neq a \in \begin{cases} \mathbb{F} & \textit{if } \ \mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\} \\ \mathrm{Ker}\,\mathbb{O} = \mathbb{R} & \textit{if } \ \mathbb{F} = \mathbb{O}\,. \end{cases}$$

They have center o. These collineations are called homotheties.

In the field case $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, the group of these homotheties is transitive on $X \setminus \{o, X \wedge W\}$. For $\mathbb{F} = \mathbb{O}$, this is not so.

(b) *The collineations of $\overline{\mathscr{A}_2}\mathbb{F}$ with axis $X = [0, 0]$ and center $v = (\infty)$ are precisely the mappings*

$$(x, y) \mapsto (x, ay), \ \ (s) \mapsto (as)$$

for $0 \neq a \in \mathbb{F}$ if $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, and for $0 \neq a \in \mathbb{R}$ if $\mathbb{F} = \mathbb{O}$.

(c) *The collineations of $\overline{\mathscr{A}_2}\mathbb{F}$ with axis $Y = [0]$ and center $u = (0)$ are precisely the mappings*

$$(x, y) \mapsto (ax, y), \ \ (s) \mapsto (sa^{-1})$$

for $0 \neq a \in \mathbb{F}$ if $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, and for $0 \neq a \in \mathbb{R}$ if $\mathbb{F} = \mathbb{O}$.

Remark. The difference between $\mathbb{O}$ and the fields $\mathbb{R}, \mathbb{C}, \mathbb{H}$ exhibited here reflects the non-associativity of $\mathbb{O}$, as will become clear in the proof. Synthetically, this difference is expressed by the fact that Desargues' theorem for non-incident centers and axes, which holds in planes over (not necessarily commutative) fields, is not valid in the octonion plane. The fundamental interrelations between transitivity properties of groups of homologies, the validity of Desargues' theorem, and associativity properties of the coordinate domain are due to Baer [42] Theorem 5.1, p. 146 and Theorem 6.2, p. 151, cf. also (23.22) and the books mentioned there.

Proof of (12.13). (a) The collineations in question are precisely the transformations of the form stated in (12.7) with $\gamma = \mathrm{id}$, i.e., the collineations

$$\delta : (x, y) \mapsto (x^\alpha, y^\beta); \;\; \delta|_W = \mathrm{id}$$

with bijections α, β of $\mathbb{F}$ satisfying

$$(*) \qquad (sx)^\beta = s \cdot x^\alpha$$

for all $s, x \in \mathbb{F}$. With $x = 1$ and $a := 1^\alpha$ this gives $s^\beta = sa$; and putting $s = 1$ we obtain $\beta = \alpha$. Thus, δ is of the form

$$(1) \qquad \delta : (x, y) \mapsto (xa, ya) \,,$$

and $(*)$ requires that

$$(2) \qquad (sx)a = s(xa)$$

for all $s, x \in \mathbb{F}$.

Now, if $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, then every element a satisfies (2) because of associativity; if $\mathbb{F} = \mathbb{O}$, then (2) holds precisely for the elements a of the kernel $\operatorname{Ker} \mathbb{O} = \mathbb{R}$, see (11.20). Thus, the first part of assertion (a) is proved.

For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, the easy direction of the preceding argument would suffice, namely the verification that the homotheties μ_a are indeed collineations for all $a \in \mathbb{F} \setminus \{0\}$. The proof can then be completed by the following standard transitivity argument. Obviously, in these cases, the set $\{\, \mu_a \mid a \in \mathbb{F} \setminus \{0\} \,\}$ of homotheties is transitive on $X \setminus \{o, X \wedge W\} = \mathbb{F} \times \{0\} \setminus \{(0, 0)\}$; one now observes that, by the uniqueness properties (12.4c), the group $\mathsf{A}_{[o,W]}$ of all homologies with center o and axis W is sharply transitive there and thus consists precisely of the homotheties as stated. See also (23.9).

A further proof for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ can be obtained from the fundamental theorem of affine geometry (12.10). The collineations fixing the origin are given in (12.11) together with their actions on W; the collineation $\gamma_{\alpha,\varphi}$ described there has axis W if, and only if $c + d \cdot s^\alpha = s(a + b \cdot s^\alpha)$ for all $s \in \mathbb{F}$. By putting in turn $s = 0, s = 1$, and $s = -1$ one sees that this is equivalent to $c = 0 = b$, $d = a$ and $s^\alpha = a^{-1}sa$ for all $s \in \mathbb{F}$. In this situation, $\gamma_{\alpha,\varphi}$ maps (x, y) to $\varphi(x^\alpha, y^\alpha) = (a \cdot x^\alpha, a \cdot y^\alpha) = (a \cdot (a^{-1}xa), a \cdot (a^{-1}xa)) = (xa, ya)$; this proves our proposition anew. Note that, in the non-commutative case $\mathbb{F} = \mathbb{H}$ at hand, $\gamma_{\alpha,\varphi}$ is not a linear transformation (of the affine point set $\mathbb{H}^2$ considered as a *right* $\mathbb{H}$-vector space)!

(b) may be proved analogously. The collineations with axis X and center v are the mappings

$$(x, y) \mapsto (x, y^\beta), \quad (s) \mapsto (s^\gamma)$$

with bijections β, γ of $\mathbb{F}$ satisfying $(sx)^\beta = s^\gamma x$ for all $s, x \in \mathbb{F}$. With $s = 1$ and $c := 1^\gamma$, we obtain $x^\beta = cx$. Putting $x = 1$, we infer $\gamma = \beta$. Thus, the collineations in question are the transformations $(x, y) \mapsto (x, cy)$, where $c \neq 0$ satisfies

$$c(sx) = (cs)x \quad \text{for all } s, x \in \mathbb{F}\,. \tag{3}$$

For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, the latter condition is trivially fulfilled because of associativity. For $\mathbb{F} = \mathbb{O}$, by applying conjugation to (3), we obtain the equivalent condition $(\bar{x}\,\bar{s})\bar{c} = \bar{x}(\bar{s}\bar{c})$ for all $s, x \in \mathbb{F}$, which says that $\bar{c}$, and hence c, belongs to the kernel $\mathbb{R}$ of $\mathbb{O}$.

In the field cases $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, analogous variations of the proof as in (a) are possible, of course.

(c) is obtained from (b) by conjugation; one applies the conjugation formula (12.4b) to the reflection $(x, y) \mapsto (y, x)$, $(s) \mapsto (s^{-1})$ of (12.6), which interchanges the axis X with Y and the center u with v. □

Collineations of $\mathscr{A}_2\mathbb{O}$, Moufang identities, and triality

Finally, we proceed to a more detailed study of the stabilizer ∇ of the triangle $o = (0, 0)$, $u = (0)$, and $v = (\infty)$ in the collineation group of the projective octonion plane $\overline{\mathscr{A}_2\mathbb{O}}$. The group ∇ can also be viewed as the stabilizer of the coordinate axes $X = ou$ and $Y = ov$ in the group of collineations of the affine plane $\mathscr{A}_2\mathbb{O}$. This stabilizer is an easily accessible part of the collineation group. A study of the whole collineation group of the projective plane $\overline{\mathscr{A}_2\mathbb{O}}$ will be the subject of separate sections (17 and 18).

First, we exhibit special collineations which generate ∇ and which are closely related to the Moufang identities (12.15). By an idea of Salzmann [59c], these collineations can be constructed very easily as compositions of elations which are known from (12.5). When translating the geometric fact that these transformations are collineations into algebraic language, one immediately obtains the Moufang identities. They express other facets of the weak associativity properties of $\mathbb{O}$ besides alternativity (11.13). Actually, they are inherent in alternativity, although this is not entirely obvious. Here, this fact may be obtained by inspection of our argument. For purely algebraic proofs, see Pickert [75] 6.1 no. 2, p. 160, Schafer [66] III.1, p. 28, Zhevlakov et al. [82] Lemma 2.7, p. 35.

With the special collineations mentioned above at our disposal, it is then easy to determine ∇. The structure of ∇ is closely related to the triality principle (12.18).

As a special trait of our presentation, triality does not appear as an algebraic fact having geometric applications, but is intertwined with geometric phenomena right from the start.

Our arguments would be valid more generally for any alternative field instead of $\mathbb{O}$. In particular, they work over (not necessarily commutative) fields, as well; but then, the collineations in question may also be obtained directly from the fundamental theorem of affine geometry (12.10), see (12.19), and the Moufang identities are trivially satisfied because of associativity.

12.14 Proposition. *The following mappings are collineations of* $\overline{\mathscr{A}_2\mathbb{O}}$, *for every* $a \in \mathbb{O} \setminus \{0\}$:

$$\begin{aligned} \gamma_a &: (x, y) \mapsto (a^{-1}x, ay), \ (s) \mapsto (asa) \\ \gamma'_a &: (x, y) \mapsto (axa, ya), \ \ (s) \mapsto (sa^{-1}) \\ \gamma''_a &: (x, y) \mapsto (xa, aya), \ \ (s) \mapsto (as) \ . \end{aligned}$$

Proof. We use the elations $\sigma_a : (x, y) \mapsto (x, y + ax)$, $\sigma'_a : (x, y) \mapsto (x + ay, y)$ from (12.5b and c) and check that γ_a is the following product of such elations:

$$(*) \qquad \gamma_a = \sigma'_1\, \sigma_{a-1}\, \sigma'_{-a^{-1}}\, \sigma_{a-a^2} \ .$$

The biassociativity of $\mathbb{O}$ (11.9) will be essential for the necessary calculations. Under the given product of elations, an affine point is mapped as follows:

$$\begin{aligned} (x, y) &\mapsto (x + y, y) \\ &\mapsto (x + y, y + (a - 1)(x + y)) = (x + y, a(x + y) - x) \\ &\mapsto \big(x + y - a^{-1}(a(x + y)) + a^{-1}x, a(x + y) - x\big) = \big(a^{-1}x, a(x + y) - x\big) \\ &\mapsto \big(a^{-1}x, a(x + y) - x + (a - a^2)(a^{-1}x)\big) = \big(a^{-1}x, a(x + y) - x + x - ax\big) \\ &= (a^{-1}x, ay) \ . \end{aligned}$$

The line $[s, 0]$ joining $(0, 0)$ and $(1, s)$ is mapped onto the line joining the image points $(0, 0)$ and (a^{-1}, as); this line is $[asa, 0]$, since $(asa)a^{-1} = as$ by biassociativity, so that indeed $(a^{-1}, as) \in [asa, 0]$. The projective extension thus maps (s) to (asa), and $(*)$ is proved.

