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Unitary Representation Theory of Exponential Lie Groups

by

Horst Leptin
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Walter de Gruyter · Berlin · New York 1994

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Preface

The reason for and the purpose of this book is to present a full proof of Theorem 1 in chapter III: The Kirillov map K from the space $\mathfrak{g}^*/G$ of coadjoint G-orbits onto the unitary dual $\widehat{G}$ of the connected, simply connected exponential Lie group G with Lie algebra $\mathfrak{g}$ is a homeomorphism.

This theorem solves one of the basic problems of the unitary representation theory of locally compact groups for the class of exponential Lie groups, namely the concrete and explicit determination of the unitary dual $\widehat{G}$ of the group G as a topological space. It is, in a sense, the widest possible extension of the Kirillov–I. Brown theorem for the nilpotent case. Let us briefly sketch its content:

Let G be a connected, simply connected solvable Lie group, $\mathfrak{g}$ its Lie algebra and E the exponential mapping from $\mathfrak{g}$ into G. We say that G is *exponential*, if E is a diffeomorphism from $\mathfrak{g}$ onto G. Let l be a real linear form on $\mathfrak{g}$, i.e. an element of the linear dual $\mathfrak{g}^*$ of $\mathfrak{g}$.

A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called *subordinate* to l, if $l(\mathfrak{h}') = 0$, $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$ the commutator algebra of $\mathfrak{h}$. In this case the formula

$$\chi(E(x)) = e^{il(x)}, \qquad x \in \mathfrak{h},$$

defines a unitary abelian character of the subgroup $H = E(\mathfrak{h})$ corresponding to $\mathfrak{h}$. We set

$$\pi(l, \mathfrak{h}) = \operatorname{ind}_H^G \chi$$

for the unitary representation of G, induced from the character χ of H.

After some basic work of Dixmier, A. A. Kirillov published in 1962 his fundamental paper [17], in which he proved:

If G, resp. $\mathfrak{g}$ is nilpotent, then

- $\pi(l, \mathfrak{h})$ is irreducible if and only if $\dim \mathfrak{h}$ is maximal among the dimensions of subalgebras of $\mathfrak{g}$ subordinate to l.
- If $\pi(l, \mathfrak{h}) \in \widehat{G}$, then $\pi(l, \mathfrak{h}) = \pi(l)$ is independent of $\mathfrak{h}$.
- G acts on $\mathfrak{g}^*$ via the coadjoint representation. The elements $l, l' \in \mathfrak{g}^*$ are in the same G-orbit if and only if $\pi(l)$ and $\pi(l')$ are equivalent.
- Every $\pi \in \widehat{G}$ is of the form $\pi(l)$ for some $l \in \mathfrak{g}^*$.

These results can be expressed in the following more concise form:

For $l \in \mathfrak{g}^*$ let $\Omega_l \subset \mathfrak{g}^*$ be the orbit of l and set

$$K(\Omega_l) = \pi(l) \in \widehat{G}\ .$$

Then $K : \Omega_l \to K(\Omega_l)$ is a bijection from the orbit space $\mathfrak{g}^*/G$ onto the unitary dual $\widehat{G}$ of G.

The mapping K (or sometimes its inverse) is called the *Kirillov map.* Kirillov's result parametrizes set–theoretically in a most satisfactory way the dual $\widehat{G}$. However, $\mathfrak{g}^*/G$ and $\widehat{G}$ both bear natural topologies and already Kirillov himself had proven that K is continuous. His *conjecture*, namely that K is a homeomorphism, was proved only in 1973 by I. D. Brown [6]. It is easy to see that, in the nilpotent case, it suffices to prove the conjecture only for q–step nilpotent algebras $\mathfrak{g}$ with k free generators (see [6]). The core of Brown's proof consists in the fact, that these algebras, resp. groups have an abundance of symmetries, more precisely: Every linear automorphism of $\mathfrak{g}/\mathfrak{g}'$ extends to an algebra automorphism of $\mathfrak{g}$.

Already in 1957 Takenouchi had shown that also for an exponential group G every $\pi \in \widehat{G}$ is monomial, i.e. of the form $\pi(l, \mathfrak{h})$ for some $l \in \mathfrak{g}^*$ and certain l–subordinate algebras $\mathfrak{h} \subset \mathfrak{g}$ [34]. Later, in 1965, Bernat was able to extend Kirillov's theorem to general exponential groups: There exists a canonical bijection $K : \mathfrak{g}^*/G \to \widehat{G}$, see [1].

While for nilpotent groups the representation $\pi(l, \mathfrak{h})$ is irreducible if and only if $\dim \mathfrak{h}$ is maximal among the l–subordinate subalgebras $\mathfrak{h}$, already the "$ax + b$"–algebra $\mathbb{R}a \oplus \mathbb{R}b$ with $[a, b] = b$ shows, that this is no longer true for exponential groups. The answer to this irreducibility question was given in 1967 by L. Pukanszky: $\pi(l, \mathfrak{h})$ is irreducible if and only if the linear submanifold $l + \mathfrak{h}^\perp$ of $\mathfrak{g}^*$ is contained in the orbit Ω_l [28]. One year later Pukanszky proved the continuity of the Kirillov map K for general exponential groups [29]. Left open for a long time was the question whether K was also open, i.e. a homeomorphism from $\mathfrak{g}^*/G$ onto $\widehat{G}$. The first substantial result in this direction came 1984 from H. Fujiwara [10]. He proved that $\mathfrak{g}^*/G$ contains a dense open set, which K maps homeomorphically onto a dense open set of $\widehat{G}$. This book, finally, contains in its chapters II and III the complete proof of the Kirillov conjecture: For all exponential groups, K is a homeomorphism from $\mathfrak{g}^*/G$ onto $\widehat{G}$.

It is worth mentioning that the analogous problem for the primitive ideal–space of the enveloping algebras of exponential Lie algebras recently has been solved by O. Mathieu [22].

Contrary to the nilpotent case there are no "free models" for general exponential groups and in general the inner structure of these groups is on the one hand various, on the other rather rigid, which implies that there is no way to extend Brown's method directly to the exponential case. This led Jean Ludwig to the idea to force the necessary amount of flexibility of the objects by extending the category of groups and algebras to the category of *variable* objects. This idea and the basic steps of the actual proof in chapter III are due to him. Ludwig reported on it first at the conference on "Harmonische Analyse und Darstellungstheorie topologischer Gruppen" in Oberwolfach, summer 1987.

The textbook literature on solvable Lie groups and their representations is very limited, still the 1972–volume [2] by Bernat et al. is the main source in this field and apparently no monograph exists, which exposes the theory beyond the type I–groups, in particular the fundamental work of Pukanszky [30], [31]. In this context we point the interested reader to the excellent survey article of C. C. Moore in the Proceedings of the 1972 Williamstown conference [23]. In any case, this situation caused us to include a relatively long first chapter in this book, the content of which will be sketched below.

Chapter I starts in § 1 with a general discussion of exponential Lie groups. § 2 contains fundamental facts on homogeneous spaces, quasiinvariant measures and group algebras. In § 3 we define and study induced representations. The central result is Mackey's imprimitivity theorem. We give a complete proof of the theorem in its general form for arbitrary locally compact groups, with a minor, unessential restriction: For the sake of transparency we suppose that on the homogeneous space G/H there exists a relatively invariant measure. Also included in § 3 are results on intertwining operators and an irreducibility criterion for induced representations. In § 4 we come back to exponential Lie groups. After studying polarizations we prove Bernat's results, i.e. the bijectivity of the Kirillov map. § 5 contains two theorems with the precise description of the kernels of the restriction $\pi|_H$ of an irreducible representation of G onto a closed subgroup H, resp. of the induction $\operatorname{ind}_H^G \pi$ of $\pi \in \widehat{H}$. This problem, resp. the problem of decomposing $\pi|_H$ and $\operatorname{ind}_H^G \pi$ of course has been studied extensively in the literature; from the many papers by Corwin, Greenleaf, Grélaud, Lipsman and others we cite only Fujiwara [10], who proves a precise formula for the decomposition of $\pi|_H$ into irreducible representations of H.

