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For Jindra, Veronica, and Peter

Preface

> Intellectual adventure is nowadays best defined as treating respectfully that which was accepted as a truism only a few generations ago.
>
> William F. Buckley, Jr

Fate has not been kind to Gottlob Frege and his work. His logical achievement, which dwarfed anything done by logicians over the preceding two thousand years, remained all but ignored by his contemporaries. He liberated logic from the straight-jacket of psychologism only to see others claim credit for it. He expounded his theory in a monumental two-volume work, only to find an insidious error in the very foundations of the system. He successfully challenged the rise of Hilbert-style formalism in logic only to see everybody follow in the footsteps of those who had lost the argument.

Ideas can live with lack of recognition. Even ignored and rejected, they are still there ready to engage the minds of those who find their own way to them. They are in danger of obliteration, however, if they are enlisted to serve conceptions and purposes incompatible with them.

This is what has been happening to Frege's theoretical bequest in recent decades. Frege has become, belatedly, something of a philosophical hero. But those who have elevated him to this status are the intellectual heirs of Frege's Hilbertian adversaries, hostile to all the main principles underlying Frege's philosophy. They are hostile to Frege's platonism, the view that over and above material objects, there are also functions, concepts, truth-values, and thoughts. They are hostile to Frege's realism, the idea that thoughts are independent of their expression in any language and that each of them is true or false in its own right. They are hostile to the view that logic, just like arithmetic and geometry, treats of a specific range of extra-linguistic entities given prior to any axiomatization, and that of two alternative logics—as of two alternative geometries—only one can be correct. And they are no less hostile to Frege's view that the purpose of inference is to enhance our knowledge and that it therefore makes little sense to infer conclusions from premises which are not known to be true.

We thus see Frege lionized by exponents of a directly opposing theoretical outlook. Theorists whose unavowed view it must be that the celebrated master got hardly anything right, nevertheless claim inspiration from him. (The best known contemporary Frege scholar is, in his spare time, a dedicated advocate of antirealism and intuitionistic logic, both of which would have been anathema to Frege.) G.P.Baker and P.M.S.Hacker in their recent monograph rightly point to the oddity of this situation and highlight the gulf separating Frege's doctrine from what they call 'the wisdom of the 20th century.'

The logical wisdom of this century consists, in a nutshell, in trading things for symbols. It consists in relinquishing thoughts in favour of sentences, logical objects in favour of connectives and operators, and truth in favour of derivability from axioms. Baker and Hacker rightly argue that Frege is not the originator of this approach, and that modern logic is the result of logicians' turning their backs on Frege, rather than following him.

Twentieth-century logicians turned away from Frege not because they refuted his arguments, but because they decided to ignore them. (As George Santayana once remarked, we no longer *refute* our predecessors, we simply wave them good-bye.) Word somehow got around that looking at linguistic expressions (that is, at strings of typographical characters) is more illuminating than looking at what they represent. A new paradigm arose; and paradigms, of course, do not assert themselves through rational argument but through intellectual stampede.

I beg to be excused from joining the stampede called symbolic logic. Turning logic into the study of an artificial language (which nobody speaks) does not strike me as the height of wisdom. A formula of symbolic logic, just like a piece of musical notation, is utterly uninteresting in its own right. Its interest stems exclusively from its ability to represent something other than itself. But if so, it is difficult to see what advantage can come from focusing on the formulas in preference to, and to the exclusion of, what is represented by them. For if the formulas are perspicuous then what they represent cannot be more complex, or more difficult to handle, than the formulas themselves. The disadvantage of the approach, on the other hand, is obvious: once the entities represented by the formulas are lost sight of, they cannot be quantified over. Nor can such quantification be mimicked by quantifying over the corresponding formulas. The enterprise of logic (and mathematics) is thus radically stunted.

Both Frege and Russell took, inconsistencies notwithstanding, an objectual view of logic. They both devised and *used* ingenious symbolic languages, whose various modifications were to become the stock in trade of symbolic logic. Yet they themselves were not symbolic logicians; a symbolism to them was not the *subject matter* of their theorizing but a mere shorthand facilitating discussion of extra-linguistic entities.

The theories of Frege and Russell are far from 'noble ruins', interesting only from an historical point of view. They are, rather, the most advanced theories of objectual logic we have. Those who believe that there is more to logic than the study of finite strings of letters, have to go back to where Frege and Russell left off and go on from there.

This, at any rate, is what I propose to do in this book. I shall assume that in its general thrust and philosophical underpinnings Frege's doctrine is sound. As it stands, however, Frege's theory *is* seriously flawed. I do not mean just the formal inconsistency discovered by Russell in *Grundgesetze*. There are serious inconsistencies and ambiguities in the very ontological and semantic foundations of the system. Hence a great deal of the exegesis which follows will have to be critical. Remedy, however, will not be sought in the conventional linguistic approach, but in the direction of Russell's Ramified Theory of Types. It will be sought, in other words, within the broad objectual research programme that Frege's theory was part of.

Russell's logic suffers from ambivalence no less than Frege's does. What is more, the ambivalence has the same main source: a failure to devise a viable objectual account of the variable. It was this failure which forced both authors to deviate in crucial points from their own objectual approach and to resort to linguistic ascent. And it is these deviations which provide the present-day advocates of the linguistic approach with an excuse for claiming Frege and Russell as their spiritual forefathers.

It is one of the aims of the present work to propose a non-linguistic theory of the variable and to give a consistently objectual version of Russell's Ramified Theory of Types. I will argue that the 'hierarchy of entities' which results from this rectification of Russell's system is not only a useful tool for diagnosing the flaws and ambiguities in Frege's logic but also the right medium for modelling our whole conceptual scheme.