According to (12.7), the fact that γ_a is a collineation is expressed algebraically by the identity $a(sx) = (asa)(a^{-1}x)$. Substituting $z = a^{-1}x$, $az = x$ yields the first *Moufang identity*

$$(1) \qquad a(s(az)) = (asa)z \ .$$

Applying conjugation on both sides of (1) and renaming, one obtains the second Moufang identity

$$(2) \qquad ((ba)x)a = b(axa) \ .$$

With $s = ba$, $sa^{-1} = b$ this is transformed into the identity $(sx)a = (sa^{-1})(axa)$, from which in turn, by applying (12.7) backwards, it follows that γ'_a is a collineation. An immediate verification with the help of biassociativity shows that $\gamma''_a = \gamma_a\gamma'_a$, so that this is a collineation as well. By (12.7) once again, the latter fact is equivalent to the third Moufang identity

$$(3) \qquad a(sx)a = (as)(xa) .$$

We collect these identities as a corollary to the proof.

12.15 The Moufang identities. *For all* $a, b, c \in \mathbb{O}$, *the following hold*:

$$\begin{aligned} a(b(ac)) &= (aba)c \\ ((ab)c)b &= a(bcb) \\ (ab)(ca) &= a(bc)a . \end{aligned}$$

□

12.16 Lemma. *The group of collineations generated by the collineations* γ_a *and* γ'_a *of* (12.14) *for* $a \in \mathbb{O} \setminus \{0\}$ *fixes the vertices of the triangle* o, u, v *and is transitive on the set* $(\mathbb{O} \setminus \{0\}) \times (\mathbb{O} \setminus \{0\})$ *of affine points not incident with the sides of this triangle.*

Proof. For $a, b \in \mathbb{O} \setminus \{0\}$, we show that the point $(1, 1)$ may be mapped to the point (a, b) by a product of collineations of the specified kind. Indeed, since ba belongs to a subfield of $\mathbb{O}$ isomorphic to $\mathbb{C}$, see (11.7), there is $c \in \mathbb{O}$ with $ba = c^3$. Now, using alternativity (11.8), one easily verifies that $\gamma_{c^{-1}}\gamma'_c\gamma_b$ maps $(1, 1)$ to (a, b). □

In the following proposition, we use the notation for orthogonal groups introduced in (11.21).

12.17 Proposition: The stabilizer of the coordinate axes.

(a) *The group* ∇ *of collineations of* $\overline{\mathcal{A}_2\mathbb{O}}$ *leaving the points* o, u, *and* v *fixed consists of the transformations*

$$(A, B \mid C) : (x, y) \mapsto (Ax, By), \ (s) \mapsto (Cs) ,$$

where A, B, C *belong to* $\mathrm{GO}_8^+\mathbb{R}$ *and satisfy the* triality condition

$$(*) \qquad B(s \cdot x) = Cs \cdot Ax \quad \text{for all } s, x \in \mathbb{O} .$$

(b) *The group* ∇ *is the direct product of the subgroup consisting of the transformations*

$$\mu_{r,t} : (x, y) \mapsto (rx, ty), \ (s) \mapsto (tr^{-1} \cdot s) \quad \text{for } 0 < r, t \in \mathbb{R}$$

by the subgroup

$$\mathrm{S}\nabla = \{ (A, B \mid C) \in \nabla \mid A, B, C \in \mathrm{SO}_8\mathbb{R} \} .$$

(c) *The projection homomorphisms* $\mathrm{pr}_\nu : \mathrm{S}\nabla \to \mathrm{SO}_8\mathbb{R}$ *for* $\nu = 1, 2, 3$ *defined by* $\mathrm{pr}_\nu(A_1, A_2 \mid A_3) = A_\nu$ *are surjective. Their kernels are of order* 2 *and are generated by the reflections*

$$\begin{aligned}
\iota_v &= \mu_{1,-1} : (x, y) \mapsto (x, -y), \quad (s) \mapsto (-s) \text{ with center } v, \text{ axis } X = [0, 0] \\
\iota_u &= \mu_{-1,1} : (x, y) \mapsto (-x, y), \quad (s) \mapsto (-s) \text{ with center } u, \text{ axis } Y = [0] \\
\iota_o &= \mu_{-1,-1} : (x, y) \mapsto (-x, -y), \quad (s) \mapsto (s) \quad \text{with center } o, \text{ axis } W = [\infty] .
\end{aligned}$$

These reflections are the non-trivial elements of the center $\mathbb{Z}_2 \times \mathbb{Z}_2$ *of* $\mathrm{S}\nabla$.

Proof. 1) According to (12.7), the transformations of the form $(A, B \mid C)$ for $A, B, C \in \mathrm{GO}_8^+\mathbb{R}$ are collineations if, and only if, the triality condition $(*)$ is satisfied; they obviously fix o, u, and v. Let Ψ be the subgroup of ∇ consisting of these transformations. By (11.22), the collineations given in (12.14) belong to Ψ. From (12.16), we know that Ψ is transitive on $(\mathbb{O} \setminus \{0\}) \times (\mathbb{O} \setminus \{0\})$. The Frattini argument (91.2a) yields that $\nabla = \nabla_e \cdot \Psi$, where $e = (1, 1)$. According to (12.8), the stabilizer ∇_e of o, u, v, e consists of the collineations $(x, y) \mapsto (x^\alpha, y^\alpha)$, $(s) \mapsto (s^\alpha)$ for $\alpha \in \mathrm{Aut}\,\mathbb{O}$. Now $\mathrm{Aut}\,\mathbb{O}$ is contained in $\mathrm{SO}_8\mathbb{R}$ by (11.31e). Thus $\nabla_e \subseteq \Psi$ and $\nabla = \Psi$, as asserted in (a).

2) We now prove (b). For an element $(A, B \mid C)$ of ∇, let $A = rA'$, $B = tB'$, $C = qC'$ with $0 < r, t, q \in \mathbb{R}$ and $A', B', C' \in \mathrm{SO}_8\mathbb{R}$. Then, in the equation

$$tB'(s \cdot x) = qC's \cdot rA'x$$

obtained from $(*)$, we consider the norms of both sides. As A', B', C' are orthogonal maps and as the norm is multiplicative (11.14), we see that $t^2\|s\|^2\|x\|^2 = q^2\|s\|^2 r^2\|x\|^2$, whence $q = tr^{-1}$. This shows that $(A, B \mid C) = \mu_{r,t} \cdot (A', B' \mid C')$.

3) Surjectivity of pr_ν follows from the fact that the collineations γ_a, γ'_a, and γ''_a constructed in (12.14) for $a \in \mathbb{O} \setminus \{0\}$ belong to $\mathrm{S}\nabla$ if $\|a\|^2 = 1$. Indeed, the corresponding transformations $x \mapsto ax$, $x \mapsto xa$, and $x \mapsto axa$ belong to $\mathrm{SO}_8\mathbb{R}$ and generate this group by (11.22).

The kernel of pr_3 consists of homologies with axis W and center o, i.e., of real homotheties according to (12.13a). Since $\mathrm{S}\nabla$ induces an orthogonal group on the coordinate axes, it follows that $\ker \mathrm{pr}_3 = \{(x, y) \mapsto (\pm x, \pm y)\}$. Analogously, $\ker \mathrm{pr}_1 = \mathrm{S}\nabla_{[v,X]}$ and $\ker \mathrm{pr}_2 = \mathrm{S}\nabla_{[u,Y]}$ can be obtained from the description of homologies in (12.13b and c).

have order 2, the center of $\mathrm{S}\nabla$ cannot have more than four elements, so that it consists of id, ι_v, ι_u, and ι_o. □

Reformulating certain aspects of the preceding geometric result (12.17) in algebraic terms, one obtains the following well-known statement.

12.18 Triality principle. *Among the triples of transformations $A, B, C \in \mathrm{SO}_8\mathbb{R}$ satisfying*

$$(*) \qquad B(s \cdot x) = Cs \cdot Ax \quad \textit{for all } s, x \in \mathbb{O}\,,$$

each of A, B, or C may take on every value in $\mathrm{SO}_8\mathbb{R}$, and, in such a triple, each of the elements A, B, C determines the other two uniquely up to sign.

This is an immediate consequence of the information in (12.17c) about the surjective homomorphisms pr_ν and their kernels. By (12.7), the triality condition $(*)$ just expresses the fact that $(A, B \mid C)$ is a collineation.

For other approaches to the triality principle, see e.g. van der Blij–Springer [60] and Harvey [90] p. 275 ff.

12.19 Remarks. 1) The study of the stabilizer ∇ and of certain of its subgroups will be continued in Section 17, see (17.11 through 16). For instance, the homomorphisms pr_ν will be used to show that $\mathrm{S}\nabla$ is the universal (two-fold) covering group $\mathrm{Spin}_8\mathbb{R}$ of $\mathrm{SO}_8\mathbb{R}$. The discussion of the latter topic in (17.13) does not make essential use of Sections 13–16, so the reader may continue right there if he wishes.

2) Over $\mathbb{R}$ and $\mathbb{H}$ instead of $\mathbb{O}$, the group ∇ may be described in complete analogy with (12.17). Over $\mathbb{C}$, the collineations analogous with those given in (12.17) only constitute the $\mathbb{C}$-linear part of ∇.

Over $\mathbb{F} \in \{\mathbb{R}, \mathbb{H}\}$, even the proof of (12.17) carries over verbatim. On the other hand, the result may be obtained directly from the fundamental theorem (12.10) of affine geometry, according to which the affine collineations fixing the coordinate axes $X = \mathbb{F} \times \{0\}$ and $Y = \{0\} \times \mathbb{F}$ are the maps

$$(x, y) \mapsto (ax^\alpha, dy^\alpha)\,,$$

for $a, d \in \mathbb{F}^\times$ and $\alpha \in \mathrm{Aut}\,\mathbb{F}$. The only automorphism of $\mathbb{R}$ is the identity, and the automorphisms of $\mathbb{H}$ are precisely the inner automorphisms (11.25). Thus, the collineations in question, with their actions on the line at infinity according to (12.11), are the mappings of the form

$$(x, y) \mapsto (axc, dyc), \ \ (s) \mapsto (dsa^{-1})\,.$$

(This collineation is composed of the collineation $(x, y) \mapsto (c^{-1}xc, c^{-1}yc)$,

$(s) \mapsto (c^{-1}sc)$ obtained from the inner automorphism $x \mapsto c^{-1}xc$, and of the collineation $(x, y) \mapsto (acx, dcy)$, whose action on the line at infinity is $(s) \mapsto (dcs(ac)^{-1}) = (dcsc^{-1}a^{-1})$.) Now, the transformations $\mathbb{H} \to \mathbb{H} : x \mapsto axc$ constitute the group $\mathrm{GO}_4^+\mathbb{R}$, see (11.23), so that we arrive at a description of ∇ for the plane over $\mathbb{H}$ which is analogous to (12.17).

