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Editors: Gregory R. Baker, Walter D. Neumann, Karl Rubin

Topology '90

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Preface

This volume consists of contributions from participants in a Research Semester in Low Dimensional Topology which took place under the auspices of the International Mathematical Research Institute at Ohio State University from February through June 1990. The Research Semester was funded by The Ohio State University through a grant from University Challenge and included an international conference in March 1990.

The main topics of the Research Semester included: the geometry and topology of 3-manifolds, with particular emphasis on hyperbolic 3-manifolds and their interactions with number theory; the "new" invariants of 3-manifolds related to quantum field theory; plane algebraic curves.

A number of long term visitors (2–3 months) were in residence at any given time. These visitors were: L. Siebenmann (Orsay), T. Yoshida (Tokyo), B. Apanasov (Novosibirsk, now Oklahoma), V. Turaev (Leningrad, now Strasbourg) and S. Orevkov (Moscow). In addition each week saw additional short term visitors. A list of these visitors and all talks is given after this Preface.

The Research Institute. The International Mathematical Research Institute at Ohio State University was founded in 1989 to support a program of visiting research scholars in mathematics at Ohio State and to run Workshops and Special Emphasis Programs on topics of particular importance and timeliness. The Research Semester on Low Dimensional Topology was the first major program of the Institute. Since then the Institute has supported workshops on, among others, Nearly Integrable Wave Phenomena in Nonlinear Optics, Quantized Geometry, Arithmetic of Function Fields, and L-Functions Associated to Automorphic Forms, and a workshop on Geometric Group Theory will take place from May to June 1992. The Institute is currently supporting about 20–30 other research visitors (mostly short term) per year. The Institute publishes a preprint series as well as this book series, which is devoted to research monographs, lecture notes, proceedings, and other mathematical works arising from activities of the Research Institute.

Acknowledgements. First and foremost, the editors thank The Ohio State University for its support of this program through the Research Institute. We thank our visitors and our fellow topologists at Ohio State for their contributions to the success of this project. We also thank the non-academic staff of the Mathematics Department for their help in the organization and running of the Research Semester, particularly Marilyn Howard (administration and visas), Marilyn Radcliff (expenses), Gena Dacons (administration) and Terry England (typing).

All contributions to this volume were refereed, and the editors thank the referees for their invaluable service. Many authors helped us by preparing their contributions in TeX. The TeX macros were written by Larry Siebenmann and edited by Walter Neumann, who benefited from the TeXpertise of many unnamed people.

Walter D. Neumann and Alan W. Reid, for the editors, March 1992.

Program of the Research Semester

February to June 1990

The visitors to the program included the speakers listed below and additional short term visitors: A. Broughton, T. Chinburg, M. Feighn, J. Gilman, C. Hodgson, K. Millet, U. Oertel, J. Harer, J. Ratcliffe, M. Scharlemann, R. Skora, A. Thompson, J. Weeks.

In addition to the talks listed below, a seminar was run through weeks 13–19 on the new 3-manifold invariants of Turaev, Reshetikhin, Viro, and Witten, with talks by D. Burghelea, V. Turaev and D. Yetter.

Week 1 (February 4–10)
H. M. Hilden, Universal Groups (3 lectures).

Week 2 (February 11–17)
L. Siebenmann, Knot complements—the Tietze conjecture revisited.
C. Frohman, Knot invariants via intersection homology.

Week 3 (February 18–24)
R. Lee, Topological invariants from conformal field theory (2 lectures).
B. Apanasov, Hyperbolic cobordisms and conformal structures, I.

Week 4 (February 25–March3)
W. Whitten, Imbeddings of 3-manifold groups.
L. Mosher, Dynamical systems and the homology norm of a 3-manifold (2 lectures).
F. Bonahon, The circle at infinity of a surface and applications.
B. Apanasov, Hyperbolic cobordisms and conformal structures, II.

Week 5 (March 4–10)
Y. Xia, Tate-Farrell cohomology of mapping class groups.
B. Apanasov, Hyperbolic cobordisms and conformal structures, III.

Week 6 (March 11–17)
D. McCullough, A conjectural picture of 3-manifold mapping class groups.
Lee Rudolph, Generalized Jones' polynomial, symplectic topology, and complex plane curves.
R. Meyerhoff, Anti-length spectrum of hyperbolic 3-manifolds.

Week 7 (CONFERENCE IN LOW-DIMENSIONAL TOPOLOGY, March 17–20)
C. Adams, Noncompact hyperbolic 3-orbifolds of small volume.
B. Apanasov, Nonstandard conformal 3-manifolds and 4-dimensional topology.
M. Bestvina, The boundary of negatively curved groups.
L. Kauffman, Combinatorial version of the SL(2)q 3-manifold invariant—spin networks and quantum groups.

D. Long, Peripheral separability.

P. Melvin, Evaluations of the 3-manifold invariants of Witten and Reshetikhin-Turaev.

A. Reid, Commensurators of hyperbolic 3-manifolds.

N. Reshetikhin, Invariants of 3-manifolds connected with finite dimensional Hopf algebras.

H. Rubinstein, Polyhedral metrics of non-positive curvature on three and four-manifolds.

P. Shalen, Patterson measures, Margulis numbers, and volumes of hyperbolic 3-manifolds.

O. Viro, Combinatorial construction of quantum invariants of 3-manifolds.

T. Yoshida, Floer homology and splittings of manifolds.

Week 8 (March 25–31)

R. Brooks, Low eigenvalues of arithmetic manifolds.

R. Brooks, Isospectral manifolds.

W. Menasco, Developing a calculus on links in S^3.

Week 9 (April 1–7)

V. Poenaru, The 3-dimensional Poincaré conjecture, I, II.

V. Poenaru, Almost convex groups, combable groups and π_1^∞ of universal covering spaces of 3-manifolds.

C. Maclachlan, Fuchsian subgroups of Bianchi groups.

Week 10 (April 8–14)

V. Poenaru, The 3-dimensional Poincaré conjecture, III, IV.

V. Poenaru, Killing stable 1-handles in π_1^∞ of open 3-manifolds.

J. Przytycki, Skein module of handlebodies.

J. Hass, Flows and intersections of curves on surfaces.

P. Scott, Least area surfaces in 3-manifolds (2 lectures).

H. Rubinstein, Cubulated 3-manifolds (2 lectures).

T. Yoshida, A splitting formula of spectral flow and calculation of Floer homology of some special homology 3-spheres, I.

Week 11 (April 15–21)

C. McA. Gordon, Reducible manifolds and Dehn surgery.

T. Yoshida, A splitting formula of spectral flow and calculation of Floer homology of some special homology 3-spheres, II.

Week 12 (April 22–29)

S. Morita, On the structure of the mapping class group and the Casson Invariant.

M. Shapiro, Automatic structures and 3-manifold groups.

Week 13 (April 30–May 5)

V. G. Turaev, New invariants of links and 3-manifolds.

T. Yoshida, On ideal points of deformation curves of hyperbolic 3-manifolds with 1 cusp.

Week 14 (May 6–12)

M. Baker, Finding homology in covers of 3-manifolds.

M. Baker, Reminisces on the Cuspidal Cohomology Problem and arithmetic links.

Week 15 (May 13–19)

A. Libgober, Topology of affine surfaces and trigonometric sums.

S. Orevkov, Fundamental group of plane curve complements and the Zariski conjecture, I.

S. Kerckhoff, Local rigidity and the representation space of link complements.

Week 16 (May 20–26)

A. Pazhitnov, Morse-Novikov theory for closed 1-forms.

S. Orekov, Fundamental group of plane curve complements and the Zariski conjecture, II.

S. Kaliman, On the classification of polynomials in 2 variables.

D. Yetter, Tangles in cobordisms.

Week 17 (May 27–June 2)

S. Orevkov, Some approaches to the Jacobian conjecture.

J. Corson, Two-complexes of groups.

W. Neumann, Amalgamation and the invariant trace-field of Kleinian groups.

Week 18 (June 3–9)

No talks

Week 19 (June 10–16)

R. Penner, Decorated Teichmüller theory (2 lectures).

Contents

Noncompact Hyperbolic 3-Orbifolds of Small Volume

*Colin C. Adams**

Abstract. We determine the six noncompact orientable and the six noncompact nonorientable hyperbolic 3-orbifolds of least volume. This extends previous results of Meyerhoff where he determined the unique noncompact orientable and nonorientable hyperbolic 3-orbifolds of least volume. Our results are obtained through analysis of horoball diagrams for rigid cusps.

1. Introduction

In **[6]**, the noncompact orientable and nonorientable hyperbolic 3-orbifolds of least volume were determined. We extend those results here to determine the six noncompact orientable and nonorientable hyperbolic 3-orbifolds of least volume. See Theorem 6.1 and Corollary 6.2 for the volumes; the corresponding orbifolds are described in the paragraphs preceding the statements of Theorems 3.3, 4.2, and 5.2.

The idea is to analyze horoball patterns in order to list all the possible volumes of maximal cusps below a given volume. These results, in conjunction with known cusp density results, then yield the noncompact hyperbolic 3-orbifolds of least volume. Similar techniques were used in **[3]** to determine the three smallest limit volumes of hyperbolic 3-orbifolds.

In what follows, v_0 will denote the volume of an ideal tetrahedron in $\mathbb{H}^3$ with all dihedral angles equal to $\pi/3$. In particular, $v_0 = 1.01494146\ldots.$ We will let v_1 denote the volume of an ideal tetrahedron in $\mathbb{H}^3$ with dihedral angles $\pi/2$, $\pi/4$ and $\pi/4$. In particular, $v_1 = 0.91596544\ldots.$

We will work in the upper-half space model of hyperbolic 3-space, denoted $\mathbb{H}^3$. We will think of a hyperbolic 3-orbifold as being obtained by taking the quotient of $\mathbb{H}^3$ by a discrete group G of hyperbolic isometries. Since all of the orbifolds which we will be interested in have a single cusp, we will always assume that the single point at ∞ on the boundary of the upper-half-space model of $\mathbb{H}^3$ is a parabolic fixed point for this cusp.

The pre-image in $\mathbb{H}^3$ of the cusp will be a set of disjoint horoballs, including one centered about ∞. (The center of a horoball is the point on the boundary of hyperbolic space where the horoball is tangent.) The horoball centered about ∞ appears as a horizontal plane with positive z coordinate together with all the points above it.

We will maximize the single cusp in the orbifold by expanding it until it touches itself on the boundary. This corresponds in $\mathbb{H}^3$ to expanding the set of horoballs equivariantly until they first touch one another.

* Supported in part by NSF Grants DMS-8711495 and DMS-9000937.

It is convenient to normalize the picture so that the bounding plane of the horoball centered about ∞ has Euclidean height 1 above the $x-y$ plane. Since the cusp has been maximized, there are horoballs tangent to the horoball centered about ∞. These horoballs will have Euclidean diameter 1 and will be the horoballs of largest Euclidean diameter. We call these horoballs ***full-sized horoballs***.

We call the view of the set of all horoballs looking from ∞ down toward the $x-y$ plane a ***horoball diagram*** or a ***cusp diagram***. See **[4]** for examples of horoball diagrams corresponding to knots.

Additional background on hyperbolic 3-orbifolds appears in **[8]**. Additional background on horoball diagrams and definitions of terms can be found in **[1]**, **[2]** and **[3]**. All of the hyperbolic 3-orbifolds under consideration will be assumed orientable until the last section.

We will utilize the following lemma which follows from work of Robert Meyerhoff in **[5]**.

Lemma 1.1. *If a finite volume hyperbolic 3-orbifold Q contains a set of cusps with disjoint interiors such that their total volume is β, then* $\mathrm{vol}(Q) \geq \beta(2v_0/\sqrt{3})$.

We include one geometric lemma from **[3]** which will be helpful in what follows.

Lemma 1.2. *If two horoballs H_x and H_y have Euclidean diameters a and b respectively and their centers x and y are a distance c apart, then there exists a horoball with diameter ab/c^2.* □

Proof. The shortest hyperbolic distance between the two horoballs can be determined by rotating $180°$ about a geodesic which is a semi-circle of radius c with one endpoint at y and with its high point directly above x. Since the hyperbolic distance from H_x to the high point on the geodesic is $\ln(c/a)$, H_x is sent by the rotation to a horoball centered at ∞ with boundary a horizontal plane at height c^2/a. The rotation fixes H_y. The shortest distance between the two horoballs is unaffected by the rotation and is therefore $\ln(c^2/ab)$.

