Frege: Importance and Legacy

Edited by
Matthias Schirn

Walter de Gruyter · Berlin · New York

1996

Printed on acid-free paper which falls within the guidelines
of the ANSI to ensure permanence and durability.

Library of Congress Cataloging-in-Publication Data

Frege : importance and legacy / edited by Matthias Schirn,
p. cm. – (Perspektiven der Analytischen Philosophie =
Perspectives in analytical philosophy ; Bd. 13)
"Papers, most of which were presented at the international
conference Foundational Problems in Frege and in Modern Logic,
held in Munich from 8 to 13 July, 1991" – Pref.
Includes bibliographical references and indexes.
ISBN 3-11-015054-9
1. Frege, Gottlob, 1848–1925 – Congresses. 2. Logic, symbolic
and mathematical – Congresses. I. Schirn, Matthias. II. Series:
Perspectives in analytical philosophy ; Bd. 13.
B3245.F24F68 1996
193–dc21 96-39046
CIP

Die Deutsche Bibliothek – CIP-Einheitsaufnahme

Frege : importance and legacy / ed. by Matthias Schirn. – Berlin
; New York : de Gruyter, 1996
(Perspektives in analytical philosophy ; Bd. 13)
ISBN 3-11-015054-9
NE: Schirn, Matthias [Hrsg.]; Perspektiven der analytischen Philosophie

Printed in Germany
Typesetting and Printing: Saladruck, Berlin
Binding: Lüderitz & Bauer, Berlin
Cover design: Rudolf Hübler, Berlin

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Preface

This book is a mixed selection of papers, most of which were presented at the international conference "Foundational Problems in Frege and in Modern Logic", held in Munich from 8 to 13 July, 1991. The papers deriving directly from the conference are those of Michael Resnik, Bob Hale and Crispin Wright, Christian Thiel, Peter Simons, Franz von Kutschera, Eva Picardi, Gottfried Gabriel, Terence Parsons, and Bob Hale.

The conference was organized by the editor of this volume and sponsored by the Deutsche Forschungsgemeinschaft, to whom I express my gratitude. I am grateful to Bosch GmbH (especially to Dr. Marcus Bierich) and IBM Deutschland GmbH for their financial support. Special thanks are due to Uwe Lück for his valuable help during the entire conference. I would also like to thank the participants for much stimulating discussion throughout the conference. Last but not least, I wish to thank an anonymous referee for his detailed and useful comments on earlier versions of most of the essays collected in this volume.

The principal purpose of this collection is to display both the breadth and the significance of current Frege research. Frege's importance for the development of modern logic as well as for the formation of analytic philosophy can hardly be overestimated. And, what is more, his work continues to be much debated. There is, in particular, a renewed and profound interest in his philosophy of mathematics. Michael Dummett's superb study *Frege: Philosophy of Mathematics* (Duckworth, London 1991) is one prominent example. The second claim made by the title of this volume is that Frege's work has left a legacy, that is, a set of questions yet to be answered. I, for my part, believe that this claim is supported by most of the essays here collected.

I am very sad to have to report that George Boolos died of cancer on 27 May, 1996 at the age of 55. I knew George personally since July 1991 when he came to Munich to read a paper in our Institute and at the Frege conference. Afterwards we met in Boston and Cambridge, Massachusetts and again in Munich in the summer of 1993 during another conference I had organized. I very much admire George's work, especially his subtle and profound studies on Frege's logic and philosophy of mathematics. He will be greatly missed both as a person and as a scholar. I dedicate this volume to the memory of George Boolos.

Munich, 30 May, 1996 Matthias Schirn

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Introduction: Frege on the Foundations of Arithmetic and Geometry

MATTHIAS SCHIRN

Logicians and philosophers nowadays largely agree that Frege did most important work in logic, the philosophy of mathematics and the philosophy of language. This is true even though some of his doctrines have been the target of sharp criticism. In my view, to reveal errors or shortcomings in Frege's work may well go hand in hand with admiration for his major achievements, the power and depth of his argument and the lucidity and precision of both his exposition and his style.

This collection is divided into three parts. The first is entitled "Logic and Philosophy of Mathematics", the second "Epistemology" and the third "Philosophy of Language". In what follows, I shall mainly give a kind of overview of some selected topics in Frege's philosophy of mathematics. In doing so, I shall place emphasis not only on his project of laying the logical foundations of number theory and analysis, but also on his remarks on the foundations of geometry which are scattered throughout a number of his writings. My motive for considering geometry is twofold. Firstly, Frege's "philosophy of geometry", although it is probably of minor importance compared with his philosophy of arithmetic, nonetheless deserves close attention. Moreover, investigating the former may shed considerable light on some aspects of the latter. Secondly, none of the papers collected in this volume actually deals with Frege's ideas about geometry. Of course, I do not mean to fill the gap here, but would be pleased if my account makes it feel a trifle narrower, at least for those who agree with me about the need to account for geometry in Frege's philosophy of mathematics. I follow my discussion of arithmetic and geometry with a brief assessment of current Frege research, and conclude with remarks about the essays in this collection.[1]

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[1] I use the following abbreviations for references to Frege's works: BS: *Begriffsschrift. Eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*, Louis Nebert, Halle a. S. 1879; GLA: *Die Grundlagen der Arithmetik. Eine logisch mathematische*

Number theory and analysis

To Frege we owe the first systematic construction of an axiomatized, complete and consistent calculus of first-order logic with identity which encompasses the classical propositional calculus. The calculus of first-order logic was developed in his *Begriffsschrift* of 1879. In this small book, Frege also in effect used second-order logic serving to introduce concepts such as *following in a sequence* and *heredity of a property in a sequence,* required for laying the logical foundations of arithmetic. By way of integrating the propositional and predicate calculus, and, in particular, by solving the problem of multiple quantification, Frege went far beyond Boole's logic.

The *Begriffsschrift* presented ground-breaking achievements in logic. Frege's second book, *Die Grundlagen der Arithmetik* of 1884, was intended to make plausible the claim that the truths of arithmetic are analytic, that is, that they can be derived exclusively from primitive truths of logic and definitions. The investigation of the concept of number is carried out within the framework of natural language and employs only a few technical devices. Frege's main concern in the critical part of *Grundlagen* is a vigorous attack on two rival theories of arithmetic, the physicalistic and the psychologistic. In the final chapter, he takes a third rival theory to task, Hankel's (and Kossak's) so-called formal theory of negative, fractional, irrational, and complex numbers.

In the constructive part of *Grundlagen,* Frege immediately turns to the central task he has set himself, namely to frame a definition of number in purely logical terms. After exploring two unsuccessful definitions, he proceeds to define the number which belongs to the concept F (symbolically: $N_xF(x)$) as an equivalence class of the second-level relation of equinumerosity. (Henceforth, "E" is to abbreviate "equinumerous".) However, this explicit definition (call it (D)) rests on the assumption that we intuitively know what the extension of a concept in general is. To be sure, at the time when Frege wrote *Grundlagen* he could not rely on a commonly accepted view of the nature of extensions of concepts, let alone of the nature of numbers. Thus, his assump-

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Untersuchung über den Begriff der Zahl, W. Koebner, Breslau 1884; GGA: *Grundgesetze der Arithmetik. Begriffsschriftlich abgeleitet,* Vol. I, H. Pohle, Jena 1893, Vol. II, H. Pohle, Jena 1903; KS: *Kleine Schriften,* ed. I. Angelelli, G. Olms, Hildesheim 1967; NS: *Nachgelassene Schriften,* eds. H. Hermes, F. Kambartel, and F. Kaulbach, Felix Meiner, Hamburg 1969; WB: *Wissenschaftlicher Briefwechsel,* eds. G. Gabriel, H. Hermes, F. Kambartel, C. Thiel, and A. Veraart, Felix Meiner, Hamburg 1976.

tion seems to jeopardize the viability of his foundational project as outlined in *Grundlagen*. At any rate, (D) falls short of solving what has come to be known as "the Julius Caesar problem". The problem is this: the criterion of identity embodied in the tentative contextual definition (C) "$N_x(F(x) = N_xG(x) := E_x(F(x),G(x))$" is powerless to decide whether Julius Caesar is identical with, say, the number of churches in Rome. Unfortunately, (D) does not lead us out of the impasse, because we do not know whether Julius Caesar is a number unless we know whether or not he is an extension. In *Grundlagen*, Frege lays down identity-conditions not for extensions of concepts in general, but only for various kinds of equivalence classes. And, as I have just said, his assumption concerning extensions must be regarded as disputable. We are thus left with the problem that (D), like (C), fails to fix uniquely the reference of the cardinality operator "$N_x\varphi(x)$". I, for my part, believe that the prospects for removing the pervasive referential indeterminacy of "$N_x\varphi(x)$" within the setting of *Grundlagen* are poor, unless Frege were to have contrived appropriate additional stipulations.

Returning to the line of thought in *Grundlagen*, we see that Frege, once he has established definition (D), proceeds to define equinumerosity in terms of one-one correspondence. Note that the latter definition is conceptually prior to (D). Frege is then able to give his definition of *n is a number*. In what follows, he has to establish the systematic fruitfulness of his logicist definition of number by deriving from it (augmented by further definitions, for example, of the successor relation and of the ancestral) familiar laws of number theory. Somewhat surprisingly, Frege actually uses (D) only in his sketchy proof of the equivalence "$N_xF(x) = N_xG(x) \equiv E_x(F(x),G(x))$"; let us call this equivalence "(T)". The theory obtained by adjoining (T) to second-order logic enjoys consistency.[2] Having derived (T), Frege never appeals to (D) again. Instead, he derives fundamental theorems of number theory from (T) together with his other definitions. In the literature on Frege, the derivability of number theory from (T) within second-order logic was first noted by Charles Parsons in his essay 'Frege's Theory of

[2] Cf. J. P. Burgess, Review of C. Wright, *Frege's Conception of Numbers as Objects*, Aberdeen University Press, Aberdeen 1983, *The Philosophical Review* **93** (1984), 638–640; G. Boolos, 'The Consistency of Frege's Foundations of Arithmetic', in J. J. Thomson (ed.), *On Being and Saying. Essays for Richard Cartwright*, The MIT Press, Cambridge, MA 1987, 3–20.

Number'[3]. It has received due attention only in recent years, though, notably in Crispin Wright's study *Frege's Conception of Numbers as Objects* (1983) and in several articles by George Boolos[4].

In *Grundlagen*, Frege confined himself to indicating briefly how the proofs of certain arithmetical laws proceed. In *Grundgesetze der Arithmetik* (Vol. I 1893, Vol. II 1903), he set out to demonstrate the validity of logicism beyond doubt by producing gapless chains of inference. He firmly believed that only in this way do we gain a secure basis for assessing the epistemological status of the law that is proved. Frege emphasizes that his (so-called) Begriffsschrift has undergone a number of internal changes due to a far-reaching development of his logical views. The signs familiar from *Begriffsschrift* that appear outwardly unchanged in *Grundgesetze*[5], and whose "algorithm" has also scarcely changed, are said to be provided with different explanations. Here, then, are what appear to be more or less thoroughgoing changes: the decomposition of what was earlier called "judgeable content" into *thought* and *truth-value*, as a consequence of the distinction between the *sense* and the *reference* of a sign[6]; the introduction of the two truth-values as the references of assertoric sentences (or more exactly: as the references of object-names which have the syntactic structure of sentences) and the related conception of concepts and relations as (monadic or dyadic) functions from suitable arguments to truth-values; the clear-cut distinction between the functional expression and the function as its reference, which in turn goes hand in hand with a sharper characterization of the nature of functions as opposed to objects; the explicit distinction between functions of first, second, and third level as well as between equal-levelled and unequal-levelled functions (relations); the treatment of identity as an object language predicate; the introduction of the content-stroke "–" as a (primitive) function-sign (now called the "horizontal"), which undergoes a drastic reinterpretation; the introduction of a sign designed to serve as a substitute for the definite article of ordinary

[3] In M. Black (ed.), *Philosophy in America*, Cornell University Press, Ithaca, NY 1965, 180–203.

