Postoptimal Analyses, Parametric Programming, and Related Topics

Tomas Gal

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Degeneracy, Multicriteria Decision Making, Redundancy

Second Edition

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With 185 tables and 46 figures

♾ Printed on acid-free paper which falls within the guidelines of the ANSI to ensure permanence and durability.

To my grandchildren
Thomas, Anna, Sandra, and Anique

Foreword

For some considerable time, linear programming has been one of the methods of operations research which has been widely known and much applied in Germany too, both in the literature of the subject and in practice. Its applications range from production planning through finance planning and from optimization of traffic networks through urban planning. The literature of linear programming includes textbooks of a strictly mathematical nature as well as programmed textbooks for those with no previous knowledge of mathematics.

An objection frequently heard to more extensive dissemination of the theories of linear programming in the practical field has been that data which are available in practice are at once too inexact and too unreliable to provide the basis for the application of "exact" procedures like linear programming. This problem is the starting point of the present volume.

The inexactitude and unreliability of existing data often cannot be disputed. Using conventional planning methods, the determination of the effects of these inaccuracies is frequently very difficult, if not impossible. In most cases, this is, however, possible using sensitivity analysis in the widest sense (i.e., including postoptimal analysis and parametric programming), and the amount of effort involved is reasonable. My colleague, Professor Gal has been involved in research in this field for many years. His decision to undertake the writing of an introduction to and interpretation of the area of linear programming which enables us to make statements on the possible effects of data changes, data inaccuracies, and decision changes on operational and other problems is, therefore, to be welcomed. Sensitivity analysis, as interpreted in the present volume, has been seen by experts – correctly, in my opinion – as the bridge between pure dissemination of information and decision making. There is, thus, ample justification for including a volume on such an important subject in the "Operations Research" series.

As the present volume is likely to interest both those working in linear programming and research mathematicians, the chapters have been written primarily with practical application in mind, and an abridged mathematical version has been appended to each of them. It is hoped that this will increase the usefulness of the volume for a wide range of readers.

Professor Dr H.-J. Zimmermann
Aachen, July 1973

Preface to the (second) English edition

This edition aims at to bringing the book up-to-date and correcting some errors which – in spite of almost endless efforts proofreadings again and again – still are found in the (first) English edition. Also, in the organization of References a change has been made, which should, hopefully, be of advantage to the reader: direct quotations of literature are put together as References to each of the chapters. At the end of the book a Bibliography is to be found, in which the quoted publications and such which deal directly or indirectly with the subject of the book, are listed. To offer the reader another help, an annotated bibliography is added.

In order to keep the size of the book in some limits, the author decided to omit Section 10-8 (of the previous edition) "Parametric programming in the Transportation Problem" hinting, of course, to the relevant literature in the References to Chapter 10.

The author notices with pleasure that his hope, as stated in the Preface to the (first) English edition, namely that "… this will not remain the only work on parametric programming in English and … inspiring others to write new and more comprehensive monographs on the subject", has been fulfilled. The reader will find several monographs on parametric programming written in English in the Bibliography at the end of the book (see also the References to Chapter 4).

The organization and the structure of the book has been described and explained in the Preface to the German edition. From this point of view nothing has been changed.

The author apologizes in this place for being wrong in the Preface to the (first) English edition (written in 1978). First, as is pointed out in a study on the history of parametric programming[1], Gass and Saaty have not been the very first authors dealing with parametric programming. Second, claiming, in 1978, that "… there has never been a conference on parametric programming anywhere in the world …" was wrong: In July 1977, A.V.Fiacco organized the first Conference on Data Perturbation in Washington, D.C., which is ever since being held every year.

I should take this opportunity of thanking my new/old publisher, W. de Gruyter, for taking care of publishing this new edition, while McGraw Hill forgot to notify me that the book had been sold out in 1983. I am indebted to Dr. F. Geue for his carefully reading the proofs and to Mr Th. Hanne for unifying the bibliography. My thanks go also to my second wife, Gisela, who helped me to formulate in some

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1 See [11, 12] in Chapter 4.

places the English text and for whom I spoiled our stay in Spain by intensively working on this new edition instead of sitting on the beach.

Tomas Gal
Benidorm, Spain /Hagen, Germany, January 1994

Preface to the (first) English edition

Parametric programming was first developed over 20 years ago (the first papers by Saul I. Gass and Thomas L. Saaty date back to 1955). Since then, theories of linear and nonlinear parametric programming have been worked out, solution procedures for various cases have been developed, and parametric programming has also been used for solving problems of mathematical programming, such as decomposition, quadratic-, fractional-, and nonlinear programming. Parametric programming has also been applied in various fields of economics, chemistry, technology, agriculture, etc. It is, therefore, rather odd, that there are very few courses on parametric programming in existence and that, to the author's knowledge, there has never been a conference on parametric programming anywhere in the world; also, there has been no previous monograph on parametric programming in English.

The manuscript of this book was originally written in Czech in the years 1968 through 1969. In 1973 it appeared in German. That is why the main theme of the work is linear parametric programming. The aims and objectives in writing this book are discussed in the Preface to the German Edition. The reader will also find there an explanation of why each chapter is divided into two parts.

The original German edition of this book has here been revised with the aim of correcting errors and bringing the new English edition up-to-date. Since the first chapter of the German edition (on "Systems, Models and Systems Analysis") is now out of date, it has been omitted here. In its place appears a new chapter (Chap. 9) on "Multicriteria linear programming". Minor alterations have been in almost all chapters, but Chap. 2 (Chap. 1 in the present English edition), in particular, has been extensively revised to include some relations between convex polyhedrons and graphs as well as a newly developed consideration of degeneracy. Chap. 10 now includes a section on parametrization of transportation problems. The author would like to thank Dr W. Hummeltenberg for preparing this section.

The writer regards it as a honor to find himself the author of the first monograph on parametric linear programming in English. It is to be hoped that this will not remain the only work on parametric programming in English and that the present book will make some contribution to the development, application, and dissemination of parametric programming, while at the same time inspiring others to write new and more comprehensive monographs on the subject.

The bibliography has been something of a problem. The original German version contained a list of 392 works with an appendix containing a further 27. Since the appearance of the German book, something like 300 further titles have appeared. It was quite simply impossible to include these with the earlier titles in a single list. This would have meant altering all references throughout the book.

The author, therefore, opted for presenting the new titles in a separate list, in which each number is indicated by a prime. Almost every day some new title appears. In view of the sheer number of journals and books appearing, it is impossible ever to produce a complete list of references, let alone find all the relevant publications. We shall simply add new titles to the end of the list until such time as the book has actually gone to press.

By now there exist about 500 direct or indirect references to parametric programming. The bibliography at the end of the book has been subdivided and is designed to provide the reader with a survey of relevant literature from various aspects. We make no claim to having provided a complete list. It has been put together in a way that seemed appropriate to the author, although this is possibly not the best conceivable arrangement.

I should like to take this opportunity of thanking my publisher, McGraw-Hill, for their interest and their cooperation, especially Mr A. von Hagen, Ms B. Scholtz, his editorial asistent, and their colleagues. My thanks also go to Dr Geoffrey V. Davis for his careful translation and his patience in discussing the whole text with me. I am indebted to my wife Dana for reading the proofs.

Tomas Gal
Aachen, October 1977

Preface to the German edition

Economists and specialists in other fields often meet with failure when they first attempt to introduce linear programming (LP) into their operations. The reason for this is frequently one of the following factors:

1. The difficulties which have to be overcome in devising a suitable model.
2. The uncertainty and inaccuracy of the intial data, the linearizing of the – *de facto* – nonlinear relationships, the neglect of time (dynamic) factors, the determining of originally stochastic data, etc.
3. The problem of evaluation and interpretation as well as the application and exploitation of the results in practice.

The aspects mentioned under 2 may possibly inspire specialists with a deep distrust of the result of a solved LP problem. It is for this reason that the optimal solution of a linear program may also be considered the first step towards the solution of a given operational or similar problem.

The question arises as to whether and to what extent the results of the solution of an LP problem may be of practical use, in spite of the "disadvantages" mentioned. A first reply to this would be that the optimal solution of the problem as such does, in fact, have more of an informative character.