March 1988 Pavel Tichý

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Pavel Tichý

Contents

Chapter One: Constructions

1. Entities, constructions, and functions

When one travels from Los Angeles to New York, going, say, by way of St. Louis, Chicago, and St. Louis again, one's *destination* and the *itinerary* one follows to get there are clearly two distinct items. There is no sense in which Los Angeles, St. Louis, or Chicago are parts, or constituents, of New York. Each of the three cities, on the other hand, is an inalienable constituent of the circuitous itinerary in question, and the removal of any of them produces a different itinerary. An itinerary is a compound in which a number of locations occur, some of them possibly more than once, as St. Louis does in our example.

An arithmetical calculation is much like an itinerary. When one multiplies two by two and subtracts three from the result, one makes an intellectual journey whose destination is the number one. This number is no more to be confused with the calculation that yields it than New York is with any particular route leading to it. There is no sense in which the numbers two and three or the operations of multiplication and subtraction are parts, or constituents, of the number one. Each of them, on the other hand, is an inalienable part of the calculation, or, as I shall also say, *construction*, in question. A numerical construction is a compound in which several numbers and operations occur, some of them possibly more than once, as two does in our example.

Arithmetical expressions represent, or depict, constructions. The construction consisting in multiplying two by itself and subtracting three from the result, for example, finds its linguistic representation in the term '(2.2)–3'. The primitive symbols '2', '3', '.', and '–' represent the primitive constituents of the construction (namely the numbers two and three and the operations of multiplication and subtraction respectively) and the way the symbols are arranged into the term is exactly parallel to the way those numbers and operations are organized into the construction. For example, the two occurrences of the numeral '2' in the expression correspond to the two appearances of the number two in the construction, and the parentheses group the symbols the way the corresponding entities are grouped in the construction: they indicate that the multiplication of two by two is a self-contained stage of the construction, which (say) the subtraction of three from two is not.

The number two, which occurs twice in our model construction, can be removed from it and replaced by another number. If it is replaced by three we get the construction which consists in multiplying three by three and subtracting three from the result; if it is replaced by four we get the construction which consists in multiplying four by four and subtracting three from the result, and so on. The part which all these constructions have in common is an *incomplete* construction, a constructional torso as it were. It will be convenient to speak of it, occasionally, as the construction consisting in multiplying an *unspecified number* by itself and subtracting three from the result. This *façon de parler*, however, is not to be taken literally. It is not meant to imply that apart from specific numbers there are also unspecified ones. It is just a short way of saying that the construction is incomplete in the sense of containing a gap, and that a complete construction can be obtained from it by filling the gap with any arbitrary (but every time perfectly specific) number.

Incomplete constructions can also be represented linguistically. It is usual to set certain letters aside and use them exclusively as indicators of constructional gaps. Using 'x' in this way we can represent the incomplete construction discussed in the foregoing paragraph by the expression '$(x.x)–3$'.

Every time the gap in that incomplete construction is filled by a particular number the construction which is obtained yields a unique number. When it is filled by two, three, four etc. the resulting complete constructions yield, respectively, the numbers one, six, thirteen etc. The incomplete construction thus induces, or effects, a specific association between numbers, whereby the number one, for example, is associated with itself, three with six, four with thirteen, and so on.

We have noted that the gapless construction consisting in multiplying two by itself and subtracting three from the result is not to be confused with the number, one, produced by it. Quite analogously the gappy construction consisting in multiplying an unspecified number with itself and subtracting three from the result, is not to be confused with the association induced by it. For the very same association is induced by many other incomplete constructions, for example, by the one depicted by the term '$(x+x^2)–(3+x)$'. The two constructions are unmistakably distinct: the gap (or 'unspecified number'), for instance, appears twice in the first and thrice in the second. But the association they effect is one and the same.

What I have been calling an association between numbers goes in modern mathematics by the name of *function*. This use of the term 'function', however, is relatively recent. To those who first used it in a mathematical context, a function was more of a mathematical *formula* containing 'variable' *letters* like 'x' and 'y'. Bernoulli, for example, regarded a function as an expression made up of a variable and some constants, and Euler as any equation or formula involving variables and constants.

But despite the way they expressed themselves, it is unlikely that these early function theorists would have insisted that a function is literally nothing but a string of letters, such as the string consisting of '(', 'x', '.', 'x', ')', '–', and '3'. In the mind of the eighteenth-century mathematician an expression was not yet sharply separated from its significance, from what it represents. Now what is directly signified or represented by an expression like '$(x.x)$–3' is clearly a calculation scheme—an incomplete construction consisting of a sequence of operations which can be carried out starting with any arbitrary number. If asked directly, Bernoulli and Euler would undoubtedly have equated the function with the calculation scheme and agreed that the seven-term string of symbols is just a way of representing the calculation diagrammatically on a piece of paper. But they would have deemed the questioner something of a scholastic who harps on a difference that does not really make any difference. In view of the perfect isomorphism between a calculation and the formula which represents it, it matters little whether one takes himself to be concerned with one or the other in discussing mathematics. A musician would be equally impatient with someone who made heavy weather of the difference between a piece of musical notation and the sound structure it represents. The piece of notation and the sound structure are, to be sure, two different things, but because they are perfectly isomorphic little harm can come from failing to keep them strictly apart in discussing music. It does not matter much whether by 'note' or even 'music' one means something that impinges on the eardrums or something that is written on a sheet of paper.

It is thus not far from the truth to say that originally functions were understood as particular ways or methods of proceeding from numbers to numbers, i.e., as incomplete numerical constructions.