[4] 'The Consistency of Frege's Foundations of Arithmetic', *op. cit.;* 'Saving Frege from Contradiction', *Proceedings of the Aristotelian Society,* 1986–87, 137–151; 'The Standard of Equality of Numbers', in Boolos (ed.), *Meaning and Method: Essays in Honor of Hilary Putnam,* Cambridge University Press, Cambridge 1990, 261–277.

[5] There is one exception: "$\equiv$" is replaced by "$=$".

[6] In retrospect, Frege is inclined to regard the judgeable content primarily as what he then calls the thought; cf. WB, 120.

language; and finally the introduction of the courses-of-values of functions, which, apart from achieving greater simplicity and flexibility, Frege considered to be of vital importance for carrying out his logicist programme: all numbers were to be defined as extensions of concepts.[7] Let us now take a closer look at some issues involved in his introduction of courses-of-values.

In *Grundgesetze,* Frege seems to be well aware that a methodologically sound introduction of courses-of-values as logical objects cannot proceed via an elucidation of the primitive, second-level function $\dot{\varepsilon}\varphi(\varepsilon)$, the so-called course-of-values function; more specifically, it cannot proceed via an elucidation modelled upon the pattern of the semantic explanations provided for the other primitive functions of his system. The simple reason is that an elucidation along these lines, let us say: "The value of $\dot{\varepsilon}\varphi(\varepsilon)$ for every monadic first-level function $\Phi(\xi)$ as argument shall be the course-of-values of $\Phi(\xi)$", presupposes that we already know what courses-of-values are. Remember that it is precisely this unwarranted assumption, relating to the more special case of extensions of concepts, that seems to overshadow the programme of *Grundlagen.* While Frege feels entitled to proceed from the assumption that in our ordinary practice of judging and asserting we are already familiar with the objects the True and the False, he refrains from assuming that the reader of *Grundgesetze* is sufficiently acquainted with courses-of-values. That the two truth-values are, indeed, distinguished logical objects for Frege emerges very clearly both from the way he introduces the eight primitive functions of his system and from his attempted proof of referentiality for all well-formed names of his formal language (cf. GGA I, § 31). If you allow an analogy: the True and the False are in the domain of objects of Frege's logical theory what the primitive functions are in the domain of functions. The truth-values may thus properly be called the *primitive objects of logic.*

In § 3 of *Grundgesetze,* Frege introduces courses-of-values of functions, which comprise extensions of concepts and of relations as special cases, by means of an informal stipulation corresponding to the ill-fated Axiom V of his logical system. Cardinal numbers are now defined as

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[7] As we shall see later, Frege intended to define the real numbers as *"Relationen von Relationen"*. He uses the term "Relation" to refer to special double courses-of-values, namely to courses-of-values of dyadic (first-level) functions whose values are, for every admissible pair of arguments, either the True or the False, in short: to extensions of (first-level) relations [*Beziehungen*]. Cf. GGA II, §§ 162, 245.

extensions of *first*-level concepts. Since, in contrast to Frege's view in *Grundlagen*, it is the extension of a concept that is construed as the "bearer" of number, the cardinality operator appears as a *first*-level function-name. While in *Grundlagen* Frege does not appeal to extensions of concepts, once he has derived (T) from (D)[8], in *Grundgesetze* matters stand differently. Throughout this book, he makes extensive use of courses-of-values, and he does so for reasons of economy and convenience. In § 34 of *Grundgesetze*, he defines membership in such a way that with its help one may employ, in place of second-level functions, first-level functions in his logical calculus. This "level-reduction" is rendered possible through the following device: first-level functions which appear as arguments of second-level functions are represented by their courses-of-values – "though of course not in such a way that they give up their places to them, for that is impossible" (GGA I, 52). Yet such "representational" uses of courses-of-values, however attractive they may have appeared to Frege, can easily be dispensed with. By contrast, the use of courses-of-values in the proof of (T) is indispensable.

In *Grundgesetze*, when it comes to the construction of arithmetic, Frege proceeds in much the same way as in *Grundlagen*. Here, too, he first derives (T) from his explicit definition of number and subsequently deduces the basic laws of cardinal arithmetic from (T) within axiomatic second-order logic. In *Grundgesetze*, (T) is presented in a somewhat different fashion, but the new version is otherwise equivalent to the old one in *Grundlagen*. Until recently it has not been noted, at least not in print, that Frege in fact provided an axiomatization for arithmetic. In his paper 'The Development of Arithmetic in Frege's *Grundgesetze der Arithmetik*'[9], Richard Heck draws attention to this. Although Frege proves each of the Dedekind-Peano axioms, he is primarily concerned to prove his own axioms for arithmetic. The latter are, of course, equivalent to the former. In his second paper on *Grundgesetze* (this volume), as well as in his third, entitled 'The Finite and the Infinite in Frege's *Grundgesetze der Arithmetik*'[10], Heck presents further important mathematical results of *Grundgesetze*, explains how

[8] It is not until Section 83 of *Grundlagen* that Frege employs the term "extension of concept" again. In this Section, he completes his sketch of the proof that every finite number has a successor.

[9] *The Journal of Symbolic Logic* 58 (1993), 579–601.

[10] Forthcoming in M. Schirn (ed.), *Philosophy of Mathematics Today*, Oxford University Press, Oxford 1997.

these are established by Frege and puts them into historical perspective. One point made in the last-mentioned paper is that Frege discovered, around 1892, at least the axiom of countable choice, if not the full axiom of choice.

When Frege introduces courses-of-values in the way hinted at earlier, he encounters a variation of his old indeterminacy problem from *Grundlagen*, now clad in formal garb. Construed as a means of fixing the references of certain abstract singular terms, Basic Law V, in view of its close affinity to (T) (actually, the exact structural analogue of (T)!), was likely to arouse suspicion anyway. The criterion of identity for courses-of-values incorporated in Basic Law V takes care of the truth-conditions of only those equations in which both related terms are of the form "$\grave{\varepsilon}\Phi(\varepsilon)$". Yet the criterion fails to determine the truth-value of "$\grave{\varepsilon}\Phi(\varepsilon) = q$" if "q" is not of the form "$\grave{\varepsilon}\Psi(\varepsilon)$". Frege proposes to remove the referential indeterminacy of course-of-values terms "by determining for every function, as it is introduced, what values it obtains for courses-of-values as arguments, just as for all other arguments" (GGA I, § 10). At the stage of § 10, the proposed procedure eventually boils down to the determination of the values of the identity relation. Somewhat surprisingly, Frege confines himself to determining its values only for courses-of-values and the two truth-values as arguments, contrary to what the phrase "just as for all other arguments" seems to suggest. He explains (or attempts to justify) this restriction by mentioning the fact that up to § 10 no other objects have been introduced. At any rate, Axiom V is powerless to decide whether or not either truth-value is a course-of-values.

Having set out what has come to be known as his *permutation argument*, Frege feels free to make a stipulation which he regards as the key for resolving the referential indeterminacy of a term "$\grave{\varepsilon}\Phi(\varepsilon)$": he identifies the True and the False with their own unit classes. The formal legitimacy of this "transsortal" identification, namely its consistency with Axiom V, is established by the permutation argument.[11] However, the technical details of this argument need not concern us here. Rather, I wish to consider the famous second footnote to § 10 of *Grundgesetze* as well as to provide a response to the question whether Frege succeeds

[11] T. Parsons ('On the Consistency of the First-Order Portion of Frege's Logical System', *Notre Dame Journal of Formal Logic* 28 (1987), 161–168, here 165) shows that, contrary to what Frege claims, it is not always possible to stipulate that an arbitrary course-of-values shall be the True and another the False.

in fixing completely the references of course-of-values terms, and thus, in justifying the use of these terms in his formal theory. In doing so, I shall prescind from the inconsistency of that theory.

The second footnote is both intricate and puzzling. Moreover, it is of considerable importance for assessing the overall strategy in § 10. Frege considers here the possibility of generalizing his stipulation concerning the two truth-values so that all objects whatsoever, including those already given to us as courses-of-values (i.e., referred to by course-of-values names), are identified with their own unit classes. The suggestion goes awry. Frege rejects it on the grounds that it may contradict the criterion of identity embodied in Axiom V, if the object to be identified with its own unit class is already given to us as a course-of-values. At the same time, he jettisons the intuitively appealing proposal of identifying with their unit classes all and only those objects which are given independently of courses-of-values.[12] He does so by using what is basically the same argument as that in § 67 of *Grundlagen,* saying that the mode of presentation of an object must not be regarded as an invariant property of it, since the same object can be given in different ways.[13]

There are several difficulties looming here. To begin with, Frege fails to spell out his motive for examining the possibility of generalizing the stipulation of § 10. We are only told that the identification of every object Δ with $\dot{\varepsilon}(\Delta = \varepsilon)$ would be a natural suggestion. But in what sense does it suggest itself? What would we achieve with it if it were formally sound? Now, it is quite true that even if Frege does regard the domain of the first-order variables as limited to such objects as are required to exist by the axioms of his logical system, and thus to truth-values and courses-of-values, he appears to have envisaged extensions of the system to include, say, geometry or physics. Such an extension having been made, versions of the Julius Caesar problem would then emerge at once.

Likewise, Frege appears to have been aware of what we might describe as follows. Suppose that his envisaged foundation of number theory and analysis required the introduction of a new, third type of

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[12] The proposal is intuitively appealing, because it may seem that we can decide, by invoking Axiom V, whether a given course-of-values is identical with an object Δ, not referred to by a course-of-values name, once we have identified Δ with its own unit class.

[13] In fact, within the system of *Grundgesetze*, the extension of the concept $-\xi$, for instance – under it falls the True alone – could be designated not only by course-of-values terms (e.g., by "$\dot{\varepsilon}(\varepsilon = (\varepsilon = \varepsilon))$"), but also by truth-value names (e.g., by "$\forall x(x = x)$"), and definite descriptions (e.g., by "$\backslash\dot{\varepsilon}(-\varepsilon)$"). Outside the system, $\dot{\varepsilon}(-\varepsilon)$ could be referred to as "Frege's favourite object", for example.

objects and, correspondingly, a new category of singular terms to refer to those objects. In that case, too, the referential indeterminacy of course-of-values terms would arise anew. An extension of the domain could proceed in the following way. In addition to the second-level concept of generality $\forall x\varphi(x)$ and the course-of-values function $\grave{\varepsilon}\varphi(\varepsilon)$, a third primitive second-level function, say, $\hat{\alpha}\varphi(\alpha)$, is introduced through the following stipulation: its value for every monadic first-level function as argument shall be an object of the third kind. I am aware that Frege might be reluctant to endorse this way of introducing a function $\hat{\alpha}\varphi(\alpha)$, on the grounds that it illegitimately presupposes an acquaintance with the function-values qua objects of a third kind. However, let us assume, for the sake of argument, that he would accept the proposed introduction of objects of a hypothetical third kind.