However, there do exist other possibilities of utilizing this solution and, in a certain sense, of eliminating the "disadvantages" mentioned in 2. Such is, for example, the aim of sensitivity analysis, by means of which (among other things) one may test in what region the values, say, of the right-hand side of the restrictions can be changed, so as to maintain the optimality of the optimal solution obtained. The extension of sensitivity analysis into parametric programming also enables us to compute all existing optimal basic solutions in relation to their dependence on the values of the components of the right-hand side.

Approaches to the solution of these problems have been provided by the development of systems theory. On this theory are based systems analysis, systems synthesis, systems engineering, and other developments. Nevertheless, there is considerable confusion over the symbolism and vocabulary of these branches of systems theory. This makes it difficult to characterize them and distinguish them from one another in a few words. This is discussed more fully in the first chapter.[1]

By way of a summary, let us consider a firm as a system: using the methods of systems analysis, it is then possible to devise a linear model which can serve as the basis for setting up of the LP model. The methods of sensitivity analysis or of parametric programming enable us to follow up the connection between the firm,

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1 See the Preface to the (first) English edition.

the model, and the optimal solution. From the point of view of an expert (of an economist, for example), it is doubtless both important and necessary to be able to take into account as many of the changes occuring in the initial factors during the course of time as possible. The methods of parametric linear programming and sensitivity analysis, in association with the ideas of systems analysis, are particularly suited to this purpose. These methods are, thus, no longer the object of the investigation; they become it means.

In the practical application of LP, three main complexes of problems may be distinguished.

1. The setting-up of the linear model.
2. The computation of the optimal solution.
3. The evaluation, interpretation, and analysis of the optimal solution.

The systems analysis approach is of assistance with the first and third complexes of problems. If we are dealing with a "normal" linear program, the computation of the optimal solution requires nothing but a computer.

Sensitivity analysis and parametric programming are mainly, though not exclusively, of use in dealing with the third complex of problems. Moreover, parametric linear programming is an instrument which may possibly be applied to all three complexes of problems.

It is the aim of this book to elucidate the various methods of sensitivity analysis and parametric linear programming. The main emphasis, however, lies on the application of these methods to the most diverse purposes involved in the analysis of a model and/or of the corresponding optimal solution, as well as on the combinations of different approaches to the solution of practical problems. We shall also indicate other possible theoretical and practical applications of the methods described.

It is assumed that the reader is familiar with the fundamentals of linear programming and, therefore, also with linear algebra (vector and matrix calculus, the theory of systems of linear equations, and inequalities). As an additional aid, the fundamental principles of LP have been briefly recapitulated in Chap. 2.[2]

Each chapter is divided into two parts. In the first part, the problems are discussed with the aid of examples; the second part is then an abridged mathematical presentation which, however, makes no claim to being exhaustive. It is left to the reader to decide whether he[3] wishes to read both parts or only that which corresponds more closely to his[4] own particular interests.

The examples have been chosen with a view to helping the reader to arrive at a better understanding of the methods and problems described. They contain few unknowns and a few constraints.

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2   See the Preface to the (first) English edition
3   Or she
4   Or her.

The appendix offers an extensive list of references. Not all the authors and titles mentioned there have been quoted in the text. The aim was rather to provide a survey of the literature which is concerned directly or indirectly with the theory, solution procedures, and applications of sensitivity analysis and parametric programming. A selection of works has been included, in addition, which deal with the theory and application of systems analysis as well as a number of textbooks on linear programming which make reference to sensitivity analysis and parametric programming.

The author of this book formerly worked in Prague and, after a short stay at the University of Louvain in Belgium, took up his present post at the Rhine-Westphalian Technical University of Aachen in the autumn of 1970. The preparation of the German text of this book from an earlier rough manuscript involved certain difficulties of a linguistic nature, which were overcome with the assistance of several members of the staff of the Department of Operations Research in Aachen. I should particularly like to thank Messrs. H. Gehring, W. Hummeltenberg and Dr U. Eckhardt, as well as Miss I. Teutsch. I am also much indebted to the Head of the Department of Operations Research at the University of Aachen, Professor Dr H.-J. Zimmermann, for his friendly support during the preparation of this book and for enabling me to devote a considerable amount of effort to this time-consuming work. Finally, I should like to express my thanks to my publisher, de Gruyter, for their cooperation and their understanding of my particular situation .

Tomas Gal
Aachen, March 1972

List of symbols

Symbol	Meaning
$\mathbb{R}^n$	n-dimensional real space
$\mathbf{a} = (a_1, ..., a_n)^T = \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix}$	column vector, $\mathbf{a} \in \mathbb{R}^n$
$\mathbf{a}^T = (a_1, ..., a_n)$	row vector – vector $\mathbf{a}$ transposed
$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$	an (m, n) matrix with elements a_{ij}, $i = 1, ..., m, j = 1, ..., n$, technological matrix, matrix of the coefficients of the variables
$\mathbf{o} = (0, ..., 0)^T$	null vector, $\mathbf{o} \in \mathbb{R}^n$
$\mathbf{0} = \begin{pmatrix} 0 & \cdots & 0 \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ 0 & \cdots & 0 \end{pmatrix}$	(m, n) null matrix
$\mathbf{e}^k = (\overbrace{0, ..., 0, 1}^{k}, 0, ..., 0)^T$	unit vector, $\mathbf{e}^k \in \mathbb{R}^n$
$I = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$	(m, m) identity matrix
$\mathbf{x} = (x_1, ..., x_n)^T$	vector of variables
$\mathbf{c} = (c_1, ..., c_n)^T$	vector of objective function coefficients, or, briefly, cost vector
$z = \mathbf{c}^T \mathbf{x} = \sum_{j=1}^{n} c_j x_j$	objective function
$\mathbf{b} = (b_1, ..., b_m)^T$	right-hand side of the constraints $(A\mathbf{x} = \mathbf{b})$, $\mathbf{b} \in \mathbb{R}^m$

$\mathbf{a}^j$	the jth column of matrix $\mathbf{A}$; the jth activity; coefficient vector of the jth variable x_j
$\rho = \{j_1, \dots, j_m\}$	basic-index; index-set of basic variables
φ	index-set of nonbasic variables
$\mathbf{B} = (\mathbf{a}^{j_1}, \dots, \mathbf{a}^{j_m})$	basis
$\mathbf{B}^{-1} = (\boldsymbol{\beta}^1, \dots, \boldsymbol{\beta}^m)$	inverse to $\mathbf{B}$
$\boldsymbol{\beta}^k = (\beta_{1k}, \dots, \beta_{mk})^T$	kth column of matrix $\mathbf{B}^{-1}$
$\mathbf{x}_B = (y_1, \dots, y_m)^T$	basic solution, $\mathbf{y} = \mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_B^{(o)}$	complete basic solution associated with ρ
$\hat{\mathbf{x}}_B$	general solution associated with ρ
$\bar{\mathbf{x}}_B$	reduced general solution associated with ρ
$\mathbf{c}_B = (c_{j_1}, \dots, c_{j_m})^T$	basic cost vector associated with ρ
$z_{max}^{(\rho)}, z_{min}^{(\rho)}$	maximal (minimal) value of the objective function associated with ρ ; ρ is the optimal basic-index
$^{\rho}\mathbf{z}^T$	criterion row; reduced costs row
$\Delta z_j = {}^{\rho}z_j - c_j$	criterion elements, reduced costs
$\mathbf{u}^T = (u_1, \dots, u_m)$	dual solution
$\Omega_{min}^{(k)}$	upper endpoint of the feasible interval of the nonbasic variable x_k
$X = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} \le \mathbf{b}, \mathbf{x} \ge \mathbf{o}\}$	set of feasible solutions
$\bar{X} = \{\bar{\mathbf{x}} \in \mathbb{R}^{m+n} \mid \bar{\mathbf{A}}\bar{\mathbf{x}} = \mathbf{b}, \bar{\mathbf{x}} \ge \mathbf{o}\}$	augmented set of feasible solutions
$\bar{\mathbf{A}} = (\mathbf{A} \mid \mathbf{I})$	augmented matrix $\mathbf{A}$
$\bar{\mathbf{x}} = \begin{pmatrix} \mathbf{x} \\ \mathbf{s} \end{pmatrix}$	augmented vector $\mathbf{x}$; $\mathbf{s}$ vector of slack variables
λ or $\boldsymbol{\lambda}$	scalar or vectorparameter for the right-hand side
$\mathbf{F} = (f_{ik})$	(m, s) matrix of the coefficients of $\boldsymbol{\lambda}$
$\mathbf{f}$	vector of coefficients of λ
$^{\rho}\mathbf{F} = (^{\rho}f_{ik})$	matrix $\mathbf{F}$ transformed into $\mathbf{B}$ associated with ρ
$\mathbf{Y} = \mathbf{B}^{-1}\bar{\mathbf{A}}$	matrix $\bar{\mathbf{A}}$ transformed into $\mathbf{B}$ associated with ρ
$\underline{\lambda}_k^{(\rho)}, \bar{\lambda}_k^{(\rho)}$	critical values of the scalar parameter λ_k associated with ρ
$\underline{\lambda}^{(\rho)}, \bar{\lambda}^{(\rho)}$	critical values of the scalar parameter λ associated with ρ