§ 6 is more or less independent of the rest of this book. Based on Ludwig's paper [21] it treats the following problem: Let $\pi = \operatorname{ind}_P^G \chi \in \widehat{G}$ be given. The quotient G/P is diffeomorphic with some $\mathbb{R}^m$ and the representation space of π can naturally be identified with $L^2(\mathbb{R}^m)$. Then for any $f \in L^1(G)$ the operator $\pi(f)$ is a kernel operator, i.e. there exists a function K_f on $\mathbb{R}^m \times \mathbb{R}^m$ such that $(\pi(f)\xi)(x) = \int_{\mathbb{R}^m} K_f(x, y)\xi(y)dy$. Problem: Which kernels occur in this fashion? For nilpotent G it is known that for every Schwartz function $F \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m)$ there exists $f \in L^1(G)$ with $F = K_f$, see [14]. For exponential groups such a result cannot be expected, for the following reason: The parametrization of G/H via $\mathbb{R}^m$ depends substantially upon the choice of "coexponential bases" in $\mathfrak{g}$ for the polarization $\mathfrak{p}$. In the nilpotent case a change of the basis induces a bipolynomial diffeomorphism of the parameter space $\mathbb{R}^m$, which is compatible with polynomial decay and Fourier transform. In the general exponential case however exponential functions enter the picture and these are incompatible in particular with Fourier transform. Whether our theorem 12 is optimal we do not know, in any case it is the best we could prove for the moment. Certainly it guarantees the existence of finite rank operators in the image $\pi(L^1(G))$ for every $\pi \in \widehat{G}$. For more applications of this theorem see [21]. It is in order to mention here also the remarkable result of D. Poguntke [27]: Let G be a connected locally compact group and $\pi \in \widehat{G}$.

If $\pi(C^*(G))$ contains the compact operators, then $\pi(L^1(G))$ contains rank–one projections. This is false for non connected groups [26].

Chapter II contains the foundation of the theory of variable structures, in particular of variable groups and algebras and their representation theory. Obviously this theory has intimate connections with other existing theories, e.g. with the modern theory of groupoids, with bundles and above all with contractions. Here we do not discuss these connections and don't even cite the papers of Renault, Fell, Saletan etc. In § 3 we show that there exists a functor $\mathbf{l}$ from the category of variable Lie groups into (but not onto!) the category of variable Lie algebras. For every variable Lie group $\mathbf{G}$ with algebra $\mathbf{l}(\mathbf{G})$ there exists the exponential mapping $E : \mathbf{l}(\mathbf{G}) \to \mathbf{G}$ which also in the variable case is a local homeomorphism. For exponential $\mathbf{G}$ the map E is a global homeomorphism.

In chapter III finally we prove the Kirillov conjecture for general variable exponential groups. § 1 contains the continuity part of the proof. Already here the use of variable structures simplifies matters substantially. In order to show that also the inverse K^{-1} of the Kirillov map is continuous we of course apply induction on $\dim G = \dim \mathfrak{g}$, where G, resp. $\mathfrak{g}$ is the underlying manifold, resp. vector space of the variable structure. With relatively simple arguments the problem is reduced to the case where all algebras ${}_p\mathfrak{g}$, constituting the variable Lie algebra $\mathbf{g} = P \times \mathfrak{g} = \bigcup_{p \in P} p \times {}_p\mathfrak{g}$, have the same one dimensional center $\mathfrak{z}$. Then, as usually for real solvable Lie groups, one has to distinguish two cases: Case 1: Infinitely many quotients ${}_p\mathfrak{g}/\mathfrak{z}$ contain a one dimensional minimal ideal; or Case 2: Infinitely often there exist minimal two dimensional ideals in ${}_p\mathfrak{g}/\mathfrak{z}$. Although the basic ideas of the proof appear already in Case 1, treated in § 2, considerable not only technical complications occur in Case 2, which is treated in § 3.

The process of writing this book was for more than one reason complicated and non linear. Certainly this will have left traces, although we have tried to do our best in detecting, correcting and removing misprints, inconsistent notations etc; in this respect we just hope for the lenience of the reader. The same wish we have with respect to colleagues who feel that their work is insufficiently or even not at all regarded. As one can see we have quoted only such facts and results which are indispensable for our purposes, in particular for our proofs. Moreover, to keep the book selfcontained, in particular in chapter I, we gave our own proofs whenever possible. This means under no circumstances that e.g. we claim priority. We hope that this attitude is justified by the fact that this book is not a monograph on an established, well defined mathematical field, but the representation of new methods and results in such a field.

Bielefeld and Metz, December 1993

Horst Leptin
Jean Ludwig

Chapter 1

Solvable Lie Groups, Representations

§ 1 Bases in solvable Lie algebras, exponential groups

We assume the reader to be familiar with the basic facts of the theory of Lie groups and their algebras: If G is a real n dimensional Lie group then its Lie algebra $\mathfrak{g}$ is the n dimensional vector space of the left translation invariant vector fields on G. For every $x \in \mathfrak{g}$ there exists a unique one parameter subgroup in G, which we denote by $e^{\mathbb{R}x}$ or $\{e^{tx}; t \in \mathbb{R}\}$. Thus $t \to e^{tx} \in G$ is the smooth homomorphism from $\mathbb{R}$ into G, for which

$$(xf)(g) = \tfrac{d}{dt} f(ge^{tx})|_{t=0} \ .$$

The *exponential mapping*

$$E : \mathfrak{g} \to G$$

is then defined by $E(x) = e^x$. We will also use the notation exp. It is known that E is an analytic mapping. If $\{b_i\}_1^n$ is a linear basis for $\mathfrak{g}$, then

$$\mathbb{R}^n \ni \xi = (\xi_1, \xi_2, \cdots, \xi_n) \to E\left(\sum_1^n \xi_i b_i\right) \in G$$

defines a local coordinate system in a neighborhood of the unit element e of G, the so called *canonical coordinates of the first kind.*

More important for many purposes are the canonical coordinates of the second kind. These are defined by the mapping

$$(\xi_1, \xi_2, \cdots, \xi_n) \to E(\xi_1 b_1) E(\xi_2 b_2) \cdot E(\xi_n b_n) \ .$$

They are particularly usefull, when the basis $\mathcal{B} = \{b_1, b_2, \cdots, b_n\}$ of $\mathfrak{g}$ is a *Malcev basis* or a *Jordan–Hölder* basis: Let

$$\mathfrak{g}_k = \mathfrak{g}_k(\mathcal{B}) := \sum_{i=k}^{n}{}^{\oplus} \mathbb{R}\, b_i$$

be the subspaces generated linearly by $b_k, b_{k+1}, \cdots, b_n$.