Plainly, from what I have just speculated to be an extension of the domain of Frege's formal theory, it would follow that the intended complete specification of the references of course-of-values terms by adjoining the stipulation of § 10 to Axiom V could not be secured.[14] For, from Frege's point of view, we could not rule out that an object of the third kind, which $\hat{\alpha}\varphi(\alpha)$ assigns to a suitable argument, in fact coincides with a course-of-values. In order to overcome this iterated indeterminacy, Frege would be compelled to make a further stipulation, designed to guarantee that the truth-value of every equation in which the sign of identity connects a course-of-values term with a term of the new category is determined. And, of course, every additional extension of the domain would require further stipulation.

If, on the one hand, we focus on the way Frege actually proceeds in § 10, and more importantly in § 31 of *Grundgesetze*, we should assume that he does regard the domain of first-order quantification as restricted to truth-values and courses-of-values. Canvassing the possibility of generalizing the stipulation of § 10 could then be construed as an attempt to supply a general solution to versions of the Julius Caesar problem that would inevitably arise for every possible extension of the domain. On the other hand, there are remarks in *Grundgesetze* suggesting that Frege is after all operating with an all-encompassing domain. Yet if the domain comprises all the objects there are, then the second footnote to § 10 might rather be seen as reflecting Frege's resi-

[14] I assume here, for the sake of argument, that the domain of Frege's system is (initially) limited to truth-values and courses-of-values. The question whether it is plausible to assume that the domain is all-inclusive anyway will be discussed later.

dual uneasiness about the restriction he imposes on the range of the arguments when he comes to determine the values of $\xi = \zeta$. So, we face what appears to be a head-on conflict; I shall spell it out more fully in a moment. But let me first draw attention to two further difficulties to which Frege's line of argument in the footnote gives rise.

Suppose that Frege considers the domain of his formal theory to contain only truth-values and courses-of-values. Suppose further that he regards course-of-values terms as referentially indeterminate, even after having identified the True and the False with their own unit classes. One obvious reason for his taking their references to be indeterminate would be that at a later stage the domain has actually undergone an extension. Consider now, in the light of these assumptions, the tentative stipulation $\grave{\varepsilon}(\Delta = \varepsilon) = \Delta$, and recall that it is supposed to embrace not only objects given independently of courses-of-values, but also objects referred to by course-of-values terms. But why should objects already given to us as courses-of-values be taken into account at all? If I am right, then the question as to whether the stipulation $\grave{\varepsilon}(\Delta = \varepsilon) = \Delta$ can be consistently extended to objects already given to us as courses-of-values proves to be irrelevant for any attempt to eliminate the referential indeterminacy of course-of-values terms. For clearly, the question whether, e.g., $\grave{\varepsilon}(\grave{\varepsilon}(\varepsilon = (\varepsilon = \varepsilon)) = \varepsilon)$ coincides with Julius Caesar poses the same problem as the question whether $\grave{\varepsilon}(\varepsilon = (\varepsilon = \varepsilon))$ is identical with the Roman general who crossed the Rubicon.[15]

Another difficulty may be described as follows. At the outset of the footnote, Frege claims that the stipulation $\grave{\varepsilon}(\Delta = \varepsilon) = \Delta$ is possible for every object given to us independently of courses-of-values on the same basis as he has observed with the truth-values. His subsequent argument calls this claim into question, however. For the argument seems to convey that we may have to recognize any object Δ not given to us as a course-of-values as being a course-of-values. Yet if Δ is a course-of-values, then we cannot identify Δ with its own unit class without offending against Basic Law V, unless Δ is the extension of a concept under which Δ alone falls.

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[15] We cannot rule out, prior to the stipulation made in § 10, that, e.g., $\grave{\varepsilon}(\varepsilon = (\varepsilon = \varepsilon))$ coincides with Julius Caesar. But are we better off after the True has been identified with its own unit class? Frege would presumably insist that it is by virtue of our intuitive familiarity with the two truth-values – something which we are lacking with courses-of-values – that we can distinguish safely the True and the False from Julius Caesar. However this may be, it seems awkward to say that once the True has been identified with $\grave{\varepsilon}(\varepsilon = (\varepsilon = \varepsilon))$ we know that $\grave{\varepsilon}(\varepsilon = (\varepsilon = \varepsilon))$ is distinct from Julius Caesar.

How does the stipulation in § 10 fare in the light of what appears to be Frege's argument in the footnote? Should we deny that the latter has any repercussions at all on the former? Or should we maintain the other extreme, and perhaps say that the identification of the True and the False with their own unit classes is after all nothing but a flash in the pan? On the face of it, it seems consistent for Frege (1) to dismiss as indefensible the general proposal of identifying with their unit classes all and only those objects which are not given to us as courses-of-values and yet (2) to allow certain particular identifications which the general proposal, if accepted, would also license. On closer reflection, however, this is less clear. As was said earlier, the stipulation in § 10 is indeed consistent with Axiom V; yet following Frege's line of thought in the footnote, it seems that, before we make the stipulation, we are bound to rule out that the True and the False are courses-of-values "containing more than one object". For according to the argument given there, the fact that an object Δ is not given to us as a course-of-values does not imply that it is not one; in particular, we have no guarantee that Δ is not a course-of-values distinct from its unit class. But why should this argument not apply to the object referred to by "$\forall x(x = x)$", for instance? And if it does, how can we then legitimately identify the True with $\acute{\varepsilon}(-\varepsilon)$, i.e., with its own unit class?

It is generally agreed that it is essential for Frege's foundational project, resting on a classical logic with a classical semantics as it does, to secure a reference for every well-formed expression, especially for every well-formed formula of his Begriffsschrift. To establish beyond doubt that every well-formed expression is referential is precisely what his proof in § 31 of *Grundgesetze* is supposed to accomplish. As a matter of fact, Frege confines himself to demonstrating that every primitive function-name of his system has a reference, apparently relying on the assumption that if the primitive function-names are referential, his formation rules will bequeath the property of being referential to every name formed in accordance with them. Now, we may certainly grant that in attempting to secure a reference for every well-formed Begriffsschrift expression, Frege is at liberty to make certain additional stipulations. Yet, whenever he thinks it is convenient to make one, he ought to take care that it tallies with any thesis he propounds or advocates in the relevant context.

To conclude my assessment of § 10 and its attendant footnote, let me return to what I called a head-on conflict. Frege's remark in § 9 of *Grundgesetze,* that by introducing his notation for courses-of-values we also extend the domain of what can appear as the argument of a (first-level) function, might be taken to suggest that it is the expressive power or the referential repertoire of his Begriffsschrift that determines the range of what can appear as the argument of a first-level function. Seen from this angle, Frege probably thinks it is sufficient to determine the values of the relation of identity for courses-of-values and truth-values as arguments, because the language of his formal theory does not and need not contain any means to refer to other objects. Admittedly, it was said above that the existence of any objects other than truth-values and courses-of-values is not required by the axioms of Frege's theory. Yet limiting the determination of the values of $\xi = \zeta$ to just those objects to which we can refer by means of well-formed function-value names of his Begriffsschrift seems to be incompatible with several remarks by Frege, which I shall now present.

In § 2 of *Grundgesetze,* Frege emphasizes that the domain of what is admitted as arguments of type 1, i.e., objects, must be extended to objects in general. I term this demand "Frege's non-exclusiveness doctrine". Correspondingly, he elucidates the primitive first-level functions of his system for all suitable arguments whatsoever, i.e., for an all-embracing domain of objects, and defines certain logically complex functions of first level, in accordance with his principle of completeness, "for all possible objects as arguments" (GGA I, 52 f.). Finally, the free individual variables a, b, ..., which Frege uses in his formal theory, have the task of indicating objects in general, not only those of a domain with fixed boundaries (cf. GGA II, 78). If we take these remarks at face value – and I think we should – then we may plausibly assume that in *Grundgesetze* Frege takes the individual variables to range over all the objects there are. To respond to this by saying that the formal language of *Grundgesetze* (presumably) does not contain names for Julius Caesar or the Eiffel Tower, or that such spatio-temporal objects need not fall within the domain of a model of the axioms of the system would lack any force. For if the domain of first-order quantification encompasses all objects whatsoever, then Frege faces the question (and, indeed, has to answer it) whether a course-of-values included in the domain coincides with, say, Julius Caesar or the Eiffel Tower. The reason is that formulae of the form "$\forall x(x = \grave{\varepsilon}\Phi(\varepsilon) \rightarrow p)$" will not have been provided with a determinate reference, i.e., truth-value, unless "$x = \grave{\varepsilon}\Phi(\varepsilon)$", for Julius Caesar (or the Eiffel Tower) taken as value of "x", has been given a reference.[16] Thus, if the domain embraces all the objects there are, Axiom V together with the additional stipulation in § 10 fails to fix completely the references of course-of-values terms.

The conflict under discussion is even exacerbated, when we take a look at Frege's (abortive) attempt to prove that every well-formed name of his formal language has a reference. The proof, in which he invokes both Axiom V and the stipulation of § 10 when he attempts to show that the primitive function-name "$\grave{\varepsilon}\varphi(\varepsilon)$" has a reference, relies crucially on the assumption that only truth-values and courses-of-values are in the domain. For, if we do not make this assumption, then sentences of the form "$x = \grave{\varepsilon}\Phi(\varepsilon)$" will not have been given a reference for every assignment of members of the domain to "x", and the proof would not even get off the ground.[17] I, for my part, fail to see how the two positions regarding the size of the domain, which Frege appears to endorse in *Grundgesetze,* could be reconciled. To claim that he cannot have been blind to the above-mentioned aspect of his attempted proof of referentiality, and, therefore, did not consider the domain of his system to include all the objects there are, hardly resolves the discrepancy.

So much to Frege's introduction of courses-of-values and his indeterminacy problem.[18] It remains to cast a glance at his work on the foundations of analysis in the second volume of *Grundgesetze.* Let us then do this.

Frege's way of discussing analysis in *Grundgesetze* is, in a sense, akin to his treatment of the natural numbers in *Grundlagen.* In both cases, he does not begin by presenting his own theory, but rather by launching an attack on rival theories. In the second volume of *Grundgesetze,* the main targets are Heine's and Thomae's radical version of formalism, Cantor's theory of real numbers as well as Weierstraß's conception of numbers. The reader of this volume who is expecting a thorough examination of the theory of irrational numbers "of such a distinguished mathematician as Weierstraß" (GGA II, 149) will be disappointed. Frege takes the easy route. He basically confines himself to making critical remarks, spiced with plenty of irony, about Weierstraß's treatment of the natural numbers[19] and eventually tries to convince us

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[16] Thanks to Richard Heck at this point.

[17] I am grateful to Richard Heck for pointing this out to me.

[18] For a more detailed investigation see my paper 'Referential Indeterminacy in Frege's Philosophy of Arithmetic' (forthcoming).

[19] One might speculate what Frege himself would have answered if he had been roused in the night with the question: "What is a number?"

that, due to its shaky foundations, Weierstraß's theory of irrational numbers need not be discussed in greater detail. Likewise, Frege pays comparatively little attention to Dedekind's theory, though he praises his sharp distinction between sign and reference [*Bedeutung*] and his taking numbers to be what numerical signs refer to. However, himself endorsing arithmetical platonism, Frege naturally finds fault with Dedekind's creation of new mathematical objects.