$\Lambda_k^{(\rho)}$	critical interval of the scalar parameter λ_k associated with ρ
$\Lambda^{(\rho)}$ or $\mathbf{\Lambda}^{(\rho)}$	critical interval or region of the scalar or vector parameter λ or $\boldsymbol{\lambda}$
K	region of admissible parameters
$z_{max}(\lambda)$	optimal value function for $\lambda \in K$
$\mathbf{t} = (t_1, \ldots, t_s)^T$	vector parameter for the cost vector
$\mathbf{H}$	(s, N) matrix of coefficients of the vector parameter $\mathbf{t}$
$\mathbf{h}^j$	jth column of matrix $\mathbf{H}$
$\mathbf{H}_B$	submatrix of $\mathbf{H}$ which is assigned to the basic cost vector $\mathbf{c}_B$
$\mathbf{h}_B$	analogous to $\mathbf{c}_B$
$^{\rho}\mathbf{h}^T$	parameter criterion row
$\Delta h_j = {}^{\rho}h_j - h_j$	parameter criterion elements
$\underline{t}_k^{(\rho)}, \bar{t}_k^{(\rho)}$	critical values of the scalar parameter t_k associated with ρ
$\underline{t}^{(\rho)}, \bar{t}^{(\rho)}$	critical values of the scalar parameter t associated with ρ
$T_k^{(\rho)}$	critical interval of t_k associated with ρ
$T^{(\rho)}$ or $\mathbf{T}^{(\rho)}$	critical interval or region for t or $\mathbf{t}$ associated with ρ
$\overset{K}{\underset{k=1}{\gamma}}(\mathbf{v}^k)$	convex hull of $\mathbf{v}^1, \ldots, \mathbf{v}^K, \mathbf{v}^k \in \mathrm{I\!R}^n$
$G = (S, \Gamma)$	undirected graph for parametric programming
$G = (S, V)$	undirected representation graph of a polytope
σ	degeneracy degree
$G_+^o(\mathbf{x}^o) := G_+^o$	positive degeneracy graph of the σ-degenerate vertex $\mathbf{x}^o$

Contents

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Chapter one

1 Basic concepts and notation in linear programming

This Chapter contains a brief survey of the basic concepts and notation of linear programming (LP for short)[1]. This Chapter and those following have each been divided into two parts. The first part is based on illustrative examples, while the second part presents an abridged mathematical version of the same information.

Column vectors are represented by lower-case boldface letters, e.g., $\mathbf{x}$, $\mathbf{a}^j$, $\mathbf{c}$. The transpose into row vectors is indicated by the symbol T, e.g., $\mathbf{x}^T$, $\mathbf{c}^T$. The symbol $\mathbf{x} \in \mathrm{IR}^n$ means that vector $\mathbf{x}$ is an element of a real n-dimensional space; $\mathbf{x}$ may then be represented as an ordered m-tuple $(x_1, \ldots, x_m)^T$ of real numbers $x_1, \ldots, x_m$. The vector $\mathbf{x}$ is also called an m-vector, and the numbers $x_1, \ldots, x_m$ are also known as the elements or the coordinates of the vector $\mathbf{x}$. Matrices will be represented by upper-case boldface letters, e.g., $\mathbf{A}$, $\mathbf{B}$. The transpose of a matrix will be indicated in the usual manner by T, e.g., $\mathbf{A}^T$, $\mathbf{B}^T$. If there are indices needed, vectors are characterized by superscripts, e.g., $\mathbf{a}^j$, $\mathbf{x}^i$, scalars by subscripts, e.g., x_j, s_i.

At this point, several special vectors and matrices will be noted which frequently occur in the text later on. Thus,

$$\mathbf{e}^k = (0, \ldots, 1, 0, \ldots, 0)^T = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \qquad \text{a unit vector, } \mathbf{e}^k \in \mathrm{IR}^n;$$

$$\mathbf{o} = (0, 0, \ldots 0)^T, \mathbf{o} \in \mathrm{IR}^n \qquad \text{a null vector;}$$

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1   The reader is referred to the (selected) textbooks for a full discussion of LP and/or linear algebra: [1, 3, 8, 9]. Note that these references are found at the end of this Chapter; in the Bibliography at the end of this book there are some more textbooks listed.

$$\mathbf{I} = \begin{pmatrix} 1\ 0\ 0\ \ldots\ 0\ 0 \\ 0\ 1\ 0\ \ldots\ 0\ 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ 0\ 0\ 0\ \ldots\ 1\ 0 \\ 0\ 0\ 0\ \ldots\ 0\ 1 \end{pmatrix} \quad \mathbf{I} \in \mathbb{R}^{m \times m} \qquad \text{an } (m, m) \text{ identity matrix;}$$

$$\mathbf{0} = \begin{pmatrix} 0\ \ldots\ 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ 0\ \ldots\ 0 \end{pmatrix} \quad \mathbf{0} \in \mathbb{R}^{m \times m} \qquad \text{an } (m, m) \text{ null matrix.}$$

1-1   The simplex tableau

All necessary explanations can be illustrated from the following example.

Example 1-1   Maximize the following objective function (linear form):

$$z = 3x_1 + 8x_2 \tag{1-1}$$

subject to

$$\begin{aligned} x_1 +\ \ x_2 &\le 13 \\ 5x_1 -\ 4x_2 &\le 20 \\ -8x_1 + 22x_2 &\le 121 \\ 4x_1 +\ \ x_2 &\ge 8 \end{aligned} \tag{1-2}$$

$$x_1 \ge 0, x_2 \ge 0 \tag{1-3}$$

Note that (1-3) is called the nonnegativity conditions.

The coefficients of the respective variables x_1, x_2 form a column vector, also called the activity vector. Thus, to x_1 the vector $\mathbf{a}^1 = (1, 5, -8, 4)^T$ is assigned, to x_2 the vector $\mathbf{a}^2 = (1, -4, 22, 1)^T$. The elements of the right-hand side of the constraints (1-2) also form a column vector $\mathbf{b} = (13, 20, 121, 8)^T$. Those variables which may be considered to be the "bearers" of the quantity of an actual or real activity (thus, in our case, x_1, x_2) are called real variables. Hence, the variable vector (of real variables) is $(x_1, x_2)^T \in \mathbb{R}^2$, i.e., in general $\mathbf{x} \in \mathbb{R}^n$, whereas matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$. In our specific case $m = 4$, $n = 2$. The inequality system (1-2)

is transformed by means of the slack and surplus variables x_3, x_4, x_5, and x_6[2] into the following system of equations:

$$
\begin{aligned}
x_1 + \ \ x_2 + x_4 &= 13, \\
5x_1 - \ 4x_2 + x_5 &= 20, \\
-8x_1 + 22x_2 + x_6 &= 121, \\
4x_1 + \ \ x_2 - x_3 &= 8, \\
x_j \geq 0, j = 1, \ldots, 6. &
\end{aligned}
\tag{1-4}
$$

Note that after having transformed our inequalities into equations we have $N = m + n$, i.e., $4 + 2 = 6$ variables altogether.