Since all horoballs are identified in the quotient, there is an isometry in the orbifold group which sends H_x to H_∞. It must then send H_y to a horoball which is a hyperbolic distance $\ln(c^2/ab)$ from H_∞. Since the boundary of H_∞ is a plane at height 1, H_y must be sent to a horoball of Euclidean diameter ab/c^2. □

2. Rigid cusps

A cusp in a hyperbolic 3-orbifold is called ***rigid*** if Dehn filling cannot be performed on it and otherwise it is called ***non-rigid***. In **[3]**, it was shown that a cusp is rigid if and only if there are singular curves of order other than two going all the way out the cusp.

Since a cusp is isometric to the quotient of the set of all of the points above a horizontal plane in the upper-half-space model of hyperbolic space by a Euclidean group of transformations, there are exactly three possibilities for the structure of the singularities in a rigid cusp. In all three cases, the cusp contains three singular axes. Denoting the orders of the three singular axes as $\{a,b,c\}$, the possibilities are $\{3,3,3\}$, $\{4,4,2\}$ or $\{6,3,2\}$.

Lemma 2.1. *If the action of the orbifold subgroup fixing ∞ identifies all of the full-sized horoballs in the horoball diagram of the orbifold, then every point of tangency between two horoballs lies on the axis of an order two elliptic isometry in the orbifold group such that the axis is tangent to both the horoballs.*

Proof. Let H_x be a full-sized horoball centered at the point x in the $x-y$ plane and let H_∞ be the horoball centered at ∞. Then there is an isometry j in the fundamental group of the orbifold such that $j(x)=\infty$. Let $y=j(\infty)$. Then $j(H_\infty)$ is a full-sized horoball, denoted H_y. By hypothesis, there is an isometry k which fixes ∞ and which sends y to x.

Then $k\circ j$ sends x to ∞ and ∞ to x, while fixing the point of tangency of H_x and H_∞. Thus $k\circ j$ is the order two elliptic isometry with axis through the point of tangency between H_x and H_∞. Since elements of the fundamental group of the orbifold identify all tangency points between pairs of horoballs to tangency points on H_∞, there must be corresponding order two elliptic axes through all points of tangency. □

Note that once we have normalized so that the boundary of the horoball centered at ∞ has Euclidean height one, the volume of the cusp is one half of the area of a fundamental domain in the $x-y$ plane for the Euclidean subgroup of the orbifold group which fixes ∞.

In the next three sections, we will only be interested in orbifolds with volume less than $v_0/4$. By Lemma 1.1, a maximal cusp in such an orbifold has volume less than $\sqrt{3}/8$. Hence, our goal will be to list those cusps with volumes less than $\sqrt{3}/8$.

The singularities in the rigid cusps will restrict the placement of the full-sized horoballs in the horoball diagrams. Since none of the horoballs overlap in their interiors, the only way a vertical singular axis in the upper-half-space model can intersect the interior of a horoball is if the horoball is centered at the end of the vertical axis in the $x-y$ plane.

3. $\{6,3,2\}$-cusp

We examine first the case where the cusp under consideration is a $\{6,3,2\}$-cusp. If there is more than one equivalence class of full-sized horoballs under the action of the subgroup of isometries fixing ∞, the least volume for the cusp occurs when a pair of full-sized horoballs are centered at the 3-fold and 6-fold singularities, yielding a volume in the cusp of at least $\sqrt{3}/8$. Hence, we will assume from now on that there is only one equivalence class of full-sized horoballs in the cusp diagram.

If there is a full-sized horoball which is not centered at any of the singular points in the cusp diagram, the least volume cusp appears as in Figure 1(a), with a cusp volume of $(3/2+\sqrt{3})/6 = .5386\ldots$.

If there is a full-sized ball centered at the 2-fold singularity, then the smallest possible area for the cusp diagram occurs when the full-sized horoballs at each of the 2-fold singularities touch each other as in Figure 1(b). This gives an area for the fundamental domain in the plane of at least $1/\sqrt{3}$ and a cusp volume of at least $\sqrt{3}/6 = .2887\ldots$.

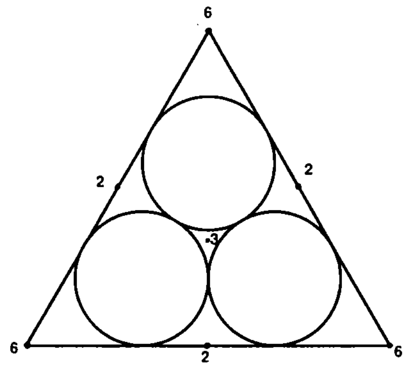

Figure 1(a)

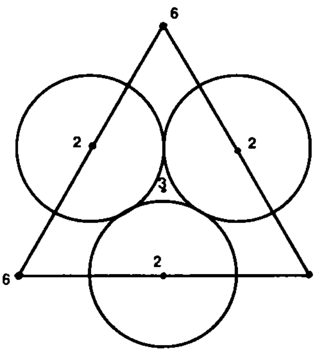

Figure 1(b)

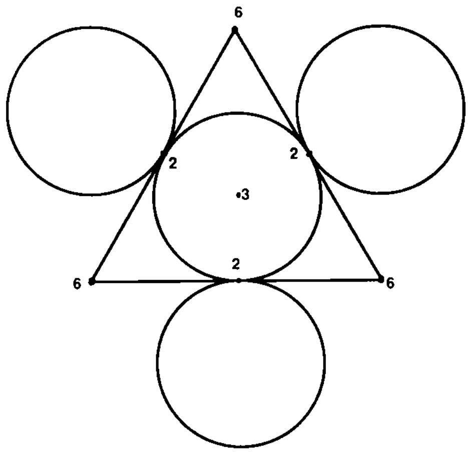

Figure 1(c)

If there is a full-sized ball at the 3-fold singularity, the volume is at least $\sqrt{3}/8$ as in Figure 1(c). Note that this volume is also the least volume for the case that we have full-sized balls at both the 6-fold and 3-fold singularities.

The rest of our time will be spent on the case that there is a full-sized horoball at the 6-fold singularity and nowhere else. The least such volume that can occur is in the case that these full-sized balls touch one another, yielding a cusp volume of $\sqrt{3}/24$. This is the smallest possible volume of an orientable cusp in an orbifold and it occurs in the smallest noncompact orientable orbifold, which was determined in **[4]**. It is not hard to prove that this smallest noncompact orientable orbifold is unique.

Now, we look at the situation where the full-sized balls centered at the 6-fold singularities do not touch. Let the shortest distance between the centers of a pair of full-sized balls be given by d. Throughout the rest of the section, we are assuming $d > 1$. We will find the following lemma useful.

Lemma 3.1. *If $d > 1$, there cannot be a set of three horoballs which are pairwise tangent.*

Proof. If there were such a set of three balls, an isometry exists which would take the center of one of them to ∞ and therefore send the other two balls to two full-sized horoballs which are tangent to each other. This implies that the minimal tangency distance d is in fact equal to 1, contradicting our assumption that $d > 1$. □

Since we are assuming that there is only one full-sized horoball up to the group action, we know from Lemma 2.1 that there is an order two axis perpendicular to the order six axis and through the point of tangency at height 1. Rotation about the order two axis sends the full-sized ball to a ball centered at ∞ and sends the ball centered at ∞ to the full-sized ball. It also sends the six neighboring full-sized balls, each a distance d away from our original full-sized ball, to six smaller balls. We call these smaller balls $(1/d)$-***balls*** as each of their centers will be a distance $1/d$ from the center of the full-sized ball. See Lemma 4.3 and the paragraphs following Lemma 4.6 of **[3]** for more details. The Euclidean diameter of these $(1/d)$-balls is $1/d^2$ by Lemma 1.2.

Define the distances u, v and w as in Figure 2. We can determine each of them in terms of d and β utilizing the law of cosines.

By symmetry, we will always assume that $0 \le \beta \le \pi/6$. We obtain the following equations:

$$u^2 = d^2 + 3/d^2 - \sqrt{3}\sin\beta - 3\cos\beta \tag{1}$$

$$v^2 = 4\left((d/2)^2 + 1/d^2 - \cos\beta\right) \tag{2}$$

$$w^2 = d^2 + 1/d^2 - 2\cos\beta. \tag{3}$$

If the center of a $(1/d)$-ball comes within a distance 1 of the center of a full-sized ball without the two balls touching or if the centers of two $(1/d)$-balls come within $(1/d)$ of each other without the two balls touching, then when one of the two balls is put at ∞, the other will form a ball intermediate in size between a full-sized ball and a $(1/d)$-ball, by Lemma 1.2. Since $(1/d)$-balls are the biggest balls tangent to full-sized balls, there must then be a ball tangent to no ball bigger than itself. The upper hemisphere of this ball

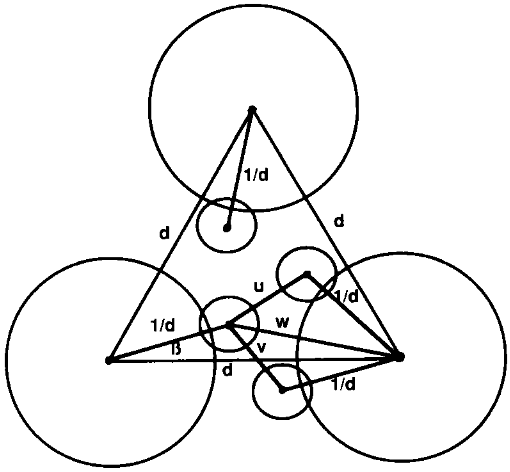

Figure 2

forms a disk of no tangency. (See the proof of Lemma 4.1 in [2] or Lemma 4.6 and the proof of Lemma 4.8 in [3]). Hence, there is the equivalent of another full-sized ball in the diagram. The least possible volume results when the additional ball is centered at the 3-fold singularity, yielding a volume of $\sqrt{3}/8$.

From now on, we will only investigate the situation where no such disk of no tangency exists. Suppose first that a single $(1/d)$-ball is shared by a pair of neighboring full-sized balls and the $(1/d)$-ball does not touch a third full-sized ball. Since the angles between the centers of the $(1/d)$-balls as measured from the center of a full-sized ball that they touch must be a multiple of $\pi/3$, it is easy to see that the center of the single $(1/d)$-ball is in line with the centers of the two full-sized balls that it touches. Hence $d = 2/d$, yielding $d = \sqrt{2}$ and a cusp volume equal to $\sqrt{3}/12 = .1443\ldots$. In this case, the horoball diagram appears as in Figure $3(a)$.

Suppose now that a single $(1/d)$-ball is shared by three full-sized balls. Then $1/d = d/\sqrt{3}$ yielding $d = \sqrt[4]{3}$. The cusp volume is then $1/8 = 0.125$, with a horoball diagram as in Figure 3(b).

From this point on, we assume that each $(1/d)$-ball touches a unique full-sized ball. In order that no intermediate balls are created, it must be that the center of each $(1/d)$-ball stays a distance at least 1 away from the center of the nearest full-sized ball. This forces $w \geq 1$ and hence

$$\cos\beta \leq \frac{d^2 + 1/d^2 - 1}{2}. \tag{4}$$

Since we are assuming by symmetry that $0 \leq \beta \leq \pi/6$, we have $\sqrt{3}/2 \leq \cos\beta \leq 1$. This can only occur when

$$d \geq \frac{\sqrt{1 + \sqrt{3} + \sqrt{2}\sqrt{3}}}{2} = 1.51546\ldots.$$

We will restrict ourselves to these values of d.