Frege begins the constructive part of *Grundgesetze II* by contrasting the reals with the cardinal numbers. He claims that the domain of the latter cannot be extended to that of the former. Cardinal numbers, by their very nature, answer to questions of the kind "How many objects of a certain kind are there?". By contrast, the reals are to be construed as ratios of quantities; they measure how large a given quantity is compared with a unit quantity. Thus, in Frege's view, the mode of application of the reals differs fundamentally from that of the cardinal numbers. And just as in *Grundlagen* he attempted to account for the application of the natural numbers in counting in their definition, so in *Grundgesetze* he takes pain to secure that the application of the real numbers in measurement is appropriately built into their definition.

The method of introducing the real numbers proposed by Frege lies between the traditional geometrical approach and the theories developed by Cantor, Weierstraß, and Dedekind. From the geometrical approach Frege adopts the characterization of the reals as ratios of quantities or, as he also says, as "measurement-numbers". Following Cantor, Weierstraß, and Dedekind, he detaches the reals from all special types of quantity. The rationale for doing this, so we are told, is that the application of the real numbers is not restricted to any special types of quantity, but rather relates to the domain of the measurable which encompasses all types of quantity whatsoever.

Frege observes that all previous attempts to define the notion of quantity have miscarried, because they posed the question wrongly. Instead of asking "What properties must an object have in order to be a quantity?" we ought to ask: What properties must a class (extension of a concept) have in order to be a quantitative domain? Frege adds that something is a quantity not in itself, but only in so far as it belongs with other objects to a class which is a quantitative domain.

The quantities to be considered are extensions of relations [*Beziehungen*] which Frege calls for short *"Relationen"*. (Henceforth, I use the term "Relation" for "extension of a relation".) Frege thus takes the reals or ratios of quantities to be Relations of Relations. Quantitative domains are classes of Relations, namely extensions of concepts subordinate to the concept *Relation.* Frege then turns to the question of where to obtain the quantities whose ratios are irrational numbers. Plainly, they have to be non-empty Relations, since with the aid of the empty Relation one cannot define a real number at all. If q is the empty Relation, then both the inverse of q and the composition of q with its inverse coincide with q. The upshot so far is this: "We thus need a class of objects which stand to one another in the Relations of our quantitative domain, that is, this class must contain infinitely many objects" (GGA II, § 164). Frege observes that the required class must have a cardinality greater than the class of natural numbers, and further that the number of the concept *class of natural numbers* is in effect greater than the number of the concept *natural number.* Surprisingly, Cantor's proof that for any set M, $\mathfrak{P}$M is always of a power greater than M itself is passed over in silence.

Having arrived at this point, Frege sketches his plan for the envisaged introduction of the real numbers. In order to make his exposition more readily understood, he temporarily assumes the irrational numbers known. Every positive real number a can be represented in the form

$$r + \sum_{k=1}^{k=\infty} \left\{ \frac{1}{2^{n_k}} \right\}$$

where r is a positive integer or 0, and n_1, n_2, ... form an infinite monotone increasing sequence of positive integers. To every positive rational or irrational number a there belongs an ordered pair <r,R>, where r is a positive integer or 0, and R an infinite class of positive integers (class of the n_k). If instead of the integers we take cardinal numbers, then to every positive real number there belongs an ordered pair whose first member is a cardinal number and whose second member is a class of cardinal numbers which does not contain the cardinal number 0.[20] Suppose that a, b and c are positive real numbers and that a + b = c holds. Then for every b there is a relation holding between the pairs belonging to a and c. This relation is said to be definable without presupposing the real numbers. Thus, we have relations, each of which in turn is characterized by a pair (belonging to b), to which we add their inverses. As

[20] For simplicity's sake, I do not use Frege's special notation for the cardinal numbers (cf. GGA II, § 157).

Frege points out, the extensions of these relations (i.e., these Relations) correspond one-one to the positive and negative real numbers; and to the addition of the numbers b and b′ corresponds the composition of the corresponding Relations. He further observes that the class of these Relations is a domain which suffices for his plan, but hastens to add that "it is not thereby said that we shall hold precisely to this route".

Before embarking upon the formal development – that part is entitled "The theory of quantity" [*Die Größenlehre*] – Frege draws attention to two points he considers to be important. First, neither the classes of natural numbers nor the ordered pairs mentioned above, nor the Relations between these pairs are irrational numbers. Second, it is possible to define the aforementioned relations between the pairs without invoking any acquaintance with the irrational numbers. "In this way, we shall succeed in defining the real numbers purely arithmetically or logically as ratios of quantities which can be shown to be available, so that no doubt can remain that there are irrational numbers" (GGA II, 162). Frege's formal account, as far as it goes, can only be presented in crude outline here.

Frege begins by asking: What properties must a class of Relations possess in order that in it the commutative and associative laws for the composition of Relations hold? The proof of the associative law (Theorem 489) requires that several sentences about the identity of Relations be derived (Theorems 485–487). Unlike the associative law, the commutative law does not hold in general. Frege first proves it for members of a sequence like K, K|K, K|(K|K), ... (Theorem 501). (Here I use the symbol | for composition.) On the strength of Theorem 501, the class of the members of such a sequence can be taken to be a quantitative domain, and every positive rational number can be defined as a ratio of two quantities belonging to that domain. The negative rational numbers could be introduced by adding the inverses of Relations. When it comes to the irrational numbers, Frege observes that "they can be obtained only as limit", which in turn can be defined only in terms of the *greater than* relation. For reasons of convenience, he wants to reduce this relation to the notion of the positive: a is greater than b if and only if the Relation composed of a and the inverse of b is positive. Now, if a positive class P is at hand, the quantitative domain associated with it can be determined as follows: to it belongs every Relation which either is a member of P, or is the inverse of a Relation belonging to P, or is composed of a Relation belonging to P and its inverse (cf. GGA II, § 173).

Having defined the notion of a quantitative domain, Frege outlines the following strategy. The central concept of a positive class cannot be defined directly. One must rather take a roundabout route, that is, one must first introduce the wider concept of what Frege calls a *positival class.* Equipped with the latter we can define the notion of least upper bound, and with the least upper bound we arrive at the notion of a positive class. A positival class S is a class of Relations satisfying the following five conditions: (1) Each Relation belonging to S is one-one; (2) the composition of such a Relation with its inverse does not belong to S; (3) if the Relations p and q are members of S, then the composition of p with q is in S; (4) if p and q are in S [and $p \neq q$], then the Relation composed of p and the inverse of q belongs to the quantitative domain of S; (5) if p and q are in S, then the composition of the inverse of p with q is in the domain of S (GGA II, § 175, Definition Ψ).

Frege emphasizes that in his definition of the notion of a positival class he has taken pains to include only those clauses that are independent of one another. He claims, however, that their mutual independence cannot be proved, and expresses the belief that especially clause (5) cannot be dispensed with. Naturally, the question arises here as to whether the mutual independence of the clauses, in particular the independence of clause (5) of the other four (if it does exist), can in fact be proved, contrary to what Frege claims. It seems unsatisfactory to say: I have tried repeatedly, but in vain, to reduce, say, clause (5) to any of the other clauses. Hence, clause (5) is likely to be independent of the other four (cf. GGA II, 172).[21]

Frege must have felt uneasy about the way he presents his "independence problem", as is evident from a footnote at the very end of his formal account (cf. GGA II, 243). There he suggests that his earlier claim that the mutual independence of the clauses of definition Ψ is unprovable ought not to be construed in an absolute sense. He doubts, however, that at the stage he has reached in § 175 it should be possible to give examples of classes of Relations to which all clauses of definition Ψ except one apply, without presupposing geometry, the rational and irrational numbers, or even empirical facts.

Frege proves a number of theorems concerning the notion of a positival class and then proceeds to define the notion of least upper

[21] On Frege's "independence problem" see S. A. Adeleke, M. Dummett, and P. Neumann, 'On a Question of Frege's about Right-Ordered Groups', in Dummett, *Frege and Other Philosophers,* Oxford University Press, Oxford 1991, 53–64.

bound of a class of relations in a positival class (he says: d is an S-limit of Q). The Relation d is a least upper bound of a class Q in a positival class S if and only if the following conditions are satisfied (GGA II, § 193, Definition *AA*): (1) S is a positival class; (2) d belongs to S, (3) every Relation in S which is smaller than d belongs to Q; (4) every Relation in S which is greater than d is greater than at least one Relation in S which does not belong to Q.

The requirements for defining the notion of a positive class are now met. A class S must have the following properties to be a positive class (GGA II, § 197, Definition *AB*): (1) S must be a positival class; (2) for every Relation in S there must be a smaller Relation in S; (3) if there is a Relation in S which is such that in S every smaller Relation belongs to a class Q, while there is a Relation in S which is not a member of Q, then Q must have a least upper bound in S.

Frege's next objective is to prove the Archimedian Law: For any two Relations in a positive class there is a multiple of one which is not smaller than the other (Theorem 635). (The proof is carried out in GGA II, §§ 199–214). In what follows, Frege turns to the task of proving the commutative law. He first proves it in a positive class (Theorem 674) and then for the entire quantitative domain of a positive class (Theorem 689) (cf. GGA II, §§ 215–243). In the concluding § 245, Frege describes in a few sentences what he thinks has to be done next. First, he plans to prove the existence of a positive class, as indicated in § 164, and then he will define the real numbers as ratios of quantities of a domain belonging to a positive class. He adds that this will enable him to prove that the real numbers themselves belong as quantities to the domain of a positive class. With these remarks, Frege's formal account breaks off, overshadowed by Russell's paradox.[22]

In their paper 'On a Question of Frege's about Right-ordered Groups' (*op. cit.*, 64), S. A. Adeleke, M. Dummett, and P. M. Neumann have observed that Frege "treated the applications of the real numbers as far more decisive for the way they should be defined than they are in other theories of the foundations of analysis. Mathematically, his construction of the real numbers, uncompleted because of the disaster

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[22] For more information about Frege's theory of real numbers and, in particular, for possible ways of completing that theory see F. von Kutschera, 'Frege's Begründung der Analysis', *Archiv für mathematische Logik und Grundlagenforschung* 9 (1966), 102–111; P. Simons, 'Frege's Theory of Real Numbers', *History and Philosophy of Logic* 8 (1987), 25–44; M. Dummett, *Frege: Philosophy of Mathematics, op. cit.*, chapter 22.

wrought by Russell's contradiction, was a pioneering investigation of groups with orderings. [...] It is an unjustice that, in the literature on group theory, Frege is left unmentioned and denied credit for his discoveries."

Besides Frege's attempt to demonstrate that arithmetic (number theory and analysis) is a branch of logic, the debate with his antagonist Hilbert on the axiomatic method has gained a fair amount of attention in the history and philosophy of mathematics. The key issues of this controversy are the methodological status of the axioms and definitions of a mathematical theory, the (alleged) necessity to distinguish sharply between these two types of sentence, the consistency and independence of an axiom system, and the problem of how the primitive terms of a mathematical theory are to be given a meaning. Some interpreters have maintained that Frege failed to appreciate the innovative character of the axiomatic method, while others have claimed that in his correspondence with Hilbert and his series of papers entitled 'Über die Grundlagen der Geometrie' (1903 and 1906) he demonstrated a subtle understanding of some important features of this method. The debate between Frege and Hilbert is only one of several issues controversially discussed by Frege scholars.