The cost coefficients[3] of the slack variables are zero (cf., for example [2, 3, 5]). Each solution of the system (1-4) is a solution of the system (1-2), as will be shown later.

It is apparent that, with the variable x_3, the coefficient vector is the vector $-\mathbf{e}^4$; in the case of the variables x_4, x_5, x_6, it is the vectors $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$.

As we know, an identity matrix is required for the initial tableau. According to (1-4), only the $(\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3)$ part is present. The system (1-4), therefore, has to be augmented by an artificial variable p, so that, from (1-4) there follows

$$
\begin{aligned}
x_1 + x_2 + x_4 &= 13, \\
5x_1 - 4x_2 + x_5 &= 20, \\
-8x_1 + 22x_2 + x_6 &= 121, \\
4x_1 + x_2 - x_3 + p &= 8.
\end{aligned}
\tag{1-5}
$$

The nonnegativity conditions are to be valid for all variables, i.e.,

$$
x_j \geq 0, j = 1, \ldots, 6, p \geq 0.
\tag{1-6}
$$

If the original objective function $z = 3x_1 + 8x_2$, which contains the real variables, is augmented by the slacks, then the values of the slacks have no effect on the value of the objective function, since, as mentioned above, the cost coefficients of the slacks are zero.

If the objective function is further augmented by an artificial variable, then the value of the objective function is apparently influenced by the value of the artificial variable, since this variable is added with the coefficient one.[4]

The augmented objective function in Ex. 1-1 thus reads

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2   In this example x_3 is a surplus variable, x_4, x_5 and x_6 are slack variables. We shall not distinguish between these kinds of variables throughout the text and shall use only the term slack variables or simply slacks.

3   We shall use the term "cost coefficient" rather than the more clumsy "objective function coefficent"

4   This is true as long as the artificial variable p is basic variable. When p is eliminated from the basis, its value becomes zero and it does not influence any more the value of the objective function (see also Sec. 1-4).

$$z_E = 3x_1 + 8x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 + p. \tag{1-7}$$

A frequently used concept is "basis"; this term is used to denote a set of m linearly independent non-null vectors $\mathbf{v}^j \in \mathbb{R}^m$. Linear independence means that none of the given vectors $\mathbf{v}^j$ can be expressed as a linear combination of the other vectors (for an exact description, see, for example, [5, 8, 9]).

The so-called initial basis is formed by the columns (column vectors) of the identity matrix. In our particular case, these are the vectors $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$, $\mathbf{e}^4$, where $\mathbf{e}^j \in \mathbb{R}^4$, $j = 1, \ldots, 4$. Any vector from $\mathbb{R}^4$ can be uniquely expressed as a linear combination of the basic vectors. In this way $\mathbf{a}^1$, for example, can be uniquely expressed as

$$\mathbf{a}^1 = \mathbf{e}^1 + 5\mathbf{e}^2 - 8\mathbf{e}^3 + 4\mathbf{e}^4,$$

i.e.,

$$\begin{pmatrix} 1 \\ 5 \\ -8 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - 8\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

In this connection, it should be mentioned that the slack and artificial variables can be used to form an initial basis. Those variables, the coefficient-vector of which form the basis, are called basic variables. The other variables are called nonbasic variables. Setting the nonbasic variables equal to zero gives us a basic solution. It follows from (1-5) that the initial basic solution in our example is

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 13, x_5 = 20, x_6 = 121, x_7 = p = 8.$$

A solution is said to be feasible if it satisfies the constraints. In our specific case, the initial basic solution is feasible, since it satisfies all the equations of the system (1-5) and the values of all the basic variables are nonnegative (conditions (1-6)).

If the given problem is solved by the simplex method, the calculations can be arranged in tabular form. There are no fixed rules for the setting-up of such tableaux. Table 1-1 represents the initial solution and Table 1-2 the optimal solution. All the symbols used in the tableaux will be explained in due course.[5]

The basic variables can easily be recognized in each of the tables from the unit vectors associated with them. In accordance with Table 1-2, these variables are, therefore, x_1, x_2, x_3, and x_5. If, in the light of the basic variables from Table 1-2, the corresponding coefficient vectors of the basic variables are selected from Table 1-1, the following matrix results:

---

[5] Note that in almost any professional software the initial data are arranged in a suitable form of the initial tableau by means of a matrix generator (for more details, see, for example, [7]).

Table 1-1   Initial tableau for Ex. 1-1

Col. No	1	2	3	4	5	6	7	8	9	10
	$\mathbf{c_B}$	c_j	3	8	0	0	0	0		**b**
		ρ_I \ j	1	2	3	4	5	6	7	
	0	4	1	1	0	1	0	0	0	13
	0	5	5	−4	0	0	1	0	0	20
	0	6	−8	22	0	0	0	1	0	121
	1	7(p)	4	1	−1	0	0	0	1	8
$-\sum\limits_{i=1}^{1}\rho_i$			−4	−1	1	0	0	0	0	−8

Table 1-2   Final tableau for Ex. 1-1

Col.-No	1	2	3	4	5	6	7	8	9	10
	$\mathbf{c_B}$	c_j	3	8	0	0	0	0		$\mathbf{x_B}$
		ρ \ j	1	2	3	4	5	6	7	
	3	1	1	0	0	11/15	0	−1/30	0	11/2
	8	2	0	1	0	4/15	0	1/30	0	15/2
	0	3	0	0	1	16/5	0	−1/10	−1	43/2
	0	5	0	0	0	−13/5	1	3/10	0	45/2
Δz_j			0	0	0	13/3	0	1/6	0	153/2

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 5 & -4 & 0 & 1 \\ -8 & 22 & 0 & 0 \\ 4 & 1 & -1 & 0 \end{pmatrix} \tag{1-8}$$

In column 2 of Table 1-2, the indices of the basic variables (or the initial basic vectors) stand below the symbol ρ. The set $\rho = \{1, 2, 3, 5\}$ is then called the basic-index. If ρ is associated with an optimal basic solution, then we speak of an optimal basic-index. The matrix (1-8) is called basis for short and denoted by

Table 1-3   Abbreviated form of the final tableau

c_B	ρ	0 4	0 6	0 7	x_B
3	1	$0.7\overline{3}$	$-0.0\overline{3}$	0	5.5
8	2	$0.2\overline{6}$	$0.0\overline{3}$	0	7.50
0	3	3.2	-0.1	-1	21.5
0	5	-2.6	0.3	0	22.5
Δz_j		$4.\overline{3}$	$0.1\overline{6}$	0	76.5

B, where $\mathbf{B} = (\mathbf{a}^1, \mathbf{a}^2, -\mathbf{e}^3, \mathbf{e}^2)$. If we are dealing with several bases, we simply number the basic-indices in order $\rho_1, \rho_2, ..., \rho_k, ...$ The corresponding bases are then marked accordingly by the corresponding subscripts: $\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_k, ...$.

In column 1 of Table 1-2, those cost coefficients which are associated with the basic vectors stand below the symbol $\mathbf{c}_B$.

In the initial tableau (Table 1-1) the basic-index is denoted by ρ_I, where $\rho_I = \{4, 5, 6, 7\}$[6].

The index-set of all variables is denoted by $J = \{j \mid j = 1, ..., 6\}$[7]. A subset of this set is the index-set ρ of the basic variables. The complementary subset φ is the index-set of all nonbasic variables; thus, in our particular case, $\varphi = \{4, 6\}$.

The tableau can be written in abbreviated form by simply omitting the unit vectors associated with the basic variables. The abbreviated form of the tableau Table 1-1 is shown in Table 1-3.