Definition. The basis $\mathcal{B}$ of $\mathfrak{g}$ is a *Malcev basis* or briefly an *M* basis, if all $\mathfrak{g}_k$ are subalgebras of $\mathfrak{g}$ and $\mathfrak{g}_{k+1}$ is an ideal in $\mathfrak{g}_k$ for $k = 1, 2, \ldots, n$, with $\mathfrak{g}_{n+1} = 0$. $\mathcal{B}$ is a *Jordan–Hölder basis* or JH basis, if $\mathcal{B}$ is a Malcev basis and the sequence $\{\mathfrak{g}_k\}_k$ is a refinement of a composition series.

So an M basis $\mathcal{B}$ is a JH basis, if for $k = 1, 2, \ldots, k$ either $\mathfrak{g}_k$ is an ideal in $\mathfrak{g}$ or $\mathfrak{g}_{k+1}$ is an ideal and $\mathfrak{g}_{k-1}/\mathfrak{g}_{k+1}$ is an irreducible $\mathfrak{g}$ module. Evidently M bases, resp. JH bases exist if and only if $\mathfrak{g}$ is solvable.

It is clear that for every normal series in $\mathfrak{g}$, i.e. every series $\mathfrak{g} = \mathfrak{h}_1 \supset \mathfrak{h}_2 \supset \cdots \supset \mathfrak{h}_m \supset 0$ of ideals $\mathfrak{h}_j$ in $\mathfrak{g}$, there exists a JH basis $\mathcal{B}$ such that $\mathfrak{h}_j = \mathfrak{g}_{k_j}(\mathcal{B})$ for suitable k_j, $j = 1, 2, \ldots, m$. To see this we may assume that $\{\mathfrak{h}_j\}$ is a composition series. This implies that all quotients $\mathfrak{h}_j/\mathfrak{h}_{j+1}$ are $\mathfrak{g}$ irreducible, hence dim $(\mathfrak{h}_j/\mathfrak{h}_{j+1}) \leq 2$. If dim $(\mathfrak{h}_j/\mathfrak{h}_{j+1}) = 2$, then $[\mathfrak{h}_j, \mathfrak{h}_j] \subset \mathfrak{h}_{j+1}$ and any subspace $\mathfrak{a}$ with dim $\mathfrak{a} =$ dim $\mathfrak{h}_{j+1} + 1$ and $\mathfrak{h}_j \supset \mathfrak{a} \supset \mathfrak{h}_{j+1}$ is an ideal in $\mathfrak{h}_j$. Adding such subspaces to the chain of the $\mathfrak{h}_j$ we obtain a refinement $\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_n$ of the series $\{\mathfrak{h}_j\}$. Then any set $\mathcal{B} = \{b_k\} \subset \mathfrak{g}$ with $\mathfrak{g}_k = \mathbb{R}\, b_k \oplus \mathfrak{g}_{k+1}$ is an JH–basis with $\mathfrak{h}_j = \mathfrak{g}_{k_j}(\mathcal{B})$ for suitable k_j.

Now let $\mathcal{B} = \{b_j\}$ be a fixed M–basis of the solvable Lie algebra $\mathfrak{g}$. Then the following proposition holds:

The mapping

$$E_{\mathcal{B}} : \xi = (\xi_1, \xi_2, \ldots, \xi_n) \to E_{\mathcal{B}}(\xi) = E(\xi_1 b_1)E(\xi_2 b_2) \ldots E(\xi_n b_n)$$

is bijective and bianalytic from $\mathbb{R}^n$ onto the connected, simply connected solvable Lie group G, defined by $\mathfrak{g}$.

The well known proof is an immediate consequence of the following, equally well known fact: Let $\mathfrak{h}$ be an ideal of codimension 1 in $\mathfrak{g}$ and H the corresponding analytic closed normal subgroup of G. If $\mathfrak{g} = \mathbb{R}\, a \oplus \mathfrak{h}$ and $R = E(\mathbb{R}\, a)$ is the one parameter subgroup defining a, then $G = R \ltimes H$ is the semidirect product of R and H and $(\xi, x) \to E(\xi a)x$ is a bianalytic map from $\mathbb{R} \times H$ onto G.

If a Malcev basis $\mathcal{B}$ is fixed, we will frequently identify $\xi \in \mathbb{R}^n$ with its image $E_{\mathcal{B}}(\xi) \in G$, so G, as a manifold, "is" $\mathbb{R}^n$ and the one dimensional subspace $\{x \in \mathbb{R}^n\, ,\ x_i = 0 \text{ for } i \neq j\}$ is the one parameter group $e^{\mathbb{R}\, b_j}$. In this sense we can write

$$E\left(\sum_1^n \xi_j b_j\right) = (y_1(\xi), y_2(\xi), \ldots, y_n(\xi)) \tag{1}$$

where the y_j are real analytic functions in ξ. Assume that $\mathfrak{h} = \mathfrak{g}_k$ for some k is an ideal, hence $H = \{x \in \mathbb{R}^n\, ;\ x_i = 0, i < k\}$ is the corresponding normal subgroup in G. Then the functorial properties of the exponential mapping imply, that the coset $E(x)H$ for $x \in \mathfrak{g}$ depends only on the coset $x + \mathfrak{h}$ in $\mathfrak{g}$. Consequently the functions $y_j(\xi)$ in (1) depend for $j < k$ only on the ξ_i, $i < k$. This implies

(2)**Lemma.** *Let $\mathfrak{B}$ be a JH basis and $j_1 = 1, j_2, \ldots, j_l$ the sequence of indices, for which the $\mathfrak{g}_{j_k}$ are ideals, thus $j_{k+1} \leq j_k + 2$. Then E is a local diffeomorphism in $x^0 = \sum \xi_j^0 b_j \in \mathfrak{g}$ if and only if all determinants $D_k = \frac{\partial y_{j_k}}{\partial \xi_{j_k}}\,(x^0)$ $(j_{k+1} = j_k + 1)$, resp. $D_k = \left(\frac{\partial y_{j_k}}{\partial \xi_{j_k}} \frac{\partial y_{j_k+1}}{\partial \xi_{j_k+1}} - \frac{\partial y_{j_k}}{\partial \xi_{j_k+1}} \frac{\partial y_{j_k+1}}{\partial \xi_{j_k}}\right)(x^0)$ are nonzero.*

This is clear, because the product $\prod_{k=1}^{l} D_k$ is the determinant of the Jacobian $\left(\frac{\partial y}{\partial \xi}\right)$ of E in x^0.

We will now study the conditions, under which the exponential mapping is a global diffeomorphism, i.e. G is an exponential group in the sense of the

Definition. An *exponential group* is a connected, simply connected solvable Lie group G, for which the exponential map E is a diffeomorphism from the Lie algebra $\mathfrak{g}$ of G onto G.

We start with the indecomposable groups of dimension 2 and 3.

$n = 2$: There exists exactly one algebra

$$\mathfrak{s}_2 = \mathbb{R}\, a \oplus \mathbb{R}\, b \text{ with } [a, b] = b$$

with the group

$$S_2 = \mathbb{R} \ltimes \mathbb{R} \text{ with } (x_1, x_2)(y_1, y_2) = (x_1 + y_1, e^{-y_1} x_2 + y_2)\ ,$$

thus $a = \partial_1 - x_2 \partial_2$, $b = \partial_2$. Here (and in the sequel) we use the notation $\partial_j = \frac{\partial}{\partial x_j}$.

So clearly $e^{\mathbb{R} a} = \mathbb{R} \times \{0\}$, $e^{\mathbb{R} b} = \{0\} \times \mathbb{R}$ and

$$E(\xi_1 a) E(\xi_2 b) = (\xi_1, \xi_2)\ .$$

Let

$$\tilde{e}(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} = \frac{e^z - 1}{z}\ .$$

Then one verifies easily that in S_2 the coordinates of the first kind are given by

$$E(\xi_1 a + \xi_2 b) = (\xi_1, \tilde{e}(-\xi_1) \xi_2)\ ,$$

which shows that S_2 is exponential.