Geometry

As promised at the outset, I now turn to geometry. I begin by listing several theses which Frege propounded in his writings and philosophical correspondence between 1873 and about 1914; the theses partly overlap. Let me add that the following list does not aim at completeness. As will become obvious, the most important source for assessing Frege's philosophy of geometry is *Grundlagen.* In this book, his observations about geometry are almost exclusively motivated by the desire to contrast it with arithmetic. This may partly explain why he does not go to great lengths to explain the nature of geometrical knowledge *per se.*

(1) The whole of geometry rests ultimately on axioms which derive their validity from the nature of our intuitive faculty (KS, 1).

(2) There is a remarkable difference between geometry and arithmetic in the way in which their fundamental principles are grounded (KS, 50).

(3) In geometry general sentences are derived from intuition (GLA, § 13).

(4) Geometrical truths govern the domain of the spatially intuitable (GLA, § 14; cf. KS, 104; WB, 164).

(5) Euclidean space is the only space of whose structure we have any intuition. In non-Euclidean geometry we completely abandon the base of intuition (GLA, § 14).

(6) Everything geometrical must be originally intuitable (GLA, § 64).

(7) The axioms and theorems of Euclidean geometry are synthetic a priori (GLA, §§ 14, 89; cf. WB, 163).

(8) We cannot know whether space appears the same to one man as to another. Yet there is something objective in it all the same; everyone recognizes the same geometrical axioms, and must do so if he is to find his way about the world. What is objective in space is what is subject to laws, what can be conceived, judged, expressed in words. What is purely intuitable is not communicable (GLA, § 26).

(9) Our knowledge of the axioms of geometry flows from a source very different from the logical source, a source which might be called spatial intuition (WB, 63, cf. 70; KS, 262).

(10) The sense of the geometrical terms "straight line", "parallel" and "intersect" is inseparably connected with Euclid's parallels axiom (NS, 266).

(11) No man can serve two masters. Whoever acknowledges Euclidean geometry to be true must reject non-Euclidean geometry as false, and whoever acknowledges non-Euclidean geometry to be true must reject Euclidean geometry (NS, 183 f.).

Near the end of his life, Frege abandoned the idea of logicism, having convinced himself of its irremediable failure. He then turned, albeit in a rather fragmentary fashion, to a geometrical foundation of arithmetic, thus giving up another conviction he had defended from the beginning of his career, namely that the principles of arithmetic and geometry are to be justified in fundamentally different ways. Apart from (2), he did not expressly relinquish any of the remaining theses listed above. In particular, it is plausible to assume that Frege always held that geometrical truths are synthetic a priori and that our knowledge of them is based upon spatial intuition.

Frege opens his doctoral thesis *Über eine geometrische Darstellung der imaginären Gebilde der Ebene* (1873) by propounding thesis (1). This raises the question as to the sense we may attach to imaginary forms, since we attribute to them properties which clash with our intuitions. To make this plain, Frege appeals to points at infinity which likewise are non-intuitable. He not only seeks to treat these "improper elements" in the same way as the proper ones, (i.e., to calculate with them in the same way), but he also wishes to make them amenable to intuition, to have them before his eyes. For points at infinity in the plane this is easily achieved by projecting the plane on a sphere from a point of the sphere which is neither the nearest nor the furthest.

Reading through Frege's dissertation, it springs to mind that by the act of making visible ("sichtbar machen") or of illustrating ("veranschaulichen") improper elements he understands a geometrical construction with a pair of compasses and ruler.[23] I venture to surmise that the word "intuitive faculty", as it is used in thesis (1), is meant to refer to a faculty of visualizing geometrical configurations in a way which is essentially the same for all or most human beings. The particular intuitions which we have of geometrical figures or our making these figures visible are to be construed as realizations of this faculty. If this is correct, one might suggest that for the early Frege our intuitive ability as regards geometry consists in the ability to visualize with closed eyes, as it were, spatial configurations (call this faculty visual imagination or imaginative representation) as well as to carry out visualizations according to the rules of geometrical constructions. Undoubtedly, thesis (1) has a Kantian ring to it. It would be unjustified, however, to infer from this that in his doctoral dissertation Frege had adopted the point of view of transcendental idealism.

At the outset of his post-doctoral dissertation *Rechnungsmethoden, die sich auf eine Erweiterung des Größenbegriffs gründen* (1874), Frege aims at illustrating what I referred to as thesis (2) by investigating the notion of quantity. This notion is said to have gradually freed itself from intuition and made itself independent. Its range of application is indeed so comprehensive that Frege is certainly right in denying that it derives from intuition. "The elements of all geometrical constructions are intuitions, and geometry refers to intuition as the source of its axioms" (KS, 50).

In his first dissertation, Frege mentions these elements by appeal to the foundations of analytic geometry. "The equation of a straight line is derived with the aid of sentences about the similarity of triangles and about the angles formed by parallel lines. From the same sentences we can infer Pythagoras's theorem which in turn gives us the expression for the distance between two points. These are the elements from which

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[23] Frege uses "sichtbar machen" and "veranschaulichen" in the same sense; cf. KS, 26, 31, 37. For lack of a better word, I have rendered "veranschaulichen" as "illustrate".

all geometrical constructions are composed" (KS, 2). If we disregard the real numbers, it is only the similarity of triangles and the parallel lines that remain as purely geometrical elements. The existence of similar triangles can be inferred from Euclid's parallels postulate, and vice versa. Yet Frege holds that everything geometrical must originally be given in intuition. In § 64 of *Grundlagen,* he disavows that anyone has an intuition of the direction of a straight line, but claims that we do have an intuition of parallel straight lines. This claim may be doubted, on the grounds that to obtain the concept of parallelism concerning straight lines some mental activity connected with intuition is required, as is the case when the concept of direction is to be obtained. Especially when it comes to Euclid's parallels postulate, the question arises as to whether our spatial intuition is exact enough to yield it.

So much to Frege's early views on geometry. How about his early views on the foundations of arithmetic? His answer in *Rechnungsmethoden* is this. Since we have no intuition of the object of arithmetic, its principles cannot rest on intuition either. We are, however, not told from which source of knowledge they are supposed to originate. Although logic is not even mentioned in this work, stressing the comprehensive range of application of the concept of quantity, as Frege does, seems to foreshadow his later argument from the universal applicability of arithmetic to its purely logical nature. In part III of *Begriffsschrift,* entitled "Einiges aus der allgemeinen Reihenlehre", Frege derives a number of sentences about sequences to provide a general idea of how to handle his Begriffsschrift and underscores the extensive applicability of the sentences obtained. He makes it clear that the range of validity or application of truths is as wide as the scope of the source of knowledge from which they derive. Finally, in *Grundlagen* and 'Über formale Theorien der Arithmetik' (1885) the truths of arithmetic are said to govern the domain of what is countable. According to Frege, this is the widest domain of all; in fact, it is all-embracing, because everything thinkable belongs to it. A source of knowledge more restricted in scope, like sense perception or spatial intuition, would not suffice to guarantee the universal applicability of arithmetical truths.

I shall now focus on the theses on geometry which Frege puts forward in *Grundlagen.* Let me begin with thesis (5). Does Frege's claim that in non-Euclidean geometry we leave the base of intuition entirely behind carry as much conviction as he wishes to make us believe? One might question his view by drawing attention to an argument of his contemporary Hermann von Helmholtz, for example. By way of describing several non-Euclidean situations, Helmholtz seeks to demonstrate that the objects in a space of negative curvature (he calls it "pseudospherical space") can well be intuited, visualized or represented by the mind's eye, or more specifically: that they satisfy his definition of what it means to visualize or imagine an object that we have never encountered in our visual experience.[24] Thus he assumes, for instance, that a convex mirror maps an open region of ordinary space into an imaginary space. The mapping is injective, and every straight line of the outer world is represented by a straight line in the image, and likewise every plane by a plane.[25] A similar example is Eugenio Beltrami's representation of pseudospherical (hyperbolic) space in a sphere of Euclidean space to which Helmholtz appeals repeatedly and with predilection.[26] Beltrami's model enables us indeed to describe in fairly precise terms how the objects of a pseudospherical world would appear to an observer who could enter in it, assuming that he has gained both his sense of proportion and his visual experiences in Euclidean space. It enables us to do this, because the metric in the centre of the "Beltrami-sphere" is approximatively Euclidean and straight lines actually appear as such. Helmholtz emphatically gainsays that we should be able to visualize a four-dimensional space, however.[27] He points out, moreover, that if the thesis that the Euclidean axioms provide the only proper foundation of geometry is to be sustained, our inner intuition of the straightness of the lines, of the equality of distances or of angles ought to be absolutely exact.[28] It is undeniable, however, that our visualiz-

[24] Cf. H. von Helmholtz, 'Über den Ursprung und die Bedeutung der geometrischen Axiome', 'Über die Tatsachen, die der Geometrie zum Grunde liegen' and 'Über den Ursprung und den Sinn der geometrischen Axiome: Antwort gegen Professor Land', all reprinted in von Helmholtz, *Über Geometrie*, Wissenschaftliche Buchgesellschaft, Darmstadt 1968; see especially 25 f., 28, 64, 73.

[25] Cf. von Helmholtz, 'Über den Ursprung und die Bedeutung der geometrischen Axiome', 28.

[26] Cf. E. Beltrami, 'Saggio di interpretazione della geometria non-euclidea', *Giornale di matematiche* **6** (1868), 284–312; reprinted in Beltrami, *Opere matematiche*, Ulrico Hoepli, Vol. I, Milan 1902, 374–405.

[27] See in this connection Hans Reichenbach's arguments regarding the visualization of non-Euclidean geometries in his book *Philosophie der Raum-Zeit-Lehre*, Berlin, Leizpig 1928, § 11. He also deals with the question as to whether we could, in principle, visualize a space of say, four dimensions (cf. 329).

[28] In my view, there are good reasons for assuming that Kant, if he had been confronted with non-Euclidean geometry, would have rejected it; see my 'Kants Theorie der geometrischen Erkenntnis und die nichteuklidische Geometrie', *Kant-Studien* **82** (1991), 1–28.

ation of geometrical objects lacks the absolute precision required by Helmholtz, especially concerning their metrical properties.[29]

It is time to say a little about Frege's relation to Kant as far as geometrical knowledge is concerned. In § 13 of *Grundlagen,* Frege maintains that it is only when several points, lines or planes are simultaneously grasped in a single intuition that we distinguish them. "When in geometry general sentences are derived from intuition, it is evident from this that the points, lines, or planes that are intuited are not really particular ones and hence can serve as representatives for the whole of their kind." At first glance, this may be reminiscent of Kant's dictum that mathematical knowledge, construed as knowledge gained by reason from the construction of concepts, considers the universal in the particular. One might thus be tempted to establish a parallel between the view Frege expresses in the passage quoted above and Kant's construction of geometrical concepts in spatial intuition, conceived of as an exhibition a priori of the intuition (or object) which corresponds to the concept. Let us see whether Frege follows indeed in Kant's footsteps.