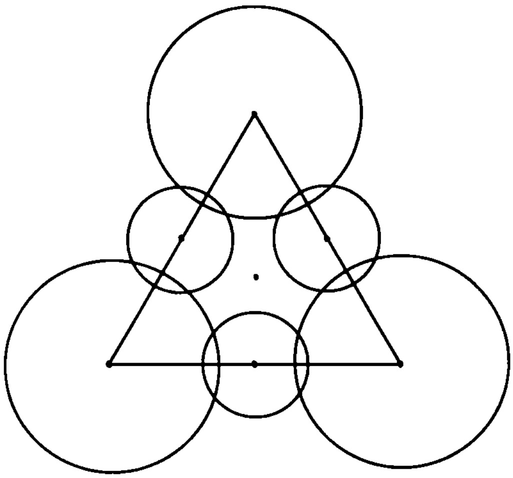

Figure 3(a)

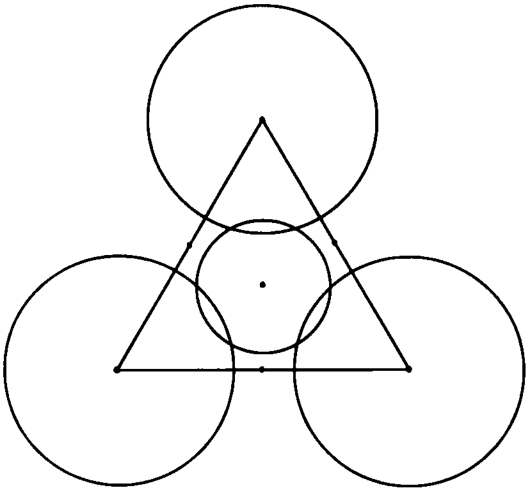

Figure 3(b)

Suppose first that a pair of $(1/d)$-balls touch each other and no other $(1/d)$-ball touches this pair. The point of tangency of the two balls must then be directly over a 2-fold singularity. It follows that

$$\cos\beta = d^2/4 + 1/d^2 - 1/(4d^4). \tag{5}$$

Then (4) and (5) together yield the inequality

$$d^6 - 2d^4 - 2d^2 + 1 \geq 0. \tag{6}$$

This has no solutions for $1 < d < (1+\sqrt{5})/2$. In order that two $(1/d)$-balls touch each other, it must be that $2/d + 1/d^2 \geq d$. This forces $d \leq (1+\sqrt{5})/2$. Thus, the only possible solution is $d = (1+\sqrt{5})/2$ yielding a cusp volume of $\sqrt{3}(3+\sqrt{5})/48 = 0.18894\ldots$.

Suppose now that three $(1/d)$-balls are tangent in pairs. Lemma 3.1 yields an immediate contradiction.

We now suppose that each $(1/d)$-ball touches a unique full-sized ball and that none of the $(1/d)$-balls touch each other. In order that no intermediate sized balls are created, it must be that the $(1/d)$-balls stay a distance at least 1 from the full-sized balls that they are not touching and that the $(1/d)$-balls stay a distance 1/d from each other. Hence, $w \geq 1$, yielding Equation 4, and both $v \geq 1/d$ and $u \geq 1/d$, yielding:

$$\cos\beta \leq d^2/4 + 3/(4d^2) \tag{7}$$

$$d^2 + 2/d^2 - \sqrt{3}\sin\beta - 3\cos\beta \geq 0. \tag{8}$$

From (8) we have the following possibilities

$$\cos\beta \leq \frac{3(d^2+2/d^2) - \sqrt{24 - 3d^4 - 12/d^4}}{12} \tag{9}$$

or

$$\cos\beta \geq \frac{3(d^2+2/d^2) + \sqrt{24 - 3d^4 - 12/d^4}}{12}. \tag{10}$$

These two equations yield no restriction once d reaches a size of $1.65289\ldots$. Since we are assuming $\cos\beta \geq \sqrt{3}/2$, (9) can only hold for $d \geq \sqrt{(1+\sqrt{3})} = 1.65289\ldots$. For any $d \geq \sqrt{2}$, (10) does not contradict the fact $\cos\beta \leq 1$, so in the range $1.51546\ldots \leq d \leq 1.65289\ldots$, we can ignore (9) and only utilize (10).

However, when $d > 1$, (10) and (7) cannot hold simultaneously unless $d \geq \sqrt[4]{7} = 1.62658\ldots$. Hence, the cusp volume in this case must be at least $\sqrt{21}/24 = 0.19094\ldots$.

In Figure 4, we see the case with the $(1/d)$-balls each touching a unique full-sized ball. Applying an order two elliptic isometry with axis tangent to the horoball centered at ∞ and tangent to the full-sized horoball on the lower left sends the $(d, w, 1/d)$-triangle to a similar triangle with edge lengths $1/w, 1/d$ and $1/wd^2$. The old $(1/d)$-ball is sent to a new ball of diameter $1/w^2d^2$ by Lemma 1.2.

Call this new ball a $(1/w)$-ball. Suppose first that the three $(1/w)$-balls corresponding to three distinct $(1/d)$-balls are in fact the same ball. Then the single $(1/w)$-ball has its center at the center of the equilateral triangle with edge lengths d. Hence, it must be that $1/w = \sqrt{3}/d$. This fact, together with the law of cosines and the law of sines then forces $d = \sqrt[4]{7}$. The resulting cusp has volume exactly $\sqrt{21}/24$, and is depicted in Figure 5.

We assume now that the $(1/w)$-balls are distinct. If the $(1/w)$-balls touched full-sized balls, this would force two $(1/d)$-balls to touch, contradicting our assumption that they do not. The three $(1/w)$-balls cannot touch each other pairwise by Lemma 3.1.

In order to prevent the creation of any intermediate sized balls and the resulting disks of no tangency, it must be that the $(1/w)$-balls stay a distance at least $(1/w)$ from the full-sized balls and that the $(1/w)$-balls stay a distance $1/dw^2$ from each other by

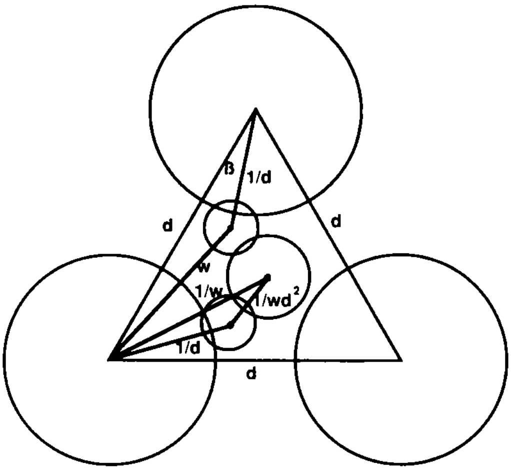

Figure 4

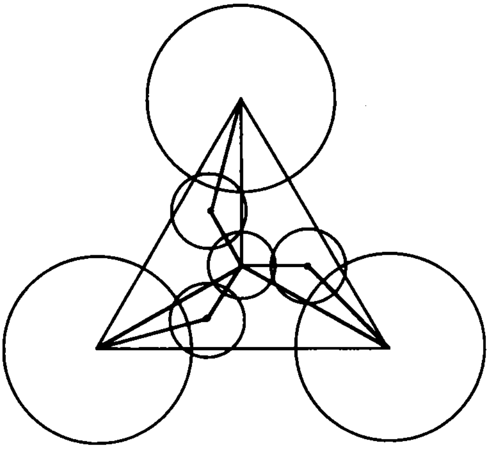

Figure 5

Lemma 1.2. This yields the following two equations:

$$\cos\beta \leq 1/2 + 1/d^2 \tag{11}$$

$$36 - 228cd^2 + 93d^4 + 484c^2d^4 - 376cd^6 - 392c^3d^6 + 70d^8 + 420c^2d^8 + \\ 112c^4d^8 - 138cd^{10} - 152c^3d^{10} + 13d^{12} + 72c^2d^{12} - 14cd^{14} + d^{16} \geq 0, \tag{12}$$

where $c = \cos\beta$

In fact, (12) will not be utilized in the argument. Since we are assuming that $\cos\beta \geq \sqrt{3}/2$, (11) implies that $d \leq \sqrt{1+\sqrt{3}}$. However, (11) together with (10) yields

$$48 + 72d^2 - 24d^4 - 36d^6 + 12d^8 \geq 0. \tag{13}$$

This forces $d \geq \sqrt{1+\sqrt{3}}$. Thus, it must be that $d = \sqrt{1+\sqrt{3}}$, yielding a cusp volume of $(\sqrt{3}+3)/24$. However, in this case, one can check that in fact the three $(1/w)$-balls at the center of the triangle are pairwise tangent. Hence, Lemma 3.1 implies no such orbifold can exist.

We summarize the results obtained in this section so far in the following theorem.

Theorem 3.2. *A maximal* $\{6,3,2\}$*-cusp in a hyperbolic* 3*-orbifold has volume either* $\sqrt{3}/24$, $\sqrt{3}/12$, $1/8$, $\sqrt{3}(3+\sqrt{5})/48$, $\sqrt{21}/24$ *or at least* $\sqrt{3}/8$. □

We would now like to determine the orbifolds which have a $\{6,3,2\}$-cusp with one of these possible cusp volumes. Let Q be an orbifold with at least one $\{6,3,2\}$-cusp. For each of the cusp volumes listed in Theorem 3.2, there is a unique corresponding horoball diagram. In each case, Lemma 2.1 and the horoball diagram are enough to determine the singular set and a fundamental domain for the corresponding orbifold. The singular axes determine how faces on the fundamental domain must be glued in order to yield the corresponding orbifold.

In the case that the $\{6,3,2\}$-cusp has volume $\sqrt{3}/24$, the corresponding horoball diagram forces Q to be the quotient of an ideal regular tetrahedron by its orientation-preserving symmetry group, yielding a volume for the orbifold of $v_0/12$. The orbifold is uniquely determined.

If instead, the $\{6,3,2\}$-cusp has volume $\sqrt{3}/12$, the horoball diagram must appear as in Figure $3(a)$. The corresponding orbifold is again uniquely determined and corresponds to the quotient of an ideal cube, with all dihedral angles $\pi/3$, by the orientation-preserving symmetry group of the cube. The resulting orbifold has volume $5v_0/24$.

Suppose now that the $\{6,3,2\}$-cusp has volume $1/8$. Then the horoball diagram appears as in Figure 3(b). The action of the orbifold group tiles $\mathbb{H}^3$ with tetrahedra, all of angles $\pi/6, \pi/6$ and $2\pi/3$. The volume of such a tetrahedron is $2v_0/3$. The quotient of such a tetrahedron by its symmetry group yields $v_0/6$ as the volume of the corresponding orbifold.

If the $\{6,3,2\}$-cusp has volume $\sqrt{3}(3+\sqrt{5})/48$, the orbifold is the quotient of an ideal regular dodecahedron by its orientation-preserving symmetry group, yielding a volume of $0.3430\ldots$.

If the $\{6,3,2\}$-cusp has a volume of $\sqrt{21}/24$, the horoball diagram appears as in Figure 4. The action of the orbifold group tiles all of $\mathbb{H}^3$ with two ideal tetrahedra. The first tetrahedron has dihedral angles a, b and c where $\cos(a) = 5/(2\sqrt{7})$ and $\cos(b) = 9/(2\sqrt{21})$. The second tetrahedron has dihedral angles d, e and f where $d = \pi/3$ and $e = a$. A fundamental domain is obtained by taking half of each of these two tetrahedra, yielding a volume of approximately 0.47 .

Theorem 3.3. *A hyperbolic* 3*-orbifold with a* $\{6,3,2\}$*-cusp has volume either* $v_0/12$, $v_0/6$, $5v_0/24$ *or at least* $v_0/4$.

Proof. If such a hyperbolic 3-orbifold does not have one of the first three volumes, it has a cusp volume of either $\sqrt{3}(3+\sqrt{5})/48$, $\sqrt{21}/24$, or at least $\sqrt{3}/8$. In the case that the cusp volume is $\sqrt{3}(3+\sqrt{5})/48$ or $\sqrt{21}/24$, we have seen that the corresponding orbifold has volume $0.3430\ldots$ or $.47\ldots$, both of which are greater than $v_0/4$. Otherwise, the cusp volume must be at least $\sqrt{3}/8$, which by Lemma 1.1 implies that the corresponding orbifold has volume at least $v_0/4$. □

4. {3,3,3}-cusp

A fundamental domain in the x-y plane for a $\{3,3,3\}$-cusp is a rhombus. Assume first of all that there is a full-sized horoball which is not centered at one of the singular points in the plane. Then the shortest distance between two singular points which are identified by the Euclidean group action is at least 2. The resulting cusp has volume at least $\sqrt{3}/3$.