The matrix $\mathbf{B}$ (cf. (1-8)) possesses all those properties which are required for the existence of an inverse matrix[8]. The inverse is denoted by $\mathbf{B}^{-1}$. Hence,

$$\mathbf{B}^{-1}\mathbf{B} = \mathbf{B}\,\mathbf{B}^{-1} = \mathbf{I}. \tag{1-9}$$

The matrix $\mathbf{B}^{-1}$ can always be found in those columns of the final tableau in which the matrix $\mathbf{I}$ stood in the initial tableau. In our case and in accordance with Table 1-2, this is, therefore,

$$\mathbf{B}^{-1} = \begin{pmatrix} 0.7\overline{3} & 0 & -0.0\overline{3} & 0 \\ 0.2\overline{6} & 0 & 0.0\overline{3} & 0 \\ 3.2 & 0 & -0.1 & -1 \\ -2.6 & 1 & 0.3 & 0 \end{pmatrix}. \tag{1-10}$$

---

6 The index 7 stands for the artificial variable p.

7 The index 7 of the artificial variable p is not included in the set J.

8 The matrix $\mathbf{B}$ is a quadratic nonsingular matrix.

The elements of matrix $\mathbf{B}^{-1}$ are denoted by β_{ij}. The vector β^k denotes the kth column of matrix $\mathbf{B}^{-1}$.

According to (1-5), for instance, the vector $\mathbf{b}$ in our example can be expressed by the vectors of the initial basis as follows:

$$\mathbf{b} = 13\mathbf{e}^1 + 20\mathbf{e}^2 + 121\mathbf{e}^3 + 8\mathbf{e}^4. \tag{1-11}$$

We also say that, with respect to the initial basis, the coordinates of the vector $\mathbf{b}$ are 13, 20, 121, 8. Multiplying the vector $\mathbf{b}$ from the left by the matrix $\mathbf{B}^{-1}$ (cf. (1-10)) gives us the coordinates of the vector $\mathbf{b}$ with respect to the basis $\mathbf{B}$. Hence,

$$\mathbf{B}^{-1}\mathbf{b} = \mathbf{x}_B, \tag{1-12}$$

where

$$\mathbf{x}_B = (5.5, 7.5, 21.5, 22.5)^T, \tag{1-13}$$

in general,

$$\mathbf{x}_B = (y_1, ..., y_m)^T. \tag{1-13a}$$

In other words, the vector $\mathbf{b}$ can be uniquely expressed by the basic vectors associated with the basic-index $\rho = \{1, 2, 3, 5\}$ as follows:

$$\mathbf{b} = 5.5\mathbf{a}^1 + 7.5\mathbf{a}^2 + 21.5(-\mathbf{e}^4) + 22.5\mathbf{e}^2. \tag{1-14}$$

The vector $\mathbf{x}_B$ is then called the vector $\mathbf{b}$ transformed into basis $\mathbf{B}$.

Similarly, with other vectors:

$$\mathbf{B}^{-1}\mathbf{a}^1 = \mathbf{y}^1 = \mathbf{e}^1; \mathbf{B}^{-1}\mathbf{a}^2 = \mathbf{y}^2 = \mathbf{e}^2; \mathbf{B}^{-1}(-\mathbf{e}^4) = \mathbf{y}^3 = \mathbf{e}^3;$$
$$\mathbf{B}^{-1}\mathbf{e}^1 = \mathbf{y}^4 = \beta^1; \mathbf{B}^{-1}\mathbf{e}^2 = \mathbf{y}^5 = \beta^2 = \mathbf{e}^4; \mathbf{B}^{-1}\mathbf{e}^3 = \mathbf{y}^6 = \beta^3;$$
$$\mathbf{B}^{-1}\mathbf{e}^4 = \beta^4 = -\mathbf{e}^3; \mathbf{B}^{-1}\mathbf{b} = \mathbf{x}_B = (y_1, ..., y_m)^T,$$

where

$$\mathbf{y}^j = (y_{1j}, ..., y_{mj})^T, j = 1, ..., 6.$$

The vector $\mathbf{x}_B = (5.5, 7.5, 21.5, 22.5)^T$ in which only the values of the basic variables are collected is called the basic solution. The vector $\mathbf{x}_B^{(o)} = (5.5, 7.5, 21.5, 0, 22.5, 0)^T$, which contains the values of all variables (except, of course that of the artificial variable), i.e., also the zeros assigned to the nonbasic variables, is called the complete basic solution.

Let us denote by $\bar{\mathbf{x}}$ the vector that consists of (all) the variables x_j (i.e., of the real and slack variables), and by $\bar{\mathbf{c}}$ the vector consisting of all cost coefficients; then we obtain

$$z = \bar{\mathbf{c}}^T\bar{\mathbf{x}}, \quad \text{i.e.,} \quad z = \sum_{j=1}^{N} \bar{c}_j\bar{x}_j. \tag{1-15}$$

In accordance with (1-15) and with reference to Table 1-2, the maximal (optimal) value $z_{max}^{(\rho)}$ of the objective function associated with ρ is given by

$$z_{max}^{(\rho)} = (3, 8, 0, 0, 0, 0)(5.5, 7.5, 21.5, 0, 22.5, 0)^T = 3 \times 5.5 + 8 \times 7.5 = 76.5.$$

It is, therefore, superfluous to perform a scalar multiplication of the whole vector $\bar{\mathbf{c}}$ by the vector $\mathbf{x}_B^{(o)}$, since all nonbasic variables are zero. In order to determine the optimal value $z_{max}^{(\rho)}$ of the objective function, it is, thus, sufficient to carry out a scalar multiplication of the basic cost vector $\mathbf{c}_B$ by the basic solution vector $\mathbf{x}_B$. Hence,

$$z_{max}^{(\rho)} = \mathbf{c}_B^T \mathbf{x}_B, \quad \text{i.e.,} \quad z_{max}^{(\rho)} = \sum_{i=1}^{m} c_{Bi} y_i. \tag{1-16}$$

Applied to our example,

$$z_{max}^{(\rho)} = (3, 8, 0, 0)^T (5.5, 7.5, 21.5, 22.5)^T = 76.5.$$

The last row of Table 1-2 (or Table 1-3) is denoted by Δz_j with

$$\Delta z_j = z_j - c_j \quad \text{for all} \quad j \in J \tag{1-17}$$

and

$$z_j = \mathbf{c}_B^T \mathbf{y}^j. \tag{1-18}$$

Since, for example, $z_4 = (3, 8, 0, 0)(0.7\bar{3}, 0.2\bar{6}, 3.2, -2.6)^T = 4.\bar{3}$ and $c_4 = 0$, the result is $\Delta z_4 = 4.\bar{3}$.[9]

For all $j \in J$[10], the row Δz_j is designated the criterion row with the criterion elements Δz_j, which are sometimes called reduced costs. In maximization problems, we speak of an optimal solution if $\mathbf{x}_B \geq \mathbf{o}$, i.e., $y_i \geq 0$ for all i, and $\Delta z_j \geq 0$ for all j. We sometimes speak of Δz_j, all j, as the (complete) dual solution and of $\mathbf{x}_B$ as the primal solution. The assertion "optimal solution" means that the solution is primal ($\mathbf{x}_B \geq \mathbf{o}$) and dual ($\Delta z_j \geq 0$) feasible (cf. also Sec. 1-5).

In this connection, it should be noted that a basis $\mathbf{B}$ is called an optimal basis, if the solution associated with $\mathbf{B}$ is optimal.

Notation note   From, e.g., $\mathbf{B}^{-1}\mathbf{F}$ it follows that $\mathbf{F}$ is being transformed into basis $\mathbf{B}$ to which the basic-index ρ is assigned uniquely. In order to keep in mind that this transformation is applied on $\mathbf{F}$, we mark this $\mathbf{F}$ by a left superscript "ρ" to show that $\mathbf{F}$ is being transformed into $\mathbf{B}$. Hence,

$$\mathbf{B}^{-1}\mathbf{F} = {}^\rho\mathbf{F}.$$

---

9   Note that in the abbreviated tableau Table 1-3 Δz_j is "listed" only for $j \notin \rho$ (i.e., $j \in \varphi$).

10   In the abbreviated Tableau, for all $j \notin \rho$.

Analogously with some other notation; if there cannot arise any confusion we keep the "normal" notation without the left superscript, e.g., Δz_j. Only if it is inevitable, we shall use the left superscript, i.e., for example, $^P\Delta z_j$.