To attain synthetic a priori knowledge it is not mandatory, according to Kant, that the construction qua exhibition a priori be carried out in pure intuition; under certain conditions, empirical intuition may serve the purpose as well. The particular geometrical figure, say an obtuse-angled triangle, which we draw is empirical; nonetheless, it expresses "universal validity" for all possible intuitions which fall under the concept *triangle* (cf. *Kritik der reinen Vernunft,* B 741 f.), or in Frege's wording: it serves as a representative for the whole of its kind. Kant tells us that this is possible, because the geometer abstracts from the accidental properties of the particular triangle (magnitudes of the sides and of the angles) and focuses entirely on his act of construction as determined by certain general conditions. In this way, he is supposed to arrive at *general* synthetic sentences. Now, despite first appearances it seems rather unlikely that Frege tacitly adopted Kant's method to form his

[29] Felix Klein regards spatial intuition as something that is essentially imprecise. By a geometrical axiom he understands the demand by virtue of which he makes exact statements out of inexact intuition; cf. Klein, *Gesammelte mathematische Abhandlungen,* Vol. I, eds. R. Fricke and A. Ostrowski, Berlin 1921, 381 f. Klein jettisons here Moritz Pasch's idea developed in *Vorlesungen über neuere Geometrie* (Leipzig 1882) that the geometrical axioms express the "facts" of spatial intuition in a way so complete that in our geometrical considerations we need not rely on intuition. Klein, for his part, considers geometrical considerations to be impossible unless he has constantly before his eyes the figure in question. However, his view about the relation between geometrical axioms and (inexact) spatial intuition lacks, I think, persuasive power.

own conception of how we attain geometrical knowledge. First, neither in *Grundlagen* nor in any other of Frege's writings is the construction of geometrical concepts à la Kant at issue. Second, and more important, Kant's method rests crucially on the results achieved in his Transcendental Aesthetics. Yet Frege's remarks on space and spatial intuition in § 26 of *Grundlagen* are clearly at variance with Kant's metaphysical and transcendental exposition of the "concept" of space.

Although in *Grundlagen* Frege expressly endorses Kant's view that the truths of Euclidean geometry are synthetic a priori and our knowledge of its axioms rests on pure spatial intuition, he in no way subscribes to Kant's transcendental idealism.[30] According to Kant, space is a pure intuition; it is facts of the world of appearances that make geometrical sentences true, facts that do not exist independently of human beings. For Frege, by contrast, space is objective in so far as it is independent of our sensation, intuition, and imagination; it is objective, because we can express its properties in words possessing a meaning which is the same for all who are able to grasp it.[31] Frege maintains that the axioms of Euclidean geometry do not state facts about our intuition, but express states of affairs about space obtaining quite independently of our spatial intuition. Otherwise, it could well happen that one and the same geometrical axiom is acknowledged to be true by one person and rejected to be false by another.

The question forcing itself upon us is whether Frege regarded intuition as justifying geometrical knowledge. As a matter of fact, he makes some remarks that seem to suggest a positive answer (cf. WB, 63, 70 and theses (1) and (3)). Moreover, in two of his late fragments Frege understands by (spatial) intuition the *geometrical source of knowledge*, that is, the source from which the axioms of geometry flow (cf. NS, 286, 292 ff., 297 ff.). Yet a source of knowledge is explained as justifying the recognition of truth, the judgment (cf. also WB, 63, 163 f.).

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[30] In Frege's view, the fact that we can always consistently deny the axioms of Euclidean geometry suggests that they are independent of one another and the primitive laws of logic, and are therefore synthetic. He also regards them as a priori, and he probably does so on the grounds that they rest on pure intuition.

[31] Note in this connection that in *Grundlagen* and subsequent writings Frege by no means uses the word "objective" in the same sense as Kant. For Kant, space is a subjective condition of our outer intuition; it is ideal as regards objects when they are considered in themselves through reason, but at the same time it is objective, i.e., empirically real with respect to outer appearances. If that were not so, Kant would be at a loss to explain the universal and necessary validity of the truths of geometry. Here I naturally cannot analyze further his notion of objectivity.

In *Grundlagen* Frege underlines the subjectivity, privacy and incommunicability of our spatial intuitions, which they share with our ideas, sensations and imagination. We cannot, in order to compare them, lay one man's intuition of space beside another's. To regard spatial intuition thus characterized as justifying our geometrical knowledge appears to be at odds or at least in tension with Frege's firm belief that the axioms of geometry are objective. If one man did not intuit or visualize spatial configurations in essentially the same way as another, then it would be hard to understand how the axioms of three-dimensional Euclidean geometry could derive their validity from our intuitive ability. To be sure, in projective geometry, where the principle of duality holds, it is perfectly intelligible to suppose that two rational beings, who connected different intuitions with the word "plane", for instance, would nevertheless be in complete agreement over all geometrical theorems (cf. GLA, § 26). In Euclidean geometry, however, Frege wishes to rule out this possibility.

Let me add one more brushstroke to the picture we have gained so far of Frege's views of geometry. Unfortunately, both his writings and his philosophical correspondence provide scarcely a clue about to what extent he kept abreast with developments in geometry in the second half of the nineteenth century. In all likelihood, he was familiar with Georg von Staudt's investigations on projective geometry[32] as well as with Bernhard Riemann's famous essay (*Habilitationsvortrag*) 'Über die Hypothesen, welche der Geometrie zugrunde liegen'[33]. We may further quite safely assume that Frege knew some of the work on geometry by Felix Klein. To my mind, it is astonishing that the influential work on the foundations of geometry by Riemann and Helmholtz, in particular their arguments against the a priori character of geometry, are completely passed over in silence in Frege's writings.[34] This might be due partly to the disparaging attitude, if not disdain, that Frege appears

[32] G. von Staudt, *Die Geometrie der Lage*, F. Korn, Nürnberg 1847 and *Beiträge zur Geometrie der Lage*, Bauer and Raspe, Nürnberg 1856–1860 (3 fascicles).

[33] *Göttinger Abhandlungen* 13 (1854), 133–152.

[34] As far as I know, Frege refers only in one place to a work by Helmholtz. In the second volume of *Grundgesetze* (139 f., footnote 2), he finds fault with Helmholtz's intention to provide an empirical foundation of arithmetic in the essay 'Zählen und Messen erkenntnistheoretisch betrachtet'. Frege regards Helmholtz's approach as confused and concludes: "I have hardly ever seen anything less philosophical than this philosophical essay, and hardly ever has the sense of the epistemological problem been more misunderstood than here." So it seems that Frege had a low opinion of Helmholtz's philosophical talents.

to have entertained towards non-Euclidean geometry. Speculations aside, it is true that he fails to work out any solid argument for the claim that geometrical truths are known a priori. By contrast, Riemann and Helmholtz in effect adduce powerful arguments in favour of the empirical nature of geometry. The idea that the existence of consistent theories of non-Euclidean geometry had an impact on their view seems to make sense, and I do not think that it can be dismissed.

In his aforementioned essay, Riemann holds that an n-fold extended quantity admits different metric relations, so that space constitutes only a special case of a threefold extended quantity. He concludes from this that the sentences of geometry cannot be derived from general quantitative concepts. Instead, there is supposed to be ample evidence that those properties by virtue of which space differs from other threefold extended quantities can only be ascertained by experience. Riemann tries to make it plain that the simplest facts which serve to determine the metric relations of space are not necessary (or a priori), but possess only empirical certainty. This is why he calls them *hypotheses*. According to Riemann, all we can say about space without invoking experience is that it is one among many possible kinds of manifolds.

I have already mentioned two objections which Helmholtz directs against Kant's thesis that the geometrical axioms originate from an a priori source of knowledge. In addition, Helmholtz argues that the geometrical principles belong not only to the pure theory of space, but deal also with quantities. Yet the introduction of quantities is said to make sense only if we provide suitable procedures of measurement for them. Every measurement of space, and therefore every quantitative concept applied to space, presupposes the possibility of spatial figures moving without change of form or size. By way of adjoining to the geometrical axioms sentences relating to the mechanical properties of natural bodies we arrive, Helmholtz claims, at a set of sentences which can be confirmed or refuted by experience, but just for the same reason can also be gained by experience.

I leave it to the reader to judge whether Frege's characterization of our knowledge of Euclidean geometry as synthetic a priori must be regarded as a retrograde step, especially in the light of the work of Riemann, Helmholtz and other contemporaries. To repeat, unlike Kant, Frege spares himself the trouble of buttressing such characterization by detailed argument; instead, he seems to take it more or less for granted. Furthermore, to stigmatize non-Euclidean geometry as a pseudoscience, as Frege tends to do in the fragment 'Über Euklidische Geometrie' (cf. NS, 182–184), appears to have been an aberration on his part. It seems that Frege was unwilling to realize that the existence of consistent theories of non-Euclidean geometry by no means compels us to acknowledge one geometry – Euclidean or non-Euclidean geometry – as the only true one. His belief that one cannot consistently recognize both Euclidean and non-Euclidean geometry as true presumably derives from his assumption that the primitive geometrical terms allow only for one interpretation, namely the Euclidean one. Yet even if Frege had accepted the legitimacy of endowing geometrical terms with a non-Euclidean interpretation, he would probably have been inclined to assign a definite priority to Euclidean over non-Euclidean geometry. For it seems that he never abandoned his conviction that everything geometrical must be originally intuitable and that non-Euclidean geometry leaves the basis of intuition entirely behind.[35]

Frege research

Of course, I cannot claim complete knowledge of the developments in Frege research over, say, the last fifteen years. It seems clear, also from what I have said before, that the discussion of Frege's work during that period has various motives. I should like to mention the following six: (i) to locate his work more accurately in the history of logic, mathematics and philosophy; (ii) to bring into sharp focus and reassess both his logicism and his arithmetical platonism, also in the light of most recent work in the philosophy of mathematics; (iii) to examine thoroughly particular aspects of his logical theory, such as his so-called permutation argument, his attempted proof of referentiality for the formal language of *Grundgesetze* or the question as to what really caused the inconsistency of his system; (iv) to analyze his mathematical work in *Grundgesetze;* (v) to investigate the various facets of his epistemology; (vi) to provide a systematic account of his semantics and to develop further certain central ideas of it.

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[35] On Frege's reflections on geometry see M. Dummett, 'Frege and Kant on Geometry', *Inquiry* 25 (1982), 233–254; M. Wilson, 'Frege: The Royal Road from Geometry', *Noûs* 26 (1992), 149–180; J. Tappenden, 'Geometry and Generality in Frege's Philosophy of Arithmetic', *Synthese* 102 (1995), 319–361.

To my mind, the most important work on Frege in recent years has been on his philosophy of mathematics.[36] Despite his pioneering work in the field of semantics, which still has a considerable bearing upon present discussions, Frege was, in his own express opinion, a logician and philosopher of mathematics, albeit a logician with a profound interest in dealing with topics we nowadays assign to the philosophy of language. Compared with the elaboration of the logicist programme, to which Frege devoted more than twenty years of his academic career, his work on the philosophy of language in a narrow sense plays a rather subordinate role. It would be short-sighted, however, to regard his theory of sense and reference merely as an appendix to his philosophy of mathematics. Undoubtedly, this theory plays a crucial role in the construction of the logical system of *Grundgesetze*.

Needless to say, to stress the importance of Frege's philosophy of mathematics is by no means to disparage the achievements brought about in other areas of Frege research. And to be sure, especially questions concerning Frege's epistemology have quite naturally been involved in one or the other investigation of his philosophy of mathematics. In any case, I do think that the distribution of the papers in the present volume can be seen as reflecting altogether the prevailing tendency in current Fregean scholarship. It is mainly for this reason that in my preceding account I have tried to throw some light on what I take to be interesting topics in Frege's philosophy of mathematics.