Suppose now that there is a full-sized ball at one of the three 3-fold singularites and nowhere else. The pattern of $(1/d)$-balls will resemble the pattern of $(1/d)$-balls which we obtained for the $\{6,3,2\}$-cusp when the cusp has one full-sized ball located at the 6-fold singularity. Hence we will have exactly twice the cusp volumes we found in that case.

Corollary 4.1. *A maximal* $\{3,3,3\}$*-cusp in a hyperbolic 3-orbifold has volume either* $\sqrt{3}/12$, $\sqrt{3}/6$, $1/4$, $\sqrt{3}(3+\sqrt{5})/24$, $\sqrt{21}/12$ *or at least* $\sqrt{3}/4$. □

For each of these possible volumes of a $\{3,3,3\}$-cusp, there is a unique orbifold, each such orbifold corresponding to the double cover of an orbifold in the $\{6,3,2\}$-cusp case. The following theorem is then immediate.

Theorem 4.2. *A hyperbolic* 3*-orbifold with a* $\{3,3,3\}$*-cusp has volume either* $v_0/6$, $v_0/3$, $5v_0/12$ *or at least* $v_0/2$. □

5. {4,4,2}-cusp

Suppose first that there is a full-sized ball in the horoball diagram which is not centered at one of the singularities. The volume in the cusp is then at least $1/2$ as in Figure 6(a). If instead, there is one full-sized ball at the 2-fold singularity, the volume in the cusp is at least $1/4$, as in Figure 6(b).

Note that if there is more than one full-sized horoball in the cusp, then the least volume occurs when there are two full-sized balls and they occur at the two 4-fold singularities, giving a volume of at least $1/4$, as in Figure 6(c).

Henceforth, we will assume that there is one full-sized ball which is centered at one of the 4-fold singularities. Suppose first that the full-sized balls in the cusp diagram touch each other. The volume in the cusp is then exactly $1/8$.

From now on, we will assume that the full-sized balls do not touch. Let d again be the shortest distance between full-sized horoballs. Then there is a set of four $(1/d)$-balls touching each full-sized ball.

Suppose first that four full-sized balls share a single $(1/d)$-ball which is centered at the other 4-fold singularity. Then $2/d = \sqrt{2}d$ and $d = \sqrt[4]{2}$. The volume of the corresponding cusp is then $\sqrt{2}/8$.

If instead, a $(1/d)$-ball is shared by exactly two full-sized balls, it must be that $2/d = d$ and $d = \sqrt{2}$, yielding a volume of $1/4$.

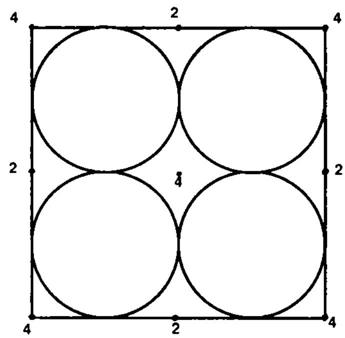

Figure 6(a)

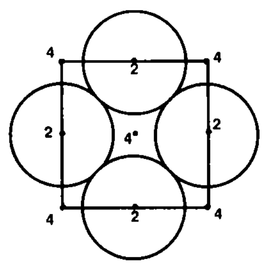

Figure 6(b)

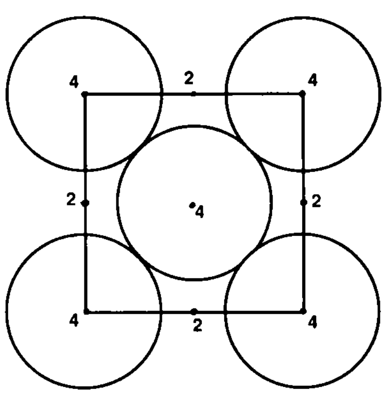

Figure 6(c)

From now on, we will assume that no $(1/d)$-balls are shared by full-sized balls. In order that disks of no tangency are not created, we will assume that all $(1/d)$-balls stay a distance at least 1 from any full-sized balls they are not tangent to. Thus, (4) from Section 3 holds.

Suppose first that a pair of $(1/d)$-balls corresponding to two distinct full-sized balls are tangent, and no other $(1/d)$-ball is tangent to the pair. Then (5) from Section 3 must hold. As in Section 3, the only possible value for d is $(1+\sqrt{5})/2$. The resulting cusp has volume $(3+\sqrt{5})/16 = 0.32725\ldots$.

Suppose now that there are four $(1/d)$-balls such that each is tangent to two others and they are symmetrically placed around one of the 4-fold singularites. Apply an isometry of the group which takes one of the $(1/d)$-balls to the horoball at ∞. The two $(1/d)$-balls which were tangent to the first $(1/d)$-ball are sent to full-sized balls. The fourth $(1/d)$-ball is sent to a horoball tangent to each of these full-sized balls with center in line with their centers. Since the centers of the four $(1/d)$-balls formed a square, this new ball will have center a distance $1/\sqrt{2}$ from each of the full-sized balls. Hence the full-sized balls have centers a distance $\sqrt{2}$ apart. However, this forces $d = \sqrt{2}$ and $1/d = 1/\sqrt{2}$. Thus, this fourth ball must be a $(1/d)$-ball. This contradicts the fact that every $(1/d)$-ball is tangent to two others.

Suppose now that none of the $(1/d)$-balls touch each other. Then, in order that no intermediate sized balls are introduced, it must be that the centers of the $(1/d)$-balls stay a distance at least $1/d$ apart by Lemma 1.2. This yields both (4) from Section 3 and

$$d^4 - 2(\sin\beta + \cos\beta)d^2 + 1 \geq 0. \tag{14}$$

This last equation becomes

$$\cos\beta \leq \frac{d^2 + 1/d^2 - \sqrt{6 - d^4 - 1/d^4}}{4} \tag{15}$$

or

$$\cos\beta \geq \frac{d^2 + 1/d^2 + \sqrt{6 - d^4 - 1/d^4}}{4}. \tag{16}$$

By symmetry, we are only interested in the cases where $0 \leq \beta \leq \pi/4$ so $\sqrt{2}/2 \leq \cos\beta \leq 1$. Applying this restriction to (15), we find that (15) does not apply until $d = 1.5537743$ at which time, (15) and (16) no longer restrict $\cos\beta$. Hence, in the range $1 < d \leq 1.5537743$, (16) must hold.

Comparing (16) with (4) yields a contradiction unless

$$d \geq \frac{\sqrt{1+\sqrt{3}+\sqrt{2}\sqrt{3}}}{2} = 1.51546\ldots.$$

This yields a cusp volume of at least $(1+\sqrt{3}+\sqrt{2}\sqrt{3})/16 = 0.28690\ldots$.

Combining the results obtained so far in this section, we have the following.

Theorem 5.1. *A maximal* $\{4,4,2\}$*-cusp in a hyperbolic* 3*-orbifold has volume either* $1/8$, $\sqrt{2}/8$ *or at least* $1/4$. □

The only way for a $\{4,4,2\}$-cusp to have volume $1/8$ is if the corresponding orbifold is the quotient of an ideal regular octahedron by its orientation preserving symmetry group. A fundamental domain will be one sixth of an ideal tetrahedron with dihedral angles $\pi/4$, $\pi/4$ and $\pi/2$. Hence, the orbifold has volume $v_1/6$.

If the $\{4,4,2\}$-cusp in a hyperbolic 3-orbifold has volume $\sqrt{2}/8$, the corresponding orbifold must come from the quotient of an ideal tetrahedron with dihedral angles $\pi/4$, $\pi/4$ and $\pi/2$ by its orientation-preserving symmetry group, yielding a volume of $v_1/4$.

Otherwise, the $\{4,4,2\}$-cusp has a volume of at least $1/4$. Lemma 1.1 then implies that the orbifold has a volume of at least $v_0/(2\sqrt{3})$, yielding the following theorem.

Theorem 5.2. *A hyperbolic 3-orbifold with a $\{4,4,2\}$-cusp has volume either $v_1/6$, $v_1/4$ or at least $v_0/(2\sqrt{3})$.* □

With an analysis of densities of horoball packings, we expect that the lower bound on volumes of $v_0/(2\sqrt{3})$ given in the above theorem could be improved to $v_1/3$.

6. Conclusions

Utilizing the results from Sections 3, 4 and 5 we have the following theorem.

Theorem 6.1. *The six noncompact orientable hyperbolic 3-orbifolds of volume less than $v_0/4$ have volumes $v_0/12$, $v_1/6$, $v_0/6$, $v_0/6$, $5v_0/24$ and $v_1/4$.*

Proof. A noncompact hyperbolic 3-orbifold must have at least one cusp. If any of the cusps are non-rigid, the results of **[3]** show that the orbifold has a volume of at least $v_1/3$. We can therefore assume all the cusps are rigid. However, a rigid cusp must be either a $\{6,3,2\}$-cusp, a $\{3,3,3\}$-cusp or a $\{4,4,2\}$-cusp. The theorem then follows immediately from Theorems 3.3, 4.2 and 5.2. □

Corollary 6.2. *The six noncompact nonorientable hyperbolic 3-orbifolds of volume less than $v_0/8$ have volumes $v_0/24$, $v_1/12$, $v_0/12$, $v_0/12$, $5v_0/48$ and $v_1/8$.*

Proof. The six orientable orbifolds from Theorem 6.1 all double cover nonorientable orbifolds of half their volumes. Any other noncompact nonorientable orbifold will be double covered by an orientable orbifold of volume at least $v_0/4$, and hence will have volume itself of at least $v_0/8$. □

An investigation into the volumes of hyperbolic 3-orbifolds with multiple cusps will appear in a subsequent paper.

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Combinatorial Cubings, Cusps, and the Dodecahedral Knots

I. R. Aitchison and J. H. Rubinstein

Abstract. There are finitely many tessellations of 3-dimensional space-forms by regular Platonic solids. Explicit examples of constant curvature finite-volume 3-manifolds arising from these are well-known for all possibilities, except for the tessellation $\{5,\ 3,\ 6\}$. We introduce the dodecahedral knots D_f and D_s in S^3 to fill this gap. Techniques used illustrate the results on cusp structures and π_1-injective surfaces of alternating link complements obtained by Aitchison, Lumsden and Rubinstein **[ALR]**.

The Borromean rings and figure-eight knot arise from the tessellation of hyperbolic 3-space by regular ideal octahedra and tetrahedra respectively. We produce exactly four new links in S^3, corresponding to the tessellations $\{4,\ 3,\ 6\}$ and $\{5,\ 3,\ 6\}$ of $\mathbb{H}^3$, and united by a canonical construction from the Platonic solids.

The dodecahedral knot D_f is the third in an infinite sequence of fibred, alternating knots, the first member of which being the figure-eight. The complements of these new links contain π_1-injective surfaces, which remain π_1-injective after 'most' Dehn surgeries. The closed 3-manifolds obtained by such surgeries are determined by their fundamental groups, but are not known to be virtually Haken.

1. Introduction

Regular tessellations of space-forms by Platonic solids have played a significant rôle in the exploration and exposition of 3-dimensional geometries and topology. Table 1, derived from Coxeter **[Co1]**, **[Co2]**, gives all such tessellations, including those by solids with deleted vertices.

Remark 1.1. The tessellations $\{3,\ 3,\ 3\}$, $\{4,\ 3,\ 4\}$ and $\{5,\ 3,\ 5\}$ are self-dual. The links of vertices are respectively tetrahedra, octahedra and icosahedra, the Platonic solids with triangular faces. The corresponding edge degrees — 3, 4 and 5, the most famous Pythagorean triple — encapsulate the notions of positively-curved, flat and negatively-curved geometry.

The corresponding symmetry groups are well-understood in the spherical and flat spaceforms, as are the subgroups acting without fixed points. For the hyperbolic tessellations, finite-index torsion-free subgroups exist by Selberg's theorem, with corresponding quotient 3-manifolds having finite volume. Infinitely many such subgroups exist.