It is often claimed that the modern notion of function has developed from this original one by gradual generalization. This, however, is a misleading way to tell the story. It is true enough that the early theorists thought of functions as calculation schemata involving only a limited range of basic arithmetical operations—addition, multiplication, and a few others—and that the range was later gradually broadened. But the introduction of the modern notion of function was not just one more step in this liberalizing process. It was rather a clean break with the underlying idea that a function is a particular *method* of getting from arguments to the corresponding values. A function was redefined as the *correspondence* between numbers which may be induced by such a method. But since *any combinatorically possible* correspondence counts as a function in the modern sense—whether or not an acceptable method of getting from the arguments to the values is known or even exists—the question as to what kinds of computational steps are admissible does not even arise.

In order to properly grasp the modern notion of function one must keep it strictly apart from the notion of schematic calculation. Although a function is often defined by means of a specific method for calculating its values from its arguments, one must always remember that the method is extraneous to the function itself. Just as the number (one) which is generated by the complete calculation (2.2)–3 bears no traces of this particular way of calculating it, so the function induced by the schematic construction $(x.x)$–3 bears no traces of this particular way of inducing it. Since the very same function is induced by many other incomplete constructions (for example, by the one symbolized by '$(x+x^2)-(3+x)$') no *particular* calculating method can be recovered from the function as such. Briefly, functions in the modern sense are individuated *extensionally*: functions which associate the same values with the same arguments are *identical*. Nothing of the sort is true of incomplete constructions.

2. Two views of arithmetic

There are two possible views one can take on the role of constructions in arithmetical discourse; I shall call them View A and View B.

On View B the proper subject matter of arithmetic is numbers and numerical functions (in the modern sense). Complete and incomplete constructions may perhaps serve to *pick out* individual numbers and functions, but they are not what the arithmetician's statements are *about*. The term '(2.2)–3' is simply a name of the number one. It may name it *through*, or *by means of*, a certain construction, the construction which finds its representation in the syntax of the term. But when the arithmetician uses the term it is not the construction that he refers to but the number (one) which the construction produces. An expression like '$(x.x)$–3' is, on View B, an incomplete number name. It contains a syntactic gap and therefore names, as such, nothing at all. But when a definite numeral fills the gap the result names a definite number. In this sense, the term '$(x.x)$–3' specifies, or indicates, a definite function. The function may be specified *through*, or *by means of*, a certain incomplete construction, the construction which is depicted in the syntax of the gappy term. But again, when the arithmetician uses the term, he is talking not of the incomplete construction but at best of the corresponding function. This is what I will call View B.

On View A arithmetical constructions constitute the proper subject matter of arithmetic. They are not just *tools* but *targets* of the arithmetician's reference. The term '(2.2)–3' is a name of the construction, or calculation, which consists in multiplying two by two and subtracting three from the result. When the mathematician uses the term it is this construction he wants to tell us something about, not the number to which it leads us. An expression like '$(x.x)$–3' is, on View A, not an incomplete number name but a complete name of an incomplete construction, of a calculation scheme. The letter 'x' occurring in it does not constitute a syntactic gap, but serves as a name of the gap in the construction. The gappy construction specifies, of course, a definite function (mapping), but when the arithmetician uses the term it is not the *function* he wants to discuss but the particular incomplete *construction* by which the function is induced.

Which of the two views is correct?

Common sense, for what it is worth, is undoubtedly on the side of View A. When told that (2.2)–3 is odd the layman will not take himself to be receiving information about the number one, which he may well know to be odd already. He will naturally assume that he is being told something about the calculation, namely that it yields an odd number, whatever particular number that may be. If he finds the piece of information interesting at all, it will be because of its labour-saving potential: should anything in the future hang on whether the calculation yields an odd number, he will be spared the trouble of carrying the calculation out to see which particular number it produces. Similarly, when he is informed that (2.2)–3 equals (8–6)/2, he will construe the informer as trying to draw his attention not to the self-identity of the number one but to the congruence of two constructions, to the fact that they yield one and the same number (whatever particular number that may be). Finally, when informed that $(x+y).(x-y)$ always equals x^2-y^2, he will construe the informer as trying to draw his attention not to the self-identity of a certain parabolic function but to the congruence of two schematic calculations. From then on he will know that instead of multiplying the sum of two numbers with their balance he can subtract the square of the second number from the square of the first, and the result will be the same. He will take himself to have learned something about two numerical constructions.

3. The linguistic turn

Modern semantic theory deems all these judgments naive and wrong. It urges us to ignore common sense and look on '(2.2)–3' as a name of a number and on '$(x.x)-3$' either as a name of nothing at all, or as a way of indicating a function (in the modern sense of the term). View A has been rejected so radically that it is not even considered worth arguing against.

The reasons for this rejection are largely ideological. Modern philosophy is agitated by a passion for ontological parsimony and by the prejudice that looking at linguistic expressions is more enlightening than looking at what they represent. Now an arithmetical formula, as we have seen, is isomorphic to the construction it represents. For many purposes, therefore, constructions can be studied indirectly by looking at the formulas themselves. This, to a parsimoniously inclined semanticist, is reason enough to apply Occam's razor and disown constructions altogether. He may perhaps concede that each of the symbols '2', '.', '–', and '3' severally represents a mathematical entity, but he will deny that the formula as a whole might represent any sort of a structure in which those objects are organized analogously to the way the corresponding symbols are organized syntactically into the expression '(2.2)–3'. The notion of calculation has dropped, in fact, from the ontology of modern metamathematics altogether.