The essays in this collection

Michael Resnik's essay 'On Positing Mathematical Objects' opens the section on logic and philosophy of mathematics. It connects his own postulational version of structuralism (and the modal approaches of Hartry Field and Geoffrey Hellman) with the views of Dedekind, Cantor and Hilbert which Frege criticized with great effect. Resnik starts with a review of the criticisms levelled by Frege against the

[36] I should like to draw attention to William Demopoulos's very useful collection *Frege's Philosophy of Mathematics*, Harvard University Press, Cambridge, MA 1995. Several of the aforementioned essays on Frege are reprinted in this volume. See also the section 'Frege and the Foundations of Arithmetic' in my forthcoming collection of articles by different hands *Philosophy of Mathematics Today* (*op. cit.*).

method of creating or postulating mathematical objects by means of definition or abstraction. It is argued that despite the force of some of Frege's objections he apparently failed to distinguish clearly between two views of introducing mathematical objects. The first endorses the erroneous device of creative definitions, while the second subscribes to the defensible method of mental constructions or acts of postulation. Resnik believes that Frege's inclination to lump together these views kept him from appreciating fully the mathematical achievements of Cantor, Dedekind, and Hilbert. As to Dedekind, it is held that he suggested creating number systems by creating new structures in which the numbers in question are nothing but positions in those structures. In Resnik's interpretation, Dedekind thought that the creation of such structures needs to be justified by showing that they can be abstracted from a suitable system of "thought objects". The serious drawback of this position, we are told, is that it merely transfers the problem of mathematical existence to that of sets. As regards Hilbert's axiomatic method, Resnik maintains that although it appears to be akin to Dedekind's structuralism, a more thorough investigation would reveal significant differences between their views. It is further suggested that (the early) Hilbert could be seen as attempting to replace Dedekind's ontological thesis that we create mathematical structures by means of abstraction with the epistemological thesis that we can recognize new structures by postulating them through axioms.

As far as Resnik's own position is concerned, he weds a realist mathematical structuralism to a postulational epistemology. From Dedekind he takes structuralism which in his hands takes on a form that parallels Hilbert's view on mathematical existence and truth. Since Resnik's postulational epistemology commits him to acknowledge the independent existence of mathematical objects, it seems that postulationism is compatible with realism. In the remainder of his paper, he shows in what sense the two doctrines can actually be combined.

In the secondary literature, Frege's polemical remarks in *Grundlagen* on various doctrines of his predecessors and contemporaries such as Baumann, Cantor, Jevons, Locke, Leibniz, Mill, Schröder, Thomae, and others on the concept of number and on unity are often said to be devastating. W. W. Tait, in his essay, does not share this view. On the contrary, he holds that Frege's scrutiny of the work of his fellow mathematicians is often characterized by lack of charity, and, what is worse, marred by serious defects and misinterpretation. Tait has accordingly set himself the task of reassessing some of Frege's criticisms and to compare Frege's views on the concept of number with those of Cantor and Dedekind. In addition, Tait juxtaposes a number of critical observations on Dummett's views in *Frege: Philosophy of Mathematics.* The central objection against Dummett seems to be that in comparing Frege's and Dedekind's treatment of the foundations of arithmetic he has failed to do justice to Dedekind. In what follows, I shall confine myself to summarizing some of the arguments that Tait advances against Frege's critique of a prominent conception of number, a version of which was also defended by Cantor: it is the conception that identifies cardinal numbers with sets of pure or featureless units (cf. Frege, GLA, §§ 29–45; KS, 163–166; NS, 76–80).

The vulnerable spot of this conception is, according to Frege, that it fails to reconcile identity of units with distinguishability and, furthermore, that every attempt to resolve this difficulty is doomed to failure: "If we try to make the number originate from the combination of distinct objects, we obtain an agglomeration comprising the objects with just those properties which serve to distinguish them from one another; and that is not the number. If, on the other hand, we try to form the number by combining identicals, this constantly coalesces into one, and we never arrive at a plurality" (GLA, § 39). In this connection, Tait criticizes Frege for conflating two questions which ought to be distinguished clearly: (1) What are the things to which numbers apply? (2) What are numbers? Tait claims that the first horn of the dilemma as laid out by Frege concerns (1). He argues further that the things to be numbered are not agglomerations, but sets, which indeed originate from the combination of distinct objects. The fact that these sets were called *numbers* by some of Frege's contemporaries, he considers to be one source of Frege's (alleged) confusion. The second horn of the dilemma concerns the conception of numbers as sets of pure units. Tait finds it ill-conceived, but, unlike Frege, in no way incoherent. Frege discusses several suggestions which might lead out of the quandary, but rejects them all. One proposal is to invoke instead of spatial or temporal order a more generalized concept of series (cf. GLA, § 42). Tait accuses Frege of conflating here the notion of a series with that of a linearly ordered set $\langle A,>\rangle$ (where, for x and y in A, $x < y$ implies that x and y are distinct).

Cantor construed the cardinal number $\bar{M}$ of a given set M as a definite set, comprised of nothing but units, which exist in our mind as an intellectual copy or a projection of M. $\bar{M}$ is obtained by carrying out the process of abstraction from both the nature of the elements of M and their order.[37] Not surprisingly, Frege considered this view to be a thorn in the flesh and commented on it with sarcasm. Tait, for his part, regards Cantor's view as unattractive, but not as indefensible, provided that it is interpreted as follows: the abstraction concerns not the individuating properties of the elements relative to one another, as Frege assumes, but rather the individuating property of the set M itself. On this interpretation, the "cardinal set" (i.e., the cardinal qua set of pure units) C "corresponding to a set M is to be constituted of unique elements, specified in no way other than that they are elements of C and that C is equipollent to M. Thus, the cardinal sets are not sets of points in Euclidean space or of numbers or of sets, or of apples or etc." So much to Tait's vindication of Cantor.

In my own contribution, I examine Frege's conception of numbers as objects in *Grundlagen* and argue that it suffers from a number of defects. One objection is that he fails in his attempt to analyze what he calls "ascriptions of number" [*Zahlangaben*] such as "The number 9 belongs to the concept planet" in such a way that cardinal numbers emerge as self-subsistent objects. Another point made is that Frege could have acknowledged ascriptions of number as numerical statements in their own right instead of construing them as equations in which the number words flanking "=" function as singular terms. Harold Hodes has claimed that Frege's method of bestowing a definite reference upon a numerical singular term, by fixing the sense of all relevant sentences in which it occurs, fails to explain the "microstructure" of reference, for instance, to cardinal numbers. I try to show that this argument rests partially on an outright misinterpretation of Frege's context principle and its relation to his thesis that a thought is built up out of parts which correspond to the parts out of which the sentence expressing the thought is built up. I conclude the first half of my paper by claiming that unless someone has succeeded in refuting Paul Benacerraf's ontological argument against number-theoretic platonism, the conception of numbers as objects remains a dogma bequeathed to us by Frege. In the second half, I canvass Frege's three attempts to define number in *Grundlagen* and argue that he falls short of resolving the pervasive indeterminacy of reference affecting the cardinality operator.

Although Bob Hale's and Crispin Wright's essay is not directly concerned with Frege, it can be seen as standing in the tradition of his phil-

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[37] Cf. G. Cantor, *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, ed. E. Zermelo, Berlin 1932, 283, 387, 411 f.

osophy of arithmetic. It is probably correct to say that in their attempt to undermine the viability of Hartry Field's nominalism, a version of Fregean mathematical platonism which they accept figures in the background, as it were. Field's rather unorthodox defence of nominalism as a philosophy of mathematics accepts a platonist account of the truth-conditions of purely mathematical statements, that is, an account which discerns in them purported reference to or quantification over mathematical entities of various kinds. In the same breath, Field maintains that such statements are false – or at least never non-vacuously true – one the grounds that there are simply no such objects as numbers, sets and the like. The key idea in his advocacy of this thesis is that we can avoid wholesale rejection of standard mathematical theories by holding that such theories have a property falling well short of truth, but akin to consistency, namely *conservativeness.* According to Field, a mathematical theory T is conservative if, for any nominalistic assertion A and any body of such assertions N, A is not a consequence of N + T, unless A is a consequence of N alone. A seemingly serious difficulty for this position, pointed out by Hale and Wright in earlier writings, is that it appears to commit Field to maintaining that the falsehood of standard mathematical theories is at worst a contingent matter. This appears open to the objection, crudely stated, that Field should be able to explain, in a nominalistically acceptable way, why the putative contingency is resolved, as, in his opinion, it is, but can provide no suitable such explanation. Field has sought to fend off this line of objection by claiming that it rests on an equivocation over the notion of contingency. Hale and Wright, for their part, try to show that this response is ineffective, though they admit that the objection, as originally presented, is unsatisfactory. Their principal aim is to arrive at a reformulation of the objection which captures its core, while relying on principles governing the notion of contingency which should command general assent.

The mathematical argumentation in Frege's *Grundgesetze* has been largely ignored by his interpreters. Richard Heck is convinced that this lack of esteem or interest is unjust, and in his paper 'Definition by Induction in Frege's *Grundgesetze der Arithmetik*' he tells us why. In it, he discusses at length Frege's proof of Theorem 263, which amounts to a proof that all structures satisfying certain conditions are isomorphic. These conditions Heck takes to be Frege's own axioms for arithmetic. It is argued that Theorem 263 is one of the central results of *Grundgesetze* and that Frege's proof of it can be reconstructed in

Fregean Arithmetic (FA), with or without the use of the ordered pair axiom. As a matter of fact, Frege proves Theorem 263 in FA, augmented by the ordered pair axiom, although, in Heck's opinion, he knew that he could have carried out the proof without it. (Heck reconstructs Frege's proof of Theorem 263 using the definition of the 2-ancestral, rather than ordered pairs.) Heck speculates that Frege's reluctance to dispense with ordered pairs when he comes to prove Theorem 263 may have had two reasons. Firstly, although the use of ordered pairs was fairly common among mathematicians of Frege's day, no suitable definit of them was at hand when he set about writing the first volume of *Grundgesetze.* Providing such a definition is in the spirit of Frege's claim to be able to formalize, in his Begriffsschrift, classical mathematics in its entirety. Secondly, using ordered pairs in this context spares him the trouble of having to do two things: first, to set up a new definition of the ancestral and, second, to prove several theorems which are analogues of ones he had already proven. The result that all "simple" and "endless" sequences, which are models of Frege's own axioms for arithmetic, are isomorphic is closely analogous to one of the theorems for which Dedekind's study *Was sind und was sollen die Zahlen*[38] is celebrated; it is Theorem 132 (§ 10) stating that all "simply infinite systems", that is, structures which satisfy the Dedekind-Peano axioms, are isomorphic. In the course of the proof of his Theorem 263, Frege proves a generalization of another of Dedekind's important results, namely the so-called recursion theorem for ω (referred to by Dedekind as Theorem 126; cf. § 9), justifying the definition, by induction, of a function defined on the natural numbers. Heck concludes that the fact that Frege proved such results may have a considerable impact on our understanding of his philosophy of mathematics.

The paper by George Boolos addresses the question as to what gave rise to the inconsistency of Frege's logical system. For many years, it has been taken more or less for granted, following Frege's own assessment, that it is Axiom V of *Grundgesetze* which is to be held responsible for the contradiction. As far as I know, this view was challenged for the first time by Christian Thiel in his largely neglected article 'Zur Inkonsistenz der Fregeschen Mengenlehre'[39]. In chapter 17 of his book *Frege: Philosophy of Mathematics,* Michael Dummett has argued that

[38] Vieweg, Braunschweig 1888.