$n = 3$: There are two different types: The three dimensional Heisenberg algebra

$$\mathfrak{h}_1 = \mathbb{R}\, a \oplus \mathbb{R}\, b \oplus \mathbb{R}\, c \quad \text{with} \quad [a, b] = c$$

and group

$$\mathbb{H}_1 = \mathbb{R}^3 \text{ with } (x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2) ,$$

thus $a = \partial_1$, $b = \partial_2 + x_1 \partial_3$, $c = \partial_3$,

$$E(\xi_1 a + \xi_2 b + \xi_3 c) = (\xi_1, \xi_2, \xi_3 + \tfrac{1}{2}\xi_1 \xi_2) ,$$

and $\mathbb{H}_1$ is exponential.

Now let $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}^\times$. We set

$$\mathfrak{s}(\lambda) = \mathbb{R}\, b_1 \oplus \mathbb{R}\, b_2 \oplus \mathbb{R}\, b_3$$

with

$$\begin{aligned} [b_1, b_2] &= \lambda_1 b_2 + \lambda_2 b_3 \\ [b_1, b_3] &= -\lambda_2 b_2 + \lambda_1 b_3 . \end{aligned}$$

and group

$$S(\lambda) = \mathbb{R} \times \mathbb{C} \text{ with } (x, u)(y, v) = (x + y\, ,\; e^{-\lambda y} u + v)$$

which means that $S(\lambda) = \mathbb{R} \ltimes \mathbb{C}$ is a semidirect product with minimal normal subgroup $\mathbb{C}$, if $\lambda_2 \neq 0$. With $(x_1, x_2 + i x_3) = (x_1, x_2, x_3) \in S(\lambda)$ the b_j are given by

$$b_1 = \partial_1 - (\lambda_1 x_2 - \lambda_2 x_3)\partial_2 - (\lambda_2 x_2 + \lambda_1 x_3)\partial_3\, ,\; b_2 = \partial_2\, ,\; b_3 = \partial_3$$

and the exponential mapping in the complex coordinates $\xi_2 + i\xi_3$ by

$$E(\xi_1 b_1 + \xi_2 b_2 + \xi_3 b_3) = (\xi_1, \tilde{e}(-\lambda\xi_1)(\xi_2 + i\xi_3)) . \tag{3}$$

We see that $S(\lambda)$ is exponential if and only if $\lambda_1 \neq 0$, hence $S(i)$ is (up to isomorphisms) the only nonexponential simply connected solvable Lie group in dimensions $\leqq 3$. In this case the complement of the image of E is the set of all elements $(2\pi k, 0) \in S(i)$ with $k \in \mathbb{Z} \setminus \{0\}$ and $E(2\pi k b_1 + x) = E(2\pi k b_1) = (2\pi k, 0)$ for all $k \in \mathbb{Z} \setminus \{0\}$, $x \in \mathbb{R}\, b_2 \oplus \mathbb{R}\, b_3$.

It turns out that the absence of $S(i)$, resp. $\mathfrak{s}(i)$ is responsible for a simply connected solvable group G to be exponential:

Theorem 1. *Let G be a connected, simply connected solvable Lie group of dimension n and let $\mathfrak{g}$ be its Lie algebra. The following conditions are equivalent:*

(i) *No factor group of G has closed subgroups isomorphic with $S(i)$.*

(ii) *No ad x, $x \in \mathfrak{g}$, has non zero purely imaginary eigenvalues.*

(iii) *E is injective.*

(iv) *E is surjective.*

(v) *E is a diffeomorphism from* $\mathfrak{g}$ *onto* G.

Proof. If (i) does not hold then $\mathfrak{g}$ contains an ideal $\mathfrak{h}$ and elements $x, y, z \notin \mathfrak{h}$ with

$$(4) \qquad [x, y] \equiv -z (\mathrm{mod}\ \mathfrak{h}), \ [x, z] \equiv y (\mathrm{mod}\ \mathfrak{h}) .$$

This means ad x $(y + iz) \equiv i(y + iz)(\mathrm{mod}\ \mathfrak{h})$, hence i is an eigenvalue of ad x. On the other hand, if i is an eigenvalue of ad x, then there is $y + iz$ in the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ with (ad $x - i)(y + iz) = 0$, i.e. $[x, y] = -z$, $[x, z] = y$.

Let $\mathfrak{a}$ be a maximal ideal with $z \notin \mathfrak{a}$ and $\mathfrak{b} \subset \mathfrak{g}$ an ideal with $\mathfrak{a} \subset \mathfrak{b}$ and such that $\mathfrak{b}/\mathfrak{a}$ is minimal in $\mathfrak{g}/\mathfrak{a}$. Then necessarily $z \in \mathfrak{b}$, hence $y = [x, z] \subset \mathfrak{b}$ and $(\mathbb{R}\, x \oplus \mathfrak{b})/\mathfrak{a} \cong \mathfrak{s}(i)$, which implies $J/A \cong S(i)$ for the subgroups J and A of G corresponding to $\mathbb{R}\, x \oplus \mathfrak{b}$ and $\mathfrak{a}$. Thus we proved (i)$\Leftrightarrow$(ii).

For the rest of the proof we use induction. The theorem is true for dimensions $1, 2$ and 3, so let us assume that it is true for all connected, simply connected solvable groups of dimension less than n.

Let $\mathfrak{m}$ be a nonzero minimal ideal in $\mathfrak{g}$ and M its corresponding connected normal subgroup in G. We set $\overline{\mathfrak{g}} = \mathfrak{g}/\mathfrak{m}$, resp. $\overline{G} = G/M$ for the quotients, similarly $\overline{a} = a + \mathfrak{m} \in \overline{\mathfrak{g}}$, $\overline{x} = xM \in \overline{G}$ for $a \in \mathfrak{g}$, $x \in G$. Now assume that (i) and (ii) hold.

Let $a, b \in G$ with $E(a) = E(b)$, hence $\overline{E(a)} = E(\overline{a}) = E(\overline{b})$, which implies $\overline{a} = \overline{b}$, because dim $\overline{G} \leq n - 1$ and clearly (ii) holds also for $\overline{\mathfrak{g}}$ and $\overline{G}$. But $\overline{a} = \overline{b}$ implies that $\mathbb{R}\, a + \mathfrak{m} = \mathbb{R}\, b + \mathfrak{m} = \mathfrak{h}$ is a subalgebra and dim $\mathfrak{h} \leq 3$. Certainly (i) is true for $\mathfrak{h}$, hence E is injective and so $E(a) = E(b)$ implies $a = b$, which means that (iii) is true for $\mathfrak{g}$ and G.

Now let $x \in G$. As $\overline{G}$ satisfies (i) and (ii) there exist $\overline{a} = a + \mathfrak{m} \in \overline{\mathfrak{g}}$ with $\overline{x} = xM = E(\overline{a})$, which means $x = E(a)y$ with $y \in M$. Again $\mathfrak{h} = \mathbb{R}\, a + \mathfrak{m}$ is a subalgebra of dimension ≤ 3, hence E is a bijection from $\mathfrak{h}$ onto the corresponding subgroup H. As $x \in H$ there exists $b \in \mathfrak{h}$ with $E(b) = x$. This shows that E is surjective, i.e. (iv) holds for $\mathfrak{g}$ and G.