The modern semanticist's approach is thus a version of what I have called View B. For him, the only targets of arithmetical reference are numbers and numerical functions. But he goes beyond View B in jettisoning the notion of construction altogether and relating numbers and functions directly to formulas, i.e., to linguistic expressions. Indeed, he considers it an advantage of his approach that it postulates no intermediary between '(2.2)–3' and the number one, or between '(*x*.*x*)–3' and the corresponding parabolic function. A whole category of entities can thus be disowned.

This ontological thriftiness, however, is not cost free.

One price to be paid is explanatory power. The modern semanticist of mathematical discourse is, in this respect, in the same position as the philosophical nominalist. Having disowned attributes, the nominalist, has no real answer to the question *why* the predicate 'white' applies to snow. All he has to say, by way of explanation, is that those who speak English *choose* to apply it to the stuff. But if the applicability of colour words is a matter of linguistic choice, as the nominalist suggests, why is it necessary to send space probes to Jupiter in order to determine whether the sentence 'Callisto is white' is true? Why don't we simply choose to call the satellite white, or red, or whatever?

The obvious truth of the matter is that the users of English do not make choices of this sort at all. They have never *decided* that snow should be called 'white', and they would not dream of making any such decision regarding Callisto. What they have agreed upon is that 'white' shall signify a certain *colour*. Now that this agreement has been made, the word owes its applicability to snow and non-applicability to Callisto to the brute *facts* that snow is and Callisto is not of that colour, facts which have nothing to do with language. The philosopher who takes pride in having purged his theoretical world of colours and other attributes must disagree. He must deny that things are called 'white' because they are white and insist with Nelson Goodman that 'things are white because they are so-called.'[1]

The modern semanticist of mathematics, who has purged his ontology of constructions, offers a similar answer to the question why '(2.2)–3' stands for the number one or '(2.2)–3=1' for (the truth-value) truth. All he has to say is that it has been so decided by the creators of arithmetese. The fact that the arithmetician knows how to devise recursive procedures which generate, in a uniform way, infinitely many decisions of this sort, does not make the explanation any more substantive or illuminating. Besides, it is well known that no such procedure can generate all the decisions that would need to be made. By Gödel's Incompleteness Theorem, for any effective procedure of this sort there will always be arithmetical expressions with respect to which the procedure yields no decision at all. And since there is no such thing as an absolutely undecidable formula, to every formula there must correspond

[1] Goodman[1971], p. 348.

something which *determines* what number, or truth-value, it should be associated with. It thus cannot be the case that the creators of arithmetese *decided* to link '(2.2)–3' with the number one. They decided merely what particular *calculation* —what particular way of constructing a number by way of others—it should represent. All decisions of this sort can be given in the form of one recursive definition. The term '(2.2)–3' then owes its connection with the number one to the brute non-linguistic *fact* that the construction represented by the term produces that particular number. This, however, must be denied by the linguistically oriented semanticist who sets great store by having purged his theoretical world of such abstract 'clutter' as mathematical calculations or constructions.

Another counter-intuitive consequence of this ontological thriftiness is that its advocates must ascribe to the mathematician a kind of notational parochialism he does not seem guilty of. If the term '(2.2)–3' is not diagrammatic of anything, in other words, if the numbers and functions mentioned in the term do not themselves combine into any whole, then the term is the only thing which holds them together. The numbers and functions hang from it like Christmas decorations from a branch. The term, the linguistic expression, thus becomes more than a way of *referring* to independently specifiable subject matter: it becomes *constitutive* of it. An arithmetical finding must, on this approach, be construed as a finding *about* a linguistic expression. To learn, for example, that two times two minus three is less than two is clearly not the same thing as learning that *one* is less than two. But if it is not the *construction* of multiplying two by two and subtracting three from the result that one learns something about then it must be the *expression* '(2.2)–3'. An arithmetical discovery must, on this approach, be construed as a discovery about an expression. But since an expression is always part of a particular notational system, our theorist must construe the arithmetician as being concerned specifically with a definite notation.

Now the mathematician must, to be sure, *use* a definite notation to state his findings; but the findings do not seem to be *about* that notation. When he finds, for example, what the result of dividing four by two is, he may record this by means of an equation containing the term '4÷2'; but it is not the *term* he has found something about. He could state the very same finding equally well by means of the term '4/2' or of the ordinary-English phrase 'the ratio of four and two'. The mathematician is interested in what results when a certain *number* is divided by another *number,* i.e., when a certain arithmetical *construction* is carried out. Having done away with constructions, our semanticist cannot do justice to this obvious fact. He must construe the mathematician as making a statement which is at least partly linguistic—a statement concerning a definite *term,* such as '4÷2'.

The semanticist will, no doubt, be quick to point out that '4÷2' is *translatable* into other notations. Through translation, he will suggest, the mathematician's finding can be brought to bear on other notations and thus acquire inter-notational status. This suggestion is hollow, however, unless the one who makes it has an answer to the question what it is for an expression to be a translation of another. And it is difficult to see how this question can possibly be answered without invoking the notion of construction: two expressions are clearly inter-translatable by virtue of representing the same construction, the same intellectual journey from some given objects to another.