Over $\mathbb{C}$, the collineations corresponding to the multitude of field automorphisms, see the references in (44.11), are not covered by this description; except conjugation, they are all discontinuous.

12.20 Note. For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, the group $\mathsf{A}|_W$ of transformations induced on the line at infinity by the group of all affine collineations was discussed in (12.12) and was shown to be the product $\mathrm{Aut}\,\mathbb{F} \cdot \mathrm{PGL}_2\mathbb{F}$. For the case $\mathbb{F} = \mathbb{O}$, we shall not enter into a detailed discussion of $\mathsf{A}|_W$ at this point; the usual definitions of $\mathrm{GL}_2\mathbb{F}$ and $\mathrm{PGL}_2\mathbb{F}$ for a field $\mathbb{F}$ do not make sense if $\mathbb{F}$ is replaced by $\mathbb{O}$. Instead, we shall give a different geometric description of the group $\mathsf{A}|_W$ in Section 15, which is valid uniformly for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and for $\mathbb{F} = \mathbb{O}$, see (15.6). At this stage, we shall merely add a few remarks related to the results obtained so far; these remarks will not be used in the sequel.

From what we know, it is not difficult to deduce that $\mathsf{A}|_W$ is generated by the transformations $\sigma_a|_W$ and $\sigma'_a|_W$ for $a \in \mathbb{O}$, where σ_a and σ'_a are the shears described in (12.5b and c). Indeed, the subgroup Ω generated by these transformations acts 2-transitively on W, so that it suffices to show that Ω contains the stabilizer of (0) and (∞). This stabilizer is the group $\nabla|_W$, where, as before, ∇ is the stabilizer of the points o, (0), and (∞) in the group of all collineations; recall that $\mathsf{A}|_W = \mathsf{A}_o|_W$ because of transitivity of the group of translations with axis W. Now ∇ was discussed in (12.17); arguing as in step 3) of the proof there, one sees that $\nabla|_W$ is generated by the restrictions $\gamma_a|_W$, $a \in \mathbb{O}\setminus\{0\}$, of the collineations described in (12.14). These, in turn, have been constructed explicitly as products of shears.

If $\mathbb{O}$ is replaced with an arbitrary Cayley division algebra K, then Ω need not coincide with $\mathsf{A}|_W$ any more, but still is an important normal subgroup. As shown by Timmesfeld [94], the group Ω encodes the entire geometry of the plane over K. One of the main themes of his paper are characterizations of the subgroup of A generated by the shears σ_a and σ'_a for $a \in K$. Timmesfeld calls that group SL_2K, thus giving a meaning to this otherwise undefined symbol. In the case of a field K, this is in accordance with the usual meaning and with the description of $\mathsf{A}|_W$ using the group PGL_2K as in (12.12).

The following fact will be useful in later sections when we continue the study of the collineation group of the octonion plane.

12.21 Lemma. *The group* A *of all collineations of the affine octonion plane* $\mathscr{A}_2\mathbb{O}$ *is the semidirect product of the translation group* T *described in* (12.5) *by the* *stabilizer* A_o *of the origin* $o = (0, 0)$, *and* A_o *consists of* $\mathbb{R}$*-linear transformations of* $\mathbb{O} \times \mathbb{O}$.

Proof. Since T is a normal subgroup of A and is transitive on the affine point set $\mathbb{O} \times \mathbb{O}$, we have $\mathsf{A} = \mathsf{A}_o \cdot \mathsf{T} = \mathsf{T} \cdot \mathsf{A}_o$ exactly as in (12.10), by the Frattini argument (91.2a). If one is willing to use basic information about translation planes, see (25.5), the linearity assertion is obtained readily: the collineations in A_o are known to be semilinear over the kernel of a coordinatizing quasifield, which in our case is $\mathbb{O}$. Now the kernel of $\mathbb{O}$ is $\mathbb{R}$ (11.20) and has no automorphism except the identity, so that semilinearity implies linearity.

Without using this information, a direct argument for the linearity of A_o may be given as follows. The subgroup Λ of A_o generated by the shears $(x, y) \mapsto (x, y + ax)$, $(s) \mapsto (s + a)$, $(\infty) \mapsto (\infty)$ of (12.5b) and by the reflection $(x, y) \mapsto (y, x)$, $(s) \mapsto (s^{-1})$ for $s \neq 0$, $(0) \leftrightarrow (\infty)$, obviously consists of $\mathbb{R}$-linear collineations. Moreover, Λ acts 2-transitively on the line at infinity, so that $\mathsf{A}_o = \Lambda \cdot \mathsf{A}_{o,u,v}$ by the Frattini argument. Now, finally, $\mathsf{A}_{o,u,v} = \nabla$ also consists of $\mathbb{R}$-linear transformations by (12.17a). □

13 The projective planes over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$

The projective planes $\overline{\mathscr{A}_2}\mathbb{F}$ over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ are highly homogeneous structures. For instance, the group of all collineations is transitive both on the set of points and on the set of lines, including the points at infinity and the line at infinity, see (13.5) and (17.2). This fact is obscured, however, by the representation of $\overline{\mathscr{A}_2}\mathbb{F}$ as the projective completion of $\mathscr{A}_2\mathbb{F}$; that description, given in (12.2), assigns a special rôle to the elements at infinity. The classical method to remedy this defect is to introduce homogeneous coordinates. For the planes over the fields $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, this is easy and will be our first objective here. Due to the non-associativity of $\mathbb{O}$, an equally homogeneous description of the octonion projective plane is of needs more complicated, and will be put off to Section 16. The present section continues with a proof of the fundamental theorem of projective geometry, which describes the group of all collineations of the projective plane over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ in terms of homogeneous coordinates. Finally, we examine some distinguished subgroups of those collineation groups, namely the elliptic motion groups and the hyperbolic motion groups. They can be defined as the groups of all (linear) collineations which commute with certain polarities. Our interest focusses on the transitivity properties of these groups.

13.0 General assumption. In this section, $\mathbb{F}$ shall denote one of the fields $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Many of our arguments would work for an arbitrary field.

13.1 The projective plane $\mathcal{P}_2\mathbb{F}$ over $\mathbb{F}$ is defined as follows. Consider $\mathbb{F}^3$ as a right vector space over $\mathbb{F}$. The 1-dimensional subspaces of $\mathbb{F}^3$ will be called 'points'; the 'lines' will be the 2-dimensional subspaces. Incidence between points and lines is given by inclusion. The set of points will be denoted by $\mathrm{P}_2\mathbb{F}$, and the set of lines by $\mathcal{L}_2\mathbb{F}$.

In this definition, the rôles of points and of lines are completely symmetric. If one wishes to think of lines as subsets of the set of points, one has to identify a line with the *point row* consisting of the points incident with it.

Elementary linear algebra shows that this geometry $\mathcal{P}_2\mathbb{F}$ is indeed a projective plane in the sense of definition (21.1). We shall see soon (13.3) that in fact $\mathcal{P}_2\mathbb{F}$ is isomorphic to the projective completion $\overline{\mathcal{A}_2}\mathbb{F}$ constructed in (12.2).

13.2 Homogeneous coordinates in $\mathcal{P}_2\mathbb{F}$**.** A point $p \in \mathrm{P}_2\mathbb{F}$, i.e., a 1-dimensional subspace of $\mathbb{F}^3$, is spanned by some nonzero vector $(x_1, x_2, x_3) \in \mathbb{F}^3$,

$$p = (x_1, x_2, x_3)\mathbb{F} .$$

The point p determines the coordinates $x_1, x_2, x_3 \in \mathbb{F}$ uniquely up to a common nonzero factor from the right; these coordinates are called *homogeneous coordinates* of p.

In order to introduce homogeneous coordinates for lines, we recall that the 2-dimensional subspaces of $\mathbb{F}^3$ are precisely the kernels of nonzero linear forms of $\mathbb{F}^3$. Assume that the 2-dimensional subspace U is the kernel of the linear form

$$(\,a_1 \quad a_2 \quad a_3\,) : \mathbb{F}^3 \longrightarrow \mathbb{F} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto (\,a_1 \quad a_2 \quad a_3\,) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sum_{\nu=1}^{3} a_\nu x_\nu ,$$

$$U = \operatorname{Ker}(\,a_1 \quad a_2 \quad a_3\,) ,$$

where $(a_1, a_2, a_3) \in \mathbb{F}_3 \setminus \{0\}$. Then U determines the coefficients $a_\nu \in \mathbb{F}$ of such a linear form uniquely up to a common factor from the left. These coefficients are called homogeneous coordinates of the line U. It is obvious that incidence of points and lines can be expressed in homogeneous coordinates as follows.

The point $p = (x_1, x_2, x_3)\mathbb{F}$ *is incident with the line* $\operatorname{Ker}(\,a_1 \quad a_2 \quad a_3\,)$ *if, and only if,*

$$(*) \qquad (\,a_1 \quad a_2 \quad a_3\,) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sum_{\nu=1}^{3} a_\nu x_\nu = 0 .$$

13.3 An isomorphism $\overline{\mathcal{A}_2}\mathbb{F} \cong \mathcal{P}_2\mathbb{F}$**.** In $\mathcal{P}_2\mathbb{F}$, we single out the line

$$W := \mathbb{F} \times \mathbb{F} \times \{0\} = \operatorname{Ker}(\,0 \quad 0 \quad 1\,) \in \mathcal{L}_2\mathbb{F} .$$

The affine plane $\mathscr{A}_2\mathbb{F}$ as described in (12.1) can be located within $\mathbb{F}^3$ on the 2-dimensional affine subspace

$$A := \mathbb{F} \times \mathbb{F} \times \{1\}$$

parallel to W by means of the bijection

$$\mathbb{F} \times \mathbb{F} \to A : (x, y) \mapsto (x, y, 1) .$$

This bijection maps the lines of $\mathscr{A}_2\mathbb{F}$ as defined in (12.1) onto the 1-dimensional affine subspaces of A.