To prove (iv) $\Rightarrow$ (ii) let a be an element in $\mathfrak{g}$ and $A = \mathrm{ad}\, a$. With the same notation as above we see that $E(\overline{u}) = \overline{E(u)}$ for $\overline{u} \in \overline{\mathfrak{g}}$ implies that E is surjective from $\overline{\mathfrak{g}}$ onto $\overline{G}$, hence (i) through (v) hold for $\overline{\mathfrak{g}}$ and $\overline{G}$, in particular A has no purely imaginary eigenvalues on $\mathfrak{g}/\mathfrak{m}$. Again consider the subalgebra $\mathfrak{h} := \mathbb{R}\, a + \mathfrak{m}$ with corresponding subgroup H. For $x \in H$ there exists $u \in \mathfrak{g}$ with $x = E(u)$, hence $\overline{x} = E(\overline{u})$ in $\overline{G}$. But $\overline{x} = E(t\overline{a})$ for some $t \in \mathbb{R}$, and because E is one to one on $\overline{\mathfrak{g}}$ it follows that $\overline{u} = t\overline{a}$, i.e. $u \in \mathfrak{h}$. This proves that E is surjective from $\mathfrak{h}$ to H and consequently $\mathfrak{h}$ is not isomorphic to $\mathfrak{s}(i)$, in particular A has no purely imaginary eigenvalues on $\mathfrak{m}$.

Next we show (iii)$\Rightarrow$(i): We assume that E is injective, but that there exist an ideal $\mathfrak{k}$ in $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$ with $\mathfrak{h}/\mathfrak{k} \cong \mathfrak{s}(i)$, which means that there are elements a, b, c in $\mathfrak{g}$, independent modulo $\mathfrak{k}$, with $[a, b] + c$, $[a, c] - b \in \mathfrak{k}$. We know that we can choose a and b such that for the cosets $\overline{a} = a + \mathfrak{k}$, we have

$E(\overline{a}) = E(\overline{a} + \overline{b})$ in the quotient $H/K \subset G/K$. This means that there exists $q \in K$ with $E(a)q = E(a+b)$ in G. But $\mathfrak{j} := \mathbb{R}\,a \oplus \mathfrak{k}$ is a proper subalgebra of $\mathfrak{g}$ and E is injective on $\mathfrak{j}$, hence E is bijective and consequently there exists $d \in \mathfrak{j}$ with $E(d) = E(a)q = E(a+b)$. Since E is injective this implies $d = a + b$, which is a contradiction, because $a + b \notin \mathfrak{j}$.

We have proved the equivalence of conditions (i)–(iv). Evidently (v) implies (iii) and (iv), so finally we have to show that a bijective E is a diffeomorphism, i.e. that the Jacobian of E is regular in every point $x \in \mathfrak{g}$. For this we will use lemma (2). Let $\mathfrak{m}$ be a minimal non zero ideal in $\mathfrak{g}$.

If $\mathfrak{m}$ is central we may assume that $\mathfrak{m} = \mathbb{R}\,b_n$. Then, with the notation of (1) and $\xi' = (\xi_1, \xi_2, \dots, \xi_{n-1}, 0)$, we have

$$y_n(\xi) = y_n(\xi') + \xi_n\ ,$$

hence $\frac{\partial y_n}{\partial \xi_n} = 1$. We take $\mathfrak{m}$ as last ideal of a JH–series and $\mathcal{B}$ as the corresponding JH–basis. Then we can apply induction and lemma (2): From $\dim \mathfrak{g}/\mathfrak{m} = n-1$ we see that $D_k \neq 0$ for $k < l$, while $D_l = \frac{\partial y_n}{\partial \xi_n} = 1$.

If $\mathfrak{m}$ is not central, then the centralizer $\mathfrak{c}$ is an ideal in $\mathfrak{g}$. If $\dim \mathfrak{g}/\mathfrak{c} = 2$, then also $\dim \mathfrak{m} = 2$ and we could find some $x \in \mathfrak{g}$ such that $\operatorname{ad} x|_{\mathfrak{m}}$ would have eigenvalues $\pm i$, which is excluded by (ii). Thus $\mathfrak{g} = \mathbb{R}\,a \oplus \mathfrak{c}$. Let $\{\mathfrak{h}_j\}_1^l$ be a composition series passing through $\mathfrak{c}$ and $\mathfrak{m}$, i.e. with $\mathfrak{h}_2 = \mathfrak{c}$, $\mathfrak{h}_l = \mathfrak{m}$, and let $\mathcal{B}$ be a corresponding JH–basis.

Then $\mathfrak{h}_2 = \sum\limits_{i=2}^{n} \mathbb{R}\,b_i$, $\mathfrak{m} = \mathbb{R}\,b_n$ or $\mathfrak{m} = \mathbb{R}\,b_{n-1} \oplus \mathbb{R}\,b_n$ and $\operatorname{ad} b_1$, restricted to $\mathfrak{m}$, has an eigenvalue $\lambda = \lambda_1 + i\lambda_2$ with $\lambda_1 \neq 0$. We will only treat the case $\dim \mathfrak{m} = 2$.

Because $\mathfrak{m}$ is central in $\mathfrak{h}_2$ we see that for any $x \in \mathfrak{h}_2$ the restrictions of $\operatorname{ad}(\xi_1 b_1)$ and $\operatorname{ad}(\xi_1 b_1 + x)$ onto $\mathfrak{m}$ coincide, hence for $\xi_1 \neq 0$ the subalgebra $\mathbb{R}(\xi_1 b_1 + x) \oplus \mathfrak{m}$ is isomorphic with $\mathfrak{s}(\lambda)$ and formula (3), after applying the isomorphism $\xi_{n-1} \cdot 1 + \xi_n \cdot i \to \xi_{n-1} b_{n-1} + \xi_n b_n$ from $\mathbb{C}$ onto $\mathfrak{m}$, gives for $x = \sum\limits_{i=2}^{n-2} \xi_i b_i$ the relation

$$E\left(\sum_{i=1}^{n} \xi_i b_i\right) = E\left(\sum_{i=1}^{n-2} \xi_i b_i\right) E((e_1(\xi_1)\xi_{n-1} - e_2(\xi_1)\xi_n) b_{n-1} + \\ + (e_2(\xi_1)\xi_{n-1} + e_1(\xi_1)\xi_n) b_n),$$

where $e_1(\xi_1)$, $e_2(\xi_1)$ are defined by $\tilde{e}(-\lambda\xi_1) = e_1(\xi_1) + ie_2(\xi_1)$. From (1) we get

$$E\left(\sum_{i=1}^{n-2} \xi_i b_i\right) = (y_1(\xi)\, ,\ y_2(\xi), \cdots, y_{n-2}(\xi)\, ,\ w_1(\xi')\, ,\ w_2(\xi'))$$

with $\xi' = (\xi_1, \xi_2, \dots, \xi_{n-2}) \in \mathbb{R}^{n-2}$. Hence the last two coordinates of $E\left(\sum_{i=1}^{n} \xi_1 b_i\right)$ are given by

$$\begin{aligned} y_{n-1}(\xi) &= w_1(\xi') + e_1(\xi_1)\xi_{n-1} - e_2(\xi_1)\xi_n \\ y_n(\xi) &= w_2(\xi') + e_2(\xi_1)\xi_{n-1} + e_1(\xi_1)\xi_n \,. \end{aligned}$$

This yields for the Jacobian of $(\xi_{n-1}, \xi_n) \to (y_{n-1}, y_n)$:

$$\left(\frac{\partial y}{\partial \xi}\right)_{n-1,n} = \begin{pmatrix} e_1(\xi_1)\,, & -e_2(\xi_1) \\ e_2(\xi_1)\,, & e_1(\xi_1) \end{pmatrix}$$

and

$$D_l = e_1(\xi_1)^2 + e_2(\xi_1)^2 = |\widetilde{e}(-\lambda\xi_1)|^2 > 0$$

for all $\xi_1 \in \mathbb{R}$, because $\lambda_1 \neq 0$. This ends the proof of Theorem 1.

