THE NEW ELEMENTS OF MATHEMATICS

Charles S. Peirce in his middle thirties.

(From the Charles S. Peirce Collection in the Houghton Library, Harvard University.)

THE NEW ELEMENTS OF MATHEMATICS

by

CHARLES S. PEIRCE

Edited by

CAROLYN EISELE

VOLUME II

ALGEBRA AND GEOMETRY

1976

MOUTON PUBLISHERS
THE HAGUE – PARIS

HUMANITIES PRESS
ATLANTIC HIGHLANDS N. J.

I.S.B.N. 90 279 3025 2

Printed in The Netherlands by Mouton & Co., The Hague

INTRODUCTION

It is perhaps in the writing of mathematics textbooks for possible publication by Ginn and Company that Peirce reveals most consistently the influence on his overall thought of the literature in what was even then the "new mathematics," really "new" then, but being brought to the lower school level only now, seventy-five years after Peirce had tried a lone hand at making it feasible. An incomplete letter from Peirce to William E. Story reads as follows:

160 W 87th St. New York City
1896 March 22

My dear Story:

I want to ask you a little question in the non-Euclidean geometry. I could get it out myself, but you can probably answer it offhand. Besides, I have some other counsel I should like from you which no man could so well give me.

My question is whether generally a single circle is all that can be drawn through three points. Also, what are the properties of a set of four points through which two circles can be drawn? Can a third circle be drawn through such a set generally?

The counsel I seek is this. I have spent a great deal of time writing a book called New Elements of Mathematics. Most of my labor has been to make it elementary; although I have put in a great deal of hard thinking too, — not so much mathematical as logical. It is something like Veronese's Geometry, but is (I think) far deeper logically, and certainly far simpler. Nor do I directly go in to non-Euclidean Geometry. It begins with a long logical introduction which I have written twice and am still not satisfied with it. It then takes up the theory of numbers and develops the first elements, with particular attention to logic. I then take up algebra, and develop with strict logic and in a somewhat novel way the fundamental ideas, first of rational fractions and then of continuous real quantity. My definition of continuity is some improvement upon Cantor. I only just touch upon imaginaries. But I have come to the conclusion that a Part has to be inserted here, or else appended, developing the calculus of imaginaries, the fundamentals of the theory of functions, so far as Cauchy went (without bothering with Riemann surfaces), the outlines of the calculus (the geometry could be stated without proof if this were put in here), and the elements of plane trigonometry.

There is no essential difference between pure and applied mathematics. The mathematician does not, as such, inquire into facts. He only develops ideal hypotheses. These hypotheses are all more or less suggested by observation and all depart from or transcend, more or less, what observation fully warrants. But if the hypotheses are developed with a view to ideal interests, it is *pure* mathematics. If they are made crabbed and one sided in the interest of truth, it is *applied* mathematics.

The science of Time receives a brief chapter, chiefly because it affords an opportunity for studying true continuity.

I then come to Geometry. There is Physical Geometry and Mathematical Geometry. In mathematical geometry I treat successively, 1st, Geometrical Topics (topology), the doctrine of the connections of places consequent upon the intrinsic constitution of space; 2nd, Geometrical Optics, or the doctrine of the intersections and envelopes of unlimited rays of *vision* (I do not say of *light*, because the idea of propagated light involves a metrical idea, and would make the rays stop at a real Absolute surface, so that not every triad of planes would have a common point,* etc.). (* I regard the *jenseits* of the real absolute of hyperbolic geometry as real (according to the assumption of visual optics though imaginary in metrics.)); 3rd, Geometrical Metrics, particularly the 1st and 3rd books of Euclid.

I found the problem of analyzing the topical constitution of space very puzzling; and I suppose that it is in consequence of this effect of my stupidity that I am decidedly vain of my solution of that problem. I state it in four postulates. The first two are: *Postulate* 1. Space is continuous. *Postulate* 2. Space has three dimensions. I illustrate this latter by showing that if a lump of ice sticks to the side of a bucket, and is partly immersed in water, then of the four places occupied by the wood of the bucket, the ice, the water, and the air, any pair has a common limit, a *surface*, any three a common limit, a *line*, and all four a common limit, a *point-pair*. The point-pair, that is, any two points, I call a simple *artiad* punctual locus.

By a *topical singularity*, *B*, of a locus, *A*, I mean that *B* is a locus of less dimensionality than, and contained in or at the limit of *A*, and that from *B* particles, filaments, or films can move along *A* along more or fewer tracks than from the majority of loci of like dimensionality with, and sufficiently near to, *B*. If the tracks are singularly many, I name *B* a singularity *by excess*; if singularly few, *by defect*. The numerical excess I term the index of the singularity. Thus, from an ordinary point on a line, a particle can move two ways. This is true even if it be a *prickly line*, by which I mean a line along which the furcations are crowded infinitely close together. The *extremity* of a line is a topical singularity of index −1; a *furcation* of index positive. I do not regard a cusp as a topical singularity. Am I right? If a line is in its midst punctually interrupted, the census-theorem compels us to regard this as having the index of singularity −2. It is two coincident extremities.

A surface may have both lines and points that are topically singular. From an ordinary line on a surface (even a *file-surface*, which is analogous to a prickly line) a filament may move two ways. From a bounding *edge*, it can only move one way; from a *schist*, or line of splitting, or nodal line, it can move more than two ways. I do not regard a cuspidal edge as a topical singularity. A schist may itself have furcations and extremities and partial extremities (that is where some of the sheets cease to be joined). The index of the last considered as a singularity of the surface is the mean of the indices on the two sides of it along the schist. The isolated topically singular points are either *punctures* or *stitches*. They may occur anywhere, even where there are other singularities. A stitch may unite two sheets, or may join a line to a surface. This sufficiently illustrates what I mean by a topical singularity.

Postulate 3. Space has no topical singularity.

As a corollary, space is unbounded; and because it is unbounded, it returns into itself. I warn the student against supposing that a mathematically correct inference proves anything. It may be, and *by itself*, always is, a *petitio principii*, unless we accept the postulates as "gospel truth".

I divide all loci into the *artiad* and the *perissid*. An artiad locus is one which if we interrupt by a sufficiently great number of topically non-singular loci of one less dimension, all pairs, triplets, or quartettes among them having the same finite number of points in common, that number is always even, so long as the intersections are not confined to certain points. But if this test cannot be applied, then the locus is artiad if it cuts the simplest non-singular locus which it interrupts into an even number of parts. A perissid locus is one which can be interrupted by an indefinite number of loci all pairs, triplets, or quartettes containing the same number of points which may be situated anywhere in it, and this number odd. Or if this cannot be applied, it is perissid if it divides the simplest non-singular locus in which it lies into an odd number of parts. Thus, one point is perissid; a point-pair is artiad, because it cuts a ray or any simple non-singular line into two parts. A ray is perissid, according to projective geometry, because it does not cut a plane or cone into two parts. On the same doctrine, a plane is perissid, because it does not cut space into two parts; and any number of rays in it cut one another in single points. A cone is artiad. But according to the theory of functions, quaternions, B. Peirce's *Geometry*, hydrodynamics, and probably any branch of physics that is in harmony with the science of our time (but not according to perspective, which neglects the fact that light is propagated etc.), the plane is artiad. I adhere to the projective conception, though I do not believe it is the fact of the external universe. A limited disk, a sphere, etc. are artiad. A plane with holes in it is perissid.

The Listing numbers I define a little differently from Listing, and get different numbers for space because I conceive of space as perissid while he virtually makes it artiad. When he talks of "all space," he only means the parts at finite distances. It is really limited, though he calls it unlimited, because he wrote before Riemann, and confuses the infinite with the unlimited.

The Listing numbers are the Chorisis, Cyclosis, Periphraxis, and Immensity. They are respectively the number of simplest possible, topically non-singular interruptions that, being established in a locus, less the number of such interruptions that, being abolished in the locus, just suffice to prevent its holding a simplest topically non-singular, Particle (for Chorisis), Filament (for Cyclosis), Film (for Periphraxis), or Body (for Immensity), except such as can collapse by shrinking while remaining in the locus in question.

Postulate 4. Each Listing number for space is 1.

The Chorisis is 1 because space is all one piece. The Cyclosis is 1 because a straight unlimited filament cannot by any continuous change of shape shrink to nothing; but if space is interrupted by a plane that suffices to prevent it from holding such a filament. The Periphraxis is 1 because a plane film cannot by any continuous change of shape shrink to nothing; but a ray suffices to prevent space from holding such a film. The Immensity is 1, because a solid filling all space cannot by any continuous change shrink to nothing. But if space is interrupted by a point, it can no longer hold a topically non-singular, that is, unlimited body.

My treatment of topology consists mainly of making clear the conceptions, exercising the imagination by numerous examples, and proving the census theorem, and explaining the topical singularities. I also demonstrate that if by a *system of homaloids* we mean all the points of space, together with a family of surface[s] of which there is just one containing any three points of space not contained in an innumerable multitude of those surfaces, and of which any three contain just one point in common, unless they contain an innumerable multitude; and together with all the lines of intersection of those surfaces; then there are an innumerable multitude of such systems of homaloids. I also show that the intrinsic properties of all these systems are precisely the same; so that no one can be distinguished from any other except by some extrinsic circumstance.

I begin Geometrical Optics by remarking that there is a system of homaloids which are distinguished from all others by the circumstance that they look differently. Namely, all lines of possible undisturbed vision (I do not say paths of light) lie in the lines of that system. They are called rays. This gives me

Postulate 5. (The Optical Property of Space.) All lines of vision lie in the lines of a system of homaloids.

I now develop the elements of projective geometry, especially the theory of nets by what I call the non-metrical barycentric notation, and the theory of degrees of freedom. I use this notation. A, B, C, etc. are points, a, b, c, etc. rays, α, β, γ, etc. planes. (ABC) is the plane of A, B, C. (Ab) is the plane of A and b. If a and b are coplanar (ab) is their plane. $[\alpha\beta\gamma]$, $[ab]$, etc. are points. $\{AB\}$, $\{\alpha\beta\}$, etc. are rays. I prove that if a and p, b and q, are copunctual, while a and q, b and p are coplanar, then $\{[ap][bq]\} = \{(aq)(bp)\}$, a useful theorem. I show that every four rays are cut by two rays. I name a one-sheeted hyperboloid, conceived as the locus of a ray cutting three fixed rays, a *phimus*. I define a conic as the plane section of a phimus, and thus easily get all the main propositions of conics.

Another draft of the letter carries the following non-Euclidean overtone under *Optical Property*:

I need not say that boundlessness is one thing, infinity another, nor that a point may be a perfectly real point of space although on the other side of the firmament and metrically imaginary. Thus, every pair of planes have a ray in common, and every three a point, etc., whatever system of measurement be employed. The topical properties suffice to prove that there are innumerable families of lines related to one another like rays. Hence, rays or straight lines, together with planes, can only be defined by means of some material thing. I chose for this thing light. The optical property of space is that light, or vision, is along lines which, continued without limit, are called rays. The rays can have but one point in common, etc. No need to state in full here.

Metrical Property. The absolutely rigid body moves so that in all displacements the film occupying a certain fixed plane, called the firmament, continues to occupy that plane; but there is no point of space across which a plane film (not the firmamental film) cannot move.

However, I consider this as only approximately true; for my calculations from Auwers' proper motions leave no reasonable doubt that space is really hyperbolic, or, if you please, that the proper motions have that property.

My book aims to be elementary. To render it so, I have spent a great amount of labor. There are some things in it not altogether without mathematical interest, I think. But it is the logic of it which I trust will be found interesting. That part has been very maturely considered, at any rate; and this analysis of the properties of space is one of my efforts in that line to which I attach value.

The five postulates adopted by Peirce offer clear evidence of his familiarity with the "Postulates of Euclidean Geometry" as summarized in a pamphlet by W. K. Clifford, a copy of which is to be found among the reprints in the Peirce Collection at the Houghton Library, Harvard University.

Peirce's reference in 1896 to the hyperbolic properties of space is notable, since he entertained that idea for a time in 1891, only to give it up on further investigation. His correspondence on this matter with Simon Newcomb may be found in the aforementioned correspondence between the two men ("The Charles S. Peirce — Simon Newcomb Correspondence", *Proceedings of the American Philosophical Society* 101:5, October 1957, pp. 410-433).

Although Peirce had opened the second letter above with the flattering statement "I look upon you as knowing all about the non-Euclidean Geometry" and turned to Story for advice, he was also exposed, as has been said, to the influence of George Bruce Halsted in the non-Euclidean area. For Halsted was spearheading in his publications on the new geometry the effort to bring to mathematicians in America an awareness of the revolution in mathematical thought.

Halsted had been one of the first twenty Fellows in the then new Johns Hopkins University (1876-1878) and constituted by himself the first class of Sylvester, receiving his Ph.D. from that institution in 1879. After an instructorship in post-graduate mathematics at Princeton College, his alma mater, he was called to a professorship in mathematics at the University of Texas in 1884. The following year his *Elements of Geometry* was published and he expressed his gratitude to Sylvester in a dedication that read "in grateful remembrance of benefit conferred throughout two informative years." His approach to the subject matter was untraditional and became even more so in the *Elementary Synthetic Geometry* published in 1892.

It is not surprising, then, that Peirce and he were in correspondence on geometric matters in the period when Peirce was attempting to produce a geometry textbook that would reflect the "new look" that was being advocated on high levels. In a letter to his friend Judge Russell (23 September 1894) Peirce spoke of having had "some friendly dealings with him [Halsted] about his Lobatchewsky, etc." Peirce had reviewed Halsted's translation of Lobatchewsky's work in the *Nation* 54 (11 February 1892), and had deemed the translation "excellent," making mention at the same time of "his useful bibliography of non-Euclidean geometry" which "was already well known." The following excerpts are from the correspondence in the Peirce Collection.

Halsted wrote to Peirce on 7 December 1891:

I was very much interested in your paper on "Astronomical Methods of determining the curvature of space," read before the National Academy of Sciences and if you distribute any reprints of it, I would be much obliged for a copy.

Again on 15 January 1892 Halsted wrote as follows:

I was much interested in your suggestion, in your letter of Dec. 13, of a modern-synthetic-geometry treatment of non-Euclidean geometry.

I think I could manage the analytic treatment, Cayley, Klein, etc.; but I would be much obliged to you for a few suggestions in regard to the modern-synthetic-geometry treatment.

I send you by this mail a copy of the 4th edition of my Lobatchewsky and hope in a few days to send you my translation of Bolyai.

I suppose you have noticed that in the *Century Dictionary* definitions Hyperbolic Geometry and Elliptic Geometry (or Space) are interchanged, the definition of each being given for the other.

A letter that Max Fisch dates "around 15 January 1892" was sent by Peirce to Halsted. The draft reads:

When I wrote to you about a projective treatment of non-Euclidean geometry (which I also suggested in the Nation) I had the idea it was easier than it is. Namely, I thought the whole thing would readily come out by the use of Story's definition of a circle as a conic having double contact with the absolute. But it seems that does not define a circle in the sense of the locus of points equidistant from a centre. I have been intending therefore to look up another point of view. I have not had time to do so, and therefore, wishing to show my good will by answering your letter, I am obliged to do so in such feeble way as I can, thinking as I go along and not knowing how I am to come out. This will throw me at your mercy, and I shall have to depend on your honor to give me such credit as I am entitled to for what I may suggest.

We shall naturally direct our attention to the absolute or line at infinity (beginning therefore with hyperbolic geometry, though even there the angle-absolute is imaginary). As we are to proceed projectively, let us suppose a plane projected centrally upon another, this again projected from a second centre upon a third plane, and this projected back upon the first plane so that the absolute shall coincide with its original position. Then, I take it the result is the same as shifting the original plane on itself as a rigid thing. If this is so, there is generally but one series of concentric circles whose old positions coincide with their new; and generally if any circle by three projections is brought back to coincide with itself, all concentric circles are so brought back.

Let α be our first plane, with a line a upon it and on that line the four points A, B, C, D. Let our second plane be β, calling the first in the line b. And let S be the centre of projection. Then S, A, B, C, D, will be in a plane.