Now we interpret this geometry on A in the projective plane $\mathscr{P}_2\mathbb{F}$, see Figure 13a. The vectors in A span the 1-dimensional linear subspaces of $\mathbb{F}^3$ not contained in W; these are the points of $\mathscr{P}_2\mathbb{F}$ not incident with the line W. A line L in A (a 1-dimensional affine subspace) spans a 2-dimensional linear subspace $U \neq W$ of $\mathbb{F}^3$, which is a line of $\mathscr{P}_2\mathbb{F}$; the points on this line are the 1-dimensional linear subspaces containing some vector on L, and the 1-dimensional subspace $U \cap W$. Denoting by $\mathrm{P}_2\mathbb{F} \setminus W$ the set of points of $\mathscr{P}_2\mathbb{F}$ not incident with W, we have a bijection

$$\psi : \mathbb{F} \times \mathbb{F} \to \mathrm{P}_2\mathbb{F} \setminus W : (x, y) \mapsto (x, y, 1)\mathbb{F} ,$$

and ψ is a collineation of the affine plane $\mathscr{A}_2\mathbb{F}$ onto an affine plane with point set $\mathrm{P}_2\mathbb{F} \setminus W$ whose lines are the elements of $\mathscr{L}_2\mathbb{F} \setminus \{W\}$ (disregarding their points on W); this affine plane will be denoted by $\mathscr{P}_2\mathbb{F}^W$.

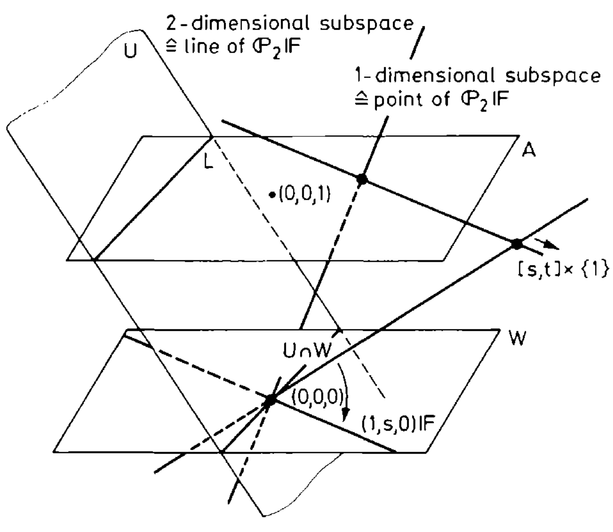

Figure 13a

We shall write down the inverse map ψ^{-1} explicitly, and incidentally prove the bijectivity of ψ once again. To do this, we use the fact that one is allowed to multiply homogeneous coordinates of a point by a common scalar from the right.

Thus, for a point $(x_1, x_2, x_3)\mathbb{F}$ not belonging to W, i.e., with $x_3 \neq 0$, we have $(x_1, x_2, x_3)\mathbb{F} = (x_1 x_3^{-1}, x_2 x_3^{-1}, 1)\mathbb{F}$. Hence,

(1) $$\mathrm{P}_2\mathbb{F} \setminus W = \{\, (x, y, 1)\mathbb{F} \mid x, y \in \mathbb{F} \,\} \ ,$$

and the inverse of ψ is the map

$$\psi^{-1} : \mathrm{P}_2\mathbb{F} \setminus W \to \mathbb{F} \times \mathbb{F} : (x_1, x_2, x_3)\mathbb{F} \mapsto (x_1 x_3^{-1}, x_2 x_3^{-1}) \ .$$

By an analogous normalization we obtain that the points on $W = \mathbb{F} \times \mathbb{F} \times \{0\}$ can be written as

(2) $$(1, s, 0)\mathbb{F} \quad \text{for } s \in \mathbb{F} \quad \text{and} \quad (0, 1, 0)\mathbb{F} \ .$$

In the affine plane $\mathcal{P}_2\mathbb{F}^W$, two different lines $L_1, L_2 \in \mathcal{L}_2\mathbb{F} \setminus \{W\}$ are parallel if, and only if, their point of intersection in the projective plane $\mathcal{P}_2\mathbb{F}$ lies on W. Thus, $\mathcal{P}_2\mathbb{F}$ is (isomorphic to) the projective completion of the affine plane $\mathcal{P}_2\mathbb{F}^W$, with W having the rôle of the line at infinity. To put it more precisely, the collineation $\psi : \mathcal{A}_2\mathbb{F} \to \mathcal{P}_2\mathbb{F}^W$ extends uniquely to a collineation $\overline{\psi}$ of the projective completion $\overline{\mathcal{A}_2\mathbb{F}}$ onto the projective plane $\mathcal{P}_2\mathbb{F}$. We now give a complete coordinate description of that collineation.

A correspondence between affine and homogeneous coordinates.
The collineation $\overline{\psi}$ of the projective completion $\overline{\mathcal{A}_2\mathbb{F}}$ onto $\mathcal{P}_2\mathbb{F}$ is given by the following maps between points and lines, respectively, of $\overline{\mathcal{A}_2\mathbb{F}}$ and $\mathcal{P}_2\mathbb{F}$.

$$\begin{array}{rlrl} (x, y) \mapsto & (x, y, 1)\mathbb{F} & [s, t] \to & \mathrm{Ker}\,(\,-s \quad 1 \quad -t\,) \\ (s) \mapsto & (1, s, 0)\mathbb{F} & [c] \to & \mathrm{Ker}\,(\,1 \quad 0 \quad -c\,) \\ (\infty) \mapsto & (0, 1, 0)\mathbb{F} & [\infty] \to & \mathrm{Ker}\,(\,0 \quad 0 \quad 1\,) = W \ . \end{array}$$

Indeed, from (1) and (2) it is immediate that the left column describes a bijection between the point sets of $\overline{\mathcal{A}_2\mathbb{F}}$ and $\mathcal{P}_2\mathbb{F}$. Furthermore, one has to check that this bijection maps the point row of every line onto the point row of the image line specified in the right column. For example, in $\mathcal{A}_2\mathbb{F}$ the point (x, y) is incident with the line $[s, t]$ if, and only if $y = sx + t$, i.e., $-sx + y - t = 0$; this is equivalent to the condition that $\overline{\psi}(x, y) = (x, y, 1)\mathbb{F}$ is incident with the line $\mathrm{Ker}\,(\,-s \quad 1 \quad -t\,)$ of $\mathcal{P}_2\mathbb{F}$, see (13.2(∗)). That line also contains the point $(1, s, 0)\mathbb{F}$ of W corresponding to the point (s) at infinity of $[s, t]$. The other verifications are just as easy. □

The isomorphism $\overline{\mathcal{A}_2\mathbb{F}} \cong \mathcal{P}_2\mathbb{F}$ described above may serve to illustrate the term 'point at infinity'. The 1-dimensional linear subspace $(1, s, 0)\mathbb{F} = \overline{\psi}((s))$ is parallel to the line $[s, t] \times \{1\} = \{\, (x, sx + t, 1) \mid x \in \mathbb{F} \,\}$ in $A \subseteq \mathbb{F}^3$ for every $t \in \mathbb{F}$. Every projective point corresponding to a point of this affine line can be rewritten as $(x, sx + t, 1)\mathbb{F} = (1, s + tx^{-1}, x^{-1})\mathbb{F}$ for $x \neq 0$. If x 'tends to infinity', this projective point 'converges' to the point $(1, s, 0)\mathbb{F}$ corresponding to the point at infinity (s), see Figure 13a. This will be made more precise in Section 14.

Collineations

13.4 Linear collineations. A linear transformation $A \in \mathrm{GL}_3\mathbb{F}$ of the right vector space $\mathbb{F}^3$ maps 1-dimensional subspaces of $\mathbb{F}^3$ to 1-dimensional subspaces and therefore induces a bijection of the point set of $\mathcal{P}_2\mathbb{F}$, given by

$$[A] : \mathrm{P}_2\mathbb{F} \to \mathrm{P}_2\mathbb{F} : x\mathbb{F} \mapsto (Ax)\mathbb{F} ,$$

where $x \in \mathbb{F}^3$. Furthermore, $[A]$ is a collineation of $\mathcal{P}_2\mathbb{F}$, because A maps 2-dimensional subspaces of $\mathbb{F}^3$ to 2-dimensional subspaces. The collineations which are obtained in this way are called *linear collineations*; they form a subgroup

$$\mathrm{PGL}_3\mathbb{F} := \{ [A] \mid A \in \mathrm{GL}_3\mathbb{F} \}$$

of the group $\mathrm{Aut}\,\mathcal{P}_2\mathbb{F}$ of all collineations of $\mathcal{P}_2\mathbb{F}$.

As to transitivity properties of $\mathrm{PGL}_3\mathbb{F}$, we consider *non-degenerate quadrangles* of $\mathcal{P}_2\mathbb{F}$, i.e., quadruples of points no three of which are collinear. One such quadrangle is the *standard quadrangle* $e_1\mathbb{F}$, $e_2\mathbb{F}$, $e_3\mathbb{F}$, $(e_1 + e_2 + e_3)\mathbb{F}$ where e_1, e_2, e_3 is the standard basis of $\mathbb{F}^3$. If p_1, p_2, p_3, p_4 is an arbitrary non-degenerate quadrangle, then the 1-dimensional subspaces p_1, p_2, p_3 generate $\mathbb{F}^3$ because they are not collinear; thus, there is a basis b_1, b_2, b_3 such that $p_\nu = b_\nu\mathbb{F}$ $(\nu = 1, 2, 3)$. A vector $x \in \mathbb{F}^3$ generating the 1-dimensional subspace p_4 may be represented as $x = \sum_{\nu=1}^{3} b_\nu\lambda_\nu$ with $\lambda_\nu \in \mathbb{F}$; and the scalars λ_ν are all nonzero because p_4 is not collinear with any two of p_1, p_2, p_3. Let $A \in \mathrm{GL}_3\mathbb{F}$ be the linear transformation mapping the standard basis e_1, e_2, e_3 onto $b_1\lambda_1, b_2\lambda_2, b_3\lambda_3$; then the collineation $[A]$ maps the standard quadrangle onto p_1, p_2, p_3, p_4. Thus we have obtained the following result.