1-2 Geometric meaning

The inequality system (1-2), (1-3) can be drawn in a system of coordinates (Fig. 1-1). An inequality defines a half-plane (in general: a half-space). If the inequality sign is replaced by an equality sign, a straight line (a hyperplane) is obtained which forms the boundary of the half-plane (half-space). The solution of a linear inequality system (with two unknowns) is then the part of the plane, which is common to all half-planes (the intersection of the given half-planes). In this way, the coordinates of the boundary points of the half-plane defined, for example, by the first inequality $x_1 + x_2 \leq 13$ satisfy the equation $x_1 + x_2 = 13$. This can be drawn and provided with a short arrow, which characterizes the corresponding half-plane. The inter section of the given four half-planes is shaded in Fig. 1-1 and forms a convex polyhedral set[11].

If the given problem is to be solved geometrically, the straight line $z_o = 3x_1 + 8x_2$ is drawn for an arbitrary value z_o. This straight line is then moved parallel to itself until it reaches the vertex[12] P of the polytope, where the distance of the straight line from the origin O is greatest. In this position, $z = 76.5$.

The set of all points of the polytope (including the boundary) is called the set of all feasible solutions or the solution set. It can be proved (cf., for example, [1, 3, 4, 5, 8]) that a basic solution is always assigned to at least one vertex of the polytope.

1-3 Types of solution vectors

The result from Table 1-3 can be expressed in the form of a linear equation system:

$$x_1 + 0.7\overline{3}x_4 - 0.0\overline{3}x_6 = 5.5$$
$$x_2 + 0.2\overline{6}x_4 + 0.0\overline{3}x_6 = 7.5$$
$$x_3 + 3.2\ x_4 - 0.1\ x_6 = 21.5$$
$$x_5 - 2.6\ x_4 + 0.3\ x_6 = 22.5 \qquad (1\text{-}19)$$

Note that the artificial variable has been omitted.

---

11 If this set is bounded it is called a convex polytope.
12 Such a point, i.e., vertex, is sometimes called extreme point.
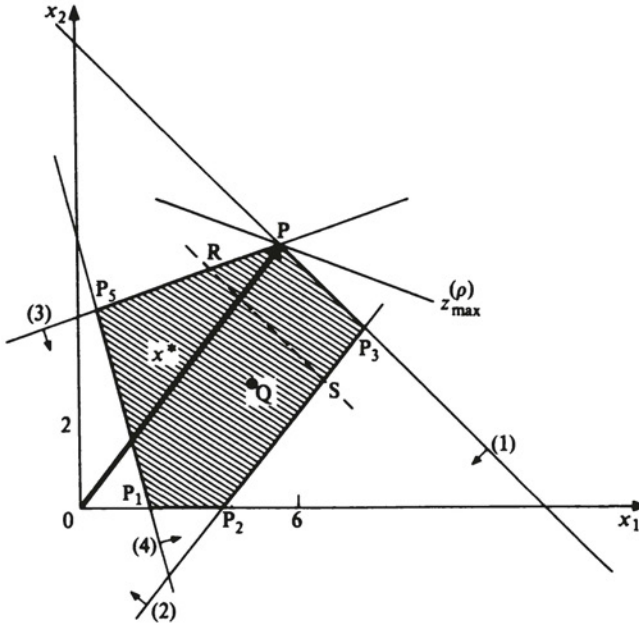
Figure 1-1

Setting $x_4 = x_6 = 0$, we immediately obtain from (1-19) not only the basic solution $\mathbf{x}_B$, but also the complete basic solution $\mathbf{x}_B^{(o)}$.

As can easily be seen from (1-19), all the basic variables depend on the nonbasic variables; namely,

$$
\begin{aligned}
x_1 &= 5.5 - 0.7\overline{3}x_4 + 0.0\overline{3}x_6 \\
x_2 &= 7.5 - 0.2\overline{6}x_4 - 0.0\overline{3}x_6 \\
x_3 &= 21.5 - 3.2\ x_4 + 0.1\ x_6 \\
x_5 &= 22.5 + 2.6\ x_4 - 0.3\ x_6
\end{aligned}
$$

This result can also be written in vector form:

$$
\hat{\mathbf{x}}_B =
\begin{pmatrix}
5.5 - 0.7\overline{3}x_4 + 0.0\overline{3}x_6 \\
7.5 - 0.2\overline{6}x_4 - 0.0\overline{3}x_6 \\
21.5 - 3.2x_4 + 0.1x_6 \\
x_4 \\
22.5 + 2.6x_4 - 0.3x_6 \\
x_6
\end{pmatrix}
\tag{1-20}
$$

The vector $\hat{\mathbf{x}}_B$ is called the general solution (associated with basis $\mathbf{B}$).

Another connection is worth mentioning. According to (1-16), we have $z_{max}^{(\rho)} = c_B^T x_B$, and, according to (1-12), we have $x_B = B^{-1}b$. Hence,

$$z_{max}^{(\rho)} = c_B^T x_B = c_B^T B^{-1} b. \tag{1-21}$$

Denoting

$$c_B^T B^{-1} = u^T, \tag{1-22}$$

we obtain

$$z_{max}^{(\rho)} = u^T b. \tag{1-23}$$

It is known that u is the dual solution (cf., for example, [1, 2, 8, 9]). If the optimal value of the objective function in the dual problem is denoted by $f_{min}^{(\bar\rho)}$,[13] then

$$z_{max}^{(\rho)} = f_{min}^{(\bar\rho)} \tag{1-24}$$

According to (1-22) and taking into account Table 1-2,

$$u^T = (4.\bar3, 0, 0.1\bar6, 0),$$

so that, from (1-23), it follows that

$$z_{max}^{(\rho)} = (4.\bar3, 0, 0.1\bar6, 0)(13, 20, 121, 8)^T = \frac{13}{3} \times 13 + \frac{1}{6} \times 121 = 76.5.$$

Note that Δz_j for all j forms the dual complete basic solution whereas u is, in analogy to x_B, the dual basic solution.

To take a particular example, let us suppose that the problem (1-1) through (1-3) is a maximization of the total profit from manufacturing two products P_1 and P_2, subject to the given constraints (1-2), (1-3). Let x_1 be the quantity of the product P_1 manufactured, x_2 the quantity of P_2. Let the coefficient $c_1 = 3$ in the objective function represent the profit (expressed in money units (MU) of, say, \$ 10) per unit of P_1, $c_2 = 8$ MU be the profit per quantity unit (QU) of P_2.

The first of the constraints is a typical capacity restriction. This inequality can be interpreted more or less as follows: a raw material R_1 is used in equal parts in both products P_1 and P_2 at the rate of 10 kg per QU. No more than 130 kg of this raw material is available. The coefficients 1, 1, and 13 are thus expressed in units of 10 kg.

The second constraint, in the manufacture of one unit of P_2, 4 kg of a joint product are obtained, which are then incorporated in the product P_1 at the rate of 5 kg per QU. A further 20 QU of this joint product are available. The production of the joint product must be at least as large as the consumption; thus,

$$4x_2 + 20 \geq 5x_1.$$

By multiplying this inequality by -1 and rearranging, we obtain

---

13   $\bar\rho$ is the basic-index of the dual solution.

$$5x_1 - 4x_2 \leq 20.$$

The third constraint: the product P_2 incorporates 22 kg of a raw material R_2 per QU. This raw material becomes available as a by-product from P_1 at the rate of 8 kg per unit produced. Raw material R_2 can also be bought on the market. But, for capacity reasons, the firm cannot use up more than 121 kg of R_2.

The fourth constraint: this is a typical requirement condition and it can be interpreted roughly as follows. For the production of one quantity unit of product P_1, 40 h on a machine M are required ; for each quantity unit of P_2, 10 h on this machine M are required. The total time during which the machine is actually to be in use for the manufacture of the products must be at least 80 h, because of difficulties in converting the machine and the like. The corresponding coefficients in the fourth constraint are, therefore, given in 10 h units.