Or say we have three planes $\alpha\beta\gamma$ with common lines $\alpha\beta = c \quad \beta\gamma = a \quad \gamma\alpha = b$. And one common point O. We also have three centres of projection X Y Z with three lines $XY = z \quad YZ = x \quad ZX = y$ and a common plane as [...]

Another letter from Halsted to Peirce on 15 February 1892 asked for aid in showing by non-Euclidean means a fallacy in a "pseudo-proof" of Euclid's axiom. The letter ended with the statement: "I will be much obliged for a simple sentence from you in regard to it. I had the pleasure a few days ago of sending you my translation of Bolyai." (1st edition).

When Halsted sent Peirce a copy of his Bolyai's *Science Absolute of Space, entirely rewritten,* he was President of the Academy of Science in Texas. The covering letter to Peirce, dated 17 December 1895, carried the news that he had been "so fortunate as to be able to announce important additions to the knowledge of the history of these most interesting researches and the lives of the two Bolyais." He signed, "With high regard." This time Halsted referred to the 4th edition.

But beyond these contacts, there was the great influence of the famous Felix Klein which is still felt in the mathematics curriculum of the western world. Although Klein in his writings had advised against the attempt to introduce non-Euclidean geometry on an elementary level, Peirce did slip it into his textbook writings without identifying it as such. Pierce was also surely familiar with Klein's address at the opening of the Mathematical and Astronomical Congress at Chicago, Illinois, printed as "The Present State of Mathematics" and published in the *Monist*, IV, no. 1 (October 1893), a very influential paper. Klein's international influence on the teaching of mathematics in his own time is incalculable for he became President of the International Committee on the Teaching of Mathematics created in 1908 after his years of missionary work in advocating reform in the way of updating the subject matter of the curriculum. Peirce not long thereafter read a paper on "Rough Notes on Geometry, Constitution of Real Space" at a meeting of the American Mathematical Society on Saturday, 24 November 1894. It was not printed later, but one of the incomplete papers in this collection (MS. 121) could well have been that talk, since Fiske had asked Peirce for a copy and Fiske's name is written in Peirce's hand on the back of it.

But it was to Story that Peirce turned for information and inspiration. During those years, Story headed the Mathematics Department at Clark University, indeed had been administrator since the founding of the institution in the fall of 1889. At the weekly mathematics conferences there, conducted by Story, non-Euclidean geometry had been the "subject of systematic discussion." Even while at the Johns Hopkins University earlier under Sylvester and in Peirce's time, Story had been working in hyper-space and non-Euclidean geometry as well as in algebraic invariants. During those years, Story published in the *American Journal of Mathematics* 4 (1881) the paper "On the Non-Euclidean Trigonometry," and in vol. 5 (1882), "On the Non-Euclidean Geometry," and "On the Non-Euclidean Properties of Conics." These papers and another by Halsted in volume 1 entitled "Bibliography of Hyperspace and Non-Euclidean Geometry" were surely known to Peirce whose formal interest in the subject is evidenced by his reading before the Metaphysical Club at the Johns Hopkins University in 1879 a paper on the "Non-Euclidean Conception of Space" written by his pupil Miss Ladd. Story, on his part, seems to have had considerable respect for Peirce's talent. At the time of the Decennial Celebration (1889-1899) at Clark University Story was active in the organization of the Celebration ceremonies. On 27 June 1899 he invited Peirce to an entertainment in his home during the celebration the following week when he planned to have as guests Émile Picard of the University of Paris and H. B. Fine of Princeton University. Peirce not only attended the meetings but reviewed the *Clark University Decennial Celebration* in *Science* n.s. 11 (20 April 1900). Story was an editor of that volume and gave a full report therein on the professional activities of the members of his mathematics department. In describing the work of Henry Taber on the theory of matrices Story made special mention of C. S. Peirce's contribution to the multiple algebras that he called quadrates. Incidentally, when Tabor asked Peirce in 1892 for a recommendation for a mathematics appointment at Columbia University, Peirce accommodated him but indicated that he would look into the vacancy for himself.

It is not surprising, then, to find Peirce telling Story about *The New Elements of Mathematics* mentioned above in the Story letter. It is indeed a manuscript of unusual interest in many ways (MS. 165) and Peirce's long struggle with its contents merely reflects his usual preoccupation with the challenges constantly raised by textbook presentation of that subject. Typical of Peirce's perseverance in the study of mathematics is his explanation to his wife in a letter of 1892 that the train was late at Port Jervis, "but I had in my pocket a book on the Geometry of the Circle which I am studying. It will bring me a few dollars immediately for a notice, but the chief reason for studying it is to keep up my acquaintance with every branch of mathematics, which gives me a reputation and is useful to me in other ways."

There were "other ways" indeed. Peirce's philosophical and logical interest in the problem of continuity made for his deepest concentration on developments within the then infant branch of mathematics — the topology of Listing and Gauss. He studied these new ideas in great detail and came to believe that topically singular points were the only ones having identity while, in general, a point on a continuous line has no individual identity at all (MS. 159). In his theory of logical criticism "the temporal succession of ideas is continuous and not by discrete steps. The flow of time is continuous in the sense of the non-discrete" (MS. 377).

Yet Peirce's purpose in writing the *New Elements of Mathematics* was not a philosophical one. After his enforced retirement in 1891 from service in the United States Coast and Geodetic Survey, he had turned to textbook writing in elementary mathematics as a means of economic survival. Then, as now, professional mathematicians enjoyed great financial success in such ventures and it was natural for Peirce to believe that his new and original approach to the teaching of foundations would attract wide and sympathetic support.

On 15 July 1894 Charles's brother, Herbert, wrote a note to him from aboard the North German Lloyd liner *Saale* on the way to St. Petersburg where he had been appointed Secretary of the United States Legation. He related that "Mr. Ginn the publisher in on board and has spoken to me about your books. He says he wants a little time to consider them.... He is really interested but wants to look into the subject himself and I think is disposed to take them."

The correspondence that followed depicts Charles's futile attempts to meet the sales-minded demands of the publisher. This effort was to result in *The New Elements of Geometry* (MS. 94), a response to Mr. Ginn's desire in 1894 for an updated version of Benjamin Peirce's *Elementary Treatise on Plane and Solid Geometry* of 1837.

It is difficult to determine which of the two manuscripts (165 or 94) was completed first. Manuscript 165 is the carefully planned and conscientiously ordered work of the Story letter. Tens of pages of worksheets, and lists of definitions, scholia, glosses, theorems, and outlines for the entire work are still extant. Yet some material was repaginated for the geometry section of MS. 165 and is now missing from MS. 94. The full outline of MS. 165 suggests in a hastily written entry the existence of Chapters X and XI which now, after a separation of years, have been reunited with the main body of that manuscript. Regardless of priority, it is the story of MS. 94 that we shall relate first.

On 5 April 1894 Charles had written to his brother James Mills (Jem) at Harvard University of the usual financial troubles.

Consequently, I have to abandon all idea of the Arithmetic for the present; and probably that means never doing it ... the advantage of the geometry is that it would soon be done, as I have most of it written already. It might bring little, but might lead to further employment. The trigonometry would be a useful book and in the long run must pay tolerably well; for it would be greatly superior to Chauvenet from every point of view. The chart which I am now trying to push might, if I am lucky, bring enough at once to enable me to set to work on the Arithmetic. The Almanac about which I am negotiating with considerable prospect of success with the Times, would take all my time, but would give me a steady though very modest salary ....

[P.S.] My notion about Father's geometry would be to make an introduction of three chapters. The first on the nature of mathematical reasoning and how to perform it. The second on projective geometry, especially perspective. The third on the metrical properties of space and the justification of the idea that an angle is a difference of directions.

I don't know that the body of the book would require any correction. The thing would be to preserve its leading characteristic, which is to lay hold of and render definite our instinctive or natural ways of considering space, especially in reference to moving about in it.

I think, that done, the work would be a masterpiece and would be much admired. Its sale would be good, too. Plimpton is anxious to have us do it. You and I should do it together, n'est-ce pas? What say you? ... I don't know that I would not reform and enlarge the part relating to plane problems and append a chapter on Descriptive Geometry, and perhaps an appendix on Modern geometry.

By November 18 Charles was writing that if Jem

would take hold of the Geom. and improve it in places, etc., we should make a fine thing of it.

I have gone on the idea the Elementary Geometry must be reformed. There are many signs it will be and we might as well take opportunity by the forelock. So far, I have not ventured to use your name on the title page and certainly if you don't do anything about it nor give your name, you won't expect any considerable share of the profits. But if you will take hold why we shan't quarrel over the division, I guess.

My notion is to make a book to teach most of those who use it all the geometry they are going to study. I decline to be tied down therefore to any idea. If a given thing is useful, — especially, if calculated to enlarge the mind, — and is sufficiently easy, I won't be deterred from putting it in because it is not what is called Elementary Geometry. At the same time that most of the students of the book will go no further, I wish to present the subject in a mathematical and modern spirit, and to teach them what mathematics is, in order that they may be able to judge whether they wish to pursue the study further or not.

Moreover, I don't teach that geometrical "demonstrations" prove anything. Having made an initial hypothesis they serve to keep the hypothesis consistent. But whether it is true or not would have to be ascertained by special observation. Ordinary observation shows it is very near true indeed; and the astronomical observations show the errors if any are extraordinarily minute.

Since writing you an account of it, I have succeeded in simplifying the hard places very much; there still exist passages which I regret the abstruseness of, and I dare say if you got hold of it, could in a few days do great things in the way of simplification.

I have introduced a little algebra. I don't see why not. The students have got to learn a little algebra sometime. They had best learn it on an occasion when they can see at once its great utility. Of course I explain the thing from the very bottom. I only ask them to use letters a little, and the *minus* quantities, both in addition, subtraction, multiplication and division, and the distributive principle. Instead of letters, I mostly use prime numbers and not performing but only indicating the operations, I say, it is evident any other numbers would do as well since we have not done anything depending on the special values of those we have used. That it is a book, sufficiently easy, really easier than the geometries of the old school, and calculated to open the mind to mathematical ideas beyond anything elementary ever done before, is my own flattering opinion of it.

Without exactly saying so, I have aimed to have the student go away with the conviction that mathematics is the *art of exact generalization.* Demonstration [in] Elementary Geometry of the past made the whole thing appear as nothing but the pavement over which the mathematician drives his team, with a goal in view and with a plan for reaching it. The three faculties I try to educate are Imagination, Concentration, Generalization.

It will occur to you that I am taking too great liberties with father's book. Especially so, since I am in places quite diffuse, being persuaded that those who have little turn for mathematics have to have their milk watered, like other babes. But I am sure that father would not at this day have approved of the kind of logic which governs a good deal of his book. I think my work is in the spirit which father always radiated; although I admit that I have not been able to attain his classical simplicity. If you would take the hold of it, I believe you would do a good deal toward that. It is in some aspects like a book you always, I thought, underrated, JMP's Analytical Geometry.

This note was sent, apparently, soon after Charles had attempted to sell his manuscript to the American Book Company. For he said that "the American Book Company are favorable to the Geometry; but have not decided as yet to take it; and it is to be sent back to me for some alterations which will only take a few days. I sent it right off to them because I could hear nothing from you and because I was greatly pressed for money."

However, after further reference to his personal difficulties, Peirce once again returned to the geometry theme. Now he says that "Ginn and Company said they were almost certain to take the Geometry and push it (they thought your name at least would appear on the title page) and certainly I do not think that written though it has been under a terrible load of misery anybody can deny it has a great deal of merit .... What I am coming to is, that I want your helping hand in perfecting the geometry very much ...."

On an undated Friday night Charles again wrote to Jem that he was seeking the latter's advice about the Geometry (and other books). "It is a new trade to me. I am most anxious to correct my notions. I am very desirous you should take time to look over the Geometry and a piece of the Elementary Arithmetic I have here. It is a most serious work."

A letter to Jem dated 2 January 1895 describes the Geometry as hanging fire, "though I was sure I was going to get it off at once." Though his personal trials had multiplied he was full of hope as he wrote: "After my Geometry, my Arithmetic of the success of which I have no doubt, and *then* the Philosophy."

It must have been in this period that Peirce sent his brother an outline of his book explaining in an undated letter that he had not yet completed it because of illness.

Great changes were necessary to make it modern. I have a preliminary Book on Logic; but don't feel sure about inserting it. Then a Book on the Fundamental Properties of Space. I. Tridimensionality. II. Continuity. III. All parts alike and that it is conceived by geometers to return into itself, etc. IV. That it has homoloidal lines and surfaces each cutting the others once, and that one system of these, namely *rays* and *planes*, are important owing to the property of light. V. That the motions of rigid bodies are such as to give certain metrical properties.

Then a Book on Topology, chiefly the connectivity of surfaces and Euler's polyhedral formula (which I give in a very extended form, applying to polyhedral nets on surfaces of any connectivity wrapped any number of times by the net, the vertices having any number of whorls and the faces any number of circuits round them). This book is quite elementary and simple and good to begin on.

Then a Book on Projective Geometry. This is as elementary as I could make it, but I arrange to make metric geometry independent of what goes before if teachers want to skip the rest. I give the elements of projective conics. I first study a "phimus" defined as the surface generated by a ray that always touches three fixed rays. This gives me the properties of conics very simply. I use a modified barycentric calculus for nets (*non metric*) and I also use this notation (ab) as the plane of the two rays a and b, (ABC) as the plane of the points A, B, C. (Ab) is the plane of point A and ray b. $[ab]$ is the point of plane a and ray b. $[ab]$ is the point of two rays a and b. $[\alpha\beta\gamma]$ is the point of intersection of planes α, β, γ. $\{\alpha\beta\}$ is the ray common to the planes α and β etc. If rays a, b both intersect both p and q, then

$$\{(ap)(bq)\} = \{[aq][bp]\}$$

This proves Pascal's theorem instantly, from the phimus.

Metric Geometry begins with a bad chapter about the motion of a fluid and how it might be measured. Then [it] shows what the peculiarity of rigid motion is. This gives me a start in the next chapter like Euclid's, putting *pons asinorum* at the base. (He was evidently a student of non-Euclidean geometry.) And I soon get to *direction* as valid in a plane and *aspect* of planes round a ray. For practical postulates I assume parallels can be drawn. It is so easy practically by a ruler and a triangle and more methodical. I also assume a right angle can be made, instead of that a circle can be drawn. But I connect these with the usual postulates. Spherical geometry (which I have not got to) will be made very simple.

The book has some things too advanced for children, but it will be arranged so that these can be skipped. It has a great deal that is very easy. As soon as I am through I shall send it to you.

That Jem was not in a position to cooperate was evident in a letter dated 21 November 1894. It reads in part as follows:

I received today your interesting account of your Geometry. It will be an epoch making book, highly original and fertile. It will be distinctly *your* book, and the embodiment of *your* ideas. It will certainly give new notions to teachers of Mathematics. How far these will stand, must remain to be seen.

For myself, it will be impossible for me to take part in the book as I have already written you. I have neither time nor strength to add that to my other labors. Whatever power I may find for writing must be devoted to Quaternions and to the other subjects which I am teaching in my courses.

It would be impossible for me also to cooperate in a work so distinctively thought out by another mind. But if I can make a suggestion here and there, I shall be glad to do so.

I know nothing of the American Book Co. If they take the Geometry, you had better let them have it at once. It should appear as *your* book and, the sooner the better. I see no reason why the Logic and the Geometry should be in any way tied together or brought out by the same publishers. I hope you can dispose of the Logic at once, and the Geometry too.

I saw Sylvester in London. He has been very ill and has aged. He probably has *Angina Pectoris.* He inquired for you, and spoke most affectionately of father and mother.

Writing in the same vein on 16 February 1895, Peirce told Jem,

Ginn wants a geometry. But a geometry with ideas in it is more than he has bargained for. A Nation's Ideals are bound to be embodied in their system of education; and a people with ideals like ours, — worshippers of Individual prosperity, — will have a system of teaching to match. Such a people rightly feels there is nothing whatever upon which they ought to be so much on their guard as the entering wedge of living thinking .... The fault of my style which I am perfectly aware of lies deeper than you think. It is aggravated by my great use of the logic of relatives which embraces everything together so easily; so that my instinct for how much people can comfortably swallow at a mouthful (never strong) is now ruined. You would have to rewrite, not merely changing wording, but ways of stating reasons.