13.5 Proposition: Homogeneity. $\mathrm{PGL}_3\mathbb{F}$ *is transitive on the set of non-degenerate quadrangles.* □

13.6 Fundamental theorem of projective geometry. *The collineations of* $\mathcal{P}_2\mathbb{F}$ *are precisely the transformations induced by semilinear bijections of* $\mathbb{F}^3$. *Explicitly, these are the transformations* $[A, \alpha]$ *defined by*

$$[A, \alpha] : \mathrm{P}_2\mathbb{F} \to \mathrm{P}_2\mathbb{F} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \mathbb{F} \mapsto A \begin{pmatrix} x_1^\alpha \\ x_2^\alpha \\ x_3^\alpha \end{pmatrix} \cdot \mathbb{F} ,$$

where A is a regular 3×3-matrix over $\mathbb{F}$, and $\alpha \in \mathrm{Aut}\,\mathbb{F}$.

Proof. By (13.5) and with the help of the Frattini argument (91.2a), the proof reduces to showing that the stabilizer of the standard quadrangle

$$e_1\mathbb{F} = (1,0,0)\mathbb{F},\ \ e_2\mathbb{F} = (0,1,0)\mathbb{F},\ \ e_3\mathbb{F} = (0,0,1)\mathbb{F},\ \ (e_1+e_2+e_3)\mathbb{F} = (1,1,1)\mathbb{F}$$

in $\operatorname{Aut}\mathcal{P}_2\mathbb{F}$ consists precisely of the transformations $[\mathrm{id}, \alpha]$, $\alpha \in \operatorname{Aut}\mathbb{F}$.

By means of the collineation $\overline{\psi} : \overline{\mathcal{A}_2\mathbb{F}} \cong \mathcal{P}_2\mathbb{F}$ described in (13.3), a collineation of $\mathcal{P}_2\mathbb{F}$ fixing the standard quadrangle can be written as $\overline{\psi} \circ \delta \circ \overline{\psi}^{-1}$, where δ is a collineation of $\overline{\mathcal{A}_2\mathbb{F}}$ that fixes the corresponding quadrangle formed by the points $u = (0)$, $v = (\infty)$, $o = (0,0)$, and $e = (1,1)$. By (12.8), the collineations of $\overline{\mathcal{A}_2\mathbb{F}}$ fixing those points are precisely the transformations $\delta : (x,y) \mapsto (x^\alpha, y^\alpha)$, $(s) \mapsto (s^\alpha)$, $(\infty) \mapsto (\infty)$ for all $\alpha \in \operatorname{Aut}\mathbb{F}$. For these, it is easy to show that $\overline{\psi} \circ \delta \circ \overline{\psi}^{-1} = [\mathrm{id}, \alpha]$ by checking that $\overline{\psi} \circ \delta = [\mathrm{id}, \alpha] \circ \overline{\psi}$. □

13.7 Remarks. Not only the collineations coming from automorphisms of $\mathbb{F}$, but in fact all collineations of $\mathcal{A}_2\mathbb{F}$ as determined by the fundamental theorem of affine geometry (12.10) are easily translated into homogeneous coordinates via the collineation $\overline{\psi} : \overline{\mathcal{A}_2\mathbb{F}} \to \mathcal{P}_2\mathbb{F}$ of (13.3). In this way, the collineation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} x^\alpha \\ y^\alpha \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

of $\mathcal{A}_2\mathbb{F}$ yields the collineation $[C, \alpha]$ of $\mathcal{P}_2\mathbb{F}$ with

$$C = \begin{pmatrix} c_{11} & c_{12} & a \\ c_{21} & c_{22} & b \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3\mathbb{F}\,.$$

For a diagonal matrix

$$D = \begin{pmatrix} d & & \\ & d & \\ & & d \end{pmatrix}, \qquad 0 \neq d \in \mathbb{F}\,,$$

the collineation $[D] = [D, \mathrm{id}]$ is the same as $[\mathrm{id}, \operatorname{int} d]$, where $\operatorname{int} d$ is the inner automorphism $\operatorname{int} d : \mathbb{F} \to \mathbb{F} : x \mapsto dxd^{-1}$; indeed, $[D, \mathrm{id}]$ maps $(x_1, x_2, x_3)\mathbb{F}$ to $(dx_1, dx_2, dx_3)\mathbb{F} = (dx_1d^{-1}, dx_2d^{-1}, dx_3d^{-1})\mathbb{F}$.

Thus, if all automorphisms of $\mathbb{F}$ are inner automorphisms, then every collineation is induced by a linear transformation of $\mathbb{F}^3$. Here, this applies to $\mathbb{R}$ and $\mathbb{H}$, see (11.26 and 28).

13.8 Corollary. *For* $\mathbb{F} \in \{\mathbb{R}, \mathbb{H}\}$, *the full collineation group of* $\mathcal{P}_2\mathbb{F}$ *is equal to* $\mathrm{PGL}_3\mathbb{F}$. □

13.9 Warning. For $\mathbb{F} = \mathbb{C}$, conjugation is an automorphism, and therefore the map

$$(x_1, x_2, x_3)\mathbb{C} \mapsto (\overline{x_1}, \overline{x_2}, \overline{x_3})\mathbb{C}$$

is an involutory collineation of $\mathcal{P}_2\mathbb{C}$, whose fixed points are precisely the points of $\mathcal{P}_2\mathbb{R}$ viewed as a subplane of $\mathcal{P}_2\mathbb{C}$. In the case $\mathbb{F} = \mathbb{H}$, the analogous definition would not even give a well-defined map. For example, with the standard Hamilton triple i, j, k of $\mathbb{H}$ (11.16), one has $(j, k, 0)\mathbb{H} = (j, ij, 0)\mathbb{H} = (1, i, 0)\mathbb{H}$ and $(\bar{j}, \bar{k}, \bar{0})\mathbb{H} = (-j, -k, 0)\mathbb{H} = (j, k, 0)\mathbb{H} = (1, i, 0)\mathbb{H} \neq (1, -i, 0)\mathbb{H} = (\bar{1}, \bar{i}, \bar{0})\mathbb{H}$. On the other hand, conjugation does give rise to a polarity of $\mathcal{P}_2\mathbb{F}$ in all cases, see (13.12).

Polarities and their motion groups

13.10 The dual plane $\mathcal{P}_2^*\mathbb{F}$. The symmetry of the rôles of points and lines in the axioms of a projective plane $\mathcal{P}$ (21.1) allows us to interchange the notions of points and lines; in this way, we obtain another projective plane, called the *dual* plane $\mathcal{P}^*$. This yields the following *duality principle*: If a statement is true in all projective planes, then its *dual* statement, obtained by interchanging the words 'point' and 'line', is equally true.

Now we redescribe the construction of the projective plane $\mathcal{P}_2\mathbb{F}$ in a way which is particularly suited to determine its dual plane; the latter will be denoted by $\mathcal{P}_2^*\mathbb{F}$. The points of $\mathcal{P}_2\mathbb{F}$ are the 1-dimensional subspaces of $\mathbb{F}^3$ as a *right* vector space, and via homogeneous coordinates (13.2) the lines correspond to the 1-dimensional subspaces of $\mathbb{F}^3$ as a *left* vector space; recall that the homogeneous coordinates of a line are determined up to a common factor from the left. Incidence is described by equation (13.2(∗)). Thus, we obtain the dual plane $\mathcal{P}_2^*\mathbb{F}$ instead of $\mathcal{P}_2\mathbb{F}$ when we reverse the order of multiplication in $\mathbb{F}$, passing to the opposite field $\mathbb{F}^{\mathrm{op}}$:

$$\mathcal{P}_2^*\mathbb{F} \cong \mathcal{P}_2(\mathbb{F}^{\mathrm{op}}) .$$

Note that in our context, for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, we have that $\mathbb{F}^{\mathrm{op}} \cong \mathbb{F}$ and therefore $\mathcal{P}_2\mathbb{F} \cong \mathcal{P}_2^*\mathbb{F}$; in general, this is true if, and only if, $\mathbb{F}$ admits an antiautomorphism.

13.11 Definitions. A *duality* of a projective plane $\mathcal{P}$ with point set P and line set $\mathcal{L}$ is a collineation of $\mathcal{P}$ onto its dual plane $\mathcal{P}^*$. A duality can be described as a bijection π of the disjoint union $P \dot\cup \mathcal{L}$ onto itself exchanging P and $\mathcal{L}$ and having the following property:

$$p \text{ is incident with } L \text{ if, and only if, } L^\pi \text{ is incident with } p^\pi$$

for all $p \in P$ and $L \in \mathcal{L}$. This condition is equivalent to

$$(pq)^\pi = p^\pi \wedge q^\pi \quad \text{and} \quad (L \wedge M)^\pi = L^\pi M^\pi$$

for distinct points p, q and distinct lines L, M. A *polarity* is a duality which as a bijection of $P \dot\cup \mathcal{L}$ is involutory, i.e., satisfies

$$(p^\pi)^\pi = p \quad \text{and} \quad (L^\pi)^\pi = L$$

for all $p \in P$ and $L \in \mathcal{L}$. The line p^π is called the *polar* of p, and the point L^π is the *pole* of L. A collineation γ of $\mathcal{P}$ is said to *commute* with the polarity π if for every point p one has $p^{\gamma\pi} = p^{\pi\gamma}$. This is equivalent to $L^{\gamma\pi} = L^{\pi\gamma}$ for every line L. The collineations commuting with a given polarity form a subgroup of the group of all collineations.

13.12 The standard elliptic polarity and the standard hyperbolic polarity of $\mathcal{P}_2\mathbb{F}$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ are constructed using the following two Hermitian forms on $\mathbb{F}^3$:

$$(\,x \mid y\,)_\pm := \overline{x_1}y_1 + \overline{x_2}y_2 \pm \overline{x_3}y_3$$

for $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{F}^3$.