It is clear from theorem 1 that closed connected subgroups and simply connected quotients of exponential groups are also exponential; moreover we have

Proposition 1. *Let G be as in theorem 1. If H is a closed connected subgroup of G with corresponding subalgebra $\mathfrak{h}$, then H is exponential if and only if $H = E(\mathfrak{h})$.*

This follows immediately from theorem 1 and the functorial properties of the exponential mapping.

Let H be a closed subgroup of the Lie group G. The coset space G/H of all left cosets $\overline{x} = xH$, $x \in G$, is also a manifold. If G is solvable, connected and simply connected of dimension n and H is connected of dimension l, then G/H is diffeomorphic with $\mathbb{R}^{n-l}$. We want to describe this diffeomorphism explicitly by means of the exponential map and suitable bases in the Lie algebra $\mathfrak{g}$ of G.

Let $\mathfrak{h}$ be the Lie algebra of the closed subgroup H. Then $\mathfrak{g}/\mathfrak{h}$ is an $\mathfrak{h}$ module, hence there exists $\mathfrak{h}$ composition series of $\mathfrak{g}/\mathfrak{h}$, i.e. $\mathfrak{h}$ invariant subspaces $\mathfrak{g} = \mathfrak{g}_1 \supset \dots \supset \mathfrak{g}_r \supset \mathfrak{g}_{r+1} = \mathfrak{h}$, such that the quotients $\mathfrak{g}_j/\mathfrak{g}_{j+1}$ are $\mathfrak{h}$ irreducible, so dim $(\mathfrak{g}_j/\mathfrak{g}_{j+1})$ is 1 or 2. Slightly incorrectly we will call such a series $\{\mathfrak{g}_j\}_{j=1,\dots,r+1}$ an $\mathfrak{h}$ composition series of $\mathfrak{g}/\mathfrak{h}$.

Definition. A *coexponential basis for* $\mathfrak{h}$ *in* $\mathfrak{g}$ is a set $\mathcal{B}_{\mathfrak{h}} = \{b_1, b_2, \dots, b_d\}$ of elements $b_j \in \mathfrak{g}$, independent modulo $\mathfrak{h}$, such that

(i) $\{\mathfrak{b}_j \oplus \mathfrak{h}\}_{j=1,\dots,d}$, with $\mathfrak{b}_j = \sum_{i=j}^{d}{}^{\oplus} \mathbb{R} b_i$, is a refinement of an $\mathfrak{h}$ composition series of $\mathfrak{g}/\mathfrak{h}$ and

(ii) the mapping

$$E_{\mathfrak{h}} : (\sum_{j=1}^{d} \xi_j b_j, h) \to E(\xi_1 b_1)E(\xi_2 b_2) \cdots E(\xi_d b_d)h$$

maps $\mathfrak{b}_1 \times H$ diffeomorphically onto G.

It follows that for a coexponential basis $\mathcal{B}_{\mathfrak{h}}$ the mapping $x = \sum_1^d \xi_j b_j \to E_{\mathfrak{h}}(x, e)H \in G/H$ is a diffeomorphism of $\mathfrak{b}_1$ onto G/H.

Proposition 2. *If G is connected, simply connected solvable, then for every subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ there exist coexponential bases.*

Proof. Let $\mathfrak{k}$ be a subalgebra with $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$. If $\{b_1, \dots, b_d\}$ is a coexponential basis for $\mathfrak{k}$ in $\mathfrak{g}$ and $\{c_1, \dots, c_k\}$ a coexponential basis for $\mathfrak{h}$ in $\mathfrak{k}$, then obviously $\{b_1, \dots, b_d, c_1, \dots, c_h\}$ is coexponential for $\mathfrak{h}$ in $\mathfrak{g}$. So we only need to consider the case that $\mathfrak{h}$ is a maximal proper subalgebra of $\mathfrak{g}$. If $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$, then $\mathfrak{h}$ is an ideal and any $b \notin \mathfrak{h}$ can be taken as a coexponential basis. Otherwise $\mathfrak{g} = \mathfrak{g}' + \mathfrak{h}$. As dim $\mathfrak{g}' <$ dim $\mathfrak{g}$, induction allows us to assume that $\mathfrak{g}'$ contains a coexponential basis $\{b\}$, resp. $\{b_1, b_2\}$ for $\mathfrak{g}' \cap \mathfrak{h}$. We claim, that this basis is also coexponential for $\mathfrak{h}$ in $\mathfrak{g}$. We consider only the case $\{b_1, b_2\}$. With $B = E(\mathbb{R}b_1)E(\mathbb{R}b_2) \subset G'$ we have $G' = B(G' \cap H)$, hence $G = G'H = BH$. From $xy = x'y'$, $x, x' \in B$, $y, y' \in H$, we get $x'^{-1}x = y'y^{-1} \in G' \cap H$, hence $x = x'$, $y = y'$, because $\{b_1, b_2\}$ is coexponential for $\mathfrak{g}' \cap \mathfrak{h}$ in $\mathfrak{g}'$. Thus $E_{\mathfrak{h}}$ is bijective from $(\mathbb{R}b_1 \oplus \mathbb{R}b_2) \times H =: \mathfrak{b} \times H$ onto G. Clearly $E_{\mathfrak{h}}$ is a local diffeomorphism from a neighborhood of $\{0\} \times \{e\}$ onto a neighborhood of the unit element e in G. Let $\{b_3, \dots, b_n\}$ be a basis of $\mathfrak{h}$, so that $x = \sum_1^n \xi_j b_j \to F(x) = E(\xi_1 b_1)E(\xi_2 b_2) \dots E(\xi_n b_n)$ is a diffeomorphism F of a neighborhood of $0 \in \mathbb{R}^n$ onto a neighborhood of e. If $q : x \to q(x) \in \mathfrak{g}$ is a polynomial mapping of $\mathfrak{g}$ into $\mathfrak{g}$ without constant and linear terms, then also $x \to E(q(x))F(x)$ is a local diffeomorphism. Now fix an element $(x^0, h_0) = (\xi_1^0 b_1 + \xi_2^0 b_2, h_0)$, in $\mathfrak{b} \times H$. Then, since the ad y for $y \in \mathfrak{g}'$ are uniformly nilpotent, it is easy to see that there exists a $\mathfrak{g}'$ valued polynomial $q(\xi_1, \xi_2)$ without constant and linear terms, such that for $x = \xi_1 b_1 + \xi_2 b_2$, $h \in H$

$$E_{\mathfrak{h}}(x + x^0, hh^0) = E(x^0)E(q(\xi_1, \xi_2))E_{\mathfrak{h}}(x, h)h^0 \ .$$

Hence $E_{\mathfrak{h}}$ is also locally diffeomorphic in (x^0, h^0).