It is obvious that Edwin Ginn on his part had been writing to the Peirce brothers. In a letter dated 14 February 1895 he was gratified to learn from Jem that Charles's book met with Jem's approval and that Jem "will be willing to aid in the revision of it. We shall be very glad to see the revision of the MS., and there can hardly be a doubt but that we shall want to publish it. If it were not in some respects a matter of form we would settle the question now."

Another letter from Charles to Jem on 3 March reveals that Charles had sent the Geometry to Ginn ten days earlier. Charles continued: "They positively promised me in writing an answer in 48 hours. They have not broken silence and I conclude they don't mean to take it." He spoke of his arithmetic as becoming profitable after a long time, of his Lowell lectures that the *Atlantic* might be induced to publish as articles, of his ability to "make the standard history of science and the possibility of adopting Marie's plan of 12 small volumes coming out singly." Then, again, there is the frustrated cry, "I see Teubner's Mitteilungen 1895 No. 1 speaks of my 'eminent vollkommenen' notation for the Logic of Relatives and gives as one *raison d'être* of Schröder's third volume, that my papers are so scattered. I had a great ambition to some day write a Popular Logic or a Logic for the Million — But I must be upon my guard against things I have an inclination for."

One page of that letter reveals that Peirce has had "a number of opportunities to see that I am a good teacher for those whom the ordinary mechanical system of pedagogy does not reach. My acquaintance with logic and psychology enables me to deal well with such a one ...."

Again on March 11, the troubled Charles states that he was anxious before he died "to get what I can from the geometry, and logic, and map ...."

As to the geometry, I sent it to Ginn and they sent back word that "it seems to be in two parts, one a sort of mathematical philosophy, and the other more properly a text-book. Now the mathematical philosophy would meet with hardly any sale ... but Book V which you call Metrics is a proper text-book in Geometry. It is not only a good revision of your father's text-book, but much more than that. Now why cannot we make this suitable for the use of schools and colleges, publishing it by itself, and test the market with it before undertaking the other?"

They had evidently merely looked at the part you looked at, which was intended merely for reference and had not examined my introductory books of Topology and Graphics. I replied saying that I fully assented to their proposal; but that at the same time I could not but think it a mistake. That nothing could be further from "mathematical philosophy" than my topology which is merely a book of the simplest definitions designed to exercise the imagination, with one single theorem, the correct form of Legendre VII.[25], which was omitted in my father's treatise. That the graphics contained merely the ordinary propositions of graphics, — the simplest of them, together with a chapter on algebra. That I thought very likely I had gone too far in Graphics, while I was led to think *more* likely not, considering most pupils would never have another book on mathematics. Mr. Ginn now writes: "I forwarded your letter to your brother, Mr. J. M. Peirce, this morning and I wish that you and he would largely work out this problem together, for I feel that you can yourselves do it more intelligently than it is possible for us to do it. What we want is a good revision of your father's book, with such additions as will make it even more popular than his was and cause it to meet with even a greater sale, perhaps."

The "problem" to which he refers is whether it is better to prefix the two books of Topology and Graphics (I begin to feel myself the "Fundamental Properties of Space" is deadweight to be thrown away or not). But his meaning is evidently vague. He means also what modifications are to be made. Now I should be glad, very glad, to have your active collaboration. But I don't want the thing to be forever coming to the *paying* point. I want the money, *bad*. If you could sit down and read the topology and then send it to me with remarks both general and particular, why I would proceed to do what seemed necessary to making it acceptable. That instruction in geometry ought to begin with awakening the geometrical imagination, both psychology and experience show. That the first example of proof offered should be a good specimen of real mathematical reasoning and not the kind of thing which astounds the pupil by demonstrating at length something obvious at a glance, wherein be it observed he is thoroughly in the right and the geometry a mere "game" as De Morgan called it, I equally have no doubt. For my part, I don't think it a matter of consequence that pupils should have the quasi-military drill of the class room about reciting this part. Let them miss it, it makes no difference; or is even *better*. It isn't in *that* direction I want their attention turned. I want to get into their vacant, wandering minds some sort of an idea of what geometry is. If I do that I have made a grand beginning, and don't care if they have not a definite idea of the proofs or do not remember the definitions.

Next comes Graphics. If you could spare the time to look over this too long "Book" and give me your opinion I should be glad. I suppose I *have* put in too much. Still, is it not well pupils who never own another book on mathematics should have a little surplusage, to refer to, even to *study* later? The teacher can skip any amount. The metrics is independent of it. But the real utility of the graphics is that the students can have it as a preparatory exercise of their faculties and if they don't carry much away, it makes no difference whatever with their progress in metrics. Will you please return the graphics when you have made your remarks?

The first part of the metrics I fear will have to be sacrificed. I think it is rather interesting; but it is too hard for young pupils. I am rather sorry, because I have spent much thought in trying to put this in a clear shape and it is no good except for an elementary book. But I feel it has got to go. I wish however you would read it *through*. After the hard part comes some easy matter perhaps worth keeping.

The first five chapters of father's book have been pretty thoroughly looked over. I have not changed much; but I have changed some things. In a general way, I thought the later chapters could go, but I daresay they should be examined more carefully. Having examined the metrics will you please send me that and all the rest?

Now should there not be an Appendix on Coördinates? Another on Trigonometry? Another on Brocard Geometry? I think so.

Please remember that Ginn means business; but he wants a *popular* book. Not a baby book nor an *easy* one necessarily, but one that people will think is a good one to teach and a good one to *keep and not sell to the next class*. Remember too that it is the first and last book on mathematics (I don't count Arithmetic) for the majority.

Remember that if I can once make a popular book, I shall have almost *carte blanche* afterwards to write all the profound ones I like. But that popular book must be made *coûte que coûte*. I want your generous aid, and *prompt* aid, about this; but I don't want you to kill yourself over it. On the whole, perhaps it will be as well to have only my name on the title page as editor, that is, if it is going to be a good book. If not, if you were with me, each could say it was all the other's fault.

I asked Ginn and Co. to forward my answer to you after reading.

Your loving
C.S.P.

What, then, had Ginn and Charles been writing to each other at this time? A letter from Charles dated only 1895 March but stamped Mar. 7 at Ginn and Co. reads:

I assent to your proposition to publish the Metrics alone, as contained in your letter which I received this PM. But I do not *approve* of it.

One book, that on the Fundamental Properties of Space has the character you mention of being concerned with the "philosophy of geometry." But it has always been considered as proper to insert in the elements the Postulates and Axioms. This book is the same thing in a modern form. I go on the principle it is wrong to teach as true what is known to be false. Among advanced mathematicians the axioms of geometry as self-evident truths have long been completely exploded. What takes their place is what I have drawn up under the head of *The Constitution of Space*. The body of the book is devoted to a clear explanation of that "Constitution." But in my Preface I expressly say that Book on the Fundamental Properties of Space should only be used for reference.

The study of the book ought to begin with Topology. There is nothing of the nature of "philosophy" in my book on topology. It consists wholly of Definitions and figures designed to awaken the pupils' mathematical imagination, together with one single proposition. That proposition is substantially that of Legendre, *Eléments de géometrie*, livre VII, Prop. XXV which Chauvenet used, *Treatise on Elementary Geometry* (1st Ed. I have not Byerly's) Book VII Prop. XXXII §97. My father omits it, probably because he thought the proof unsatisfactory. It can hardly be that in 1837 he was aware it was false. My proposition is simply the truth of the matter. It has generally been considered germane to the elements, and *should not be omitted.* I put it first as being far the simplest, and I do not make a parade of forms of demonstration, because the pupil must learn *one thing at a time*, and at this stage it is most essential that demonstration be kept in the background. Otherwise he goes through life with the absurd idea that demonstration is the main element of mathematical thought, — an idea exploded by all modern mathematicians but very rife among the ungeometrical. I have had great success in teaching dull boys, especially in geometry. The reason is I am so well acquainted with psychology and logic, and I know the first thing is to excite them to *really think*. During that part of the instruction they seem to make little progress; but it is really by far the most important step of all they are taking. The proposition I have put first is, I am fully convinced, after experiment and reflexion, quite the most suitable to be put first of all the propositions of geometry. At any rate it must come in somewhere. There is nothing "philosophical" about it. On the contrary, it is peculiarly elementary. But by giving a separate book to it, those teachers who choose to do so can make it the *last* proposition by simply teaching topology *after* metrics. I could insert a suggestion to that effect in the preface. Let me beg you to read that over again and to ask yourself if what I say of it is not true.

The other book which precedes metrics is graphics. There is certainly no "philosophy" about it. It contains merely the elementary propositions of graphics. Perhaps it goes too far. About that I was doubtful. I reached the decision I did by considering that many pupils will go no further in mathematics than this treatise and I had better insert all that will be practically useful to them and which is not too difficult. I am decidedly of [the] opinion graphics ought to be studied, because of its extreme utility, and because further exercises of the generalizing power and of the imagination are requisite to successful teaching of geometry before the element of demonstration is emphasized as much as it is in metrics. (By the way, you speak of *my* "calling" it "metrics," as if that were not the recognized term.) I am not at all unfavorable to cutting down what I have given of graphics, although my individual preference would be for retaining it.

Nevertheless, although my experience of teaching is that the worst results are obtained by beginning with metrics and the best by taking topology, graphics, metrics in that order, yet, after all, I want the book to sell, and if you think the old stupid way, which makes a large percentage of the scholars dunces, is the best way to sell the book, I consent reluctantly to the excision of the parts you do not like. All the more, that I shall expand them into another book for use by the more intelligent of the teachers. Or, if you like, those books can be made appendices in large type with a recommendation in the preface to begin with them.

I should incline to another appendix in smaller type on Brocard geometry, which is all the rage now.

I shall be happy to draw up about a hundred problems in geometrical construction. More if desired.

Now I leave you to make such decisions as you like. I dare say J.M.P. would lend his judgment if desired.

I have half a mind to insert at the end a simple treatise on trigonometry as Legendre and others have done.

I shall be glad to hear from you as soon as may be.

A footnote warns that "Almost any such change as you propose will require a not long introductory chapter to be inserted."

By 11 March 1895 Peirce wrote directly to Ginn and Co. again asking that this letter be forwarded to J.M.P. Since Jem was a successful professor of mathematics at Harvard University, Ginn was, of course, anxious to have his name on the title page. And so Charles says,

I shall be very glad if my brother's health and leisure are sufficient to enable him to collaborate in any perfectionment of my revision of my father's geometry, of which you express so favorable an opinion. I used every argument to persuade him to help me about it; and he did go so far as to promise to do so; but he found it impossible and had to give it up. I don't want him to undertake what he cannot perform promptly, because it would only be a drag upon him. If he would take hold of the thing, — although you think it so well done, — and would make such changes as he thought proper, I should think it a great advantage. He has made some examination of the MS. and has expressed his wish the thing should appear substantially as it is. But if he would even do so much as to read through my two books of Topology and Graphics, and make any positive suggestions, it would, I am sure, be very useful. You know what I think of those books. Namely, I consider the book on Topology consists of nothing but the easiest definitions with one single proposition, and that the easiest one (not puerile) in geometry. The graphics is more advanced but the bulk of it is in my opinion easier than and a proper and necessary introduction to Metrics. I may have carried it too far; but it is easy to omit the more difficult parts. I have constantly borne in mind that a geometry is the last book on mathematics most of the pupils will study, and therefore it ought to include as much that is useful and not too hard as possible, even if the teacher chooses to skip a good deal.

My father's little treatise dates from 1837. By the time I was a schoolboy, his views about teaching geometry were so decided that he interfered with the course in my school to insist that before I was put into his book or into Legendre, I should go through a book containing substantially the same matter as my books on Topology and Graphics.

I have talked much with mathematicians on the subject. I think they generally, almost unanimously, warmly agree to this.

I have repeatedly been pressed to write for educational magazines etc. on the teaching of geometry. I have always postponed it until my book should be out. I shall then let drive and shall preach to all the teacher's conventions.

The old way of teaching is a total failure with a considerable percentage of pupils. The modern method, — not absolutely new, for there are several treatises based on these views, — has never failed in my hands, though I have had some pupils considered quite incapable of mathematics.

We are a strong party. We have modern mathematicians, modern psychologists, and modern logicians with us. We will blow the old system out of water.

One firm of publishers, — I mean, a firm of powerful American schoolbook publishers, — will have a modern treatise soon.

Nevertheless, I am not obstinate if you think the additional chapters will stop the sale of the book or injure it, let us cut them out. I had that possibility constantly in view in writing the thing, and have an introductory chapter all prepared for that contingency.

I don't consent to any changes in my book on the Fundamental Properties of Space. That is meant as explained in the preface *for reference only.* I will consent to omitting it altogether, or I will substitute for it such twaddle as may be thought more salable.

I fully consent to the omission of the Topology and Graphics. I only say I am confident it will be a capital mistake. But I dare say I have gone too far in graphics. Of course, I *think* not; but it is a doubtful thing in my own mind. What I feel most misgivings about is the first part of the metrics. But the thing is written so that it can be omitted. There is just now a perfect rage for the Brocard geometry. I don't share it particularly; but I think it might perhaps be well to give a taste of it. I have lots of construction problems.

One wonders if the introductory chapter referred to above contained the bitter fragment found in MS. 153 which reads:

Circumstances have unfortunately compelled the editor to publish this edition, not as the author would have desired it revised, with introductory books upon topical and graphical geometry, but in the miserable shape in which it appears. A number of treatises already present the subject somewhat as it should be treated, and are meeting deserved success. The editor will at the earliest possible moment offer the schools his *New Elements of Mathematics*, which his experience in teaching leads him to hope may do away with the phenomenon of boys destined to distinguish themselves as sound thinkers unable with their utmost diligence to make any progress in the intelligent study of geometry.

As is now apparent Peirce's determination to give a modern rendition of the geometry could easily have been reinforced by the materials then being published in the *Bulletin of the New York Mathematical Society.* For example in vol. 1 (1891-1892) one finds "The Teaching of Elementary Geometry in German Schools," a review by Alexander Ziwet of *Inhalt und Methode des planimetrischen Unterrichts* of Dr. Heinrich Schotten. Ziwet speaks of the sixty-year-old agitation for reform in teaching of geometry in Germany, of the publicity given it there, and the hope "that the *Bulletin of the New York Mathematical Society* may, in the course of time, perform a similar service towards the improvement of mathematical instruction in this country." According to Ziwet, Dr. Schotten held that "the object of mathematics teaching in the *Gymnasium* is not to produce mathematicians but to improve the mind, not only by training in logical thinking, but by accustoming the student to precision of language in writing and speaking, by awakening his self-activity through the solution of problems, and in the case of geometry in particular, by forming and practicing the power of mental intuition ('Anschauung')." He advocated the introduction of simple ideas in the projective geometry which must be an integral part of the whole system. The *Bulletin* did take up the cause — for example in publishing T. H. Safford's article on "Instruction in Mathematics in the United States" in October 1893. Peirce was thus exposed to the new radical "reform" thinking that was already in the air at that time.

After writing the letter to Charles on 8 March which Charles had quoted to his brother on 11 March, Edwin Ginn wrote to Jem on 14 March, expressing again his hope that the brothers would cooperate on the geometry venture. Ginn continues:

I am wondering whether it would not be well for you to consult some of your friends at Harvard who have some of the Geometry work to do, and get any suggestions you can in regard to the new book. Would it not be well for us to consult some few leading teachers in other parts of the country, so that the book may not only be suited for Harvard, but be adapted to a broader field. Would it meet your views to see other people? I feel that you will be the practical person in the issue of this book, that your brother has had less to do with teaching, has less knowledge of the real wants of the schools, and is perhaps more theoretical in many ways. I hope you will be able to give the necessary time to pruning his work and putting it in such shape as to make it meet a good degree of popularity.