Carnap, as is well known, tried to define intertranslatability—without invoking constructions—by means of his concept of intensional isomorphism. According to Carnap, '9–2' is intensionally isomorphic with, say, 'minus(IX,II)' because the occurrences of the primitive symbols '9', '2', and '–' in the former expression correspond, in a one-to-one fashion, to the occurrences of the primitive symbols 'IX', 'II', and 'minus' in the latter, and the corresponding symbols denote the same entities. Here is the relevant part of Carnap's own definition:

> Let two compound designator matrices [for example, '9–2' and 'minus(IX,II)'] be given, each of them consisting of one main submatrix ['–' and 'minus' respectively]... and *n* [in the present case two] argument expressions ['9' and '2' in the former expression and 'IX' and 'II' in the latter]... The two matrices are intensionally isomorphic $=_{Df}$ (1) the two main submatrices are intensionally isomorphic, and (2) for any *m* from 1 to *n*, the *m*th argument expression within the first matrix is intensionally isomorphic to the *m*th in the second matrix ('the *m*th' refers to the order in which the argument expressions occur in the matrix).[2]

Closer examination reveals, however, that this definition is inadequate. To see this, let Ar be ordinary arithmetese and Ar* the following slight modification of it. Numerals like '9' and '2' and functors like '–' mean in Ar* exactly the same as they do in Ar, but functional application is expressed differently. While in Ar the name of the first argument of a two-argument function is written to the left of the functional sign and the name of the second argument to the right, in Ar* the reverse convention prevails; the subtraction of 2 from 9, for example, is written '2–9'. We certainly do not want '9–2' *qua* an expression of Ar to be intensionally isomorphic with '9–2' *qua* an expression of Ar*; taking 2 from 9 is not the same thing as taking 9 from 2—even the results are different! Yet it is difficult to see how the expressions can possibly fail to be intensionally isomorphic on Carnap's definition.

Carnap might defend himself by declaring that his definition is not meant to apply across the board but only to languages where the writing of arguments is governed by the left-to-right convention. This would exclude,

[2] Carnap[1947], p. 59.

inter alia, ordinary English, for an expression like 'the result of subtracting 2 from 9' fails to conform to the convention. But, more importantly, the convention—and *a fortiori* the restriction to languages which conform to it—cannot be *stated* without resort to the notion of construction. For the convention is one concerning the method of recording the application of a binary function to two arguments, i.e., of symbolizing a certain arithmetical *calculation.* The reason Ar* is disqualified is because it records *the application of the subtraction function to an ordered couple of arguments* (i.e., a certain numerical construction) in such a way that the name of the second argument comes before that of the first. It is thus no more than an illusion that Carnap's theory makes the notion of construction redundant.

The repudiation of constructions creates especially troublesome problems when partial functions are to be dealt with. In a theory which associates arithmetical terms directly with numbers, an expression like '3÷0' is associated with nothing at all: it is a semantic dangler. Yet it is an arithmetical fact, as brute as any other, that 3÷0 is, as the arithmeticians say, *undefined.* The modern semanticist cannot, however, accept the sentence '3÷0 is undefined' as an expression of that fact, for on his theory the subject term of the sentence is an empty sound designating nothing. He must reformulate the statement as one about that *term* and construe the mathematician as making a purely linguistic comment.

But even this stratagem fails when it comes to the statement, also undeniably true, that

for exactly one number *n*, 3÷*n* is undefined.

Any attempt to re-phrase this as a statement about the *term* '3÷*n*' will be frustrated by the fact that one cannot quantify into a quotation context.

The problem, however, is entirely of the theorist's own making. The statement says nothing about any linguistic expression at all. What is meant by saying that, for exactly one number *n*, 3÷*n* is undefined, is that there is exactly one number such that the *construction* or *calculation* consisting in dividing three by that number yields no number at all. By disallowing himself talk about constructions, the theorist deprives himself of this natural and satisfactory explanation.

The need for the category of construction is even more obvious in connection with statements like

(i) For any F, if F is undefined then 3÷F is also undefined,

exemplified by

(ia) If (2.2)-3 is undefined then 3÷((2.2)-3) is also undefined,

(ib) If 3÷0 is undefined then 3÷(3÷0) is also undefined, etc. The statements concern an operation, call it Φ, which takes every arithmetical construction F to the construction which consists in carrying out F and then dividing three with the result. In terms of Φ, we can express the above statements as follows:

(i*) For any F, if F is undefined then Φ(F) is also undefined,

(ia*) If (2.2)–3 is undefined then Φ((2.2)–3) is also undefined,

(ib*) If 3÷0 is undefined then Φ(3÷0) is also undefined,

etc. Φ is clearly no numerical function, but a construction-forming operation. An application of Φ to, say, 3÷0 yields a construction (namely the construction of dividing three by nought and then dividing three by the result); the application itself is thus a construction of a construction. It is this *meta*-construction which is adverted to in the consequent of (ib*). What the consequent says is that the construction produced by this meta-construction is improper in the sense of generating no number at all.

The theorist who is anxious to do without the category of construction has to rephrase those statements beyond recognition. In order to say anything remotely similar he must leave the realm of arithmetical objects altogether and speak of expressions of a particular mathematical notation. The 'F' of (i), for instance, has to be construed as a *syntactic* variable ranging over the terms of that particular notation. The rephrased statement is thus a purely syntactic one and what it says cannot even be carried over to other notations by translating. A parallel statement concerning some other notation has a completely different subject matter (expressions of the other notation) and hence cannot be intertranslatable with the original statement. Yet (i) seems to enunciate a purely mathematical, language-independent, fact: the fact that if a calculation yields nothing, then dividing 3 with what the calculation yields is another calculation which yields nothing.

How has it come about that, despite all these drawbacks, View B reigns supreme in twentieth-century logic and metamathematics?