The activity $\mathbf{a}^1$ represents the "technology" of P_1, $\mathbf{a}^2$ the "technology" of P_2. In accordance with the optimal solution, $x_1 = 5.5$ QU of product P_1, and $x_2 = 7.5$ QU of product P_2 are to be manufactured, where the optimal profit is $z_{max}^{(p)} = 76.5$, i.e., \$ 765.[14] Since $x_3 = 21.5$, the minimum total machine time required in the optimal production program is exceeded by 215 h. The slack variable $x_4 = 0$ refers to the full exploitation of the available raw material R_1. Furthermore, $x_5 = 22.5$, i.e., in the manufacture of 5.5 QU of P_1 and 7.5 QU of P_2 there are still 2.5 QU of the joint product left over for further use. From $x_6 = 0$ it follows that exactly 121 kg of the raw material R_2 have been used up.

1-4 The simplex method

The reader may ask the question why to learn, how the simplex method works, when we have powerfull computers, which are very quick and efficient. It is enough, symbolically said, "to push a buttom" and we have the result. In this book, however, we deal with postoptimal analysis of the results, so that, in order to understand what is all behind, we need to understand the main features of the principles of the simplex method.

The simplex algorithm including all special cases has been described in detail in almost all textbooks of linear programming (LP) (cf., for example, [2 – 6, 8, 9]). We shall confine ourselves to illustrating the main principles using Ex. 1-1. The problem in Ex. 1-1, augmented by slacks, can be written as follows. Maximize

$$z = \sum_{j=1}^{N} \bar{c}_j \bar{x}_j \tag{1-25}$$

---

14   These figures are, for illustration, chosen arbitrarily.

subject to

$$\sum_{j=1}^{N} \bar{a}_{ij}\bar{x}_j = b_i, \qquad i = 1, \ldots, m,$$

$$x_j \geq 0, j = 1, \ldots, N. \tag{1-26}$$

In our example $N = 6$, $m = 4$; as we know already from the preceding section, to start the calculation we need, in our case, an artificial variable, the subscript of which is 7.

After drawing up the initial tableau, the pivot element is to be determined. Two selection criteria will be used for this: the optimality criterion and the feasibility criterion. Both criteria and their use will now be described.

The augmented objective function (with artificial variables included) reads (generally):

$$z_E = \sum_{j=1}^{n} c_j x_j + 0 \sum_{j=n+1}^{t \leq N} x_j + \sum_{i=1}^{u \leq m} p_i. \tag{1-27}$$

This objective function can be divided into two "partial" functions:

$$z_N = \sum_{j=1}^{n} c_j x_j + 0 \sum_{j=n+1}^{t \leq N} x_j + 0 \sum_{i=1}^{u \leq m} p_i \tag{1-28a}$$

and

$$z_P = 0 \sum_{j=1}^{n} x_j + 0 \sum_{j=n+1}^{t \leq N} x_j + \sum_{i=1}^{u \leq m} p_i. \tag{1-28b}$$

The augmented objective function is then, evidently,

$$z_E = z_N + z_P.$$

If $p_i = 0$ for all $i = 1, \ldots, u \leq m$, then $z_P = 0$ results, so that z_E becomes the original objective function. For this reason, the simplex procedure is divided into two phases. In the first phase, $p_i = 0$ and p_i nonbasic variable have to be achieved for all i, in order to obtain $z_P = 0$. The computation of the optimal solution of the original problem follows in the second phase. Therefore, in the first phase, we minimize z_P (or maximize $-z_P$). In the initial solution (in the initial tableau), all artificial variables appear as basic variables. If the minimum $z_P = 0$ is achieved, and p_i become nonbasic variables for all i, then, evidently, $p_i = 0$ for all i, such that p_i are nonbasic variables.[15]

---

15 Note that the described procedure is only one of several possibilities to generate a feasible solution of the original problem (Phase 1). Various so called "crash methods" have been worked out and nowadays so called internal point methods are being developed [4, 7].

Table 1-4   Initial tableau

c_I	ρ_I	3 1	8 2	0 3	0 4	0 5	0 6	p	b	$\Omega_i^{(1)}$
0	4	1	1	0	1	0	0	0	13	13
0	5	5	−4	0	0	1	0	0	20	4
0	6	−8	22	0	0	0	1	0	121	—
1	←p	4*	1	−1	0	0	0	1	8	2
	−p	−4	−1	0	0	0	0	0	−8	
	Δz_j	−3	−8	0	0	0	0	0	0	

Note   It may happen that we do not succeed in eliminating all artificial variables (such that they become nonbasic variables) in the course of Phase 1. This may mean that the original system of constraints is not consistent (possesses mutually exclusive constraints). For more on this, see Sec. 10-2.

Consider now the optimality criterion as applied to our Ex.1-1. The criterion row is formulated as $-\sum_{i=1}^{u\leq m} p_{i,}$, $u = 1$, and then all negative elements in this row are found. The absolute values of these elements are then determined and the highest selected. Applied to our example,

$$\max\{\,|-4|,\,|-1|\,\} = 4.$$

The pivot column with the index $k = 1$ has now been determined. Then the feasibility criterion is applied. All positive elements in the pivot column are found and the quotients determined:

$$\Omega_i^{(k)} = \frac{b_i}{a_{ik}}, \qquad a_{ik} > 0,$$

where k = index of the pivot column. With the data of Ex. 1-1,

$$\Omega_1^{(1)} = \frac{13}{1} = 13, \Omega_2^{(1)} = \frac{20}{5} = 4, \Omega_4^{(1)} = \frac{8}{4} = 2.$$

From these quotients the smallest is chosen:

$$\Omega_{\min}^{(1)} = \min\{13, 4, 2\} = 2 \Rightarrow r = 4.$$

The pivot row with the index $r = 4$ has now been determined. At the intersection of pivot row and pivot column lies the pivot element, which in our case is $y_{41} = 4$ (marked by an asterisk in Table 1-4).

If the simplex step (also called pivot step or pivoting) described in the following is performed, the artificial variable is eliminated, so that we can go on to Phase II.

In this Phase, the criterion elements are the elements Δz_j (cf. (1-17), (1-18)). The pivot step, which follows the determination of the pivot element, can be performed by means of two equivalent procedures.

1. Let us suppose that s steps have already been performed. Let the elements of the sth tableau be denoted by ${}^s y_{ij}$, ${}^s y_i$, and let ${}^{s+1} y_{ij}$, ${}^{s+1} y_i$ denote the elements of the (s+1)th tableau, which are to be computed. Let the pivot row be the rth row and the pivot column the kth column, so that ${}^s y_{rk} > 0$ is the pivot element. Then,

$$ {}^{s+1} y_{rj} = \frac{{}^s y_{rj}}{{}^s y_{rk}}, j = 1, \ldots, N, \tag{1-29}$$

$$ {}^{s+1} y_r = \frac{{}^s y_r}{{}^s y_{rk}}, \tag{1-30}$$

$$ {}^{s+1} y_{ij} = \frac{{}^s y_{ij}{}^s y_{rk} - {}^s y_{rj}{}^s y_{ik}}{{}^s y_{rk}}, i = 1, \ldots, m, j = 1, \ldots, N, i \neq r, \tag{1-31}$$

$$ {}^{s+1} y_i = \frac{{}^s y_i{}^s y_{rk} - {}^s y_r{}^s y_{ik}}{{}^s y_{rk}}, i = 1, \ldots, m, i \neq r. \tag{1-32}$$

2. The (complete) Gauss-Jordan elimination method (elimination procedure) performed step by step. This procedure will now be briefly described using our example.

   Notes 1. Equations (1-29) through (1-32) are only a consequence of the elimination method.
   2. The last row in Table 1-4 consists of the usual criterion elements, which can either be computed via (1-17), (1-18) or simply by reversing the signs of the cost coefficients (in this simple example).

Consider Table 1-4; divide the pivot row by the pivot element, i.e.,

$$ \tfrac{1}{4} \quad | \quad 4 \quad 1 \quad \text{-}1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 8 \quad |. $$

If the division indicated is carried out, the modified pivot row, MPR for short, is obtained:

$$ | \quad 1 \quad 1/4 \quad \text{-}1/4 \quad 0 \quad 0 \quad 0 \quad 1/4 \quad 2 \quad |. $$

The MPR is entered in the new tableau (Table 1-5) in the rth (i.e., fourth) row.