Under such pressure, Charles was willing to make almost any concession to get the material printed at last. We find him writing from his home in Milford, Pa., on 22 March in these terms:

My brother sends me yours of the 18th; and both your propositions please me. The Metrics, as I have left it, is much too hard to begin with. Even my preparatory books my brother finds too hard. While I disapprove teaching geometry in that way, as do all recent writers on geometrical teaching, except that precious Society for the Improvement of the same, I have no doubt you are right that many schools will like it. However, they will *not* like, — they are just the people who won't like, — my father's short cuts in reasoning, — often logically indefensible, too (but they wouldn't always have heard that said). Therefore a new series of introductory chapters will be necessary; after which I will imitate Euclid by putting all the fundamental propositions (as I have done, but in too abstruse a way). Then most of my father's chapters with merely such changes as are necessary to meet the taste of bad teachers today. An appendix will contain about a hundred exercises.

I will abide by any understanding my brother may reach with you concerning the royalty or copyright-price. It will be convenient for him alone to sign with you. Some agreement reached, you can announce the book at once. My work is largely done in my papers here; so that the rest won't take but a short time.

Now as to the Topology and Graphics, my brother thinks parts of the former too hard. I will not have that said of it, if I have to excise the heart of it.

I shall at once proceed to rewrite it, mingling with it a boy's story. It will represent conversations on imaginary practical questions in geometry with accounts of experiments. I shall call it *Euclid Easy*. One advantage of writing in this style is that it enables me, without offence, to hint to the teacher every detail of teaching. It will afford formal statements to be learned; and that is what teachers and pupils always conceive to be the essence of learning. At the same time, the real education will be slipped in before they know it. My wife will be glad to pay for the plates, provided you will promise to take hold and push the thing energetically and in preference to any other geometry, unless the other one, which in my opinion, will soon be out of date anyway.

When it is written, I propose to get up a geometrical crusade, and I know where I can get powerful support. I want to make a hundred addresses to teachers on the subject.

I will send you a specimen from which you can judge of it, and then my wife's lawyer will send you a contract. I have tried my hand at the kind of writing I propose, to the great delight of practical teachers, and I have no doubt the thing will pay, *apart* from its use in schools. But if my wife assumes the bulk of the expense, naturally she will have to have the bulk of the profit.

Although this is not the place to discuss the purely personal circumstance in Peirce's life, let it be said that during these months the details of his personal misfortunes were harrowing and in an undated letter to Jem, obviously from this period, Charles laments, "this geometry I thought so easy has been my final ruin." In another note he complains of illness "owing to insufficient food and that is why the Geometry is not yet finished." Again, "I naturally became sick and my mind all a blur at a moment when my best powers were wanted to finish up my geometry." Yet on 30 August 1895 Ginn & Co. sent the last letter on the subject. It referred to a letter from Charles on the 27th and spoke of Charles supplying a list of professional men to whom the book might be sent and also of the preparation by Peirce of copy for the diagrams.

One senses from the letter of 22 March that negotiations were already grinding to a halt. Yet on 28 March it was suggested that C. S. P. publish the Metrics only. In that case an announcement could be put in the "forthcoming catalogue just ready for the press." The end finally came in a letter from Jem to Charles which read:

I send you a letter received today from Ginn, enclosing your MS. of "Euclid Easy." He does not appear to take much interest in it. The main idea seems to me excellent. But I can imagine that some of the decoration may appear questionable to a publisher ....

I will return the remainder of your MS. What Ginn wants is obviously merely the revised form of Father's book, and any argument on the subject from a higher standpoint (and still more from a lower one, which he thinks he understands) is thrown away on him.

I might undertake to pay for the plates of the first two books, Fundamental Properties of Space, and Topology, on condition of receiving a share of the royalty. But I must warn you that it is, in my judgment, idle to expect any large return from it. I do not understand why you have turned aside from the Arithmetic.

The letter from Ginn to Jem is strangely dated 25 April. Ginn had sought advice on the "Euclid Easy" and found it impossible to recommend the publication of a work of this kind without seeing the whole of it. "Euclid Easy" (MS. 268) seems to have terminated the Ginn involvement in the geometry publication.

Years later, as late as 4 December 1904, Charles was still writing of completing his Arithmetic and his Logic. Moreover one wonders about the extent to which MS. 137 (Topics) was involved at that time as he expressed the wishful hope: "If only I could get a month to write out my memoir on Topical Geometry people, I think, would be impressed with the desirability of having me where my work could be properly done." The following notice in the *Nation* of 14 July 1898 reveals the depth of Peirce's commitment to the cause of the proper basis for geometry instruction, a stance he maintained to the very end.

It is a little strange that, after a generation of celebrity, Reye's "Geometrie der Lage" should now be translated into English for the first time. Part I. comes to us from the Macmillan Co., Prof. Holgate of Evanston being the highly competent translator. The original has long been used in some of our American universities to great advantage. In certain respects it is a more brilliant book even than the treatise of Cremona, and it covers a somewhat wider field. But its merits are too well known to need any comment from us. Later researches into continuity go to show that Topology and not Graphic forms the real foundation and generalization of geometry; and the moment is almost at hand at which Reye's book must be superseded by one which shall lay the foundations of its logic deeper still. Meantime, this well-executed translation, with a useful preface, will serve a good purpose. We shall speak more particularly of the version in noticing the following part.

As for *The Elements of Mathematics* (MS. 165), Peirce mentions it again in MS. 229 which is included in an appendix of this collection. It is mentioned also in MS. 517 (c. 1903?) in which Peirce tells of having devoted a year to it, of its having reached three publishers, and of its having been lost. The editor believes that this carefully designed and logically perfected presentation was well advanced before the opportunity for publication of a revision of Benjamin's *Geometry* was offered by Mr. Ginn. It contains the kind of logical introduction that Peirce said had been deleted from MS. 94. The editor believes that when George A. Plimpton, an associate of Ginn at Ginn & Co., indicated to Charles that he "wants *us* to do it," Charles rewrote the geometry part of MS. 165 into the present Books of MS. 94 and attached Benjamin's book to all as a kind of Appendix. If that was the case pages were later detached from MS. 94 and inserted with new pagination into MS. 165 as is indicated in the relevant footnotes of the present edition.

CONTENTS

1

ELEMENTS OF MATHEMATICS (165)

PREFACE (1) (164)

This text-book contains all the mathematics which can conveniently be taught in the common schools, excepting practical arithmetic. It is so written that any one of its principal facts may be omitted in teaching. Teachers need not be informed that there is no elementary subject which has advanced so much of late years as geometry. No recent writers approve of putting beginners into metrics at the outset; but as circumstances sometimes compel the use of the old system, this work has been so arranged that the preliminary work, which saves a great deal of time in the end and often makes the difference between learning geometry agreeably and never learning it at all, but only learning to loathe it, can be passed by, if need be.

To what can be taught in the schools will here be found appended further information sure to be useful to the scholar in after-life, if he has prudently preserved his school-books....

PREFACE (2) (94a)

Benjamin Peirce's *Elementary Treatise on Geometry* was published in 1837, and since he had been teaching the subject in Harvard College (beginning two years before his graduation) for ten years, it is safe to infer that the substance of the work had been in his mind for a long time.

The whole aspect of the science has been metamorphosed since say 1830 or 1835. In the first place, the nature of the hypotheses (that is, the Axioms and Postulates, together with other propositions virtually taken for granted in the old books without explicit statement) is differently conceived. We all see, now, that geometry has two parts; the one deals with the *facts* about real space, the investigation of which is a physical, or perhaps a metaphysical, problem, at any rate, outside of the purview of the mathematician, who accepts the generally admitted propositions about space, without question, as his *hypotheses*, that is, as the ideal truth whose consequences are deduced in the second, or mathematical, part of geometry.

In the second place, Listing and others have created the topical branch of geometry, which studies the connection of places. This branch deals with only a portion of the hypotheses accepted in other parts of geometry; and for that reason, as well as because of its relative simplicity, it should be studied before the others. Moreover, it is most desirable that, before the scholar comes to the difficulties which in the old system meet him at the threshold of geometry, he should have had some previous training leading up to the mental effort which he is then called upon to exert.

In the third place, although in 1830 not a little had really been done by individuals toward restoring that *graphical*, or projective, or intersectional, branch of geometry, which formed so prominent a part of the ancient science, yet it remained, for the general world of mathematicians, a closed book. It has since become the most prominent part, not only of geometry, but of all mathematics. For more than forty years, now, geometers have clearly perceived that metrical geometry is but a special problem of graphical geometry. There is reason to suspect that in the school of Euclid some instruction in this branch preceded the study of the Elements. At any rate, in its modern development, some slight treatment of it, so far, at least, as to explain the practically important principles of Linear Perspective, is a needed prop[ae]deutic to metrical geometry.

In the fourth place, the whole conception of metrical geometry has been revolutionized. In 1837, the "Doctrine of Parallels" formed an urgent but unsolved problem. The earnest and persistent efforts of Legendre and of many other eminent mathematicians had been powerless to clear up its difficulties. Benjamin Peirce, in his treatise, thought to conquer them by the introduction of the idea of a *difference of direction*. There can be no intelligent question that this idea brings strong help to mathematical inquiry. Soon after its introduction by Peirce, it was taken up by Hamilton and by Grassmann with such effect as might have been anticipated. But Professor Peirce himself subsequently admitted, with the rest of the mathematical world, that there was no solution for the question of parallels except from the idea which the pupils of Gauss, Bolyai, Lobatchewsky, Riemann, derived through their master, from Lambert, and ultimately from the Italian Jesuit, Saccheri. The idea was that it is simply a question for observation of nature whether the sum of the angles of a triangle is less than, or more than, or possibly equal to two right angles. Subsequently, Cayley (in 1854) and Klein (more fully, in 1873) showed that metrical geometry is simply the geometry of the *firmament*, or *absolute*, or infinitely distant part of space, which constitutes a surface which is one or another quadric surface, according to the system of measurement adopted, that is, according to the way in which rigid bodies move. In the course of this inquiry, a fallacy in Euclid's 16th proposition was brought to light that had remained undetected for two thousand years.

In the fifth place, Georg Cantor and others have succeeded in analyzing the conceptions of infinity and of continuity, so as to render our reasonings concerning them far more exact than they had previously been; and the fundamental researches that have largely occupied mathematicians of late years, into the theory of functions, do much to render geometrical reasonings more exhaustive and precise.

All these intellectual movements ought, in the opinion of very many mathematicians, to have their effects upon the system of teaching the elements of the subject.

But this is not all. Pedagogy is an art which has come in the last sixty years to be based more and more upon modern scientific psychology, and upon modern views of logic. If diligent and intelligent youths find difficulty in understanding mathematics, teachers no longer deem it becoming to flog them or to objurgate them, as they used to do, in the days when the second theorem of Euclid received the name of the "Asses' Bridge." On the contrary, they consider the fact that a considerable percentage of the best minds imagine themselves to be utterly unable to comprehend mathematical reasoning to constitute an emphatic condemnation of that old system of teaching which had such a result.

Teachers now see that the difficulties of the first steps in mathematics must be divided and conquered, one by one. The three functions of the mind that are exercised in mathematics are exact reasoning, mathematical imagination, and complex generalization. The first of these, the logical part, is best acquired in the study of the theory of number, because that subject involves little other difficulty. It calls for but slight efforts of imagination and of generalization. Topology, or connective geometry, is the best field for the growth of imagination, demanding little logic and not very much generalization. Graphics, or projective geometry, carried far enough, affords good training in generalization. When some familiarity with the business of the mathematician has been acquired by such studies, metrical geometry may be taken up without fear that the student's mind will be confounded by its aggregation of various difficulties.

This volume is intended to contain all the mathematics (except practical arithmetic) which is necessary for a man with a good common school education, and at the same time, to give the thoughts of the student such training as may prepare him for a study of the higher mathematics.

Most of the text books of geometry have contained some algebra. Euclid's Elements is more than half devoted to that branch. Benjamin Peirce's work has a brief algebraical introduction. The present volume gives all the algebra which is indispensable to an ordinary man.

INTRODUCTION, ON MATHEMATICS IN GENERAL

Art. 1. *Definition* 1. *Mathematics* is the science which draws necessary conclusions.

Gloss 1. "Mathematics" (pronounced mathemat'iks, as it is written in Ellis's Glossic) is a noun plural in form, but now used with a verb in the singular number. A few old-fashioned people still say "the mathematics are an important science."

> The Mathematickes, and the Metaphysickes
> Fall to them as you finde your stomacke serves you.
> (*Taming of the Shrew*, I.i)

Earlier writers use the singular form, *mathematike.*

Gloss 2. Names of sciences ending in "-ics" are imitated from the neuter plurals of Greek adjectives, the word βιβλία, books, being understood. The Greek name of this science is τὰ μαθηματικά, or, more commonly, ἡ μαθηματική, where τέχνη, craft, or some such noun is understood. The adjective is formed from the noun μάθημα, lesson, from μανθάνω (μαθεῖν), learn. The root is the same that appears in the English "mind."

Gloss 3. With the Alexandrine Greeks the name mathematics (ἡ μαθηματική) came to be applied chiefly to astronomy. Thus, Ptolemy's text-book of that science, known as the Almagest, has for its genuine title Μαθηματικὴ σύνταξις (Mathematical Compendium). In the middle ages, by a "mathematician" was understood an astrologer, or even a sorcerer. So late as Queen Elizabeth's time, the learned Dr. Dee was practically driven out of England, because he was reputed a "mathematician."

In the Roman and Medieval schools, the mathematical sciences, — that is to say, the four they studied, arithmetic, geometry, music, and astronomy, — were collectively called the *Quadrivium,* a fanciful name signifying literally the four roads.

Memorandum 1. The definition of mathematics here given was originally put forward by Professor Benjamin Peirce, in 1870. It has been received with much favor by mathematicians.

The only definition which seems to have been current in the more ancient times was that of Aristotle, which was an attempt to describe the peculiar kind of abstractness and ideality of the objects of mathematical study. Its significance can only be apprehended by a person thoroughly versed in the Aristotelian philosophy. Its wording is that mathematics is the science of forms immovable but not separable from matter. This definition is laudable in recognizing that mathematics is characterized by a peculiar abstractness and ideality of its objects; but it errs in implying that metaphysics carries those qualities to a higher degree, as we shall see.

The Roman school-masters, after much of the Greek geometry had been almost forgotten, defined mathematics as the "science of quantity." But what they meant was simply that the objects which mathematics studies have magnitude, which is equally true of the objects of almost all, if not of all, sciences. The phrase subsequently became very popular as a definition of mathematics in another sense, namely, that mathematics is the science of quantity itself. It must be admitted that mathematics deals very largely with quantity, and embraces the whole science of quantity; and most mathematicians feel more at home in reasoning about quantity than about anything else. Yet two of the main divisions of geometry have nothing to do with quantity, there are other branches into which it does not enter, and several in which other elements play important parts.

During the nineteenth century one of those two branches of geometry, graphics, became the chief interest of geometers, while the other, topology, has also been zealously pursued. Moreover, the attention that came to be paid to non-quantitative considerations even in metrical subjects threw the popular definition in disfavor among mathematicians.

One of the greatest of them, Hamilton, with the concurrence of the eminent mathematical logician, De Morgan, and of others, then proposed to define mathematics as the science of time and space, being influenced in this by the philosophy of Kant. But Hamilton's definition is, by far, the most objectionable of all that have been widely in vogue. For it implicitly denies the main characteristic of mathematics, namely that this alone among the sciences makes no researches into facts, but attends solely to ideas, without seeking to establish their truth. The doctrine of Kant that Space and Time are not real entities, contrary to Newton's inference from the phenomena of dynamics, is much discredited by modern inquiries. But even according to that doctrine the properties of space and time are truths, though they be mental truths, and are not hypotheses as the mathematician treats them. Since the mathematician treats them as hypotheses, and does not inquire into their accuracy, for him they are no more than hypotheses. His studies are not inquiries into the truth about Space and Time. Hamilton called algebra the science of Time. But the most remarkable characteristic of time, namely, that the passage from the past to the future is qualitatively different from the passage from the future to the past is not represented in algebra. The chief study of the algebraists is a two-dimensional continuum, strikingly unlike time. So the higher geometry deals largely with characters to which, confessedly, nothing in real space corresponds. It is, moreover, demonstrable that arithmetic is valid, quite irrespective of whether the objects counted are in Time or Space, or not. This remains true though it be granted that Kant is right in holding that all our positive knowledge of numbers is of numbers in Time and Space.