The dodecahedral tessellation $\{5,\ 3,\ 3\}$ gives rise to Poincaré's homology sphere $\mathcal{P}^3$, a manifold ubiquitous in geometric topology, associated with problems of smoothings and triangulations of manifolds. A beautiful description of $\mathcal{P}^3$ in terms of face identifications of a dodecahedron has been given by Seifert and Weber **[SW]**, where another such

Solid	Tessellations of spaceforms by Platonic solids			
	S^3	$\mathbb{R}^3$	$\mathbb{H}^3$, compact	$\mathbb{H}^3$, ideal
tetrahedra	$\{3,\ 3,\ 3\}$	none	none	$\{3,\ 3,\ 6\}$
	$\{3,\ 3,\ 4\}$			
	$\{3,\ 3,\ 5\}$			
icosahedra	none	none	$\{3,\ 5,\ 3\}$	none
octahedra	$\{3,\ 4,\ 3\}$	none	none	$\{3,\ 4,\ 4\}$
cubes	$\{4,\ 3,\ 3\}$	$\{4,\ 3,\ 4\}$	$\{4,\ 3,\ 5\}$	$\{4,\ 3,\ 6\}$
dodecahedra	$\{5,\ 3,\ 3\}$	none	$\{5,\ 3,\ 4\}$	$\{5,\ 3,\ 6\}$
			$\{5,\ 3,\ 5\}$	

Table 1

compact 3-manifold, the hyperbolic Seifert-Weber space, is also described. The latter manifold arises from the dodecahedral tessellation $\{5,\ 3,\ 5\}$. In both cases, opposite faces are identified in a natural fashion.

The cube is distinguished in that it tessellates all spaceforms. The cubical tessellation of $\mathbb{R}^3$ gives rise to the 3-dimensional torus, with flat geometry. Euclidean space does not admit regular tessellations other than by the cube. Nonetheless, the mysterious connections between the Platonic solids allows for an intriguing manifestation of dodecahedra even in this context. Thurston **[Th]** has shown how a dodecahedron can be flattened into a cube, and then allowed to 'tessellate' $\mathbb{R}^3$. Allowing orbifold structures, Thurston then shows how this tessellation induces a singular metric on S^3, with cone angle π concentrated along the Borromean rings. Using the universality of the Borromean rings in the construction of closed orientable 3-manifolds as branched covering spaces, Hilden, Lozano, Montesinos and Whitten **[H*]** demonstrate the significance of the dodecahedral tessellation $\{5,\ 3,\ 4\}$: its group of symmetries is rich enough to produce all closed 3-manifolds.

Similarly, orbifold structures on links in S^3 arise from the Seifert-Weber manifold in the guise of the Whitehead link, and from the tessellation of $\mathbb{H}^3$ by cubes with icosahedral vertex links via the 5_2-knot (**[Be]**, **[AR1]**). Both of these links are universal.

Other *closed* hyperbolic 3-manifolds arising from tessellations of $\mathbb{H}^3$ have been described in Best **[Be]**, and in Richardson and Rubinstein **[RR]**.

The tessellations of hyperbolic space by ideal Platonic solids are of equal interest. The most famous contemporary example is $\{3,\ 3,\ 6\}$, giving the figure-eight knot complement (again universal) as quotient (**[Th]**). An example of a link complement in S^3 whose complement is the quotient of $\{4,\ 3,\ 6\}$ is described in **[AR1]**. Thurston **[Th]** also shows that two octahedra of $\{3,\ 4,\ 4\}$ form the fundamental domain for a discrete subgroup of symmetries, with quotient again the complement of the Borromean rings in S^3.

The remaining tessellation $\{5,\ 3,\ 6\}$ of $\mathbb{H}^3$ by ideal dodecahedra has not been considered previously — no *explicit* link complement in any 3-manifold is known to have such a structure. We will construct two such examples in S^3, obtaining what we call the

dodecahedral knots D_f and D_s. Whether the tessellation $\{5,\ 3,\ 6\}$ leads to as rich a domain as the other dodecahedral tessellations remains to be seen.

Complements of alternating links

In each of the cases above, the resulting link in S^3 is alternating. Investigations of the hyperbolic structures of alternating link complements have been given by Lawson **[La]**, Menasco **[Me]**, Takahashi **[Ta]**, and more recently by Weeks **[We]**, seeking to generalize the beautiful constructions of Thurston **[Th]**. In each case, the aim has been to demonstrate the existence of a complete metric of constant curvature -1 on the complement, and to calculate various invariants from such a (unique) structure. This invariably necessitates determining a combinatorial description of the link complement as the union of two 'ideal' polyhedra, with face identifications, and then decomposing these polyhedra into ideal tetrahedra whose shapes and volumes can be calculated. At this stage of the procedure, there is no canonical way to proceed, and any structure hidden in the combinatorics at the polyhedral level is lost.

That some beautiful deeper combinatorial structure may have existed has been remarked in these papers, but neither revealed nor exploited explicitly.

Retrospectively, our starting point is two remarks of Thurston **[Th]**. The first is that the figure-eight knot can be arranged on the 1-skeleton of a tetrahedron, as a 'heuristic' that the complement admits a tetrahedral decomposition. In fact, there are two simple such arrangements, and we develop the second one. Thurston's second remark is that for the Borromean rings, face identifications have a beautiful naturality: "Faces are glued to their corresponding faces with $120°$ rotations, alternating in direction like gears" **[Th]**.

We describe how, with our arrangement of the figure-eight knot on the tetrahedron, these remarks are related, and generalize to face identifications of two identical polyhedra, producing all of the examples of alternating links considered. We illustrate with each of the ideal regular tessellations of $\mathbb{H}^3$, producing 4 new links in the process. Our favourites, arising from $\{5,\ 3,\ 6\}$, are a new fibred knot D_f, and a knot D_s possessing a high degree of symmetry. The existence and simplicity of this combinatorial structure of alternating link complements is described in detail in **[ALR]**. A more general context is described in **[AR2]**.

2. The general construction for 4-valent graphs

We recall the construction of **[ALR]**. Take an arbitrary finite connected planar graph Γ, all of whose vertices having degree 4. We also require that at any vertex, all regions meeting at the vertex are distinct. Two-colour the regions of the plane checker-board fashion using white and black, with the exterior white by convention. Assign signs '$+$' and '$-$' to the white and black regions respectively. Denote the resulting combinatorial polyhedron by Π_Γ^+.

Now take an identical copy of Π_Γ^+, reverse all colours and signs, and denote the resulting polyhedron by Π_Γ^-. Each face ϕ_i of $\Pi_\Gamma^\pm$ is a combinatorial n_i-gon, with sign allocation σ_i, and we identify ϕ_i with the corresponding face ϕ_i' of $\Pi_\Gamma^\mp$ by a rotation of $\sigma_i.2\pi/n_i$, with a ' + ' sign denoting clockwise.

Denote the resulting topological space by $\overline{M}_\Gamma$, and let M_Γ denote $\overline{M}_\Gamma$ with deleted vertices. Finally, let $\mathfrak{L}_\Gamma$ denote the alternating link in S^3 canonically associated to Γ, as in Figure 1. Observe that, viewed from the center of any region, crossings are of the sign assigned to that region.

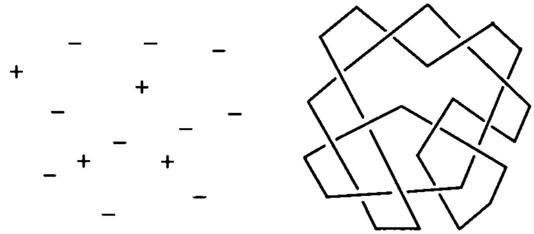

Figure 1

One of the results of [**ALR**] is

Theorem 2.1. *M_Γ is canonically homeomorphic to $S^3 - \mathfrak{L}_\Gamma$. Each edge of M_Γ is of degree 4.*

3. The six examples arising from ideal tessellations

In each case, we describe an alternating link, and face identifications of the corresponding pair of identical polyhedra. That the link complement has a complete metric of constant curvature -1 follows immediately on declaring each polyhedron to be ideal and regular in hyperbolic space.

Example 1. The Borromean rings. Applying this construction to the graph $\Gamma_{\{3,4\}}$ underlying the octahedron, we recover Thurston's description of the complement of the Borromean rings of Figure 2. The universal cover is the tessellation $\{3,4,4\}$ of $\mathbb{H}^3$.

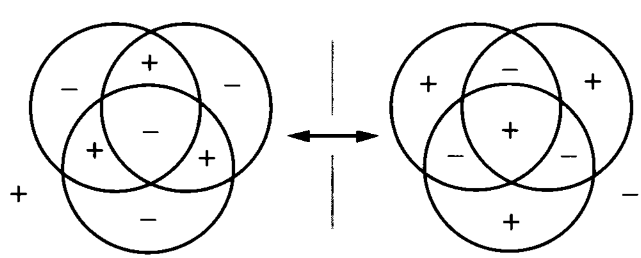

Figure 2

Example 2. The figure-eight knot. Take a tetrahedron, corresponding to the graph $\Gamma_{\{3,3\}}$ and 2-colour its faces black and white in the unique (up to symmetry) way so that no vertex is surrounded by regions all of the same colour. Assign the sign ' + ' to

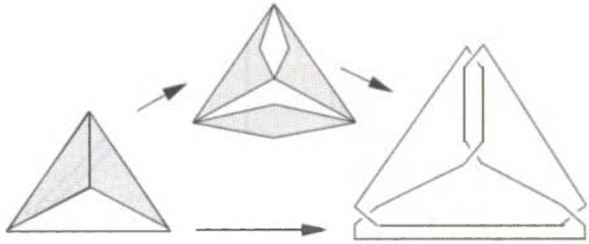

Figure 3

the white regions, ' $-$ ' to the black. Now split each edge separating regions of the same colour to obtain a 4-valent graph 2-coloured as above. (Figure 3.)

Carrying out face identifications yields the figure-eight knot complement. The two resulting 'bigons' can be squeezed back to a single edge to recapture the face identifications of tetrahedra as in Thurston's description. Note that in removing a bigon, two edges are identified in each polyhedron $\Pi_\Gamma^\pm$, from different equivalence classes. Every edge in the quotient is thus of degree 6. The universal cover corresponding to this combinatorial structure is geometrically the tessellation $\{3,3,6\}$ of $\mathbb{H}^3$, giving rise to the complete structure on the knot complement.

Examples 3, 4. Two cubical links. There are *two* ways to 2-colour the regions of the graph $\Gamma_{\{4,3\}}$. These are depicted in Figure 4.

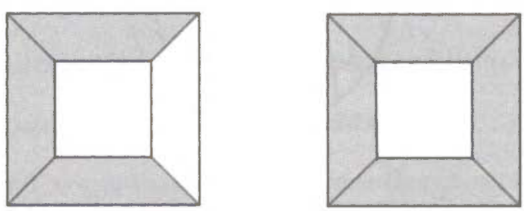

Figure 4

Proceed exactly as in the last example, observing that the introduction and deletion of bigons is unnecessary provided the link associated with such a 2-coloured trivalent graph is interpreted according to Figure 5.

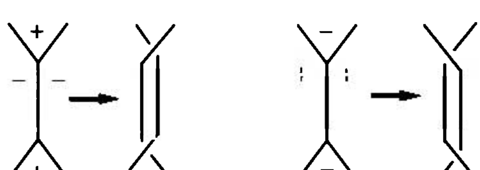

Figure 5

These two links obtained from the cube arise from the tessellation $\{4,3,6\}$ of $\mathbb{H}^3$, and are the links 8^3_4 and 8^4_1 in Rolfsen's book, depicted in Figure 6.

In **[ALR]**, 4-valent graphs admitting a collapse to a 2-coloured 3-valent graph without bigons are called 'balanced': the construction applied here works for all such graphs.

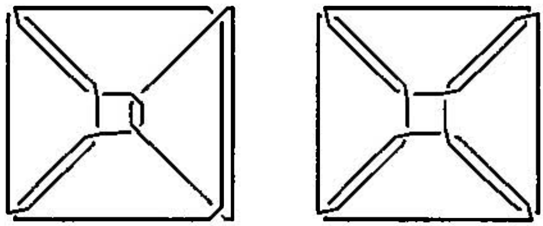

Figure 6

Remark 3.1. These two are the *only* links in Rolfsen's tables which have balanced bigons, in the sense of **[ALR]**, and no triangular regions.