We have seen that it was not the view of the pioneers of modern mathematics, like Bernoulli and Euler. To them a mathematical formula was rather like a street plan or a piece of musical notation: a *diagram*. A diagram is a graphic representation of a complex object. It need not resemble the object in appearance, but must exhibit the relationships between its various parts. Individual constituents of a street plan represent individual parts of a town, and spatial relationships between the constituents exhibit the spatial relationships between those parts. Individual note symbols represent individual components of a sound structure such as a melody, and their arrangement on the staff exhibits the way in which they fit into that structure. In general, a diagram represents, or stands for, a complex object, and its parts stand for parts of that object. Similarly for a mathematical formula: individual symbols represent mathematical objects, and the way they are knitted into the formula exhibits the way the objects fit into the mathematical construction in question. Or so a formula was understood by the founders of modern mathematics. It is because they took their formulas to be diagrams of the mathematical calculations represented by them—in other words, because they took what I have called View A—that they did not find it necessary to keep notation and significance strictly apart in discussing mathematics. A polynomial, for example, was always defined as an *expression* of a certain form; but the mathematicians working on the theory of polynomials were surely not interested in strings of numerals and letters. They were interested in a certain range of *computational methods* involving multiplication, exponentiation, and addition.

This naive, and never clearly articulated, attitude prevailed until the late nineteenth century, when attempts to put mathematics on a more rigorous basis led to a critical examination of the foundations of the whole enterprise. The attempts were motivated partly by a sheer desire to eliminate vagueness, partly by philosophical questioning, but also by certain recalcitrant technical difficulties arising in mathematical research itself. The latter difficulties were rightly diagnosed by Poincaré, and after him by Russell, as having their root in a subtle error in dealing with mathematical constructions.

A construction of a mathematical entity often involves a variable ranging over a class of entities. The error, called by Poincaré and Russell the Vicious Circle Fallacy, is committed when one imagines that the construction itself, or something else involving the very same variable, is a member of that class.

The diagnosis pointed to the need to subject the intuitive notion of construction to critical analysis, and to replace it with a rigorous notion which would preclude the fallacy. Attempts were indeed made to do just this, most notably by Russell himself. The reason they did not succeed was because nobody came up with an adequate theory of variables. Russell himself was fully aware of this.

> [T]he variable [he wrote] is a very complicated logical entity, by no means easy to analyze correctly... May some reader succeed in rendering [my analysis] more complete, and in answering the many questions which I have had to leave unanswered.[3]

In the absence of an adequate theory of the variable, theorists had to deal with variables indirectly via their linguistic representations, the *letters* 'x', 'y', etc. As a result, the notion of construction was not really extricated from the notion of a formula representing a construction.

This inconsistency was bound to be noticed and found unsatisfactory. Those who put tidiness above insight were bound to come up with the idea

[3]Russell[1903], pp. 93-4.

that *formulas*, conceived as finite sequences of symbols, are what we should focus upon, not some intangibles represented by the formulas.

The idea was irresistible. Among the philosophically inclined, abstract entities were out of fashion anyway, and any opportunity to disown them was welcome. Formulas are, of course, abstract objects in their own right, but they have always been less offensive to the nominalist than abstract objects of other sorts (the hope being that somehow or other they might be reducible to stains left on paper by evaporated ink). But the idea was also irresistible to the mathematician. Finite sequences, after all, are familiar mathematical objects, and can therefore be studied using familiar mathematical methods. Metamathematics thus became just a branch of a well-established mathematical discipline: combinatorics.

In the first surge of enthusiasm many concluded that combinatorical study of mathematical formulas is all that logic and metamathematics is about, because mathematics itself is nothing but manipulation of formulas. This extreme formalism was admittedly never quite endemic and did not survive for long. Today few will deny that there are such things as numbers and functions, and that mathematical formulas serve to inform us about such things. What will still be denied by most, however, is that numbers, functions, and other mathematical entities combine into structured wholes, and that formulas are typographical diagrams of such wholes. In other words, the notion of a mathematical calculation or construction remains beyond limits.

But what then is the object of the mathematician's beliefs, assertions, etc.? Someone who repudiates constructions is reduced to saying that to believe or assert, say, that 9–2=7 is to take an attitude to the *sentence* '9–2=7'. This, at any rate, is the solution offered by Quine: we should simply refrain from uttering sentences like 'Tom believes that 9–2=7' and say instead 'Tom believes-true "9–2=7" '.

Quine's proposal immediately elicits what seems an insurmountable objection. To believe that 9–2=7, on the one hand, and to believe that the result of subtracting two from nine is seven, on the other, are clearly *one and the same* epistemic state to be in. Yet given any attitude to *sentences*, it is clearly possible for Tom to take it to '9–2=7' without taking it to 'The result of subtracting two from nine is seven'. Quine's 'semantic reformulation' thus cannot possibly preserve meaning. Another, closely related, point is that persons do not normally take attitudes to items they are not acquainted with or aware of. But it seems undeniable that someone may believe that 9-2=7 despite being unacquainted with, and unaware of, the *sentence* '9–2=7' (perhaps because he has never been exposed to the conventional arithmetical notation). His belief thus cannot *consist* in taking an attitude to that sentence.

Quine anticipates these objections and counters them in the following way:

> The semantic reformulation is not, of course, intended to suggest

> that the subject of the propositional attitude speaks the language of the quotation, or any language. We may treat a mouse's fear of a cat as his fearing true a certain English sentence. This is unnatural without being thereby wrong. It is a little like describing a prehistoric ocean current as clockwise.[4]

Quine's analogy is apt and worth pursuing. When we describe a prehistoric ocean current as clockwise we do not mean to imply that the current's behaviour *consisted* in its relating itself to a clock. In prehistoric times, no clocks were around for ocean currents to relate themselves to. We invoke clocks merely to specify one of two *directions* a circular motion can take. It is the *direction* that we ascribe to the current rather than a liaison with a clock. Similarly, when we say (unnaturally) that

(1) The mouse fears-true 'The cat is around',

we do not mean to imply that the mouse's attitude is directed to the sentence 'The cat is around'. We merely invoke the sentence to specify a certain (actual or non-actual) *state of affairs*, that of the cat's being around. What we ascribe to the mouse is a certain attitude to that state of affairs rather than a preoccupation with a sentence. Thus (1) only makes sense if it is understood as short for

(1') The mouse fears what is *meant* by 'The cat is around'.