Scholium 1. The occasions upon which the mathematician's art is called to our aid are those upon which we find ourselves imagining so complicated a state of things that we cannot clearly make out what all the consequences would be. For example, suppose we are proposing to build a suspension-bridge, which is a long heavy structure, such that equal lengths weigh nearly equal amounts, and which is hung by numerous suspenders from cables attached to towers.[1] Suppose that, in the course of building such a bridge, we propose to allow a wire to hang in a festoon from tower to tower, so that it will touch one of the bridge-cables at its two highest points and also at its lowest point. Now, should we have occasion to know whether that festoon will touch the bridge-cable all along, or will be everywhere above it, or everywhere below it, or will cross it, we ask a question which only a mathematician can answer. If, then, we are not ourselves sufficiently expert mathematicians to solve the problem, we propose it to a mathematician. But as soon as the mathematician examines almost any of the questions that are propounded to him, he is pretty sure to find that the real facts are so vastly complicated, that the real problem can, practically, not be solved with exactitude. For

[1] This illustration calls to mind Peirce's employment as consultant by Morison in the construction of a bridge. See Volume 3, 16a. In MS. 165 Peirce has a note to "Insert Fig. 1, Parabola and Catenary" for this passage. He failed to supply the diagrams but, as is well known, the catenary idealizes the loose festoon before attaching the roadway; the parabola after the attachment.

instance, in the case supposed, there is reason to believe that the wire really consists of molecules which are not nearly in contact with one another and which are dancing about among themselves in a most intricate figure. To take account of that is beyond the mathematician's powers. Accordingly, the first thing he does is to imagine a state of things different from the real state of things, and much simpler, yet clearly not differing from it enough to affect the practical answer to the question proposed. That purely ideal state of things he proceeds to study. In the case of the bridge, the ideal wire will be continuous and homogeneous, and the ideal circumstances will differ in many other respects from the concrete reality. The mathematician now proceeds, by a method which will be described in Art. 2, to ascertain how the ideal festoon and ideal bridge-cable would be related to one another. In doing this, he not only finds out, with sensible accuracy, what will happen in the actual case, but also produces a *rule* by which other similar questions may be answered. He is, afterwards, generally led on by curiosity to consider what would happen under circumstances differing still more from the actual case.

We see, then, that the mathematician's duty has three parts, namely,

1st, acting upon some suggestion, generally a practical one, he has to frame a supposition of an ideal state of things;

2nd, he has to study that ideal state of things, and find out what would be true in such a case;

3rd, he has to generalize upon that ideal state of things, and consider other ideal states of things differing in definite respects from the first.

This description of the mathematician's duty gives the best notion of what mathematics is: *it is the exact study of ideal states of things.* To say that mathematics is the science which deduces necessary consequences comes to much the same thing; because no necessary consequence can be deduced except from an ideal state of things, and then only on condition that the state of things is generalized.

Definition 2. A *mathematical hypothesis* is an ideal state of things concerning which a question is asked.

Art. 2. *Scholium* 2. Experience shows that careful attention to the method of mathematical thought contributes not a little to success in the practice of it. It is like swimming. A person may know how swimming is done, without being able to do it; and, on the other hand, a person may swim like a South Sea Islander, without knowing how one does it. But nevertheless, a careful study of the *modus operandi* will almost always improve one's swimming, and will help very much in overcoming the first awkwardness.

One important part of mathematical thought, often mistaken for the whole, is the process of deducing the consequences of hypotheses. To show the way in which this is done, we may examine a little mathematical investigation to which many small boys upon their own motion apply themselves, — we mean the analysis of the game of Tit-tat-too. This game is played by two persons, playing alternately. We may call the first player X, the second O. They use a square board, or diagram, of nine equal square compartments. We may number these squares as shown in the figure. [Fig. 1]

1	2	3
4	5	6
7	8	9

Fig. 1

Each play consists in occupying an unoccupied square, by the player's putting his mark in it. The object aimed at is to occupy the three squares of a line, that is, one of the following triads:

1, 2, 3;
4, 5, 6;
7, 8, 9;
1, 4, 7;
2, 5, 8;
3, 6, 9;
1, 5, 9;
3, 5, 7.

As soon as a player does this, he points at the three occupied squares, one by one, saying "tit" at the first, "tat" at the second, "too" at the third. This is called "making tit-tat-too," and wins the game.

The mathematical analysis of the play consists in ascertaining the effects of the different ways of playing. It is accomplished by experiment. The first player has at the outset the choice of 9 squares, of which he can occupy any one; at the second play there are only 8 to choose from; at the third play there are only 7, etc. And thus there are only $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ ways of filling the board; or [362880] ways in all.[2] The performance of as many of these [362880] experiments as

[2] Peirce wrote 725760.

might be necessary to bring each imaginary game to a conclusion would be one way of conducting the mathematical investigation. It is the type of all mathematical deduction, the substance and staple of which is in every case *experiment*. But mathematical experiments are peculiar, in that the results of them depend upon the nature of diagrams, or "constructions," of our own creation, instead of depending upon the natures of cosmical creations.

Though we must allow the performance of those [362880] experiments would be a mathematical proceeding, it would be an unskillful one. Mathematicians always abridge such lengthy operations. This they are enabled to do by the fact that such lengthy operations always involve many repetitions of essentially the same experiment.

For instance, one experiment which we perform so quickly in imagination that we are apt to forget any experiment was needed to ascertain the result is as follows. Suppose that of a line of three squares one of the players has already occupied one, while the other two remain vacant; and suppose that that player now occupies one of the vacant squares. Then, unless the other player at once makes tit-tat-too, or else occupies the remaining vacant square, the former player can at once make tit-tat-too. This being ascertained once for all, if we bear it in mind, a vast number of experiments among the [362880] become superfluous. Such a rule, established once for all by experimentation upon diagrams, and enabling us to dispense with many repetitions of essentially the same experiment, is called by mathematicians a *theorem*. (A formal definition of this word will be given in *Art.* 11, *Def.* 8.)

If one player has occupied two squares of a tit-tat-too line, while the third is vacant, we will call that a *peril-row*, because the other player must instantly fill the vacant square, or the first can make tit-tat-too, as soon as his turn comes. Herein, this little game illustrates one of the most valuable rules of life, that a difficulty promptly met is no difficulty at all.

Let us suppose that both players use sufficient care instantly to fill peril-rows when they can. Then since the three squares of a tit-tat-too line, if occupied at all, must be occupied one after another, so that, at some stage, two of them must have been filled while the third remained vacant, it follows that tit-tat-too can never be made at all, except when situations are reached in which players *cannot* fill peril-rows. But the player whose turn it is has a right to occupy any vacant square he chooses. How, then, can it be impossible for him to fill up the one vacant square of a line? It evidently *cannot* be impossible for him to do that. Only, his adversary may have occupied a square which belongs at once to two different tit-tat-too lines, on each of which he has already occupied a square and on each of which the third square is vacant. In that case, though either peril-line may be blocked, both cannot. In that way only will tit-tat-too be possible, supposing both players fill up peril-lines when they can.

We thus see, that tit-tat-too can only be made when the party whose turn it is to play has already occupied one square upon each of two intersecting tit-tat-too lines, while the other three squares (the one at the intersection and one other on each of the lines) are vacant. This is another *theorem*, productive of great economy of experimentation.

N.B. (1) This theorem well illustrates the extreme care that is necessary in mathematical reasoning. An uncautious thinker might say that in order to make tit-tat-too, it is necessary to occupy the three squares of the line successively, and to give the adversary a turn after each one. Now, after two have been occupied, the adversary has it in his power to prevent the completion of the row, by himself occupying the third square. *Observe that this is strictly true.* Therefore, the row cannot be completed, if the adversary uses ordinary care to prevent it. Therefore, tit-tat-too cannot be made, if the adversary uses ordinary care to prevent it. All this reasoning is correct except the very last step. It is true that no one particular row can be completed, if the adversary chooses to prevent it. But it does not follow that *no* row can be completed if the adversary does his best to prevent it. For though *any one* can be blocked, it may be that it is impossible to prevent *one or other or two* from being completed.

Such slips in reasoning are hard to avoid; but they can be avoided; and the best method of avoiding them shall now be set forth. What is true is, that, taking any row whatever, a way can be found for the first party to play, so as to prevent the second party from completing that row: but what is false is, that a way can be found for the first party to play, such that, taking any row whatever, that mode of play will prevent the second party from completing that row. The point to be observed is, that a statement of the possibility of the first play's making tit-tat-too has to deal with a number of rows and a number of ways for that party to play; from the number of rows any one, *no matter what*, is to be taken, while from the number of ways of playing, a *suitable* one is to be chosen. Now, whenever we have to deal with two collections from one of which any one that comes is to be taken, while from the other a suitable one is to be chosen, it is (or may be) an advantage, serviceable to the purposes of the person who is to make that selection to postpone making it, until he sees what one of the other collection is taken. Let the reasoner, then, be always on the alert for the case of two things being said to be taken from two collections in different ways; and be careful in every such case to note which is said to be taken first. Thus, he will avoid a dangerous snare of reasoning. For example, choosing first any man whatever, a woman [is] then [to] be found (among living or dead) such that that woman was that man's mother. But it does not follow that a woman can be found such that, subsequently taking any man whatever, that woman was that man's mother. "Any man has some mother," is true; "all men have some mother," is false. "Every man loves some woman," ought to be true; "some woman is loved by every man," is another proposition.

If the reasoner is the master of this principle, as soon as he finds that the completion of any one tit-tat-too row can be prevented, he will ask, "Can there, then, be simultaneously *two* rows that require immediate blocking? If it is *not* so, tit-tat-too cannot be made, against ordinary care. But if it be so, those two rows must have reached that stage at one play, since the need for blocking each is *immediate*. Yet one play can fill but one square. Then, the square occupied by the play that brings about the fatal situation must belong at once to both rows, that is, must be their square of intersection. That is to say, tit-tat-too can only be made by a player who has occupied the square of intersection of two peril-rows, made by such occupation.

We may call such a situation, after the square of intersection is occupied, an "*A* situation," and before the square of intersection is occupied, an "*H* situation."

There will be no tit-tat-too, unless, after one party has created an *H* situation, the other plays outside of its rows so as to allow the *A* situation to be completed. Now, an *H* situation has three vacant squares in its rows, and counting the one upon which the other party plays, an *H* situation leading to tit-tat-too supposes four vacant squares. Therefore, the first player cannot make tit-tat-too unless he creates an *H* situation at his third play at latest. For then he leaves just 4 vacant squares. The second player always leaves an odd number of vacant squares; and therefore he cannot make tit-tat-too unless he creates an *H* situation at his second play. For then he leaves 5 vacant squares.

The number of experiments is greatly reduced by these considerations, and still further by considering that if there is an imaginary line such that all the squares occupied by either player are symmetrically placed to the right and left of it, it is unnecessary separately to consider the occupation of two vacant squares symmetrically placed on the two sides of that line. In such case[s] we may write the sign = between the numbers of the squares to show their equivalence.

Let us now proceed to make a partial analysis of the play, in order to show the scholar the advantage of a systematic procedure in mathematics.

We will record the supposed play by drawing a line as for a fraction and writing above it the square occupied by X and under it that occupied immediately after by O. If a play is *forced*, that is, is the obligatory filling of a peril-row, to prevent immediate tit-tat-too, we write a semicolon after its number. At the end, X!!! or O!!! shall mean that X or O makes tit-tat-too; while a curved dash shall show that neither party makes it.

H-situations are of different kinds. If the two occupied squares are on only one pair of intersecting lines elsewhere vacant, the situations may be called a *simple H* [Fig. 2].

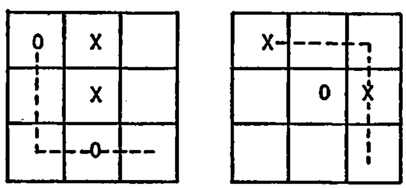


Fig. 2

If there are four such lines intersecting in two vacant squares, the situation may be called a *double H* [Fig. 3].

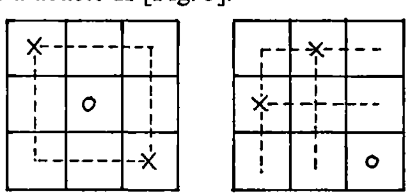


Fig. 3

If there are three lines intersecting in two vacant squares, the situation may be called a *semidouble H* [Fig. 4].

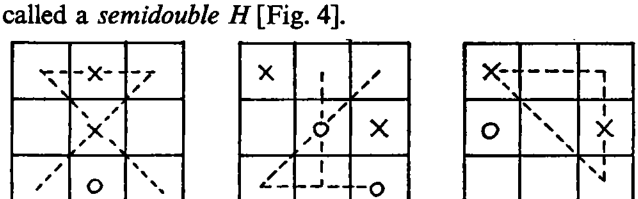


Fig. 4

If there are four such lines intersecting in three vacant squares, the situation may be called a *semitriple H* [Fig. 5].

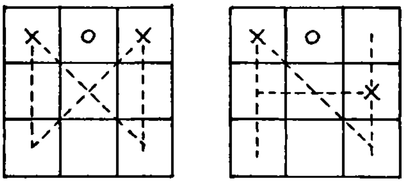


Fig. 5

If there are five such lines intersecting in four vacant squares, it may be called a *semiquadruple H* [Fig. 6].

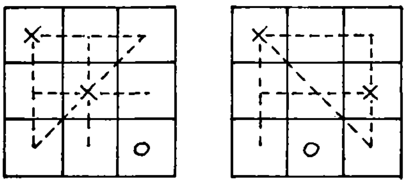


Fig. 6

If the two occupied squares belonging to an H are in one tit-tat-too row of which the third is vacant, so as to form a peril-row, it may be called a *self-reinforced H*. If then the situation be such that the adversary in blocking that peril-row does not himself create a peril-row, the other player can complete his A situation (as we will assume he always will), and the H situation may be said to be reinforced *unassailably*. [Unassailably reinforced H in Fig. 7(c); an unassailably reinforced semidouble H in Fig. 7(e).] If, however, the situation is such that the adversary in blocking the peril-row of the self-reinforced H, himself creates a new peril-row, then we have to distinguish two cases. Namely, when the creator of the H blocks that new peril-row, he may or may not thereby complete an A-situation. If he does, the original H situation was reinforced *strongly*, if not, it was self-reinforced *weakly*. [Weakly self-reinforced single H in Fig. 7(a); strongly reinforced single H in Fig. 7(b); strongly reinforced semidouble H in Fig. 7(d); strongly reinforced semitriple H in Fig. 7(f).]

If the player who creates an H, thereby also makes a peril-row, the already occupied square of which does not belong to the H, the H is said to be *externally reinforced*. Such reinforcement is always weak. [Externally reinforced H in Fig. 7(g).]

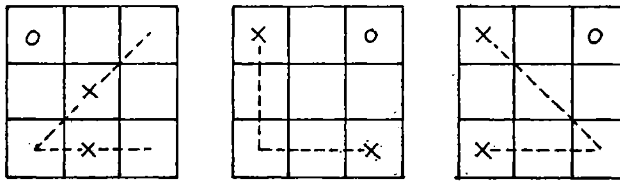


Fig. 7 (a, b, c)

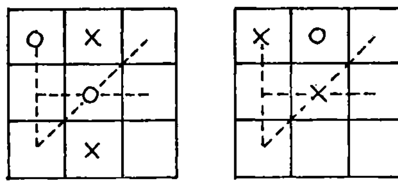


Fig. 7 (d, e)

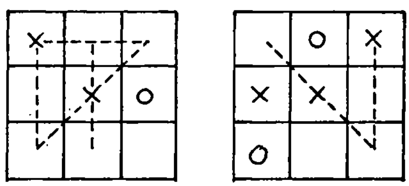


Fig. 7 (f, g)

If more than two squares are occupied by one party and each is the only occupied square of one or more tit-tat-too lines, and if these rows are all connected by vacant intersections, we may call the situation a *composite H.* [Unreinforced semidouble *H* s in Fig. 8. Unreinforced double composite *H* in Fig. 9. Unassailable semidouble composite *H* s in Fig. 10.]