The ith element of the pivot column (excluding the pivot element itself) is now taken and its sign reversed; the MPR is then multiplied by the number so obtained and the result added to the ith row of the given tableau. This procedure is repeated with all elements of the pivot column (excluding the pivot element). In order of the rows, for our example, this yields the following results.

Row 1. First element in the pivot column (in Table 1-4) is 1. Therefore,

$(-1) \times$ | 1   1/4   -1/4   0   0   0   1/4   2 |

yields

| -1   -1/4   1/4   0   0   0   -1/4   -2 |

The row so obtained is added to Row 1 of Table 1-4. Therefore,

+ row 1 | 1   1   0   1   0   0   0   13 |

yields

| 0   3/4   1/4   1   0   0   -1/4   11 |

This is row 1 in Table 1-5.

Row 2. The first element is 5. Therefore,

$(-5) \times$ | 1   1/4   -1/4   0   0   0   1/4   2 |

yields

| -5   -5/4   5/4   0   0   0   -5/4   -10 |

+ row 2 | 5   -4   0   0   1   0   0   20 |

yields

| 0   -21/4   5/4   0   1   0   -5/4   10 |

and this is row 2 in Table 1-5.

Row 3. The first element is -8. Therefore,

$8 \times$ | 1   1/4   -1/4   0   0   0   1/4   2 |

yields

| 8   2   -2   0   0   0   2   16 |

+ row 3 | -8   22   0   0   0   1   0   121 |

yields

| 0   24   -2   0   0   1   2   137 |

and this is row 3 in Table 1-5.

There is no need to show in details the computation of rows 4 and the criterion row. This follows exactly the same rules.

Note that, in Table 1-5 , the elements of the criterion row and of the feasibility criterion column, which define the pivot column and the pivot row, respectively, are also marked by an asterisk and the arrows indicate the leaving or entering variables.

Pivoting on the pivot element in Table 1-5 and using the same rules again, we obtain Table 1-6 (presented in abbreviated form).

Table 1-5   First iteration

c_B	ρ	3 1	8 2	0 3	0 4	0 5	0 6	0 p	x_B	$\Omega_i^{(2)}$
0	4	0	3/4	1/4	1	0	0	−1/4	11	44/3
0	5	0	−21/4	5/4	0	1	0	−5/4	10	—
0	←6	0	24*	−2	0	0	1	2	137	137/ 24*
3	→1	1	1/4	−1/4	0	0	0	1/4	2	8
	Δz_j	0	−29/4*	−3/4	0	0	0	3/4	6	

Table 1-6   Second iteration (abbreviated form)

c_B	ρ	3	6	7	x_B	$\Omega_i^{(3)}$
0	←4	5/16*	−1/32	−5/16	215/32	43/2*
0	5	13/16	7/32	−13/16	1279/32	1279/26
8	→2	−1/12	1/24	1/12	137/24	—
3	1	−11/48	−1/96	11/48	55/96	—
	Δz_j	−65/48*	29/96	65/48	4549/96	

Note that column 7, which is associated with the artificial variable p, is (1) equal to column 3 with opposite sign , and (2) it is not needed any more for calculating the optimum. It will, however, be needed in other connections as the 4th column of the corresponding B^{-1}.

The solution in Table 1-6 is obviously not yet optimal ($\Delta z_3 = -65/48 < 0$). It is therefore necessary to perform a further pivoting. This step will not be described here. It is worth noting however, that this step can also be carried out using the abbreviated form. The optimal solution, which we obtain as a result of the next pivoting is shown in Tables 1-2 and 1-3. As is apparent from these tableaux, the corresponding solution is indeed optimal.

1-5   The dual problem

To any linear programming problem there exists a dual problem. Alternatively, for the sake of precision, we may call the original problem the primal problem.

The primal and the dual problems are interrelated in a number of ways. The most important of these will be explained in the following example.

Example 1-2   Let the problem of Ex. 1-1 (see Sec. 1-1) be considered a primal problem. Formulate the corresponding dual problem and solve it.

For this purpose, all inequalities in (1-2) should first be of the same type:

$$x_1 + x_2 \le 13,$$
$$5x_1 - 4x_2 \le 20,$$
$$-8x_1 + 22x_2 \le 121,$$
$$-4x_1 - x_2 \le -8,$$
$$x_1 \ge 0, x_2 \ge 0.$$

Each column of these modified constraints becomes a row in the dual. It follows from this that the dual problem has four "real variables" u_1, u_2, u_3, u_4. The vector **c** of the primal problem becomes the vector **b** of the dual and the vector **b** of the primal problem becomes vector **c** of the dual. If maximization takes place in the primal problem, then there will be a minimization in the dual problem, and conversely; the type of the inequalities will also be converse.

The dual problem we obtain is, thus: minimize

$$f = 13u_1 + 20u_2 + 121u_3 - 8u_4$$

subject to

$$u_1 + 5u_2 - 8u_3 - 4u_4 \ge 3,$$
$$u_1 - 4u_2 + 22u_3 - u_4 \ge 8,$$
$$u_i \ge 0, i = 1, ..., 4.$$

The following tableau (Table 1-7) shows the calculation of the minimum of this (dual) problem.

Now, compare the results of the primal (Table 1-2) and the dual problems (step 2 in Table 1-7). Denote the dual optimal basis by $\bar{\rho} = \{1, 3\}$. It then follows that $z_{max}^{(\rho)} = f_{min}^{(\bar{\rho})} = 76.5$.

In the "criterion row" of the dual, we have the values of the primal basic variables (with opposite signs), and the basic solution of the dual yields the values Δz_j of the primal, where the subscripts j are associated with the columns of $\mathbf{B}^{-1}$ and with the nonbasic variables of the primal. Hence, (cf. (1-22) and (1-24)),

$$\mathbf{u}^T = \mathbf{c}_B^T \mathbf{B}^{-1},$$

i.e.,

$$\mathbf{u}^T = (13/3, 0, 1/6, 0).$$

That part of the criterion row in the primal which is assigned to the columns of $\mathbf{B}^{-1}$, therefore, yields the dual solution $\mathbf{u}^T$.

If the unit vectors are omitted in the final tableau, to each row of the dual a column of the primal with opposite signs is assigned and vice versa.

Table 1-7 Calculation of the minimum of the dual problem

	1	2	3	4	5	6	
step 0							
p_1	1	5	−8	−4	−1	0	3
←p_2	1	−4	22*	−1	0	−1	8
−Σp_i	−2	−1	−14	5	1	1	−11
step 1							
←p1	15/11*	39/11	0	−48/11	−1	−4/11	65/11
→3	1/22	−2/11	1	−1/22	0	−1/22	4/11
−Σp_i	−15/11	−39/11	0	48/11	1	4/11	−65/11
step 2							
→1 (x_4)	1	13/5	0	−16/5	−11/5	−4/15	13/3
3 (x_6)	0	−3/10	1	1/10	1/30	−1/30	1/6
	0	−45/2 (x_5)	0	−43/2 (x_3)	−11/2 (x_1)	−15/2 (x_2)	76.5

From this, it follows that, when calculating the primal, the results of the dual are also contained in the final tableau (and conversely).

The following should also be pointed out:

1. If any primal real variable x_k is not sign-restricted (i.e., $x_k \geq 0$ does not hold), then the kth constraint of the corresponding dual is an equation (and conversely).
2. If the rth constraint in the primal is an equation, the rth (real) variable of the dual is not sign-restricted (and conversely).

The existence of the dual problem and the relationships between the primal and dual problems give us reason to speak (in the primal tableau) of a primal and dual solution. By primal solution, we mean the solution x_B or $x_B^{(o)}$ by dual solution $\mathbf{u}^T$ or $^p\mathbf{z}^T$. A solution is considered to be optimal if, and only if (iff) it is primal and dual feasible. In maximization problems, a tableau contains the primal optimal solution if, and only if $y_i \geq 0$ for all i and $\Delta z_j \geq 0$ for all j. It contains the minimal solution iff $y_i \geq 0$ for all i and $\Delta z_j \leq 0$ for all j.