By mapping 1- and 2-dimensional subspaces of $\mathbb{F}^3$ to their orthogonal spaces with respect to either $(\ \mid\ )_+$ or $(\ \mid\ )_-$, one clearly obtains polarities π^+ and π^- of $\mathcal{P}_2\mathbb{F}$, respectively, which will be called the *standard elliptic polarity* and the *standard hyperbolic polarity*. In homogeneous coordinates, they have the following description:

$$\begin{aligned} \pi^\pm : P_2\mathbb{F} &\to \mathcal{L}_2\mathbb{F} : (x_1, x_2, x_3)\mathbb{F} \mapsto \operatorname{Ker}(\,\overline{x_1} \quad \overline{x_2} \quad \pm\overline{x_3}\,) \\ \mathcal{L}_2\mathbb{F} &\to P_2\mathbb{F} : \operatorname{Ker}(\,x_1 \quad x_2 \quad x_3\,) \mapsto (\overline{x_1}, \overline{x_2}, \pm\overline{x_3})\mathbb{F}\,. \end{aligned}$$

Note that these maps are well-defined, since conjugation is an antiautomorphism of $\mathbb{F}$. For later use, we remark that obviously

$$(1) \qquad \pi^- = \pi^+\iota_o\,,$$

where ι_o is the linear collineation

$$\iota_o = \left[\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}\right] = \left[\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}\right].$$

In affine coordinates, ι_o is given by $(x, y) \mapsto (-x, -y)$, cf. (13.7). In other words, ι_o is the unique involutory homology having the point $o = (0,0)$ as center and the line at infinity as axis (12.13).

We now want to determine the (linear) collineations which commute with these polarities. Among them are the collineations induced by linear transformations U of $\mathbb{F}^3$ which are *unitary* with respect to the Hermitian form $(\ \mid\ )_\pm$, i.e., satisfy

$$(*) \qquad (\,Ux \mid Uy\,)_\pm = (\,x \mid y\,)_\pm \quad \text{for all } x, y \in \mathbb{F}^3\,.$$

The unitary transformations form a subgroup of $\mathrm{GL}_3\mathbb{F}$, which is denoted by $\mathrm{U}_3\mathbb{F}$ in the case of $(\ \mid\ )_+$ and by $\mathrm{U}_3(\mathbb{F}, 1)$ for $(\ \mid\ )_-$. If $\mathbb{F} = \mathbb{R}$, it is more customary to call these transformations orthogonal and to write $\mathrm{O}_3\mathbb{R}$ and $\mathrm{O}_3(\mathbb{R}, 1)$ instead of $\mathrm{U}_3\mathbb{R}$ and $\mathrm{U}_3(\mathbb{R}, 1)$, respectively.

13.13 Proposition: Motion groups of $\mathcal{P}_2\mathbb{F}$. *The elliptic motion group and the hyperbolic motion group of $\mathcal{P}_2\mathbb{F}$, defined as the groups of all linear collineations which commute with the standard elliptic or hyperbolic polarity, respectively, coincide with the groups*

$$\mathrm{PU}_3\mathbb{F} = \{\, [A] \mid A \in \mathrm{U}_3\mathbb{F} \,\} \quad \textit{and} \quad \mathrm{PU}_3(\mathbb{F}, 1) = \{\, [A] \mid A \in \mathrm{U}_3(\mathbb{F}, 1) \,\}$$

of collineations induced by the respective unitary transformations.

Proof. We give a direct proof adapted to our special situation; for a more conceptual approach in a general setting see Baer [52] IV.5 Prop. 1, p. 144 ff.

To simplify notation, write $(\ \mid\ )$ for either $(\ \mid\ )_+$ or $(\ \mid\ )_-$, and let π stand for π^+ or π^-, and PU_3 for $\mathrm{PU}_3\mathbb{F}$ or $\mathrm{PU}_3(\mathbb{F}, 1)$, accordingly. By definition, $(x\mathbb{F})^\pi = x^\perp$, the orthogonal space of $x \in \mathbb{F}^3$ with respect to $(\ \mid\ )$. Let Φ be the motion group belonging to π. Obviously, $\mathrm{PU}_3 \subseteq \Phi$. We have to show the converse inclusion.

It is well known that every element $A \in \mathrm{GL}_3\mathbb{F}$ has a (uniquely determined) adjoint $A^* \in \mathrm{GL}_3\mathbb{F}$ satisfying

$$(\, A^*x \mid y \,) = (\, x \mid Ay \,)$$

for all $x, y \in \mathbb{F}^3$, see e.g. Porteous [81] Prop. 11.26, p. 207 ff. Using the adjoint, we find for $y \in \mathbb{F}^3$ that

$$\begin{aligned} (Ay)^\perp &= \big\{\, x \bigm| x \in \mathbb{F}^3,\ (\, x \mid Ay \,) = 0 \,\big\} = \big\{\, x \bigm| x \in \mathbb{F}^3,\ (\, A^*x \mid y \,) = 0 \,\big\} \\ &= \Big\{\, {A^*}^{-1} z \Bigm| z \in \mathbb{F}^3,\ (\, z \mid y \,) = 0 \,\Big\} \qquad (1) \\ &= {A^*}^{-1}(y^\perp)\,. \end{aligned}$$

Now let A be such that $[A] \in \Phi$, that is, $(Ay)^\perp = A(y^\perp)$ for all $y \in \mathbb{F}^3$. By (1), this translates into

$$A^*A(y^\perp) = y^\perp \qquad (2)$$

for all $y \in \mathbb{F}^3$. Since every line of $\mathcal{P}_2\mathbb{F}$ is of the form $y^\perp = (y\mathbb{F})^\pi$ for a suitable $y \in \mathbb{F}^3$, condition (2) says that the collineation of $\mathcal{P}_2\mathbb{F}$ induced by A^*A fixes every line and, hence, every point. In other words, A^*A leaves every 1-dimensional subspace of $\mathbb{F}^3$ invariant, and thus is of the form

$$A^*A = c \cdot \mathrm{id} \qquad (3)$$

for a suitable $c \in \mathbb{F}^\times$. Moreover, in the case $\mathbb{F} = \mathbb{H}$, it follows at this point already that c must belong to the subfield $\mathbb{R}$, because this subfield is the center of $\mathbb{H}$.

However, for other reasons, the fact that $c \in \mathbb{R}$ will presently be obtained in general, the case $\mathbb{F} = \mathbb{C}$ included.

From the definition of $(\ \mid\ )$, it is immediate that the 2-dimensional subspace $\mathbb{F}^2 \times \{0\}$ is positive definite with respect to $(\ \mid\ )$ not only in the elliptic case, but also in the hyperbolic case, in the sense that, for $0 \neq x \in \mathbb{F}^2 \times \{0\}$, the value $(x \mid x) \in \mathbb{R}$ is positive. If we choose $x \neq 0$ from the non-trivial intersection of the 2-dimensional subspaces $\mathbb{F}^2 \times \{0\}$ and $A^{-1}(\mathbb{F}^2 \times \{0\})$, then $0 < (x \mid x)$ and $0 < (Ax \mid Ax) = (A^*Ax \mid x) = (cx \mid x) = \bar{c}(x \mid x)$. It follows that $c \in \mathbb{R}$ and $\bar{c} = c > 0$.

Now, it is important that the real scalars c and $b = 1/\sqrt{c}$ belong to the center of $\mathbb{F}$ and are fixed under conjugation. The first property implies that the linear transformation $B = bA$ induces the same collineation of $\mathcal{P}_2\mathbb{F}$ as A. Using (3), we obtain, moreover, that $(Bx \mid By) = (bAx \mid bAy) = b^2(A^*Ax \mid y) = c^{-1}(cx \mid y) = c^{-1}c(x \mid y) = (x \mid y)$ for all $x, y \in \mathbb{F}^3$, which means that B is unitary. Thus, $[A] = [B] \in \mathrm{PU}_3$. □

On the subspace $\mathbb{F}^2 = \mathbb{F}^2 \times \{0\} \leq \mathbb{F}^3$, both Hermitian forms $(\ \mid\ )_+$ and $(\ \mid\ )_-$ of (13.12) induce the same Hermitian form $(a, b) \mapsto \overline{a_1}b_1 + \overline{a_2}b_2$ for $a = (a_1, a_2)$, $b = (b_1, b_2) \in \mathbb{F}^2$. The group $\mathrm{U}_2\mathbb{F}$ of unitary transformations of $\mathbb{F}^2$ with respect to this Hermitian form appears in the following description of stabilizers of the elliptic and hyperbolic motion groups.

13.14 Proposition. *The stabilizer Φ_o of the point $o = (0, 0, 1)\mathbb{F}$ in the elliptic motion group $\mathrm{PU}_3\mathbb{F}$ and the stabilizer of o in the hyperbolic motion group $\mathrm{PU}_3(\mathbb{F}, 1)$ coincide.*

The stabilizer Φ_o leaves the line $W = \mathrm{Ker}(0\ \ 0\ \ 1)$ invariant and is transitive on the set of its points. On the point set of the affine plane $\mathcal{P}_2\mathbb{F}^W \cong \mathcal{A}_2\mathbb{F}$, identified with $\mathbb{F}^2$ according to (13.3), Φ_o acts as the group

$$\left\{ \binom{x}{y} \mapsto U\binom{xc}{yc} \;\middle|\; U \in \mathrm{U}_2\mathbb{F}, c \in \mathbb{F}, \|c\|^2 = 1 \right\}.$$

Addendum. In the commutative cases $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, the homotheties $\mu_c : \binom{x}{y} \mapsto \binom{xc}{yc}$ for $c \in \mathbb{F}$, $\|c\|^2 = 1$ belong to $\mathrm{U}_2\mathbb{F}$. For $\mathbb{F} = \mathbb{H}$, the homothety μ_c is not a linear transformation of the *right* $\mathbb{H}$-vector space $\mathbb{H}^2$ unless $c \in \mathbb{R}$. The group of all homotheties μ_c with $\|c\|^2 = 1$ is isomorphic to $\mathrm{Spin}_3\mathbb{R}$ in this case (11.25). Thus

$$\Phi_o \cong \left\{ \begin{array}{l} \mathrm{O}_2\mathbb{R} \\ \mathrm{U}_2\mathbb{C} \\ \mathrm{U}_2\mathbb{H} \cdot \mathrm{Spin}_3\mathbb{R} \end{array} \right\} \quad \text{for} \quad \mathbb{F} = \left\{ \begin{array}{l} \mathbb{R} \\ \mathbb{C} \\ \mathbb{H} \end{array} \right. .$$

Proof of (13.14). The orthogonal space of $(0, 0, 1)\mathbb{F}$ with respect to both Hermitian forms $(\ \mid\ )_\pm$ is the 2-dimensional subspace $\mathbb{F}^2 \times \{0\} = \mathrm{Ker}(0\ \ 0\ \ 1)$ representing the line W. A transformation in $\mathrm{U}_3\mathbb{F}$ or in $\mathrm{U}_3(\mathbb{F},1)$ leaving $(0,0,1)\mathbb{F}$ invariant therefore leaves $W=\mathbb{F}^2\times\{0\}$ invariant, as well. Since the Hermitian forms $(\ |\ )_+$ and $(\ |\ )_-$ coincide on $\mathbb{F}^2\times\{0\}$, the transformations in question in both the elliptic and the hyperbolic case are precisely those which are represented by the matrices

$$(1) \qquad \begin{pmatrix} U & \\ & a \end{pmatrix} \quad \text{with } U\in \mathrm{U}_2\mathbb{F} \text{ and } a\in\mathbb{F},\ \bar{a}a=1.$$

The group of these transformations is transitive on the set of 1-dimensional subspaces of $\mathbb{F}^2\times\{0\}$, that is, on the set of points of the line W. A point not on W with affine coordinates $(x,y)\in\mathbb{F}^2$ has homogeneous coordinates $(x,y,1)\mathbb{F}$; the image point under the collineation induced by transformation (1) has homogeneous coordinates

$$\begin{pmatrix} U\binom{x}{y} \\ a \end{pmatrix}\mathbb{F} = \begin{pmatrix} U\binom{x}{y}a^{-1} \\ 1 \end{pmatrix}\mathbb{F} = \begin{pmatrix} U\binom{xa^{-1}}{ya^{-1}} \\ 1 \end{pmatrix}\mathbb{F}$$

and affine coordinates $U\binom{xc}{yc}$, where $c=a^{-1}$. Thus our proposition is proved.