§ 2 Invariant measures, group algebras

The existence of Haar measures, i.e. of left invariant Radon measures in Lie groups, is a simple consequence of the existence of left invariant cotangent fields on the underlying manifold, see [5],ch. III, § 3.16. The construction shows also at once that the Haar modulus Δ_G of the Haar measure dx of the group G is given by

$$\Delta_G(x) = |\det \mathrm{Ad}\, x|^{-1}$$

where Ad x is the adjoint transformation on the Lie algebra $\mathfrak{g}$ of G, defined by $x \in G$. If G is connected, then clearly $\det \mathrm{Ad}\, x > 0$ for all $x \in G$, hence $\Delta_G(x) = \det \mathrm{Ad}\, x^{-1}$.

If G is solvable and simply connected, then the Haar measure is explicitly given by $dx = d\xi_1 d\xi_2 \dots d\xi_n$, where the ξ_j are the canonical coordinates of the second kind defined by a given M basis $\mathcal{B}$ of the Lie algebra $\mathfrak{g}$ of G. This follows immediately by induction on $\dim \mathfrak{g}$ and the following trivial fact: If the locally compact group Γ is the semidirect product of the closed subgroups A and B, i.e. $\Gamma = A \ltimes B$ with normal B, then the (left) Haar measure $d\gamma$ of Γ is the product measure of the Haar measures dx of A and dy of B, i.e. $d\gamma = dxdy$.

If H is a connected closed subgroup of G and $\mathcal{B}_{\mathfrak{h}} = \{b_1, b_2, \dots, b_d\}$ a coexponential basis for $\mathfrak{h}$, then $\mathfrak{b} = \sum_1^{d}{}^{\oplus} \mathbb{R} b_j$ parametrizes the coset space $\overline{G} = G/H$. It is known that $\overline{G}$ carries an essentially unique relatively G–invariant positive Radon measure $d\overline{x}$, but this measure is in general not equivalent to the Lebesgue measure on $\mathfrak{b}$. However, for particular choices of $\mathcal{B}_{\mathfrak{h}}$ one can describe $d\overline{x}$ explicitly and simply by use of the Lebesgue measure on $\mathfrak{b}$.

Again let G, H, $\mathfrak{g}$ and $\mathfrak{h}$ be as above. As we had mentioned, the Haar moduli Δ_G and Δ_H of G and H are given by $\Delta_G(x) = \det \mathrm{Ad}_{\mathfrak{g}} x^{-1}$, $\Delta_H(y) = \det \mathrm{Ad}_{\mathfrak{h}} y^{-1}$. We set

$$\chi_H^G(y) = \frac{\Delta_G(y)}{\Delta_H(y)} = \det \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}} y^{-1}$$

for $y \in H$. Here $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(y)$ is the transformation induced by $\mathrm{Ad}_{\mathfrak{g}}(y)$ on the quotient space $\mathfrak{g}/\mathfrak{h}$. If $y = \exp h$ with $h \in \mathfrak{h}$, then $\mathrm{Ad}\, y = e^{\mathrm{ad}\, h}$, thus in this case

$$\chi_H^G(\exp h) = e^{-\mathrm{trace}\, \mathrm{ad}_{\mathfrak{g}/\mathfrak{h}} h} .$$

This shows that $\chi_H^G(\exp h) = 1$ whenever ad h is nilpotent, in particular

$$\chi_H^G(y) = 1 \text{ for all } y \in G' \cap H .$$

So χ_H^G has a unique extension to a continuous homomorphism, again denoted by χ_H^G, from $G'H$ into the positive reals:

$$\chi_H^G(xy) = \chi_H^G(y) \text{ for } x \in G' \, , \; y \in H \; .$$

If M is a closed normal subgroup in G, then $\Delta_M = \Delta_G|_M$ and $\Delta_G(x) = \Delta_{\overline{G}}(\overline{x})\Delta_M(\text{ int } x)^{-1}$ where $\overline{G} = G/M$, $\overline{x} = xM \in \overline{G}$ and $\Delta_M(\text{int } x)$ is the Haar modulus of the restriction of the inner automorphism int $x : y \to xyx^{-1}$ on M. If $M \subset H$, $\overline{H} = H/M \subset \overline{G}$, then it follows that

$$\chi_{\overline{H}}^{\overline{G}}(\overline{x}) = \chi_H^G(x) \text{ for } x \in \overline{x} \in \overline{H} \; . \tag{5}$$

Now let $\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_r \supset \mathfrak{g}_{r+1} = 0$ be a composition series of $\mathfrak{g}$, so $\mathfrak{m} := \mathfrak{g}_r$ is a minimal non zero ideal in $\mathfrak{g}$. There exists a coexponential $\mathcal{B}_\mathfrak{h} = \{b_j\}_j$ for $\mathfrak{h}$ in $\mathfrak{g}$, so that $\{\mathfrak{b}_j \oplus \mathfrak{h}\}_j$ (see the definition) is a refinement of $\{\mathfrak{g}_i + \mathfrak{h}\}_i$. This means that for every i with $\mathfrak{g}_i + \mathfrak{h} \neq \mathfrak{g}_{i+1} + \mathfrak{h}$ there is a j with $b_j \in \mathfrak{g}_i$, resp. $b_j, b_{j+1} \in \mathfrak{g}_i$ and $\mathfrak{b}_j \oplus \mathfrak{h} = \mathfrak{g}_i + \mathfrak{h} = \mathbb{R}b_j \oplus (\mathfrak{g}_{i+1} + \mathfrak{h})$, resp. $= \mathbb{R}b_j \oplus \mathbb{R}b_{j+1} \oplus (\mathfrak{g}_{i+1} + \mathfrak{h})$. Let $E_\mathfrak{h}$ be the diffeomorphism from $\mathfrak{b}_1 \times H$ onto G defined by $\mathcal{B}_\mathfrak{h}$. If $\xi = (\xi_1, \xi_2, \dots \xi_d) \in \mathbb{R}^d$ and $y \in H$, we denote the element $E_\mathfrak{h}(\sum_1^d \xi_i b_i, y)$ by (ξ, y), in particular for any function f on G we set

$$f(\xi, y) := f(E(\xi_1 b_1)E(\xi_2 b_2)\dots E(\xi_d b_d)y) \; .$$

and denote the Lebesgue measure on $\mathfrak{b}_1$, resp. $\mathbb{R}^d$ by $d\xi$, i.e. $d\xi_1 d\xi_2 \dots d\xi_d$. Then we have

Theorem 2. *With the above definitions and notations for every integrable function f on G, i.e. $f \in L^1(G)$, the following formula holds:*

$$\int_G f(x)\,dx = \int_{\mathbb{R}^d} \int_H f(\xi, y)\chi_H^G(y)\,dy d\xi.$$

For the *proof* we use induction on $n = \dim \mathfrak{g}$ and consider first the case in which $\mathfrak{m} := \mathfrak{g}_r \subset \mathfrak{h}$. Let $M = E(\mathfrak{m})$, hence $M \cong \mathbb{R}^l$, $l = 1$ or 2. We set $\overline{\mathfrak{g}} = \mathfrak{g}/\mathfrak{m}$, $\overline{\mathfrak{h}} = \mathfrak{h}/\mathfrak{m}$ etc. Then $\{\overline{\mathfrak{g}}_i\}_{1 \le i \le r-1}$ is a composition series of $\overline{\mathfrak{g}}$ and $\mathcal{B}_{\overline{\mathfrak{h}}} = \{\overline{b}_j\}$ with $\overline{b}_j = b_j + \mathfrak{m} \subset \overline{\mathfrak{g}}$ a coexponential basis for $\overline{\mathfrak{h}}$ in $\overline{\mathfrak{g}}$. We have $E(\mathbb{R}\overline{b}_j) = E(\mathbb{R}b_j)M$ in $\overline{G} = G/M$ and for any $\overline{f} \in L^1(\overline{G})$ the theorem holds. Let f be integrable on G. Then $\overline{f}$, defined by $\overline{f}(\overline{x}) = \int_M f(xz)dz$ for $\overline{x} = xM \in \overline{G}$, is in $L^1(\overline{G})$. Theorem 2,

(5) and Weil's formula [2], [32] yield

$$\int_G f(x)\,dx = \int_{\overline{G}} \overline{f}(\overline{x})\,d\overline{x} = \int_{\mathbb{R}^d} \int_{\overline{H}} \overline{f}(\xi, \overline{y}) \chi_{\overline{H}}^{\overline{G}}(\overline{y})\,d\overline{y}d\xi = \\ = \int_{\mathbb{R}^d} \int_H f(\xi, y) \chi_H^G(y) dy d\xi \ .$$

So in this case theorem 2 is proved.