There is another 3-component link also corresponding to the tessellation $\{4,3,6\}$ of $\mathbb{H}^3$, described in **[AR1]**. This does not obviously arise as part of our general construction.

Examples 5, 6. The two dodecahedral knots. The dodecahedron may be depicted combinatorially as in Figure 7.

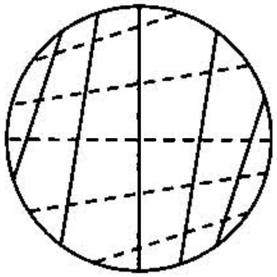

Figure 7

Up to symmetry and colour interchange, there are two allowable 2-colourings. These are depicted in Figure 8, with corresponding knots in Figure 9 denoted D_s and D_f arising from the tessellation $\{5,3,6\}$ of $\mathbb{H}^3$. The knot D_s has considerable symmetry, whereas D_f turns out to be fibred.

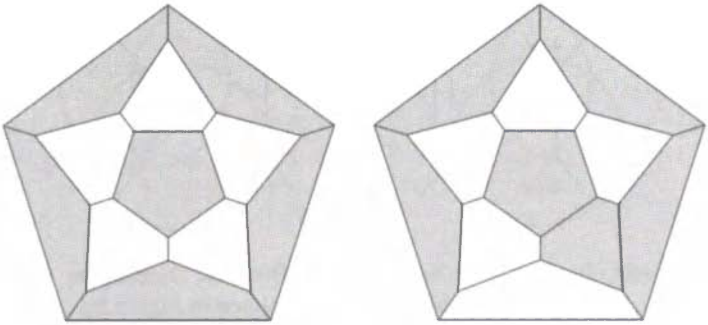

Figure 8

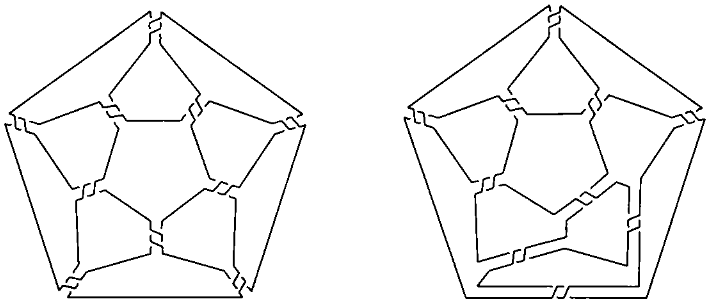

Figure 9

Added in Proof. Alan Reid and Walter Neumann have demonstrated some fascinating properties of these dodecahedral knots, in the context of their beautiful work on arithmetic structures [**NR**].

Hatcher has used similar ideas in [**Ha**], and it seems likely that Thurston is aware of the general construction, particularly since we have found the fibred dodecahedral knot in [**Ri**], referred to by Riley as 'Thurston's knot'. It is clear from the construction above that the complement of D_f admits an orientation reversing involution. The complements of both D_f and D_s contain totally geodesic immersed surfaces with respect to the complete metric of constant curvature.

4. Some fibred alternating knots from balanced links

We begin with a characterization of a class of colourable graphs.

Lemma 4.1. *Suppose Γ is a connected trivalent planar Hamiltonian graph. Then Γ can be 2-coloured with no vertex surrounded by regions of the same colour.*

Such a graph arises by drawing a circle as the equator of the sphere, and adding disjointly embedded arcs with endpoints on the equator. Colour one hemisphere white, the other black.

Remark 4.2. The 2-colourings of the cube and dodecahedron described above show that a graph with Hamiltonian circuit need not have a *unique* 2-colouring, and that the resulting alternating link may have more than one component.

A particularly nice class arises by taking the sequence of graphs Γ_t generalizing Figure 7: instead of 5 arcs in each hemisphere, take $2t - 1$ for any natural number t, with t arcs at the back meeting the equator in the left and right regions of the front. The top arc at the back meets the equator between the front t^{th}-and $(t+1)^{\text{st}}$-arcs numbered from the left. Observe that Γ_1 is a tetrahedron, whereas Γ_3 is the dodecahedron.

Proposition 4.3. *Each of the graphs Γ_t gives rise to an alternating fibred knot $K_t = \mathcal{L}_{\Gamma_t}$. The knot K_1 is the figure-eight knot, and K_3 is the dodecahedral knot D_f.*

Proof. The resulting link is fibred since the construction yields a plumbing of Hopf bands onto two sides of a disc, along the arcs of the graph. We invoke the results of Murasugi **[Mu]** and Stallings **[St]**, who show such links are fibred.

That the resulting link has one component is a simple induction on t, adding additional Hopf bands on either side of the middle edge of each side of the disc. □

5. Dehn surgeries

Every non-trivial Dehn surgery on K_t is determined by prescribing a Dehn surgery coefficient $\rho = (p, q) \neq \infty$. Denote the resulting 3-manifold by $M_{t,\rho}$.

Theorem 5.1. *For each $\rho \neq \infty$ and $t > 2$, $M_{t,\rho}$ is irreducible, has universal cover homeomorphic to $\mathbb{R}^3$, and contains an immersed π_1-injective surface satisfying the 4-plane, 1-line condition. Hence $M_{t,\rho}$ has homotopy type determined by its fundamental group.*

Sketch of Proof. Each of the trivalent polyhedra $\Pi^{\pm}_{\Gamma_t}$ has $2(2t-2)$ pentagonal faces, four $(t+2)$-gons, $(12t-6)$ edges and $(8t-4)$ vertices. Each polyhedron can be decomposed into $(8t-4)$ cubes in the standard manner (see **[AR1]** for example). After face identifications, all edges of $S^3 - K_t$ have degree $(t+2)$, 5, or 6. The former two values occur along introduced edges joining the centers of the polyhedra through points at the center of faces.

Consider a cube in the ideal cubing $\{4, 3, 6\}$ of $\mathbb{H}^3$, and bisect it symmetrically into 8 isometric subcubes by planes orthogonal at the centre, and orthogonal to the edges. Endow each of the cubes of $\Pi^{\pm}_{\Gamma_t}$ with the geometry of one of these subcubes, with the distinguished vertex at a vertex of $\Pi^{\pm}_{\Gamma_t}$. The resulting singular metric is complete, and has negative curvature at every point. The structure of the cusps is depicted in Figure 10, where there are $(16t-8)$ equilateral triangles in the decomposition of the torus. Such pictures occured originally in **[Th]**. Generators for the homology of the peripheral torus of the knot have been labelled. These are sufficiently long for $t \geq 3$ that any non-trivial Dehn surgery, in the sense of Gromov-Thurston (**[AR]**, **[GT]**) always yields a closed Cartan-Hadamard manifold with negative curvature along the core of the sewn-in solid torus, and with the metric away from the cusp remaining unaltered.

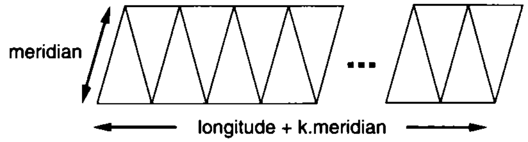

Figure 10

The immersed surface obtained by taking the union of squares bisecting each of the cubes of the decomposition of $S^3 - K_t$ is π_1-injective, being isotopic to a (singular) totally geodesic surface. Since this surface is in the 'thick' part of $S^3 - K_t$, it survives to produce an injective surface after surgery. This surface satisfies the conclusions of the theorem. For further details, see **[ALR]**, **[AR1]** and **[AR2]**. □

Remarks 5.2. The symmetric dodecahedral knot also belongs to an infinite family, obtained from the trivalent graphs $\mathcal{G}_k$, $k \geq 1$. These are obtained by drawing concentric $(k+1)$-gons in the plane, rotated relative to each other, and filling the annular region between them by $2k+2$ pentagons. The results on surgery also apply to this class, when $k > 2$. A similar argument applies to the cubical links described above.

The resulting closed 3-manifolds are not known to be virtually Haken.

Remark 5.3. The 14-sided polyhedron corresponding to $\mathcal{G}_5$ can be realized in hyperbolic space as the fundamental domain of the group action giving rise to Löbell's manifold **[Lö]**, the first closed hyperbolic 3-manifold to appear in the literature.

Acknowledgements

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Hyperbolic Cobordism and Conformal Structures

*Boris N. Apanasov**

Abstract In this paper, we briefly survey selected recent developments and present some new results in the area of uniformized conformal structures on a complete hyperbolic finite volume n-manifold (even closed) related to $(n+1)$-dimensional homology cobordisms with hyperbolic structures, especially, for the three-dimensional case.

1. Isometric and conformal group actions and maximal balls

Let $\mathbb{H}^n$ be the subspace

$$\{(x_0, \ldots, x_n) \in R^{n+1}: q(x_0, \ldots, x_n) = -x_0^2 + x_1^2 + \cdots + x_n^2 = -1\}$$

and $x_0 > 0$. The quadratic form q restricts to give a positive definite form on each tangent space of $\mathbb{H}^n$ and, consequently, endows $\mathbb{H}^n$ with a Riemannian metric. We call this Riemannian manifold the hyperbolic n-space. It has constant sectional curvature -1 and is homogeneous. Its isometry group is the real linear subgroup $SO(n,1)$ of matrices in $SL_{n+1}(R)$ preserving the form q and $\mathbb{H}^n$. A hyperbolic n-manifold M is a complete Riemannian manifold locally isometric on $\mathbb{H}^n$. In fact M is isometric to the quotient $\mathbb{H}^n/G_M$ where $G_M \cong \pi_1(M)$ is some discrete torsion free subgroup of $SO(n,1)$ determined by M up to conjugation in $SO(n,1)$.

The hyperbolic metric in $\mathbb{H}^n$ endows the $(n-1)$-sphere at infinity $\partial\mathbb{H}^n$ with a conformal structure where $SO(n,1)$ acts as the group of all conformal automorphisms of the sphere. Taking the Poincaré ball model of the hyperbolic n-space (in the unit ball $B^n(0,1)$), we have the isomorphism (see **[AP1]**):

$$\{\mathbb{H}^n,\ \partial\mathbb{H}^n,\ SO(n,1)\} \cong \{B^n,\ S^{n-1},\ \text{Möb}(n-1)\}$$

where $\text{Möb}(n-1)$ is the Möbius group generated by reflections in $(n-2)$-dimensional subspheres of S^{n-1}, the sphere S^{n-1} has the standard conformal structure induced by the Euclidean metric of $\mathbb{R}^n$ and the unit ball B^n has the Poincaré hyperbolic metric with the length differential

$$ds = \frac{2|dx|}{1-|x|^2}, \qquad x \in B^n(0,1).$$

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1.2. Now let us fix a torsion-free, finitely generated group G. An n-dimensional hyperbolic structure on G is determined by a pair $\{M, \psi\}$, where M is a hyperbolic n-manifold and $\psi: G \to \pi_1(M)$ is an isomorphism. We denote the set of all hyperbolic structures on G (defined up to isometries of hyperbolic manifolds and inner automorphisms on G) by $\mathcal{H}^n(G)$. This set is naturally identified with the set of conjugacy classes of faithful representations $\rho: G \to SO(n,1)$ with discrete image and therefore $H^n(G)$ has a natural topology, induce by the algebraic convergence topology on $\mathrm{Hom}(G, SO(n,1))$. Namely, representations are close if they are close on a finite generating set.

As an example, given a closed surface S_g of genus $g > 1$, $\mathcal{H}^2(\pi_1(S_g))$ is the Teichmüller space of $G = \pi_1(S_g)$ and is homeomorphic to $\mathbb{R}^{6g-6}$. On the other hand, given a closed (or finite volume) hyperbolic n-manifold M^n, $n > 2$, $\mathcal{H}^n(\pi_1(M^n))$ is a point due to the Mostow rigidity theorem **[MW]**.