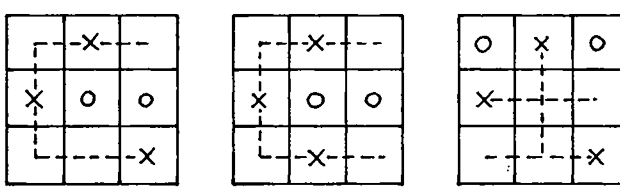


Fig. 8

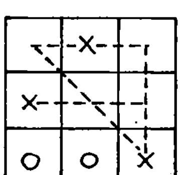


Fig. 9

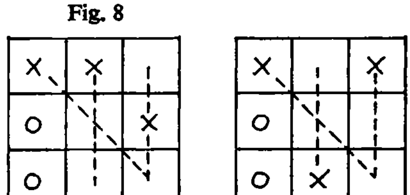


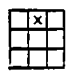
Fig. 10

The following analysis shows the result of every possible play where X first occupies square 2, supposing that each party, at each play, makes tit-tat-too if he can do so at once, and if not prevents the other from making tit-tat-too at the next play, and, if that condition allows, makes *A* at once, if he can, and if not, prevents the other party from making *A* at the next play if he can.

It is unnecessary for the pupil to go through the whole of the analysis. If he goes to case 12, it will be sufficient.

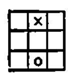
In the numbering of the cases, 135, for example, does not mean one hundred and thirty-five, but the first case, and the third subdivision of that case, and the fifth subdivision of that case. This way of numbering cases, invented by Benjamin Peirce, is of great utility [Figs. 11-74].

Case 1. $\underline{2}$



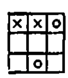
O has choice between 8 (case 11), 5 (Case 12), 1 = 3 (Case 13), 4 = 6 (Case 14), 7 = 9 (Case 15).

Case 11. $\overline{8}$



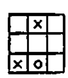
X has choice between 1 = 3 (Case 111), 7 = 9 (Case 112), 4 = 6 (Case 113), 5 (Case 114).

Case 111. $\frac{1}{\overline{3};}$



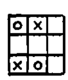
O has an unreinforced semidouble *H*, and X can only prevent the *A* by occupying 7 or 9. Neither of these gives X an *H*, and therefore there is no tit-tat-too.

Case 112. $\underline{7}$



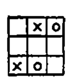
X has a semidouble *H*, and O can only prevent the *A* by occupying 1 (Case 1121) or 3 (Case 1122). O cannot make an *H*.

Case 1121. $\overline{1}$



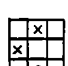
X can only make an unreinforced single *H*, and *A* would be prevented. Consequently, there is no tit-tat-too.

Case 1122. $\overline{3}$



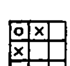
X can only make an unreinforced single H, and *A* would be prevented. Consequently, there is no tit-tat-too.

Case 113. $\underline{4}$



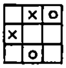
X has a single *H*, and *O* can only prevent *A* by occupying either 1 (Case 1131), 3 (Case 1132), 7 (Case 1133), or 9 (Case 1134).

Case 1131. $\overline{1}$



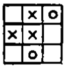
O has a single *H*. X can only prevent *A* by occupying either 7, 9, 5, or 6. The last two do not give him an *H*; so that there would be no tit-tat-too. The other plays give unreinforced single *H*s, and *A* would be prevented by O occupying 3. Consequently there is no tit-tat-too.

Case 1132. $\bar{3}$



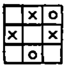
O has a semidouble *H*, and X cannot occupy 1, which would give O an *A*. X can only occupy either 5 (Case 11321), 6 (Case 11322), 7 (Case 11323), or 9 (Case 11324).

Case 11321. $\underline{5}$



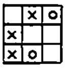
X has a weakly reinforced single *H*. There is no tit-tat-too.

Case 11322. $\underline{6}$



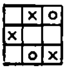
X has no *H*. There is no tit-tat-too.

Case 11323. $\underline{7}$



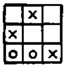
X has no *H*, and consequently there is no tit-tat-too.

Case 11324. $\underline{9}$



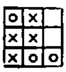
X has only an unreinforced single *H*, and *A* will be prevented. Consequently there is no tit-tat-too.

Case 1133. $\bar{7}$ $\underline{9};$



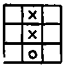
X has a double composite *H*. If O occupies either of the pair of squares 1 and 6 or of the pair 3 and 5, X makes *A* by occupying the other square of the pair, and so X makes tit-tat-too.

Case 1134. $\bar{9}$ $\underline{7};$ $\underline{5};$ $1;$



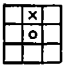
X has an *A*, and makes tit-tat-too.

Case 114. $\underline{5}$



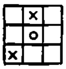
X has an unreinforced semidouble *H*, and O can only prevent *A* by occupying $1=3$. He thus gets an unreinforced single *H*, and *A* is prevented; so that there is no tit-tat-too.

Case 12. $\bar{5}$



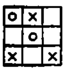
X has choice between $7=9$ (Case 121), $4=6$ (Case 122), $1=3$ (Case 123), 8 (Case 124).

Case 121. $\underline{7}$



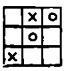
X has an unreinforced single *H*, and O, to prevent *A*, must occupy either 1 (Case 1211), 3 (Case 1212), 4 (Case 1213), or 6 (Case 1214).

Case 1211. $\bar{1}$ $\underline{9};$



There is no *H*, and consequently there will be no tit-tat-too.

Case 1212. $\bar{3}$



O has an unreinforced semidouble *H*, X can prevent *A*, but cannot get an *H* and consequently there is no tit-tat-too.

Case 1213. $\overline{4}$ $\underline{6;}$

	X	
O	O	X
X		

X has an unreinforced semidouble composite *H*. O can prevent *A* by occupying 3 or 9, and consequently there is no tit-tat-too.

Case 1214. $\overline{6}$

	X	
	O	O
X		

O has a weakly reinforced single *H*. There is no tit-tat-too.

Case 122. $\underline{4}$

	X	
X	O	

X has an unreinforced single *H*. To prevent *A*, O must occupy either 1 (Case 1221) or $3=7$ (Case 1222).

Case 1221. $\overline{1}$ $\underline{9;}$

O	X	
X	O	
		X

There is no *H*, and consequently will be no tit-tat-too.

Case 1222. $\overline{3}$

	X	O
X	O	

O has a weakly reinforced single *H*. There is no tit-tat-too.

Case 123. $\frac{1}{3;}$ $\underline{7;}$

X	X	O
	O	
X		

There is no *H*, and consequently will be no tit-tat-too.

Case 124. $\underline{8}$

	X	
	O	
	X	

O has choice between $1=3=7=9$ (Case 1241) and $4=6$ (Case 1242).

Case 1241. $\overline{1}$

O	X	
	O	
	X	

O has a strongly reinforced semidouble *H*, and makes tit-tat-too.

Case 1242. $\overline{4}$

	X	
O	O	
	X	

O has an unassailably self-reinforced semidouble *H*, and makes tit-tat-too.

Case 13. $\overline{1}$

O	X	

X has choice between 3 (Case 131), 4 (Case 132), 5 (Case 133), 6 (Case 134), 7 (Case 135), 8 (Case 136), and 9 (Case 137).

Case 131. $\underline{3}$

O	X	X

X has an unreinforced single *H*, and O, to prevent *A*, must occupy either 4 (Case 1311), 5 (Case 1312), 7 (Case 1313), or 8 (Case 1314).

Case 1311. $\overline{4}$ $\underline{7;}$

O	X	X
O		
X		

O now makes *A* and tit-tat-too.

Case 1312. $\overline{5}$

O	X	X
	O	

O has a weakly reinforced simple *H*. There is no tit-tat-too.

Case 1313. $\overline{7}$

O	X	X
O		

O has an unassailably reinforced simple *H*, and will make tit-tat-too.

Case 1314. $\overline{8}$

O	X	X
	O	

O has an unreinforced semidouble *H*. X to prevent *A*, must occupy 7 or 9. Neither way does he gain an *H*, and therefore there is no tit-tat-too.

Case 132. $\underline{4}$

O	X	
X		

X has an unreinforced simple *H*. O, to prevent *A*, must occupy either 5 (Case 1321) or 6 = 8 (Case 1322).

Case 1321. $\overline{5}$ $\underline{9}$;

O	X	
X	O	
		X

There is no *H* and consequently will be no tit-tat-too.

Case 1322. $\overline{6}$

O	X	
X		O

O has an unreinforced simple *H*. To prevent *A*, X must occupy either 5, 8, 3, or 9. The first two do not give him an *H*. The others only an unreinforced simple *H*, and *A* will be prevented. Consequently, there is no tit-tat-too.

Case 133. $\underline{5}$ $\overline{8}$;

O	X	
	X	
	O	

O has an unreinforced simple *H*. To prevent *A*, X must occupy either 4, 7, 9, or 6. The first two give him no *H*, the third only an unreinforced simple *H*, and the fourth a weakly reinforced simple *H*. Consequently there is no tit-tat-too.

Case 134. $\underline{6}$

O	X	
		X

X has an unreinforced simple *H*. To prevent *A*, O must occupy either 4 (Case 1341), 5 (Case 1342), 7 (Case 1343), or 8 (Case 1344).

Case 1341. $\overline{4}$ $\underline{7}$;

O	X	
O		X
X		

X has a double composite *H*, so that of the two pairs of squares 5 and 9, on the one hand, 3 and 8 on the other, if O occupy one of the squares X makes *A* on the other of the same pair; and X makes tit-tat-too.

Case 1342. $\overline{5}$ $\frac{9;\ 7;}{3;\ 8;}$

O	X	O
	O	X
X	O	X

There is no tit-tat-too.

Case 1343. $\overline{7}$

O	X	
		X
O		

O has a strongly reinforced semidouble *H*, and makes tit-tat-too.

Case 1344. $\overline{8}$

O	X	
		X
	O	

O has an unreinforced semidouble *H*. To prevent *A*, X must occupy either 4 (Case 13441), 5 (Case 13442), 7 (Case 13443), or 9 (Case 13444).

<table>
<tr><td>Case 13441.</td><td>$\dfrac{4}{5;}\ \dfrac{9;}{3;}\ \underline{7;}$</td><td><table><tr><td>O</td><td>X</td><td>O</td></tr><tr><td>X</td><td>O</td><td>X</td></tr><tr><td>X</td><td>O</td><td>X</td></tr></table></td><td>No tit-tat-too.</td></tr>
<tr><td>Case 13442.</td><td>$\dfrac{5}{4;}\ \dfrac{7;}{3;}$</td><td><table><tr><td>O</td><td>X</td><td>O</td></tr><tr><td>O</td><td>X</td><td>X</td></tr><tr><td>X</td><td>O</td><td></td></tr></table></td><td>There is no tit-tat-too.</td></tr>
<tr><td>Case 13443.</td><td>$\underline{7}$</td><td><table><tr><td>O</td><td>X</td><td></td></tr><tr><td></td><td></td><td>X</td></tr><tr><td>X</td><td>O</td><td></td></tr></table></td><td>X has an unreinforced simple H. A is prevented, and there is no tit-tat-too.</td></tr>
<tr><td>Case 13444.</td><td>$\underline{9}$</td><td><table><tr><td>O</td><td>X</td><td></td></tr><tr><td></td><td></td><td>X</td></tr><tr><td></td><td>O</td><td>X</td></tr></table></td><td>There is no H and consequently will be no tit-tat-too.</td></tr>
<tr><td>Case 135.</td><td>$\underline{7}$</td><td><table><tr><td>O</td><td>X</td><td></td></tr><tr><td></td><td></td><td></td></tr><tr><td>X</td><td></td><td></td></tr></table></td><td>X has an unreinforced semidouble H. To prevent A, O must occupy 5 (Case 1351) or 8 (Case 1352).</td></tr>
<tr><td>Case 1351.</td><td>$\dfrac{}{5}\ \underline{9;}$</td><td><table><tr><td>O</td><td>X</td><td></td></tr><tr><td></td><td>O</td><td></td></tr><tr><td>X</td><td></td><td>X</td></tr></table></td><td>There is no H and consequently will be no tit-tat-too.</td></tr>
<tr><td>Case 1352.</td><td>$\dfrac{}{8}$</td><td><table><tr><td>O</td><td>X</td><td></td></tr><tr><td></td><td></td><td></td></tr><tr><td>X</td><td>O</td><td></td></tr></table></td><td>O has no H, but X can form only an unreinforced simple or semidouble H. A would be prevented, and consequently there will be no tit-tat-too.</td></tr>
<tr><td>Case 136.</td><td>$\dfrac{8}{5;}$</td><td><table><tr><td>O</td><td>X</td><td></td></tr><tr><td></td><td>O</td><td></td></tr><tr><td></td><td>X</td><td></td></tr></table></td><td>O has a strongly reinforced semidouble H and will make tit-tat-too.</td></tr>
<tr><td>Case 137.</td><td>$\underline{9}$</td><td><table><tr><td>O</td><td>X</td><td></td></tr><tr><td></td><td></td><td></td></tr><tr><td></td><td></td><td>X</td></tr></table></td><td>X has an unreinforced simple H. To prevent A, O must occupy either 4 (Case 1371), 5 (Case 1372), 7 (Case 1373), or 8 (Case 1374).</td></tr>
<tr><td>Case 1371.</td><td>$\dfrac{}{4}\ \underline{7;}$</td><td><table><tr><td>O</td><td>X</td><td></td></tr><tr><td>O</td><td></td><td></td></tr><tr><td>X</td><td></td><td>X</td></tr></table></td><td>X has an unassailably reinforced semidouble composite H, and will make tit-tat-too.</td></tr>
<tr><td>Case 1372.</td><td>$\dfrac{}{5}$</td><td><table><tr><td>O</td><td>X</td><td></td></tr><tr><td></td><td>O</td><td></td></tr><tr><td></td><td></td><td>X</td></tr></table></td><td>O has an unreinforced semidouble H. X can prevent A but cannot get an H, and consequently there is no tit-tat-too.</td></tr>
<tr><td>Case 1373.</td><td>$\dfrac{}{7}\ \underline{4;}$</td><td><table><tr><td>O</td><td>X</td><td></td></tr><tr><td>X</td><td></td><td></td></tr><tr><td>O</td><td></td><td>X</td></tr></table></td><td>X has an unreinforced semidouble composite H. O can prevent A by occupying 5 or 6. No tit-tat-too.</td></tr>
<tr><td>Case1374.</td><td>$\dfrac{}{8}$</td><td><table><tr><td>O</td><td>X</td><td></td></tr><tr><td></td><td></td><td></td></tr><tr><td></td><td>O</td><td>X</td></tr></table></td><td>O has no H, and X can only get an unreinforced simple or semidouble H and A can be prevented. Consequently, there is no tit-tat-too.</td></tr>
<tr><td>Case 14.</td><td>$\dfrac{}{4}$</td><td><table><tr><td></td><td>X</td><td></td></tr><tr><td>O</td><td></td><td></td></tr><tr><td></td><td></td><td></td></tr></table></td><td>X has choice between 1 (Case 141), 3 (Case 142), 5 (Case 143), 6 (Case 144), 7 (Case 145), 8 (Case 146), and 9 (Case 147).</td></tr>
</table>

Case 141. $\frac{1}{}$

x	x	
o		

X has an unassailably reinforced simple *H* and will make tit-tat-too.

Case 142. $\frac{3}{1};$

o	x	x
o		
x		

O has a strongly reinforced simple *H* and will make tit-tat-too.

Case 143. $\frac{5}{}$

o	x	
o	x	

X has an unassailably reinforced semidouble *H* and will make tit-tat-too.

Case 144. $\frac{6}{}$

	x	
o		x

X has an unreinforced simple *H*. To prevent *A*, O must occupy either 1 (Case 1441), 3 (Case 1442), 7 (Case 1443), or 9 (Case 1444).

Case 156. $\frac{8}{5};$

	x	
	o	
o	x	

O has a strongly reinforced semidouble *H* and makes tit-tat-too.

Case 157. $\frac{9}{}$

	x	
o		x

X has an unreinforced semitriple *H*. O can prevent *A* only by occupying 1, but gets no *H* and X gets an unreinforced semidouble composite *H*. There is no tit-tat-too.