If $\mathfrak{m} \not\subset \mathfrak{h}$ we set $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$. If $\mathfrak{h} \cap \mathfrak{m} = 0$, then $\mathfrak{m} = \mathbb{R}b_d$ or $\mathfrak{m} = \mathbb{R}b_{d-1} \oplus \mathbb{R}b_d$, $K = H \ltimes M$ and $\{b_d\}$, resp. $\{b_{d-1}, b_d\}$ is coexponential for $\mathfrak{h}$ in $\mathfrak{k}$, moreover $\chi_H^K(y)$ is the Haar modulus of int y^{-1}, restricted to $M : \Delta_K(y) = \Delta_H(y)\Delta_M(\text{int } y^{-1})$ for $y \in H$. This gives immediately theorem 2 for K and H. If $\mathfrak{h} \cap \mathfrak{m} \neq 0$, then the first part of our proof applies, because $\mathfrak{h} \cap \mathfrak{m}$ is a minimal ideal in $\mathfrak{k}$. So again the claim holds for K and H.

Since $\mathfrak{m} \subset \mathfrak{k}$, the theorem is true for G and K, i.e. if $\mathcal{B}_{\mathfrak{k}} = \{b_1, b_2, \dots, b_c\}$, $c = d - 1$ or $d - 2$, is the coexponential basis for $\mathfrak{k}$, then

$$\int_G f(x)dx = \int_{\mathbb{R}^c} \int_K f(\xi, w) \chi_H^G(w) dw d\xi = \\ = \int_{\mathbb{R}^c} \int_{\mathbb{R}^{d-c}} \int_H f(\xi, \xi', y) \chi_K^G(\xi', y) \chi_H^K(y) dy d\xi' d\xi \ .$$

If $\mathfrak{m}$ is not central, in particular if $\dim \mathfrak{m} = 2$, then $\mathfrak{m} \subset \mathfrak{g}'$, hence $M \subset G'$ and χ_K^G is trivial on M. But if $\mathfrak{m}$ is central and not in $\mathfrak{g}'$, then M is a commutative direct factor of G, so again $\chi_K^G(z) = 1$ for $z \in M$. This implies $\chi_K^G(\xi', y)\chi_H^K(y) = \chi_K^G(y)\chi_H^K(y) = \chi_H^G(y)$, hence

$$\int_G f(x)dx = \int_{\mathbb{R}^c} \int_{\mathbb{R}^{d-c}} \int_H f(\xi, \xi', y) \chi_H^G(y) dy d\xi' d\xi = \int_{\mathbb{R}^d} \int_H f(\xi, y) \chi_H^G(y) dy d\xi \ .$$

So theorem 2 is proved.

It is known (see e.g. [32] ch. 3, § 3), that there exists a relatively invariant measure on the coset space $\overline{G} = G/H$. We want to express this measure explicitly in terms of the coordinates $\xi \in \mathbb{R}^d$. First we remark that we can extend χ_H^G to a continuous homomorphism χ from G into $\mathbb{R}^+$: The quotient G/G' is a real vector space with $(G'H)/G'$ as a closed subspace. Hence there exists a closed complementary subspace, i.e. a closed subgroup D in G with $G' \subset D$, $G/G' = D/G' \oplus (G'H)/G'$. It follows that $G = DH$ and that

$$\chi(xy) = \chi_H^G(y) \text{ for } x \in D\,,\ y \in H$$

defines unambigously an extension χ of χ_H^G with $\chi(x) = 1$ on D. For $f \in \mathscr{C}_0(G)$ we set

$$I_\chi(f) = \int_G f(x)\chi(x^{-1})dx$$

and as before

$$\overline{f}(\overline{x}) = \int_H f(xy)dy \text{ for } \overline{x} = xH \in \overline{G}.$$

We claim that $\overline{f} = 0$ implies $I_\chi(f) = 0$: Let e be the unit element in G. Then theorem 2 yields

$$\begin{aligned} I_\chi(f) &= \int_{\mathbb{R}^d}\int_H f(\xi, y)\chi(\xi, y)^{-1}\chi_H^G(y)dyd\xi = \\ &= \int_{\mathbb{R}^d}\int_H f(\xi, y)\chi(\xi, e)^{-1}dyd\xi = \int_{\mathbb{R}^d} \overline{f}(\overline{\xi})\chi(\xi, e)^{-1}d\xi = 0 \end{aligned}$$

with $\overline{\xi} = (\xi, e)^- = (\xi, e)H \in \overline{G}$. Since $f \to \overline{f}$ is a positive, surjective, linear mapping from $\mathscr{C}_0(G)$ onto $\mathscr{C}_0(\overline{G})$ it follows that there exists a positive Radon meassure $d_\chi\overline{x}$ on $\overline{G}$ so that

$$\int_G f(x)\chi(x)^{-1}dx = \int_{\mathbb{R}^d} \overline{f}(\overline{\xi})\chi(\xi, e)^{-1}d\xi = \int_{\overline{G}} \overline{f}(\overline{x})d_\chi\overline{x}$$

for all $f \in \mathscr{C}_0(G)$. For these f and $a \in G$ we have $\overline{f_a} = (\overline{f})_a$, where $f_a(x) = f(ax)$, $(\overline{f})_a(\overline{x}) = \overline{f}(a\overline{x}) = \overline{f}(\overline{ax})$. This implies

$$\int_{\overline{G}} \overline{f}_a(\overline{x})d_\chi\overline{x} = \int_G f(ax)\chi(x)^{-1}dx = \chi(a)\int_G f(x)\chi(x)^{-1}dx\ ,$$

hence $d_\chi(a\overline{x}) = \chi(a)^{-1}d_\chi\overline{x}$, i.e. $d_\chi\overline{x}$ is relatively invariant with modulus $\chi(a)^{-1}$. Summing up we have shown most of

Theorem 3. *Let H be a connected closed subgroup of G and let $\overline{G} = G/H$ the left coset space. There exists a coexponential basis for $\mathfrak{h}$ in $\mathfrak{g}$, which defines a global coordinate system $\mathbb{R}^d \ni \xi \to \overline{\xi} = (\xi, e)H \in \overline{G}$. In these coordinates an essentially unique relatively invariant measure $d_\chi\overline{x}$ on $\overline{G}$ is given by the formula*

$$\int_{\overline{G}} f(\overline{x})d_\chi\overline{x} = \int_{\mathbb{R}^d} f(\overline{\xi})\chi(\xi, e)^{-1}d\xi$$

in which χ is a real character of G, extending $\chi_H^G = \Delta_G|_H\Delta_H^{-1}$. For $a \in G$ one has $d_\chi a\overline{x} = \chi(a)^{-1}d_\chi\overline{x}$.