By the same token, the sentence

(2) Tom believes-true '9–2=7',

only makes sense if it is understood as short for

(2') Tom believes what is *meant* by '9–2=7'.

Just as a current can be described by reference to a clock only because there is something, a direction, the two have in common, so a mouse's fear can be described by reference to a sentence only because there is something—namely a state of affairs—which the two have in common (the fear as its object and the sentence as its meaning). The same goes for the arithmetical case. The only reason that reference to the sentence '9–2=7' can render a service in describing Tom's belief is because there is something—namely the construction consisting in subtracting two from nine and checking whether the result is seven—which the two have in common: the belief as its object and the sentence as its meaning.

Quine is in fact liberal enough to allow us to read (2) as short for (2')

[4]Quine[1956], p. 186.

thus admitting that what Tom is said to be related to is a non-linguistic entity, and that the reference to the sentence is only a means to describe that entity. But given this broadmindedness, the question arises in what sense then Quine's proposal succeeds in 'dodging' (as he puts it) the non-linguistic objects of epistemic attitudes. All Quine's proposal shows is that we can systematically refrain from naming constructions directly and use instead descriptive phrases of the form 'the item meant by "..."'. This, however, is of no more interest than the fact that we can systematically refrain from ever naming Mrs. Simpson and invariably speak, say, of 'the woman who seduced Edward VIII'. There is indeed nothing to prevent us from sticking to this rule. Only let it not be assumed that we would thereby gain ontological economy. Let it not be supposed that the lady would thereby be paraphrased away from the world and that Queen Mary's distaste for Mrs Simpson, for instance, would be explainable in terms of some attitude she took to Edward.

Failure to distinguish between entities and various ways of constructing them is an inexhaustible source of philosophical confusion and doubletalk. The notion of proposition is a typical case in point. There is an almost universal tendency to impute the structure of propositional constructions to propositions themselves. Although few would maintain that the numbers nine and two and the subtraction function are ingredients of the number (seven) denoted by '9–2', many will regard Tom, Sam, and the taller-than relation as ingredients of the proposition, or state of affairs, denoted by 'Tom is taller than Sam'. Yet the situation is completely parallel. The number seven does not contain the minus function because if it did, it would also have to contain the addition function, since seven is not just nine minus two but also three plus four. Likewise, the fact that Tom is taller than Sam does not contain the taller-than relation. If it did it would also have to contain the shorter-than relation, for Tom's being taller than Sam and Sam's being shorter than Tom are surely one and the same fact.

Propositions are also routinely spoken of as negative, disjunctive, existential, and so forth. But let us stop to reflect what can possibly be meant by a negative proposition. A number is said to be negative if it is smaller than nought. Can anything remotely similar be said of propositions? The only way of defining a negative proposition seems as a proposition which is the negation of another proposition. But to define negative propositions in this way would be just as pointless as it would be to define negative numbers as multiples of –1. For every proposition without exception is the negation of some other proposition (namely, of its own negation), just as every number n is the product of –1 and some other number (namely, $-n$). But, while the negative/non-negative distinction is completely idle when applied to propositions themselves, it is sensibly applied to propositional constructions: a proposition can be *constructed,* or arrived at, by negating another proposition. This, however, is not saying anything interesting about the proposition itself but about the particular construction of the proposition. The proposition itself bears no traces of having been arrived at in this particular way.

Occasionally, philosophers project the structure of a propositional construction not onto the proposition itself but rather—even more absurdly—onto *constituents* of the construction. Consider, for example, the following statement:

> a relation [has] a structure: in it, we have to distinguish the terms of the relation from the relation that obtains between them, and to distinguish the terms from each other.[5]

In other words, a relation is structured because it consists of three constituents, one of which is the relation itself. This mystery (rivaling that of the Holy Trinity) stems from nothing other than the absence, in the authors' conceptual scheme, of the notion of construction. As we have seen, the sentence 'Tom is taller than Sam' represents a propositional construction, a structured item consisting of two persons and a two-place relation. The authors, lacking the notion of propositional construction, cast desperately about for something to ascribe the tripartite structure to. They settle on the *taller-than* relation and find themselves saying that the relation is one of its own three constituents. One may as well claim that a billiard-ball has a tripartite structure because in it we have to distinguish the table top on which it rests, the cue that strikes it, and—itself.

[5]Hintikka and Hintikka[1986], p. 39.

Chapter Two: Fregean Functions

4. Frege and constructions

Whatever one might think of its usefulness in interpreting mathematical discourse in general, the concept of construction is indispensable in interpreting Frege's semantic theory. Although the linguistic approach, now dominating the field, is indebted to some aspects of Frege's philosophy, Frege never took the approach himself. The intuitive notion of a mathematical construction informed Frege's thought, explicitly or implicitly, throughout his career. As I hope to show in this and subsequent sections, Frege's philosophy of mathematics revolved around this notion and cannot be understood without reference to it.

The conflation of notation and significance in mathematics went unchallenged until well into the second half of the nineteenth century. One reason why Frege's work is of enormous historical significance is because it represents a divide between this naive era and the modern era which is characterized by the linguistic approach. Frege was the first to break with the long tradition which treated notation and significance as one indissoluble whole. In his *Grundgesetze* he started to distinguish scrupulously between formulas and what they stood for, populating his pages with scores of quotation marks.