SUMMARY OF THE RESULTS

$\frac{2}{1}\,\frac{9}{5}$	~	$\frac{}{5}$	~	$\frac{7}{}$	~	$\frac{3}{}$	O!!!
$\frac{}{7}$	~	$\frac{}{8}$	~	$\frac{1}{}$	~	$\frac{}{4}\,\frac{1}{}$	X!!!
$\frac{}{8}$	~	$\frac{8}{}$	O!!!	$\frac{2}{7}\,\frac{1}{}$	X!!!	$\frac{5}{}$	X!!!
$\frac{}{4}$	X!!!	$\frac{}{5}\,\frac{7}{}$	~	$\frac{5}{}$	~	$\frac{5}{}$	X!!!
$\frac{4}{}$	~	$\frac{4}{}$	~	$\frac{9}{}$	~	$\frac{9}{5}$	~
$\frac{5}{}$	~	$\frac{1}{}$	~	$\frac{6}{1}$	O!!!	$\frac{}{1}$	X!!!
$\frac{7}{}$	~	$\frac{8}{}$	O!!!	$\frac{}{4}$	X!!!	$\frac{6}{3}$	~
$\frac{6}{7}$	O!!!	$\frac{}{8}\,\frac{4}{1}$	~	$\frac{4}{8}$	O!!!	$\frac{}{9}$	~
$\frac{}{5}$	~	$\frac{}{3}$	~	$\frac{}{9}$	O!!!	$\frac{}{7}$	X!!!
$\frac{}{8}$	~	$\frac{}{7}$	X!!!	$\frac{}{5}$	~	$\frac{6}{1}$	X!!!
$\frac{}{4}$	X!!!	$\frac{}{9}$	X!!!	$\frac{}{6}$	~	$\frac{7}{}$	~
$\frac{3}{4}$	O!!!	$\frac{5}{}$	~	$\frac{8}{}$	O!!!	$\frac{8}{}$	~
$\frac{}{7}$	O!!!					$\frac{3}{}$	O!!!

If $\frac{2}{},\frac{}{1}$ gives O the best chance; for of the 7 possible plays of X, three will give the game to O. But the best play for X is then $\frac{9}{}$, when of O's four possible plays, one gives the game to X. For his first play, O may prefer $\frac{}{5}$ when of the four possible plays of X one gives the game to O. There are two first plays of O which give the game to X if he plays right. They are $\frac{}{7}$ and $\frac{}{4}$. To the former X should reply by $\frac{1}{}$.

If he plays $\overset{6}{-}$, $\overset{4}{-}$, $\overset{8}{-}$ or $\overset{3}{-}$ he gives the game to O. If O's first play is $\underset{4}{-}$, X can only lose by $\overset{3}{-}$. He should play $\overset{1}{-}$ or $\overset{5}{-}$.[3]

Art. 3. *Scholium* 2, *continued.* If the pupil goes back and looks over the analysis of tit-tat-too attentively, he will observe that the gist of the process consists in making *experiments*, or trials, systematically, and carefully *observing* and registering the results. *Let us try and see what happens must be* the motto of the mathematician.

The apparatus, or prepared instrument, upon which the experiments were made was a diagram of the tit-tat-too board. In all mathematical reasoning something like this has to be prepared; and it must be something that can be observed and that we can modify. It is a likeness, or analogue, of the imaginary thing we are reasoning about, and is called an *icon*, or image. Two kinds of icons are chiefly used by mathematicians, namely, first, geometrical *figures*, drawn with lines, and, second, *arrays* of points or letters.

In connection with our diagram, we had to use the arbitrary signs, X and O, for the marks of the two players. For such purposes, mathematicians generally use letters.

Finally, we had to draw up a careful statement of the rules of the game. The results, too, being what will *always* happen when players play in certain ways, are a sort of rules. The theorems we used were again subsidiary rules. All those rules we expressed in words. In place of words mathematicians often use other systems of symbols, having remembered rules associated with them. Such symbols have two advantages; first, they may be made clearer and more definite than words, especially for complicated cases; and second, they can be arranged so as to make *arrays*, upon which experiments and observations can be made. For such symbols, mathematicians generally employ peculiar characters, or little shapes about the size of capital letters.

Such are the tools with which the mathematician works. They have to be used intelligently. Here are some maxims which will aid your intelligence:

1. Keep your purpose in view.
2. Make all your procedure as regular, and systematic, and symmetrical as you can.
3. Be on the look out for regularities, symmetries, and systems.
4. Get rid of everything useless.

[3] Another extensive seventeen-page draft of the tit-tat-too analysis is found in MS. 1525.

5. Use every means of economizing the work of thought, not in order to think less, but in order that the energy which can be dispensed with in one place may be employed in another place where it is needed more. The theorems we introduced into the analysis of tit-tat-too were examples of such economies.

Art. 4. *Scholium* 3. Mathematical works, like others, are customarily divided into Books, the books into Chapters, and the chapters into Sections. The smallest subdivisions above paragraphs are called Articles, and are generally numbered continuously throughout the work. The formulae, or symbolically expressed rules, have to be numbered, or otherwise marked, for reference. In the present work, they will be referred to by pages, and upon each page the different formulae will be marked successively by the signs of the zodiac in their order, viz.:

♈ Aries.	♋ Cancer.	♎ Libra.	♑ Capricornus.
♉ Taurus.	♌ Leo.	♏ Scorpio.	♒ Aquarius.
♊ Gemini.	♍ Virgo.	♐ Sagittarius.	♓ Pisces.

There are also certain kinds of statements in mathematical works which have special names.

Art. 5. *Definition* 3. A *definition* is an exact description of the kind of object to which a word or phrase is exclusively applied.

Gloss 4. The word *definition* (in Chaucer and other old writers *difinicion* and *diffinitioun*) was taken directly from the medieval Latin *diffinitio*, a corruption of the classical *definitio*, having the same meaning as the English word. It was an old Latin word (from *de-* of and *finis*, a bound, from root BHID, to cleave, whence English *bite*), adopted by Cornificius* (* the supposed author of a treatise probably written about 80 B.C., perhaps the teacher of Cicero) and by Cicero to translate the Greek ὁρισμός, used by Aristotle in the same sense, and derived from ὅρος (Corcyrean ὅρϝος), a bound, also a term or general name. Euclid heads his lists of definitions ὅροι, that is, *terms*. Probably Aristotle's word was not familiar to him.

Memorandum 2. The old logics make two kinds of definitions, the *nominal*, which merely explain the use of words, and the *real* (diffinitio quid rei) which analyze the nature, i.e. the essence, i.e. the idea of the kind of thing. The kind of thing defined is called the *definitum*. A real definition ought (it was said) to state the *genus*, or general nature, and the *difference*, or specific characteristic of the kind of thing. Definitions strictly mathematical are of kinds of objects of our own creation. They ought to state the purpose or idea which governs such creation. Many mathematical kinds of objects may be differently defined from different points of view. Some mathematicians prefer *genetic* definitions, or such as show how objects of the kinds defined may be constructed. They have the advantage of showing that the kind defined is not absurd. The old logics declare that simple ideas cannot be defined; but there is nothing in mathematics which cannot be defined, assuming the person to whom the explanation is to be made has the use and experience of his eyes, muscles, muscular sense, and ordinary speech.

Axiom 1. If we know that an object is of a given kind, we may conclude that anything is true of it which is stated of that kind in its definition. This is called reasoning from definition to definition.

Axiom 2. If we know that of a certain object everything is true that is stated of a given kind in its definition we may conclude that that object is of that kind. This is called reasoning from definition to definition.

Art. 6. *Definition* 4. A *division* is an enumeration of the varieties of a certain kind, such that everything of that kind belongs to one or other of the named varieties.

Gloss 5. *Division* is another old logical word, from the Latin *divisio*, used by Cornificius in a somewhat different sense, but by Cicero nearly in the sense of our definition. The Latin verb *dīvīdĕre*, is formed from *dis-*, as under, and probably an unknown verb *vīdĕre*, which is supposed by some to mean to sunder, by others to know. It is used to translate the διαίρεσις (from διά, as under, and αἱρεῖν, to grasp) used by Plato and Aristotle in the logical sense.

Art. 7. *Definition* 5. A *postulate* is a proposition forming a part of a mathematical hypothesis.

Gloss 6. The word *postulate* (for which in Elizabethan English "petition" was used: thus Blundevile in his Logicke, 1599, speaks of "petitions, called in Latin *Postulata.*") comes from the Latin, *postulatum* used in this sense by Boethius (A.D. 500) who says it is an old word. *Postulatum* is the participle of *postulare*, to beg, frequentative from *poscĕre*, or *porcscĕre*, inceptive of *precari* from Sanskrit root PARKH, to pray, whence English *pray*. *Postulatum* is used to translate αἴτημα, used by Aristotle and Euclid, from αἰτεῖν, to beg.

Memorandum 3. This definition is intended to state what a postulate really amounts to, without implying that geometers have always understood it so. Ancient writers do not seem to agree as to what a postulate is. Aristotle, who lived before any of the important mathematicians whose works have come down to us, in the 10th chapter of the 1st book of his "Posterior Analytics," undertakes to explain the differences between the different kinds of geometrical first premises. When he speaks of postulates, he would seem to be referring to something different from the postulates of Euclid. Yet there are objections to that view. Perhaps, it is best to infer, as we may from many things in his writings, that he did not understand mathematical matters very well. He says distinctly that a postulate is *not* a hypothesis. He also says that postulates are demonstrable, δεικτά. But Aristotle sometimes uses this verbal adjective for the other; and perhaps he means δεικτέα, needing demonstration. He further says a postulate is *unlike* (ὑπεναντίον) the opinion of the learner. Now, whatever is "unlike" a universal proposition is more than *unlike*; it conflicts with it. The expression, therefore, suggests that Aristotle conceives a postulate as asserting something, not universally, but only of something or other. Other writers have likewise said that a postulate asserts that something of a given description can be constructed or found. It is true that all Euclid's postulates except one are of that description. That one (certainly in no sense a postulate) is that all right angles are equal. Supposing that conception of a postulate to have been shared by Aristotle, then, when he says postulates are demonstrable, he may mean that they are demonstrable in any particular cases, by actually constructing, or finding, an object of the sort which the postulate says can be found.

The five postulates of Euclid are as follows:

I. Let it be granted that from any point to any point a straight line be drawn;

II. And that a limited straight line be continuously produced in a straight line;

III. And that with any centre and radius a circle can be drawn;

IV. And that all right angles be equal to one another;

V. And that if a straight line falling upon two straight lines make the interior angles on the same side less than two right angles, the two straight lines, being continued to infinity, will be concurrent on the side on which are the angles less than two right angles.

Art. 8. *Definition* 6. An *axiom* is a plain truth about quantity.

Gloss 7. The word *axiom* (in Elizabethan English "Dignities and

Maximes," though *axiome* was occasionally used) is derived through the Latin form *axiōma*, from the Greek ἀξίωμα, used in the same sense, but originally meaning a dignity, or that of which one is thought worthy, from ἀξιοῦν, to esteem, from ἄξιος, worthy, ultimately from ἄγειν, to reckon, draw, from the root AG, whence English *act*, *agile*, etc. The thread of ideas is this: ἄξιος means 'equal in value to'. Thus ἄξιος βοός is "worth an ox." Hence, ἀξιοῦν means *to deem worth as much as*, and ἀξίωμα was, at first, the market value, what a thing was generally deemed worth, but later, what was generally held worth while, what was generally considered proper to be done, and finally, what was generally admitted as true. Aristotle, in the chapter referred to, speaks of "what are called the common axioms" (τὰ κοινὰ λεγόμενα ἀξιώματα). Euclid calls them, "Common Notions" (κοιναὶ ἔννοιαι), and they generally went by this name in English down to Elizabeth's reign, or later.

Memorandum 4. The axioms of Euclid as given in the best edition (Heiberg's) are as follows:

1. Things equal to the same thing are equal to one another;
2. And if to equals equals be added, the wholes are equal;
3. And if from equals equals be taken, the remainders are equal;
4. And if to unequals equals are added, the wholes are unequal (but Heiberg admits there is strong evidence this is spurious);
5. And things double the same thing are equal to one another;
6. And things half the same thing are equal to one another;
7. And things that fit over one another are equal to one another;
8. And the whole is greater than its parts;
9. And two straight lines do not enclose space. But the evidence is overwhelming that this is spurious, i.e. was inserted in the text by an ancient editor.

Art. 9. *Division* 1. The contents of a mathematical treatise should be of six kinds, to wit:

1st, *Analyses* of facts, observed or imaginary, suggesting the hypotheses;

2nd, Formal statements of the hypotheses (*postulates*);

3rd, Explanations of the meanings of

(A) the technical terms (*definitions*),

(B) the peculiar symbols employed (*notations*);

4th, Propositions borrowed from other branches of mathematics, or elsewhere, and accepted as indubitable without proofs (*axioms*);

5th, Exactly reasoned propositions, that is (see *Arts.* 11 and 12),

(A) *theorems*,

(B) *problems*;

6th, Enumerations of different cases which have to be considered (*divisions*);

7th, The exposition of the main part of the thought, depending on comparison, generalization, abstraction, etc., in short of the thought that governs the construction of the treatise;* (* In most mathematical writings, especially the old books, all this unmechanical part of the thought is left for the unaided sagacity of the reader to divine. In an elementary treatise such omission is fatal to the success of many students. In the present volume, this part is consigned to *scholia*.)

8th, Warnings against probable blunders, and notices of special stumbling-blocks (*notabenes*);

9th, Detailed *directions* for applying the knowledge gained, with recommendations of suitable methods and arrangements

(A) of drawings,

(B) of computations;

10th, *Illustrative examples* showing the meanings of statements;

11th, Immediate applications of the propositions established (*corollaries*);

12th, *Exercises* for practice; since practice, practice, practice is the one way to learn mathematics;

13th, Historical *memoranda*, to which the learner's sense of decency will surely bind him to attend, in gratitude to the great men the devotion of whose lives to the pursuit of truth, though it now contributes to our enjoyment of life, in most cases never received any other external reward.* (* *Anecdote* 1. The famous Euclid, head of the mathematical department of the University of Alexandria, was one day asked by a tyro, "What do I get by learning this?" The professor called the slave who made his disbursements and said, "Pay this gentleman three-pence, since he must gain something." It is the learned anecdotist John Stobœus who tells the story.) No other motive for remembering our benefactors of former generations need be set before American youth. But even a man devoid of the sentiment of gratitude and degraded below the lowest people living upon earth, were he one of that dreadful race which the terrible satire of Dean Swift represented as lower than horses, if such a being could be conceived to have any interest in learning anything, an acquaintance with the proceedings which have given rise to peculiar modes of thinking, is the best means [of] apprehending the significance of the thought, and affords a quite unsentimental reason for attending to [the] history of the terms and propositions of mathematics.

14th, *Comparisons* with other knowledge in respect to

(A) the substance of the hypotheses or results,

(B) the method of reasoning;

15th, *Glosses* on the pronunciation, history, derivation, and associations of the words;

16th, *Notes*, anecdotes, and pleasantries to relieve the fatigue of the brain.

Art. 10. *Definition* 7. A *demonstration* is a perfect proof of some truth.

Gloss 8. The word *demonstration*, an old word in English, was formed directly from the Latin *demonstratio*, used by Cornificius in the same sense, and formed regularly from *dēmonstrāre*, to demonstrate, compounded of *dē*, thoroughly, and *monstrare*, to show, for *mon-es-tar-are*, from *monēre*, to cause to think, from the root MAN, whence English *mind, man, mathematics*, etc. The Latin *monstrare*, compared with other words meaning to show, such as *exhibere, designare, ostendere, indicare, significare*, is found to retain the idea of causing to reflect, and not merely of causing to look. *Demonstratio* is used to translate the Greek ἀπόδειξις, the usual word for a perfect proof, which is regularly formed from ἀποδεικνύναι, to *demonstrate*, from ἀπό, *thoroughly*, and δεικνύναι, to *show, prove*, from the root DIK, to *show*, from DA, to *know*, whence English *teach, indicate, index, didactic*, etc.[4]

Art. 11. *Definition* 8. A *theorem* is a demonstrable statement amounting to the denial of the possibility of some general description of [a] thing.