If $\mathbb{F}$ is commutative, then the transformation $\binom{x}{y}\mapsto\binom{xc}{yc}=\begin{pmatrix} c & \\ & c\end{pmatrix}\binom{x}{y}$ is linear, and for $\bar{c}c=1$ it obviously belongs to $\mathrm{U}_2\mathbb{F}$. □

In the sequel, we determine various orbits of the motion groups.

13.15. *The standard elliptic motion group* $\mathrm{PU}_3\mathbb{F}$ *of* $\mathcal{P}_2\mathbb{F}$, $\mathbb{F}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$, *is flag transitive, i.e., it acts transitively on the set of all incident point–line pairs.*

Proof. For a given flag $p\subseteq L$, there is a basis (b_1,b_2,b_3) of $\mathbb{F}^3$ which is orthonormal with respect to the positive definite Hermitian form $(\ |\ )_+$ and satisfies $p=b_1\mathbb{F}$, $L=b_1\mathbb{F}+b_2\mathbb{F}$. The assertion now follows because $\mathrm{U}_3\mathbb{F}$ is (sharply) transitive on the set of orthonormal bases. □

The Hermitian form $(\ |\ )_-$ differs from $(\ |\ )_+$ in that there are 'isotropic' vectors $x\neq 0$ satisfying $(x\mid x)_-=0$. For the point $p=x\mathbb{F}$ of $\mathcal{P}_2\mathbb{F}$, this is equivalent to saying that p is incident with its polar under the standard hyperbolic polarity π^-. This situation is covered by the following general notion.

13.16 Definition. Let π be a polarity of a projective plane. A point p is said to be *absolute* if p is incident with its polar p^π; dually, an *absolute line* is a line L which is incident with its pole L^π.

Obviously, the set of absolute points and the set of absolute lines is invariant under every collineation commuting with the polarity π.

13.17 Proposition: Orbits of the hyperbolic motion group.

(a) *The set of absolute points of the standard hyperbolic polarity* π^- *of* $\mathcal{P}_2\mathbb{F}$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, *is*

$$Q = \left\{ (x, y, 1)\mathbb{F} \;\middle|\; x, y \in \mathbb{F}, \|x\|^2 + \|y\|^2 = 1 \right\} .$$

The set of exterior points, i.e., of non-absolute points which are incident with some absolute line, is

$$E = \left\{ (x, y, 1)\mathbb{F} \;\middle|\; x, y \in \mathbb{F}, \|x\|^2 + \|y\|^2 > 1 \right\} \cup \left\{ (x, y, 0)\mathbb{F} \;\middle|\; 0 \neq (x, y) \in \mathbb{F}^2 \right\} .$$

The set of interior points, i.e., of points which are not incident with any absolute line, is

$$I = \left\{ (x, y, 1)\mathbb{F} \;\middle|\; x, y \in \mathbb{F}, \|x\|^2 + \|y\|^2 < 1 \right\} .$$

(b) *The group* $\mathrm{PU}_3(\mathbb{F}, 1)$ *is transitive on* Q, E *and* I. *It is even flag transitive on the 'interior hyperbolic plane'* (*the geometry consisting of the interior points and of the lines through interior points*).

(c) *The lines through interior points are precisely the polars of exterior points.*

Remarks. 1) It is clear that the hyperbolic motion group cannot act flag transitively on the 'exterior hyperbolic plane' whose lines are all the lines through exterior points, because through every exterior point there are different kinds of lines (absolute lines, non-absolute lines without interior points, and lines with interior points), which cannot be transformed into each other by hyperbolic motions.

2) Via the collineation $\overline{\mathcal{A}_2\mathbb{F}} \cong \mathcal{P}_2\mathbb{F}$ of (13.3), the set Q of absolute points corresponds to the unit sphere in the affine point set $\mathbb{F} \times \mathbb{F} \cong \mathbb{R}^{2n}$ $(n = 1, 2, 4)$. The point set I of the interior hyperbolic plane becomes the 'interior' (= bounded) complementary component of the unit sphere, and E corresponds to the exterior component, together with the points at infinity.

Proof of (13.17). By (13.14), the orbits of the stabilizer Φ_o of $o = (0, 0, 1)\mathbb{F}$ in the hyperbolic motion group $\mathrm{PU}_3(\mathbb{F}, 1)$ are the subsets S_r corresponding to the spheres of positive radius $r \in \mathbb{R}$ in $\mathbb{F} \times \mathbb{F}$,

$$S_r = \left\{ (x, y, 1)\mathbb{F} \;\middle|\; x, y \in \mathbb{F}, \|x\|^2 + \|y\|^2 = r \right\} ,$$

and the point row $\left\{ (x, y, 0)\mathbb{F} \;\middle|\; (0, 0) \neq (x, y) \in \mathbb{F}^2 \right\}$ of the line W at infinity.

The set Q of absolute points, the set E of exterior points, and the set I of interior points are invariant under the hyperbolic motion group $\mathrm{PU}_3(\mathbb{F}, 1)$. The explicit descriptions of Q, E and I given in (a) and the transitivity assertions stated in (b) now may be inferred from the following facts, which shall be established subsequently.

1) The point $(1,0,1)\mathbb{F} \in S_1$ is absolute.
2) The point $(0,0,1)\mathbb{F}$ is an interior point, and its orbit under $\mathrm{PU}_3(\mathbb{F},1)$ contains points from every sphere S_r of radius $r < 1$.
3) The point $(0,1,0)\mathbb{F}$ on W is an exterior point, and its orbit under $\mathrm{PU}_3(\mathbb{F},1)$ contains points from every sphere S_r of radius $r > 1$.

Flag transitivity on the interior hyperbolic plane is then obtained from known transitivity properties of Φ_o. Indeed, Φ_o is transitive on the set of lines through $o = (0,0,1)\mathbb{F} \in I$, as Φ_o is transitive on the points of W.

In the following proofs of assertions 1)–3), we use the explicit description of π^- in homogeneous coordinates given in (13.12).

Proof of 1). $(1,0,1)\mathbb{F} \subseteq \mathrm{Ker}\,(\,1 \quad 0 \quad -1\,) = (1,0,1)\mathbb{F}^{\pi^-}$.

Proof of 2). The lines through $(0,0,1)\mathbb{F}$ are of the form $\mathrm{Ker}\,(\,a_1 \quad a_2 \quad 0\,)$; none of them contains its pole $\mathrm{Ker}\,(\,a_1 \quad a_2 \quad 0\,)^{\pi^-} = (\overline{a_1}, \overline{a_2}, 0)\mathbb{F}$, because $a_1\overline{a_1} + a_2\overline{a_2} > 0$. Hence, these lines are not absolute, and $(0,0,1)\mathbb{F}$ is an interior point. The matrix

$$(*) \qquad \begin{pmatrix} 1 & & \\ & \sqrt{1+t^2} & t \\ & t & \sqrt{1+t^2} \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \cosh\tau & \sinh\tau \\ & \sinh\tau & \cosh\tau \end{pmatrix}$$

with $t \in \mathbb{R}$, $\tau = \ln(t + \sqrt{1+t^2})$ belongs to $\mathrm{U}_3(\mathbb{F},1)$ and maps $(0,0,1)\mathbb{F}$ to $(0, t, \sqrt{1+t^2})\mathbb{F}$. For $t > 0$, this point can be rewritten as $(0, r^{-1}, 1)\mathbb{F}$, where

$$r = \sqrt{1 + t^{-2}}\,,$$

and $t > 0$ can be chosen such that r^{-1} is any preassigned number between 0 and 1.

Proof of 3). The point $(0,1,0)\mathbb{F}$ is not contained in its polar $(0,1,0)\mathbb{F}^{\pi^-} = \mathrm{Ker}\,(\,0 \quad 1 \quad 0\,)$, i.e., it is not absolute. It is contained in the line $\mathrm{Ker}\,(\,1 \quad 0 \quad 1\,)$, which is absolute: $\mathrm{Ker}\,(\,1 \quad 0 \quad 1\,)^{\pi^-} = (1,0,-1)\mathbb{F} \subseteq \mathrm{Ker}\,(\,1 \quad 0 \quad 1\,)$. The matrix $(*)$ maps $(0,1,0)\mathbb{F}$ to $(0, \sqrt{1+t^2}, t)\mathbb{F}$. For $t > 0$, this is the point $(0, r, 1)\mathbb{F}$, where r is defined as above; for a suitable $t > 0$, the radius r takes on any given value greater than 1.

Thus, (a) and (b) are proved. As to assertion (c), note that the interior point $o = (0,0,1)\mathbb{F}$ is contained in the line $\mathrm{Ker}\,(\,0 \quad 1 \quad 0\,)$, which is the polar of the exterior point $(0,1,0)\mathbb{F}$. Assertion (c) now follows, because, according to (b), the hyperbolic motion group is transitive both on E and on the set of lines through interior points, and because hyperbolic motions respect the polar relation. □

13.18 Notes. Here we record further polarities which, together with the standard elliptic polarity and the standard hyperbolic polarity, represent all equivalence classes of polarities of $\mathcal{P}_2\mathbb{F}$ except the discontinuous polarities of $\mathcal{P}_2\mathbb{C}$.