Also there are Morgan-Shalen-Thurston's results which say that $\mathcal{H}^n(G)$ is compact for a wider class of groups G as a weak version of Mostow rigidity for infinite volume hyperbolic manifolds; see **[MO2]**. In fact, one shows

1.3. Theorem. *Let M^3 be a hyperbolic 3-manifold, and suppose that $G = \pi_1(M^3)$ is finitely generated. Then for any $n > 2$, the space $\mathcal{H}^n(G)$ is compact if and only if G does not have a decomposition of one of the following types:*
(i) *$G = A *_C B$ with C cyclic and of infinite index in A and B;*
(ii) *$G = A *_C$ with C cyclic.* □

Notice that it follows that $\mathcal{H}^n(\pi_1(M^3))$ is compact if and only if $\mathcal{H}^3(\pi_1(M^3))$ is compact. In particular, we have the following fact.

1.4. Corollary. *If M is a finite volume hyperbolic 3-manifold, then $\mathcal{H}^n(\pi_i(M)$ is compact for all $n > 2$.* □

1.5. Given a discrete group $G \subset \mathrm{Möb}(n-1)$, we define the Nielsen hull $H_G \subset \mathbb{H}^n \cup \partial\mathbb{H}^n$ as the minimal convex (in $\mathbb{H}^n$) set containing the limit set $L(G) \subset S^{n-1} = \partial\mathbb{H}^n$. Let $\rho: \mathbb{H}^n \cup \partial\mathbb{H}^n \to H_G$ be the G-equivariant retraction where, for $x \in \mathbb{H}^n \setminus H_G$, $\rho(x) \in \partial H_G$ is the point with shortest distance to x and, for $x \in \partial\mathbb{H}^n \setminus L(G)$, $\rho(x)$ is the first point of tangency with H_G of a horosphere in $\mathbb{H}^n$ with the center at x.

For a description of the boundary of the Nielsen hull H_G, we define (following **[AP5]**) the (strictly) maximal balls in the discontinuity set $\Omega(G) \subset S^{n-1}$. Namely, an open ball $B \subset \Omega(G)$ is called a (strictly) maximal ball if the sphere of lowest dimenision containing the limit subset $\partial B \cap L(G)$ is the sphere ∂B itself.

For the case $n = 3$, the discontinuity set $\Omega(G) \subset S^2$ of any finitely generated Kleinian group G (G is discrete with non-empty $\Omega(G)$) whose limit set $L(G)$ is not contained in a circle is covered by the family $\mathcal{B}(G)$ of strictly maximal discs, finite modulo G. For the case of geometrically finite quasi-Fuchsian groups $G \subset \mathrm{Möb}(n)$, an almost similar situation holds (see **[AP5]**). There we needed the following fact (**[AP5**, Th. 6.1**]**):

1.6. Theorem. *Let $G \subset \mathrm{Möb}(3)$ be a Kleinian group having at least three maximal balls $B_i \subset \Omega(G)$ with two common limit points $x, y \in \partial B_i$ and let $\mathrm{int}(\cap B_i) \neq \emptyset$. Then* *the boundary of the Nielsen hull is a pleated 3-surface in $\mathbb{H}^4$ with a conical singularity along the geodesic with the end points x, y. Its neighborhood in this surface is an union of dihedral angles with the sum of magnitudes less than 2π.* □

The standard notion of a pleated surface (cf. **[EM]**) arrives from the Krein-Milmann theorem on extreme points of a convex hull in Euclidean space (if we take the projective Klein model of $\mathbb{H}^n$ in the ball $B^n(0,1) \subset \mathbb{R}^n$). Here the pleating locus is a geodesic lamination (partial foliation) whose leaves of co-dimension one or more may be singular (for $n = 4$, as pages of an open book).

2. Uniformized conformal structures on manifolds

2.1. Given an n-manifold M, $n \geq 3$, by a conformal structure (conformally flat structure) on the manifold M we mean a $(S^n, \text{Möb}(n))$-structure on M, i.e., a structure locally modeled on the standard conformal structure of the n-sphere $S^n \cong \mathbb{R}^n \cup \{\infty\}$. In other words, a conformal structure is a maximal atlas on M with all changes of charts in a Möbius group $\text{Möb}(n)$. Extending chart by chart in the universal covering $\widetilde{M}$ of M, we obtain the developing map $d: \widetilde{M} \to S^n$ inducing the holonomy homomorphism $d^*: \pi_1(M) \to \text{Möb}(n)$.

A conformal structure c on M will be called a uniformized structure (compare **[KP]**) if its development d is not surjective while the holonomy group $G = d^*(\pi_1(M))$ acts discontinuously in the domain $\Omega_0 = d(\widetilde{M})$, i.e., G is a Kleinian group (see **[KM]**); here the manifold Ω_0/G with the natural conformal structure is conformally equivalent to the conformal manifold (M, c). Using the fundamental group $\pi_1(M)$ for the marking of conformal structures on M, we obtain the space $\mathcal{C}(M)$ of uniformized marked conformal structures on the manifold M.

2.2. Especially, for a finite volume hyperbolic manifold M, the space $\mathcal{C}(M)$ is naturally identified with the set of conjugacy classes of faithful representations

$$\rho: \pi_1(M) \longrightarrow SO(n+1, 1) \tag{2.1}$$

with discrete image which act discontinuously somewhere in the sphere at infinity $S^n = \partial\mathbb{H}^{n+1}$. Namely, if

$$\mathcal{R}(M) \subset \text{Hom}(\pi_1(M), SO(n+1, 1)) \tag{2.2}$$

is the subspace of such representations in the representation variety, the group $SO(n+1, 1)$ acts on the representation variety by conjugation leaving the subspace $\mathcal{R}(M)$ invariant. The quotient space

$$\mathcal{T}(M) = \mathcal{R}(M)/SO(n+1, 1) \tag{2.3}$$

is the desired space of conjugacy classes of representations (2.1) and is naturally identified with the space $\mathcal{C}(M)$ via the holonomy representation d_*; see **[LK]**, **[GM]**. This yields a topology on $\mathcal{C}(M)$ defined by the topology of algebraic convergence in the representation variety $\text{Hom}(\pi_1(M), SO(n+1, 1))$. Immediately from this description, the definition of the space $\mathcal{H}^n(G)$ and Corollary 1.4 in the case $n = 4$, we obtain

2.3. Theorem. *Let M be a finite volume hyperbolic 3-manifold. Then the space $\mathcal{C}(M)$ of uniformized marked conformal structures on M has a natural compactification $\overline{\mathcal{C}(M)}$ such that each of its points corresponds to a faithful representation ρ in the corresponding compactification $\overline{\mathcal{T}(M)}$ with discrete image of $\rho(\pi_1(M)) \subset$* Möb(3). □

2.4. The space $\mathcal{C}(M)$ contains an open subspace $\mathcal{C}_q(M)$ of quasi-Fuchsian structures on the manifold M which corresponds to an open subspace $\mathcal{T}_q(M) \subset \mathcal{T}(M)$ of quasi-Fuchsian representations, i.e., quasi-conformal conjugations

$$\rho: G \longrightarrow fGf^{-1} \subset \text{Möb}(3) \tag{2.4}$$

where $\pi_1(M) \cong G \subset \text{Isom}\mathbb{H}^3 \subset \text{Isom}\mathbb{H}^4 \cong \text{Möb}(3)$ and $f: S^3 \longrightarrow S^3$ is a quasi-conformal automorphism of the sphere S^3 compatible with the action of G. This fact follows from Sullivan's stability theorem **[SU3]**; see also **[JM]**.

The first results to obtain some boundary points of the space $\mathcal{C}(M)$ as end points of smooth curves in the open subspace $\mathcal{C}_q(M)$, for the case of a closed manifold M, was Theorem B and Corollary 5.2 in **[AP2]**. These boundary points are similar to cusps on the boundary of Teichmüller space $\mathcal{T}(S_g)$ of Riemann surfaces of genus $g > 1$ (they correspond to so-called accidental parabolic elements in the holonomy groups; see **[BR]**) and were obtained as limits of bending deformations of the distinguished conformal (hyperbolic) structure on the manifold M. Here a bending deformation of the manifold $M = \mathbb{H}^n/G$, $G \subset \text{Isom}\mathbb{H}^n$, gives conformal structures $c_{\text{bend}} \in \mathcal{C}_q(M)$ which correspond to singular hyperbolic structures on M obtained by bending of M along a totally geodesic hypersurface through some angles. In fact, such a singular n-structure on M has a pleated n-plane $H_{\text{bend}} \subset \mathbb{H}^{n+1}$ as its universal covering and the structure c_{bend} corresponds to a conformal n-manifold $\Omega_0/d^*_{\text{bend}}(\pi_1(M))$ where the domain $\Omega_0 \subset \mathbb{H}^{n+1} = S^n$ is an invariant component of the holonomy group $d^*_{\text{bend}}(\pi_1(M))$ spanned on the $(n-1)$-sphere at infinity of the n-cell H_{bend}, $\partial H_{\text{bend}} = \partial\Omega_0$; see **[AP4]**, **[JM]** and **[KR]** for details. Here we give only the following well-known property of such bending structures $c_{\text{bend}} \in C_q(M)$ (see **[AP5]**):

The boundary of the Nielsen hull for the holonomy group

$$d^*_{\text{bend}}(\pi_1(M)) \subset \text{Möb}(n) \tag{2.5}$$

can be isometrically developed in the hyperbolic n-plane $\mathbb{H}^n \subset \mathbb{H}^{n+1}$.

Except bending structures in $\mathcal{C}_q(M)$, now for a 3-manifold M, only stamping structures $c_{\text{stamp}} \in C_q(M)$ obtained by stamping deformations of the distinguished structure on M along a geodesic ℓ (either with torsion around ℓ or not) are known; for definitions and details see **[AP6]**. For totally geodesic surfaces $S_i \subset M$ intersecting along the geodesic $\ell \subset M$, such a stamping deformation gives compatible simultaneous bendings along these surfaces S_i and some compression along the geodesic ℓ. The type of difference between bending and stamping structures on M is described by the conic singularity of the boundary of the Nielsen hull for the holonomy group of c_{stamp}, $d^*_{\text{stamp}}(\pi_1(M))$ —like in Theorem 1.6.

Now we formulate the following three related problems about the space $\mathcal{C}(M)$ of uniformized conformal structures (see also **[AP6**, §5**]** and **[AP8]**):

2.5. Problems. *(A) Is the space $\mathcal{C}(M)$ compact, especially for a closed manifold M? If not, what kinds of boundary points distinct from cusps are there?*

(B) Do all points of the subspace $\mathcal{C}_q(M) \subset \mathcal{C}(M)$ of quasi-Fuchsian structures become exhausted by structures obtained by bending, stamping, and stamping-with-torsion deformations of the distinguished structure on M, especially for the three-dimensional case?

(C) Is the space $\mathcal{C}(M)$ of uniformized conformal structures a connected space?

Below, we will give some advances in this direction.

2.6. Theorem. *The space $\mathcal{C}(M)$ of uniformized conformal structures on a closed hyperbolic n-manifold M, $n > 2$, with two distinct totally geodesic hypersurfaces is non-compact.*

Proof. Actually, we shall show that a limit structure on M obtained by a bending deformation of M in opposite directions along distinct totally geodesic hypersurfaces on M does not belong to $\mathcal{C}(M)$ (this is also true for $n = 2$). Let us consider the closed hyperbolic n-manifold M constructed in **[AP2**, Theorem A], i.e., a manifold with two disjoint totally geodesic hypersurfaces. As in Corollary 5.2 in **[AP2]**, by a similar way, we obtain a smooth curve

$$\beta\colon [0,1) \longrightarrow \mathcal{C}_q(M) \tag{2.6}$$

such that its end point, $\beta(1)$, is not contained in the space $\mathcal{C}(M)$ because its holonomy group $G(1)$ has the following properties:

(i) $G(1)$ is a Kleinian group in S^3 with the union of two invariant non-contractible components, Ω_0 and Ω_1, as the discontinuity set $\Omega(G)$.

(ii) The non-contractibility of these components is related to the existence of accidental parabolic elements in the group $G(1)$ which correspond to a loxodromic one in the hyperbolic co-compact group $d^*_{\beta(0)}(\pi_1(M)) \subset \mathrm{Isom}\mathbb{H}^3$, i.e., the holonomy group for the distinguished (hyperbolic) structure on M.