Per se, the step is unobjectionable. There is no virtue in pretending, as regards two different things, that they are one and the same. But, by conceptually separating linguistic compounds from the objectual compounds they represent, Frege paved the way for those who came after him and concentrated on the former to the exclusion of the latter.

Frege himself consistently adhered to the idea that the structure of an expression mirrors the structure of the mathematical entity that the expression represents. Yet he did in a way have one foot in the modern era in being hopelessly ambivalent between what I have called View A and View B. This ambivalence, I am going to argue, is the source of most of what is flawed in his semantic doctrine.

Frege was no ontological miser. To him the formalist idea that mathematics is about symbols, or even partly about symbols, was simply frivolous. Mathematical expressions were, for Frege, mere means of talking about extra-linguistic, mathematical entities. The symbols '2', '3', '.', and '–', appearing in the term '(2.2)–3', were names of numbers and numerical operations.

In his early work, Frege entertained the notion of a whole which is composed of the numbers two and three and the operations of multiplication and subtraction in the way the expression '(2.2)–3' is composed of the corresponding symbols. In *Begriffsschrift* he spoke of this whole as a particular 'mode of determination' of the number one. And since the number is determined, in the present case, *qua* the result of multiplying two by two and subtracting three from what one gets, the mode of determination was clearly nothing other than this particular calculation. It was this calculation—a whole involving two numbers and two operations—that the term stood for, at any rate when it occurred as one of the two terms flanking the equals sign, as in '(2.2)–3=1'.

It is true that what Frege actually said was that the term stood in such a context for *itself*. But one has to keep in mind that when Frege wrote *Begriffsschrift* expressions to him were still inseparable from what they signified. He emphasized himself that the syntactic difference between two names is not

> an indifferent matter of form; ... if [the names] are associated with different modes of determination, they concern the very heart of the matter.[1]

It is thus arguable that the calculation consisting in multiplying 2 by 2 and subtracting 3 from the result (a whole consisting of numbers and operations), was a constituent of what Frege called the *conceptual content* of the equation '(2.2)–3=1'; it was the conceptual content of the left-hand side of the equation.

The same follows from what Frege said about the various ways an equation could be 'split' into concepts and numbers. In his article 'Boole's Logical Calculus and the Concept Script', written roughly at the same time as *Begriffsschrift* but published posthumously, he discusses the conceptual content of the equation '$2^4=16$'. He says that the number two can be imagined as replaceable by other numbers, in which case we obtain the concept of the fourth root of sixteen, or, that the number four can be imagined as replaceable, in which case we obtain the concept of the logarithm of sixteen to the base of two.[2] Now for this to be possible, both numbers (two and four) must clearly be present in the conceptual content of the equation. The left-hand side of the equation must be looked upon as denoting a definite mode of determining the number sixteen, to wit, as the calculation which consists in applying the exponentiation operation to two and four. Thus, at least as far as identity contexts are concerned, Frege construed compound arithmetical terms along the lines of what I have called View A: as names of arithmetical constructions.

[1] Frege[1879], p. 15, Frege[1972], p. 126.

[2] Frege[1979], pp. 16-17.

It is an odd feature of *Begriffsschrift* that Frege confined this construal of compound terms to identity contexts. What he said about the role of '2^4' in '$2^4=16$' was not supposed to carry over to, say, '$2^4<17$'. In the inequation, '2^4' did not stand for a compound consisting of the numbers two and four, but simply for the number sixteen.

This asymmetry is not only inelegant but quite obviously at odds with the rest of Frege's doctrine. For what has just been said of the conceptual content of '$2^4=16$' goes, on Frege's own theory, for the conceptual content of '$2^4<17$'. The inequality can also be split into a concept and a number in at least two different ways. If we think of the number two as replaceable, we obtain the concept of a number whose fourth power is less than seventeen. If we think of the number four as replaceable, we obtain the concept of the logarithm, to the base of two, of a number which is less than seventeen. But for this to be possible, the conceptual content of '$2^4<17$' must *contain* the numbers two and four. Yet on the *Begriffsschrift* theory the conceptual content of '2^4', as it appears in '$2^4<17$', is simply the number sixteen, so that the conceptual content of '$2^4<17$' as a whole is no different from that of '$16<17$'.

This construal of '$2^4<17$' is also impossible to reconcile with the general comments concerning the notion of conceptual content that Frege made in the Preface to *Begriffsschrift*. There he said that the main purpose of the conceptual notation expounded in the book is to

> test in the most reliable manner the validity of a chain of reasoning.... For this reason, I have omitted the expression of everything which is without importance for the chain of inference. In Section 3, I have designated by *conceptual content* that which is of sole importance for me.[3]

The obvious implication is that the conceptual content of a sentence is not only bereft of all that is inferentially unimportant, but that it also retains all that is inferentially relevant. Now when it is said that the fourth power of two is less than seventeen, it is inferentially relevant that the left-hand relatum of the inequality is presented as the fourth power of two and not in some other way. For the conclusions 'There is an x such that $x^4<17$' and 'There is an x such that $2^x<17$' are correctly derived from '$2^4<17$' but not from '$16<17$'. Frege's own doctrine thus required that the conceptual content of '2^4' be a whole consisting of numbers and operations not just in identity contexts, but in general.

In later years Frege found the systematic ambiguity that the *Begriffsschrift* theory imputes to terms like '2^4' or '(2.2)-3' unsatisfactory. Yet he did not take the obvious step of generalizing the analysis he accorded to those terms in identity contexts.

[3]Frege[1972], p. 104.