Illustrations. 1. That the remainder (if there be any) after dividing by 4 the product of a number by itself, is [0 or] 1, can be demonstrated. It is, therefore, a *theorem*, since it is equivalent to saying that a remainder of 2 or 3, after such a process, is impossible.

2. If space has the properties it is generally supposed to have, and if there be 4 points all equidistant from one another, then, if a fifth point is at equal distances from the other four, the latter distance is to the former as [$\sqrt{3} : 2\sqrt{2}$].[5] This can be demonstrated. It is therefore a theorem because it amounts to denying the possibility of a fifth point

[4] Elsewhere Peirce states more simply: "Only I had better explain that a mathematical demonstration consists in contriving a *spatial* diagram of the state of things in the premiss — this diagram being a usual (a tactual, in the broad sense) *icon* or image of the form of relationship signified in the collective premiss (I mean in the two or more premisses considered as one whole) and then in observing that the diagram is a diagram of the relation signified by the conclusion of the demonstration."

[5] Peirce gives $\sqrt{3} : \sqrt{2}$.

equidistant from the others and either nearer or further than [$\frac{1}{2}\sqrt{\frac{3}{2}}$] their distance from one another.

Gloss 9. The word *theorem* is taken, perhaps through the Latin, from the Greek θεώρημα, with the same meaning, originally a *spectacle*, from θεωρεῖν, to behold, from θεωρός for θαϝρός, a spectator. There would seem to be a root ΘΑϜ.

Art. 12. *Definition* 9. A *problem* is a question how (i.e. under what circumstance) a thing of a given description, often with relations to other given things, is possible.

Gloss 10. The word *probleme* (used by Chaucer) comes probably through the French *problème*, and certainly through the Latin *problema*, from the Greek πρόβλημα, used in a geometrical application by Plato, originally something thrown forward, regularly formed from προβάλλειν, to throw forward, to put forward, from πρό, forward, and βάλλειν, to throw, from the root GAR, whence English *carbine*, *hyperbola*, *gland*, *parley*, *quell*, *volatile*, etc.

Definition 10. A *problematic proposition*, is a demonstrable statement that, accepting a certain hypothesis, something is possible. It is loosely *called* a *problem*.

Illustration 3. That there is within a circular disk a point equidistant from every point of the border is a *problematic proposition*. To find that point, that is, the question *how* that finding is possible, is a *problem*.

Art. 13. *Definition* 11. A *lemma* is a demonstrable statement not directly relating to the subject treated but introduced for the sake of facilitating the demonstration of a theorem or problem which it precedes.

Gloss 11. The word *lemma* is borrowed, probably through the Latin, from the Greek λῆμμα, where it sometimes has the same sense, originally a thing taken, and formed from a stem of the verb λαμβάνειν, to take, from the root RABH, take, from ARBH, when[ce] English *dilemma*, *syllable*, *labor*, *robust*, *elf*, etc. With Aristotle a *lemma* is an assumption or premise, with the Stoics a particular kind of premise, with later writers a proposition borrowed from another science; and it has had many other meanings.

Art. 14. *Definition* 12. A *corollary* is a statement which an immediate application of another, just proved, shows to be true.

Gloss 12. The word *corollary* can be accented either on the first or second syllable. (It is used by Chaucer and was derived from the *Latin corollarium*, which in medieval and modern Latin bears the same meaning, but in the classical language it means money given to buy a garland, a gratuity. It is formed by adding the suffix *-arium*, usually signifying a place, to *corolla*, a garland, diminutive of *corona*, a crown or garland, from the root KAR, whence English *circle*, *ring*, *cylinder*, *cycle*, *collar*, etc. It first occurs in the mathematical sense in Boethius, about A.D. 500. It translates Greek πόρισμα, used in this sense among others in mathematics. Πόρισμα according to its derivation should mean something extra supplied. It is formed by attaching the suffix *-ma*, signifying a result, to the stem of πορίζειν, to provide, from πόρος, a resource, originally a ford, from the root PAR, to cross over, to travel, whence English, *fare*, *far*, *from*, *fore*, *forth*, *for*, *fear*, *pirate*, *experience*, etc.)

Art. 15. *Definition* 13. A *proposition* is an assertion; especially applied in mathematics to definitions, divisions, postulates, axioms, theorems, problems, lemmas, and corollaries; and loosely applied to the whole matter under the head of the proposition, including the demonstration.

Gloss 13. The word *proposition*, which has been in English since Wiclif's time, was taken direct from the Latin *propositio*, used in this sense by Cornificius, formed from *propōnĕre*, to propound, compounded of *pro-*, *fore*, and *pōnĕre*, to place, contracted from *posĭnĕre*, from old *port*, at, and *sĭnĕre*, to let, lay. The word *propositio* would seem to be meant to imitate the Greek πρόθεσις, which was used in rhetoric, and in grammar, too, after that science was invented, in the same sense. But the Greek word used in Logic and Mathematics was not πρόθεσις but πρότασις. Aristotle, for instance, uses the former in his Rhetoric, the latter in the Organon. But with him the former had the present mathematical meaning, while the latter meant a premise. The word πρότασις is formed from προτείνειν, to stretch forth, from πρό, forward, and τείνειν, to stretch, from the root TAN, stretch, whence English *tense*, *tend*, *tent*, *tendril*, *tendon*, *tone*, *thin*, *dance*, etc.

Art. 16. *Scholium* 4. There has been, since the days of Euclid, a regular form, universally approved, for the demonstration of a theorem or lemma. The proposition is stated, first. Next, is described, with letters, an icon of the conditions of the proposition, which is called the beginning of the *construction*. Thirdly, the proposition is restated with reference to that construction. The construction is then completed by any additions or changes which may be necessary. Fifthly, the consequences of certain experiments upon the construction are traced out, and the proposition is thus shown to be true of it. Sixthly, this is stated to be the case; and this statement is called the proof. To that, are appended the letters

Q. E. D. for *quod erat demonstrandum*, "which had to be shown."* (* Translating Euclid's ὅπερ ἔδει δεῖξαι.)

Gloss 14. The word *construction*, used in English since the first English Euclid, in 1570, is taken from the Latin *constructio*, little used in the geometrical sense, because the Romans were not geometers. It is regularly formed from *construĕre*, to build, from *con-*, together, and *struĕre*, to pile, from the root STAR, spread, whence English, *strew*, *straw*, *star*, *sheet*, *instrument*, *latitude*, etc. In mathematics it translates the Greek κατασκευή, used in this sense, originally, furnishing, from κατά, from top to bottom, and σκευή, equipment, from the root SKU, cover, whence English, *scum*, *sky*, *house*, *hide*, *scuttle*, *cuticle*, *scutchon*, etc. [scutcheon].

Gloss 15. The word *conclusion*, used in Middle English, is taken directly from the Latin *conclusio*, used in the same sense by Cornificius, formed regularly from *conclūdĕre*, to conclude, compounded of *com-*, completely, and *-clūdĕre*, to close, to end, from the root SKLU, shut, whence English *close*, *include*, etc. *Conclusio* is used to translate Greek συμπέρασμα, used occasionally by Aristotle, and usually by his followers, in the same sense. It is formed from συμπεραίνεσθαι, to conclude, from σύν, aiding, and περαίνειν, to finish, from πέρας, end, or from περᾶν, to pass through, from the root PAR, found in πόρισμα Gloss 12.

Art. 17. *Definition* 14. A *scholium* is a remark in aid of the understanding of the course of thought, but not containing demonstrative reasoning.

Gloss 16. The word *scholium* (plural *scholia*) is used in all the English Euclids. The scholia are all written by commentators. Indeed, the word was originally understood to imply such authorship. But Legendre in his *Éléments de géomètrie* [1794] introduced "Scolies" written by himself, and succeeding authors have imitated the practice. The word is taken through modern Latin *scholium* (Cicero writes it in Greek letters), from the Greek σχόλιον, an interpretation by a commentator, from σχολή, a lecture, originally, spare time, from the root SAGH, hold in, refrain, whence English *school*, *scheme*, *sail*, *hectic*, *epoch*. Boys and girls will think it strange that *school* should come from the root in this sense, and will ask whether it means nothing else. It does mean, also, *endure*; but our word school comes from the Anglo-Saxon *scólu*, which was borrowed from the latin *schola*, meaning a school-room or lecture-room. The Greek σχολή from which it was borrowed had the same meaning; but at first it was a lecture, and before regular lectures were delivered, it was a discussion, to which the name "spare time" was given, because that was the favorite way among the Athenians of spending their spare time.

2

SEQUENCES

Art. 18. *Definition* 15. A *correspondence* is a connection established in the mind between two collections such that, considering either, every object of it is connected with the same number of objects of the other.

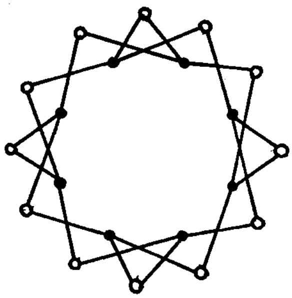


Fig. 75

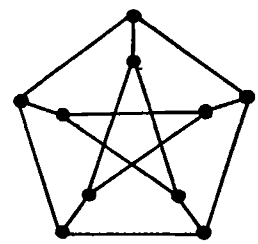


Fig. 76

N.B. (2) It is *not* said that any object of either is connected with the same number of the other collection, since the two numbers may be different.

Illustration 4. In figure 75 there is a collection of rings and a collection of dots. The lines suggest mental connections. Each dot is connected with three rings; each ring with two dots. Such a connection is called a two-to-three correspondence between the dots and rings, or a three-to-two correspondence between the rings and dots.

Scholium 5. Seeking to enlarge the idea, we remark that the two collections need not be different. Thus, in figure 76, the lines suggest a three-to-three correspondence among the dots. [...][6]

[6] The lower half of this page has been cut from Peirce's manuscript, probably in a revision by himself. Consequently the beginning of Art. 19 has disappeared carrying with it axioms 3 and 4 in that article as well as Cor. 1 at the end of Art. 18 as indicated in Peirce's Table of Contents. In a listing of corollaries Peirce indicates that in this manuscript on page 40, corollary 1 appears in terms of a "5–1 1–7 correspondence."

[*Art.* 19.

Axiom 3. The relation expressed by "after" is transitive.

Axiom 4. The relation expressed by "after" is identical.]

In that case, they would not appear as axioms. But treated as they are here, as familiar truths, they are axioms. No demonstration could make us more sure of their verity; but it would afford an insight into the manner in which different relations are connected with the relation of coming after.

A full explanation of the matter would be too long. But something may be said. Figure 77 symbolizes a relationship between four objects, *A, E, I, O*. The wedges pointing from *A* to *I* and from *E* to *O* mean

Fig. 77

that *A* is supposed to have a certain kind of a relation to *I*, like the relation of *E* to *O*. The arrows pointing from *A* to *O*, and from *E* to *I* signify that *A* has *not* that relation to *O* that is signified by the wedge, nor *E* that relation to *I*. Now suppose the relation signified by the wedge to be such that the fourfold relationship signified by the diagram is necessarily *false*, and that to make it true it is necessary to suppose that *A* that has the wedge-relation to *I* is a different thing from that *A* that has *not* the wedge-relation to *O*. For example, let the wedge mean *strong enough to lift*. Then, the diagram represents that *A* is strong enough to lift *I* but is not strong enough to lift *O*, while *E* is strong enough to

---

Another sheet summarizes the axioms in the first eleven chapters and on page 40 axioms 3 and 4 should appear.

Axiom III: The relation expressed by "after "is transitive.

Axiom IV: The relation expressed by "after" is identical.

A Scholium 6 was also lost at this point. The part of Art. 19 still extant and herein given may well be part of that Scholium. Similarly, Peirce either mislabeled or lost in revision Corollary 10 in Art. 34.

Peirce drew up elaborate summaries of the contents of MS. 165 (see the Appendix). Still extant are lists of problems, theorems, definitions, postulates, scholia, figures, exercises, glosses, illustrations, anecdotes, divisions, and notations, all of which have helped in the editing. Scholium 14 seems to have been lost to Art. 43 in Peirce's revision of page 98 of his manuscript. It concerned "the importance of algorithm for G.C.D." Problem 2 was listed on the outline as belonging to Art. 43. Yet Peirce numbered it "3" in the manuscript. The editor has attempted to adhere to Peirce's summaries where possible. As a result very few number changes have been necessary.

lift *O* but is not strong enough to lift *I*. This is manifestly absurd. Having obtained a relation of this description, namely one which thus renders the diagram absurd (the absurdity ceasing as soon as we suppose two *A*s have been confounded), the compound relation represented by a wedge and an arrow, such as the relation of *A* to *E* through *I*, like "able to lift something that cannot be lifted by," is a relation altogether analogous to the relation of coming after. Thus, corresponding to Axiom 3, we have the proposition that, to say that any man, *A*, is able to lift something that cannot be lifted by another man, *E*, is precisely equivalent to saying *A* is able to lift something that cannot be lifted by somebody who is able to lift anything that can be lifted by *E*; for *E* himself may be that somebody. Further, corresponding to Axiom 4 we have the proposition that if any man, *A*, can lift something that cannot be lifted by *somebody* that *is* able to lift anything that can be lifted by *E*, then *A* can lift something that cannot be lifted by *anybody* except a man able to lift something that cannot be lifted by *E*.

Art. 20. *Definition* 16. The objects of a collection are said to be arranged in a *sequence* when of every pair of them one object comes *after* the other.

Theorem 1. In any sequence there can be but one object that comes after no other.

Demonstration. For let *A* and *B* be any two objects of the same sequence. Then I say that one of the two objects, *A* and *B*, comes after some other.

For, by Definition 16, of the pair of objects, *A* and *B*, one of them comes after the other. Q.E.D.

Theorem 2. In any sequence there can be but one object that no other comes after.

Exercise 1. The pupil will please demonstrate Theorem 2, almost exactly as Theorem 1 is demonstrated. He must first sit down and patiently study the Demonstration of Theorem 1, in the light of Scholium 4. He must *really imagine* that which the demonstration requires him to imagine. When he catches the idea, he has to perform a similar though slightly different process with Theorem 2.

Definition 17. The *last* object of a sequence is one that none comes after; the *first* is one that comes after none.

Scholium 6. In mathematics, we must be careful not to allow our thought to be bound down to particular cases. When a sequence has a first or last, we must never forget that it might be continued further, unless it happens that the first and last are the same. In other cases, we must think of a possible continuation of the [sequence] beyond.

Definition 18. A *limit* (throughout mathematics, note this well, for future application) is a boundary between two extensions or multitudes.

Division 2. Sequences can be distinguished into such as have both *first* and *last* objects, and those which either have no *first* or no *last*. (Throughout this book the expression "either one or the other" is to be understood as meaning what some writers express by "either one, or the other, or both.")

Definition 19. A sequence which has both a first and a last is called a *bounded*, or *doubly limited* sequence. A sequence which has no first or no last is called a *boundless* [sequence].

Division 3. A boundless sequence may either be *half-bounded* (*limited one way*) or *unlimited both ways*.

N.B. (3) It is necessary to dismiss the notion that that which is unlimited is necessarily larger than that which is limited.

Memorandum 5. It was Georg Friedrich Bernhard Riemann (b. 1826-d. 1866), one of the greatest of all mathematicians, who first made this clear in 1854.[7]

Illustration 5. The different instants of a second or a year form a bounded sequence; because such a lapse of time has a first and a last instant. But the instants of a second that are *between* the first and last instants, excluding those, form a sequence unlimited both ways. Simply taking away the limits renders the sequence unlimited.

[*Illustration*] 6. The alphabet is arranged in a bounded sequence, the ordinal numbers in a sequence limited one way only. The proper vulgar fractions, if arranged in the order of their magnitudes, form a sequence unlimited both ways; for there is no greatest nor least irreducible proper vulgar fraction.

[*Illustration*] 7. If the irreducible proper vulgar fractions are arranged so that those of less denominators come before those of greater denominators and those of the same denominator, but less numerators, before those with greater numerators, they form the following sequence:

[7] A space in the MS seems to indicate that Peirce intended to add to the historical material. In several places in this section Peirce uses "series" interchangeably with "sequence," contrary to modern usage. The editor has changed "series" to [sequence] where necessary.