According to a theorem of Birkhoff and von Neumann, cf. Baer [52] IV.1 Prop. 2, p. 103 and IV.3 Theorem 1, p. 111 or Taylor [92] Theorem 7.1, p. 53, every polarity of $\mathcal{P}_2\mathbb{F}$ may be derived from some non-degenerate Hermitian form $(x, y) \mapsto f(x, y)$ on $\mathbb{F}^3$ accompanied by an involutory antiautomorphism α of $\mathbb{F}$. (Notice that the other possibility appearing in the general form of that theorem, which is formulated for projective spaces of arbitrary dimension, is excluded for projective planes. It concerns the case of a non-degenerate alternating bilinear form, instead of a Hermitian form; in particular, the underlying field then is necessarily commutative. However, every alternating form on a vector space of odd dimension is degenerate.)

For $\mathbb{F} = \mathbb{R}$, there is no (anti-)automorphism except $\alpha = \mathrm{id}$. For $\mathbb{F} = \mathbb{C}$, we are not interested in the multitude of possibilities arising from non-continuous (anti-)automorphisms, so we only consider the cases $\alpha = \mathrm{id}$ and $\alpha =$ conjugation (11.26).

For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ it is then easy to see (essentially by Sylvester's theorem) that, up to equivalence, there is only one polarity other than the standard elliptic and hyperbolic polarities of $\mathcal{P}_2\mathbb{R}$ and $\mathcal{P}_2\mathbb{C}$, namely the polarity ϱ of $\mathcal{P}_2\mathbb{C}$ obtained from the standard bilinear form

$$f(x, y) = \sum_{\nu=1}^{3} x_\nu y_\nu$$

of $\mathbb{C}^3$, see e.g. Lewis [82] Sect. 3, p. 256 and Sect. 4, p. 261. The absolute points of ϱ form a conic C, with a connected complement $P_2\mathbb{C} \setminus C$. This property distinguishes ϱ from π^+, which has no absolute points, and from π^-, whose set of non-absolute points is disconnected (13.17).

For $\mathbb{F} = \mathbb{H}$, the classification of polarities up to equivalence may be obtained from Dieudonné [71] Chap. I §§6,8 together with Dieudonné [52] §19, p. 383; the result can also be found in Lewis [82]. For $\alpha =$ conjugation there are just the standard elliptic polarity and the standard hyperbolic polarity (Lewis [82] Sect. 5, p. 263; cf. also Baer [52] Chap. IV Appendix I, Application 2, p. 130, Dieudonné [71] p. 16 (end of §8)).

A polarity of $\mathcal{P}_2\mathbb{H}$ accompanied by any other involutory antiautomorphism may also be described using a non-degenerate *skew*-Hermitian form g accompanied by conjugation, and up to equivalence there is just one possibility (Lewis [82] Sect. 6, p. 264), given by

$$g(x, y) = \sum_{\nu=1}^{3} \overline{x_\nu} \cdot i \cdot y_\nu \,.$$

For the corresponding polarity, one can again show that the set A of absolute points is non-empty and that the set of non-absolute points is connected. In contrast to the complex case, however, A is not what one would like to call a conic, because non-absolute lines carry more than two absolute points.

The two further polarities presented here, one of $\mathcal{P}_2\mathbb{C}$ and the other of $\mathcal{P}_2\mathbb{H}$, are analogous to the *standard planar polarity* of the octonion projective plane discussed in (18.28) ff. The method of classification presented there for $\mathbb{F} = \mathbb{O}$ also works for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, see (18.31). This offers an alternative approach to the classification of polarities as indicated here.

14 The planes over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ as topological planes

In this section, we study the topological properties of the projective planes over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Primarily, we shall show that they are examples illustrating the subject matter of this book, topological projective planes. This means that the geometric operations in $\mathcal{P}_2\mathbb{F}$ are continuous with respect to a topology introduced in a rather standard way. The proof of this result will work for an arbitrary topological field in place of $\mathbb{F}$.

More generally, we shall consider the point sets $\mathrm{P}_d\mathbb{F}$ of the projective spaces of arbitrary finite dimension d. We introduce their natural topologies, and we show that they are manifolds. The section closes with a description of the Hopf maps associated with the fields $\mathbb{F}$.

14.1 The topology on a projective space over $\mathbb{F}$. For $d \in \mathbb{N}$, we consider $\mathbb{F}^{d+1}$ as a *right* vector space over $\mathbb{F}$. The 'points' of *projective d-space over* $\mathbb{F}$ are the 1-dimensional subspaces of $\mathbb{F}^{d+1}$, hence the point set of this space is

$$\mathrm{P}_d\mathbb{F} := \left\{ x\mathbb{F} \;\middle|\; 0 \neq x \in \mathbb{F}^{d+1} \right\} .$$

Of course, $\mathbb{F}^{d+1}$ is also a left vector space over $\mathbb{F}$, the set of whose 1-dimensional subspaces will be denoted by

$$\mathrm{P}_d^*\mathbb{F} := \left\{ \mathbb{F}x \;\middle|\; 0 \neq x \in \mathbb{F}^{d+1} \right\} .$$

If $\mathbb{F}$ is not commutative, then $\mathrm{P}_d\mathbb{F}$ and $\mathrm{P}_d^*\mathbb{F}$ differ. The latter space is identified in a natural way with the set

$$\mathcal{H}_d\mathbb{F} := \left\{ H \leq \mathbb{F}^{d+1} \;\middle|\; \dim H = d \right\}$$

of hyperplanes of $\mathbb{F}^{d+1}$, since the hyperplanes are precisely the kernels of nonzero linear forms, and since a linear form is determined by its kernel up to a scalar factor from the left. Thus, there is a bijection

$$(1) \qquad \mathrm{P}_d^*\mathbb{F} \to \mathcal{H}_d\mathbb{F} : \mathbb{F}x \mapsto \operatorname{Ker}(\, x_1 \quad x_2 \quad \cdots \quad x_{d+1}\,) \ ,$$

where $x = (x_1, x_2, \ldots, x_{d+1}) \in \mathbb{F}^{d+1} \setminus \{0\}$, and where $(\, x_1 \quad x_2 \quad \cdots \quad x_{d+1}\,)$ is to be interpreted as the matrix representation of a linear form with respect to the standard basis.

For $d = 2$, this agrees with the construction of the projective plane $\mathcal{P}_2\mathbb{F}$ as in (13.1 and 2), with point set $\mathrm{P}_2\mathbb{F}$ and line set $\mathcal{L}_2\mathbb{F} = \mathcal{H}_2\mathbb{F}$.

In order to introduce topologies on $\mathrm{P}_d\mathbb{F}$ and $\mathcal{H}_d\mathbb{F} \mathrel{\widehat{=}} \mathrm{P}_d^*\mathbb{F}$, we use the natural topology on $\mathbb{F} = \mathbb{R}^n$ $(n = 1, 2, 4)$, defined by the norm $\|a\|^2 = \bar{a}a$, and we consider the product topology on $\mathbb{F}^{d+1}$. The spaces $\mathrm{P}_d\mathbb{F}$ and $\mathrm{P}_d^*\mathbb{F}$ will be endowed with the *quotient topologies*, cf. (92.19), determined by the canonical maps

$$\vartheta : \mathbb{F}^{d+1} \setminus \{0\} \to \mathrm{P}_d\mathbb{F} : x \mapsto x\mathbb{F}$$

and

$$\vartheta^* : \mathbb{F}^{d+1} \setminus \{0\} \to \mathrm{P}_d^*\mathbb{F} : x \mapsto \mathbb{F}x \,.$$

A topology on $\mathcal{H}_d\mathbb{F}$ is obtained by transfer via the bijection (1). This topology may also be viewed as the quotient topology determined by the map

$$\operatorname{Ker} : \mathbb{F}^{d+1} \setminus \{0\} \to \mathcal{H}_d\mathbb{F} : (x_1, x_2, \ldots, x_{d+1}) \mapsto \operatorname{Ker}(\, x_1 \quad x_2 \quad \cdots \quad x_{d+1}\,) \,.$$

With respect to these topologies, the maps ϑ, ϑ^*, and Ker are open maps. In order to see this for ϑ, one has to prove that, for an open subset $U \subseteq \mathbb{F}^{d+1} \setminus \{0\}$, the image $\vartheta(U)$ is open in $\mathrm{P}_d\mathbb{F}$. By definition of the quotient topology, this is equivalent to showing that the inverse image $\vartheta^{-1}(\vartheta(U))$ is open in $\mathbb{F}^{d+1} \setminus \{0\}$. Now, $\vartheta^{-1}(\vartheta(U)) = \bigcup \{\, \vartheta^{-1}(\vartheta(x)) \mid x \in U \,\} = \bigcup \{\, x\mathbb{F}^\times \mid x \in U \,\} = U\mathbb{F}^\times = \bigcup \{\, Ua \mid a \in \mathbb{F}^\times \,\}$ is the union of the open sets Ua. For ϑ^* and Ker, the assertion is obtained analogously. (The same argument proves a similar general assertion about the quotient topology of orbit spaces, see (96.2). The projective space $\mathrm{P}_d\mathbb{F}$ is the orbit space of $\mathbb{F}^{d+1} \setminus \{0\}$ under the action of the multiplicative group $\mathbb{F}^\times$ by scalar multiplication from the right, and similarly for $\mathrm{P}_d^*\mathbb{F}$.)

By definition, the spaces $\mathrm{P}_d\mathbb{F}$ and $\mathrm{P}_d^*\mathbb{F} \mathrel{\widehat{=}} \mathcal{H}_d\mathbb{F}$ are interchanged if we switch the right and left vector space structures of $\mathbb{F}^{d+1}$. Equivalently, we may replace the field $\mathbb{F}$ with the opposite field $\mathbb{F}^{\mathrm{op}}$, in which the order of multiplication is reversed. Therefore, the duality principle of (13.10) extends as follows: Any true statement about $\mathrm{P}_d\mathbb{F}$ depending only on properties of $\mathbb{F}$ which are shared by $\mathbb{F}^{\mathrm{op}}$ (e.g., topological properties) is equally valid for $\mathrm{P}_d^*\mathbb{F}$.

Note. A description of these topologies has been given by Pontryagin in 1938, see Pontryagin [86] Sect. 27 Example 48, p. 183. There, the topology on $\mathcal{H}_d\mathbb{F}$ is introduced by a different method, which, more generally, is used to define