(iii) The quotients $\Omega_0/G(1)$ and $\Omega_1/G(1)$ are non-compact manifolds whose (cusp) ends are conformally equivalent to ends of manifolds $\mathbb{R}^3/\Gamma$, where Γ is some discrete cyclic group of Euclidean isometries.

Therefore, for the closed hyperbolic n-manifold M, the last non-compactness property (iii) contradicts the inclusion $\beta(1) \in \mathcal{C}(M)$. This completes the proof. □

2.7. Remarks. The point $\beta(1)$ is a boundary cusp point in the sense of L. Bers **[BR]**. So the second part of the question (A) is still open. In particular, the existence of spatial degenerate groups (compare **[BR]**) is unknown.

For question (B), an expected answer will likely be positive, since the above circumscribed bending and stamping structures $c \in \mathcal{C}_q(M)$ can give any possible local behavior of the limit set (= quasisphere in S^3) for holonomy groups; see **[AP5]**, **[AP6**, §5], **[AP9]**.

The following theorem gives the advance in the problem (C).

2.8. Theorem. *On a closed hyperbolic 3-manifold M with a number $N > 70$ of disjoint totally geodesic surfaces, there exists an exotic uniformized conformal structure c^* that* *can not be approximated by bending, stamping, and stamping-with-torsion structures on* M. □

The proof of this fact is obtained in **[AP8]** where the exotic conformal structure $c^* \in \mathcal{C}(M)$ is obtained as the result of some modification of the author's Block-Building Construction for Kleinian groups in S^3 with wildly (even locally wildly) embedded 2-spheres as the limit sets. This Block-Building method has been developed in **[AT]** and **[AP7]**. Also we remark that an obstruction for approximation of c^* by bending and stamping structures on M is the property of the covering by a family $\mathcal{B}(G)$ of strictly maximal balls of the discontinuity set for the obtained exotic holonomy group $G^* \subset$ Möb(3) described as the condition of Theorem 1.6. As a result, it gives a conic singularity of the boundary of Nielsen hull H_{G^*} in $\mathbb{H}^4$.

3. Four-dimensional cobordisms with hyperbolic structures

3.1. Let us fix some closed (for simplicity) hyperbolic 3-manifold M and consider the space $\mathcal{W}(M)$ of all 4-dimensional cobordisms $(W; N_0, N_1)$, $\partial W = N_0 \cup N_1$, with the following properties:

(i) W is a geometrically finite hyperbolic cobordism: $\mathrm{int}(W)$ has a complete geometrically finite hyperbolic structure, i.e., there is a decomposition of $\mathrm{int}(W)$ into a cell by means of cutting along a finite set of totally geodesic hypersurfaces; see **[AP3**, Ch. 5**]**.

(ii) W is a homology cobordism: for its boundary components, N_0 and N_1, the relative homology groups are trivial:

$$H_*(W, N_0) = H_*(W, N_1) = 0. \tag{3.1}$$

(iii) The first boundary component of W, N_0, is homeomorphic to the closed hyperbolic manifold M and its inclusion $N_0 \subset W$ induces the homotopy equivalence:

$$\pi_*(W, N_0) = 0. \tag{3.2}$$

We note that due to A. Marden's results **[MD]**, the similar space of 3-dimensional hyperbolic cobordisms consists only of trivial cobordisms that are homeomorphic to the product of a closed surface S_g of genus $g > 1$ and a segment. After usual marking by the fundamental group, its quotient under the homotopy equivalence is isomorphic to $T(S_g) \times T(S_g)$ where $T(S_g)$ is the Teichmüller space for S_g —see Bers decomposition theory **[BR]**.

What kind of properties are there for the space $\mathcal{W}(M)$?

First, each uniformized conformal structure $c \in \mathcal{C}(M)$ with geometrically finite holonomy group $G \subset$ Möb(3) having an invariant contractible component of the discontinuity set $\Omega(G)$ corresponds to a cobordism $W \in \mathcal{W}(M)$, $\mathrm{int}W \approx \mathbb{H}^4/G$, where $M(G)$ may be non-compact, $M(G) \neq W$; see **[AT**, Theorem 3.2 and Corollary 3.3**]**, **[TE1]**, **[TE2]**.

Second, the following converse statement holds:

3.2. Theorem. *Let* M *be a closed hyperbolic 3-manifold. Then for every cobordism* $(W; N_0, N_1) \in \mathcal{W}(M)$, *there is a Kleinian group* $G \subset$ Möb(3) *with an invariant* *contractible component* Ω_0 *of the discontinuity set* $\Omega(G)$ *such that either the Kleinian manifold*

$$M(G) = [\mathbb{H}^4 \cup \Omega(G)]/G \tag{3.3}$$

is the manifold $W \in \mathcal{W}(M)$ *itself (with* $N_0 = \Omega_0/G$ *and* $N_1 = [\Omega(G)\backslash\Omega_0]/G$*) or the manifold* W *is obtained from a non-compact manifold* $M(G)$ *by the natural compactification of a finite number of its cusp-ends. These cusp-ends are homeomorphic to the product of the strip* $[0,1] \times [0,\infty)$ *and either the cylinder* $S^1 \times [0,1]$, *or the Möbius band.*

Proof. Firstly, the condition (i) for W gives us a discrete action $G \subset \mathrm{Isom}\mathbb{H}^4$ of the fundamental group $\pi_1(W)$. Moreover, this action is discontinuous on the sphere at infinity and its discontinuity set $\Omega(G)$ has a G-invariant contractible component $\Omega_0 \subset \Omega(G)$, due to the condition (iii) for W; see **[AP3**, Ch. 7**]**. Applying this fact and the Tetenov's finiteness theorem (see **[TE1]**, **[TE2**, Theorem 2**]**, or **[AT**, Theorem 3.2 and Corollary 3.3**]**), we complete the proof. □

3.3. Remark. Let $\mathcal{W}_{\mathrm{triv}}(M) \subset \mathcal{W}(M)$ be a set of cobordisms $W \in \mathcal{W}(M)$ for which the following additional condition holds:
(iv) W is homeomorphic to the product $N_0 \times [0,1]$ where $N_0 \subset \partial W$.

For the correspondence between $\mathcal{W}(M)$ and $\mathcal{C}(M)$, i.e., for the holonomy group $G \subset \mathrm{Möb}(3)$ related to a cobordism $W \in \mathcal{W}_{\mathrm{triv}}(M)$, we have that $M(G)$ may be non-compact and non-homeomorphic to W. This is realized for boundary (cusp) points of $\mathcal{C}_q(M)$ related to Kleinan groups with accidental parabolic elements — as in Theorem B in **[AP2]**. However, in the compact case, the following is true:

The holonomy group $G \subset \mathrm{Möb}(3)$ *without cusps related (by Theorem 3.2) to the trivial cobordism* $W \in \mathcal{W}_{\mathrm{triv}}(M)$ *is a quasi-Fuchsian group conjugated by a quasisymmetric embedding*

$$f\colon S^2 = \partial\mathbb{H}^3 \hookrightarrow S^3$$

with the Fuchsian group $\Gamma \subset \mathrm{Isom}\mathbb{H}^3$, $M = \mathbb{H}^3/\Gamma$, *and the limit set* $L(G)$ *is a quasisphere, i.e.,* f *is the restriction to* $S^2 \subset S^3$ *of a quasiconformal automorphism of the sphere* S^3.

This fact is a consequence of a deep result of D. Sullivan **[SU2]*** which is based on the following two conditions:
(1) local unknottedness of the limit set $L(G)$ of a Kleinian group G with two invariant components of the discontinuity set (this follows from the condition (iv) on the cobordism W);
(2) the uniform self-similarity condition for the limit set $L(G)$.

This self-similarity condition says that there exists a uniform constant $K > 0$ such that, for any point $x \in L(G)$ and any small ball $B(x,r)$ with radius $r > 0$ centered at x, there exists a K-quasi-isometry F_r which maps the set

$$h(B(x,r) \cup L(G)), \quad \text{for } h\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3\,, \;\; h(y) = x + (y-x)/r\,,$$

* For an independent approach see **[MT**, Corollary 5.9**]**.

into the limit set $L(G)$.

In particular, this condition holds for Kleinian groups with the limit sets consisting of approximation points **[SU1]**. On the other hand, the exhaustion of the present limit set $L(G)$ by the points of approximation of the group G follows from our conditions on W. Namely, the geometrical finiteness of the group G gives that the limit set $L(G)$ contains the approximation points and parabolic cusp points only (**[BM]** and **[AP3**, Ch. 5**]**); the absence in $L(G)$ of parabolic fixed points is obtained from the compactness condition. This completes the proof. □

3.4. In contrast to the 3-dimensional case, the subset $\mathcal{W}(M)\backslash\mathcal{W}_{\mathrm{triv}}(M)$ of nontrivial 4-dimensional cobordisms $W \in \mathcal{W}(M)$ is non-empty; specifically, for a closed hyperbolic 3-manifold M with a big number of disjoint totally geodesic surfaces. This fact follows from our construction in **[AT**, Theorem 5.1**]**.

Moreover, it is likely true that there exists a 1-1 correspondence between this subset $\mathcal{W}(M)\backslash\mathcal{W}_{\mathrm{triv}}(M)$ of non-trivial cobordisms and the subset $C(M)\backslash\overline{C_q(M)}$—compare Theorem 2.8.

It is interesting to correlate this conjecture with the following three points of view:

First, with the special case of S. P. Novikov's conjecture on triviality of h-cobordisms of the type $K(\pi, 1)$ obtained as the quotients

$$W = M(G) = [\mathbb{H}^4 \cup \Omega(G)]/G$$

where $\Omega(G)$ is the union of two G-invariant contractible components Ω_0 and Ω_1, $\Omega_0/G \approx M$; see **[AT**, p. 408**]**.

Second, with the theorem of F. T. Farrell and L. E. Jones **[FJ]** about K-flatness of the fundamental group $\pi_1(M)$ of a closed hyperbolic 3-manifold M, in particular.

Here a group $G = \pi_1(M)$ is called K-flat if the Whitehead group $\mathrm{Wh}(G \times C^n)$ of any group $G \times C^n$, $n \geq 0$, is trivial (C^n denotes the free abelian group of rank n). Note that $G \times C^0$ is isomorphic to G itself, $\widetilde{K}_0(\mathbb{Z}G)$ is a direct summand of $\mathrm{Wh}(G \times C^1)$ and, for $n > 0$, $K_{-n}(\mathbb{Z}G)$ is a direct summand of $\widetilde{K}_0(\mathbb{Z}(G \times C^n))$. Therefore they are trivial for a K-flat group G; see **[FJ]**.

Third, with the fact that, for the closed hyperbolic manifold $M = \mathbb{H}^3/\Gamma$ from Theorem 2.8, the Chern-Simons and η-invariants for an exotic uniformized conformal structure $c^* \in \mathcal{C}(M)$ on M are the same as for a complete hyperbolic structure on M, namely, they vanish (see **[AP9]**).

Here the Chern-Simons invariant and the η-invariant for (M, c^*) are computed in a special metric on M (we call this metric a "Kobayashi conformal metric") which induces the structure c^*. In fact, this metric corresponds to the conformally invariant metric $k(*, *)$ in the invariant contractible component $\Omega_0 \subset \Omega(G) \subset S^3$ of the holonomy group $G \subset \mathrm{M\ddot{o}b}(3)$ for $c^* \in \mathcal{C}(M)$. For any points $x, y \in \Omega_0$, the Kobayashi conformally invariant metric is defined as follows:

$$k(x, y) = \inf\left\{\sum_i d_i(x_{i-1}, x_i)\right\} \tag{3.4}$$

where we take inf over all conformal chains, i.e., couples $(x_0 = x, x_1, \ldots, x_n = y)$ of points in Ω_0 and conformal embeddings $f_i\colon B \hookrightarrow B_i \subset \Omega_0$ of the open ball $B =$

$B^3(0,1) \subset \mathbb{R}^3$ such that $x_{i-1}, x_i \in B_i = f_i(B)$ where $d_i(*, *)$ is the Poincaré hyperbolic metric in the i-th ball $B_i \subset \Omega_0$.

Details and the proof of vanishing for the Chern-Simons invariant and the η-invariant are related to the maximal ball cover of the discontinuity set component $\Omega_0 \subset \Omega(G)$ (see **[AP9]**).

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