[39] In C. Thiel (ed.), *Frege und die moderne Grundlagenforschung,* Anton Hain, Meisenheim am Glan 1975, 134–159.

the contradiction in Frege's system is primarily due to his careless treatment of the second-order quantifier in his attempted proof of referentiality for all well-formed names of his formal language (cf. GGA, § 31). Dummett regards this proof as an attempted consistency proof. In fact, Frege's first reaction to Russell's startling discovery suggests that he had more than an inkling of this interconnection (see WB, 213). At first glance, Dummett's diagnosis of what led to the inconsistency appears to be buttressed by the result that the first-order fragment of the system of *Grundgesetze* is consistent, as first established by Terence Parsons in his article 'On the Consistency of the First-Order Portion of Frege's Logical System' (*op. cit.*). Without second-order quantification, Frege's system would be "paralyzed", however, because membership would be indefinable for him. Boolos maintains that the greater the paralysis, the less plausible Dummett's view about the primary source of the inconsistency of Frege's system appears. Only if the first-order fragment had been strong enough to yield arithmetic or an interesting portion of it would it be tempting, Boolos thinks, to trace the inconsistency back to the presence of the second-order quantifier. Let me add that in Dummett's opinion Frege failed to carry out a valid consistency proof even for the first-order fragment of the system of *Grundgesetze.*

As I read him, Boolos is chiefly concerned to convince us that, contrary to what Dummett claims, everything stands as it was, though it must probably be seen in the light of a more profound and more subtle analysis: the culprit for the breakdown of the system of *Grundgesetze* is what Frege took it to be, namely Axiom V. Boolos argues, in particular, that we should not put the blame for Frege's error on the stipulations he made regarding the truth-conditions of sentences beginning with second-order quantifiers, but rather on those concerning course-of-values equations. A number of doubts in connection with Dummett's account of what caused the fatal flaw in Frege's logicist project are expressed. One point made is that Dummett has taken a "background condition" to be the cause of the contradiction. Another proviso relates to his contention that Frege appears to favour a substitutional interpretation of the second-order quantifier rather than an objectual one. Boolos thinks that to see why T. Parsons's construction of a model for the first-order portion of Frege's system cannot be extended to the full system provides further evidence for the claim that it is not any deficiency in Frege's stipulations concerning the second-order quantifier that caused the inconsistency. It is further pointed out that it is due to the "Cantor-Russell *aporia*" that any attempt to construct a model for Axiom V within Frege's full second-order language is bound to fail. Boolos concludes that in the light of possible revisions of the system of *Grundgesetze* which yield a consistent version of it, allow the construction of arithmetic and prove to be less thoroughgoing than the idea of dispensing altogether with second-order quantification, it is not the use of the latter that is to be blamed for the inconsistency of Frege's formal theory.

In spite of the arguments advanced by Boolos, Dummett, in his reply, reiterates his former contention that second-order quantification was essential for the inconsistency of Frege's logical system. Dummett is willing to concede, though, that he should not have firmly ascribed to Frege a substitutional interpretation of quantification. Nonetheless, he does not follow Boolos in attributing to Frege an objectual interpretation, because he holds that there is nothing in the system of *Grundgesetze* that compels us to do this. To demonstrate how Frege's consistency proof founders in the presence of the second-order quantifier, Dummett recalls the strategy pursued in *Frege: Philosophy of Mathematics.* It was this: to show, first, by example, without invoking Axiom V, that Frege's inductive line of argument in § 31 of *Grundgesetze* is faulty; and to show, second, that Russell's paradox can be obtained by a "modest appeal" to Axiom V. Dummett finds himself in considerable disagreement with Boolos about domains of quantification in general. Boolos repudiates Dummett's conception of "indefinitely extensible concepts", claiming that Frege "did not have the glimmering of a suspicion of the existence" of such concepts. According to Dummett, Boolos takes this denial to follow from his rejection of the view that the objects over which the individual variables of a mathematical theory range form a collection. Dummett, for his part, maintains that it is precisely the indefinite extensibility of the concept of set or class which suggests taking the objects over which the individual variables of a theory range as forming a domain or totality. Dummett concludes by emphasizing that completely unrestricted quantification is not illegitimate; what, in his view, is illegitimate is a truth-conditional interpretation of sentences involving it.

Christian Thiel, in his essay, deals with some problems deriving from the logical system of *Grundgesetze.* He begins by considering three *desiderata* of current Frege research. The first concerns Peter Aczel's claim that it is Frege's horizontal function that is to be held responsible for the derivability of Russell's paradox in the system of

Grundgesetze. The second *desideratum* is that the so-called permutation argument in § 10 of *Grundgesetze* ought to be reconsidered, even though several analyses of that argument have recently been put forward. The third *desideratum* is a more thorough examination of Frege's attempted proof in § 31 of *Grundgesetze* that every well-formed term of his formal language has a reference. Contrary to what Resnik claims in his article 'Frege's proof of referentiality'[40], Thiel maintains that a proof of referentiality, if successful, would imply the consistency of the formal theory of *Grundgesetze.* Thiel discusses the failure of Frege's attempted proof of referentiality by appeal to his essay 'Zur Inkonsistenz der Fregeschen Mengenlehre' (*op. cit.*). The pivotal point of his assessment is that the proof miscarries due to the "impredicative" nature of the rules governing the correct formation of function-names which Frege states in § 26 of *Grundgesetze,* the so-called "gap-formation principles". The latter are said to block the inductive process of transferring a reference to certain newly formed names. Thiel further analyzes, from the point of view of the inconsistency of the system of *Grundgesetze,* Frege's derivation of theorem (χ) of *Grundgesetze* which appears in the Appendix to the second volume.

One of the significant changes that Frege's Begriffsschrift had undergone between 1879 and 1893 concerned the interpretation of "–". In *Grundgesetze,* Frege introduces $-\xi$ as a primitive function (concept) mapping the True as argument on the True and every argument of type 1 (i.e., every object) distinct from the True on the False. Peter Simons, in his essay, examines both the nature and the role of "–" in *Begriffsschrift* (where "–" is called the content stroke) and especially in *Grundgesetze* (where "–" is referred to as the horizontal). In addition, he sheds light on a number of special features of the logical system of *Grundgesetze.* Simons argues that despite its obscure theoretical status in *Begriffsschrift,* "–" played an important heuristic role in Frege's logic of 1879. Matters are said to stand differently in *Grundgesetze.* Due to the thoroughgoing reinterpretation that Frege imposed in that work on the notation stemming from his *Begriffsschrift,* "–" is now given substantial work to do in the logical calculus. Simons points out that Frege was quite aware that he could have eliminated the horizontal function (and also negation). In his doctoral dissertation *On the Primitive Term*

[40] In L. Haaparanta and J. Hintikka (eds.), *Frege Synthesized. Essays on the Philosophical and Foundational Work of Gottlob Frege,* D. Reidel, Dordrecht, Boston 1986.

of Logistic of 1923[41], Tarski showed how to define conjunction in Lesniewski's protothetic in terms of material equivalence and universal quantification. Simons uses an analogue of Tarski's simplest definition and shows that Frege in effect could have reduced the number of his primitives to four, dispensing entirely with the horizontal, negation and the conditional. It is claimed that nothing illustrates more conspicuously how much Frege's logic had lost the hierarchical structure of *Begriffsschrift* and had become much more of a piece in *Grundgesetze.* In the remainder of his paper, Simons presents several further results about Frege's logic. One is that Frege's treatment of the two truth-values as objects imparts features to his formal system which brings it into the vicinity of many-valued propositional logics.

Franz von Kutschera's contribution is an historical footnote on the development of systems of natural deduction. He shows that in the first volume of *Grundgesetze* Frege formulates a calculus which, in a sense, is intimately related to Gentzen's classical calculus of sequents in his 'Untersuchungen über das logische Schließen'[42]. The crucial difference between the two approaches resides in the fact that while Frege states elimination rules for the succedent, Gentzen employs introduction rules for the antecedent. Clearly, Frege and Gentzen pursued different goals. Frege only aimed at establishing simple inference rules for the manipulation of antecedents and the succedent in implicational formulae. Gentzen, by contrast, intended to do justice to the "real" deductive practice in mathematical proofs. Von Kutschera emphasizes that the two methods rest, after all, on the same fundamental idea, namely of stating sufficient and necessary conditions for the introduction or elimination of logical operators. By confining himself to introduction rules, Gentzen paved the way for arriving at his *Hauptsatz,* which played an important role in proof theory.

Eva Picardi's essay 'Frege's Anti-Psychologism' is one of three in this volume dealing with issues belonging directly or indirectly to epistemology. Picardi explores what she believes Frege considered to be the main defect of psychologism: to rely on a picture of language which turns both the objectivity of sense and the communication of thoughts into a mystery. Her central thesis I take to be twofold. Firstly, there is a close link between the attack Frege mounts on psychologistic concep-

[41] In A. Tarski, *Logic, Semantics, Metamathematics,* second edition (ed. J. Corcoran), Hackett, Indianapolis 1983, 1–23.

[42] *Mathematische Zeitschrift* **39** (1934), 176–210, 405–431.

tions of logic on the one hand and the sharp criticisms he levels against psychologistic accounts of meaning on the other. Secondly, this link is to be found in Frege's realistic conception of truth. Picardi claims that due to the fact that Frege's anti-psychologism is essentially semantic in nature, little is to be gained by discussing it in the context of his conception of epistemology. In particular, she argues that a certain thesis propounded by Philip Kitcher and Hans Sluga should be rejected. The thesis is that Frege tacitly adopted a form of Kantian transcendentalism as a safeguard against psychologism.

In his paper 'Frege's 'Epistemology in Disguise'', Gottfried Gabriel attempts to determine the role that epistemology plays in Frege's philosophy. Gabriel holds that Frege had a direct interest both in logic and epistemology, but only an indirect interest in the philosophy of language. The main focus of the paper is the relationship between logic and epistemology. Gabriel pays close attention to the notion of apodictic statement in *Begriffsschrift* and the way it is related to the notion of a priori truth as defined in *Grundlagen.* It is argued that the former notion involves only necessity relative to general premises whose status may range from logically true to a posteriori true. Frege applies the latter notion or that of a priori knowledge, so we are told, only when the provability of the premises can be traced back to truths which neither need nor admit of proof. Gabriel puts it succinctly: the "proof-theoretical" difference between the metapredicates "apodictic" and "a priori" resides in the fact that the first embodies only relative provability, while the second expresses absolute provability. He points out that the difference between logic and epistemology comes out clearly in Frege's distinction between "reasons for something's being true" and "reasons for our taking something to be true".

Tyler Burge begins his essay by stating what he considers to be a puzzle: On the one hand, the principal aim of Frege's project is to explain the foundations of arithmetic in such a way as to enable us to understand the nature of our knowledge of arithmetic. On the other hand, Frege says strikingly little about our knowledge of the foundations of arithmetic. For Burge, the short solution of the puzzle, though leaving out a great deal, is that Frege thought he had little to add to the traditional view according to which the primitive truths of geometry and logic are taken to be self-evident. Burge argues in considerable detail for the claim that Frege was a platonist as regards abstract (i.e., non-spatial, atemporal, causally inert) entities such as logical objects, functions and thought contents. Frege's platonism is said to show itself in two ways: First, unlike an idealist, he takes the objectivity of abstract entities to be fundamental. Second, Frege believes that the assumption of the relevant abstract entities serves to explain both the objectivity and the success of science and communication. Burge stresses that there is nothing in Frege's work which might remotely indicate that he regarded either the physical world or the realm of abstract entities as dependent upon any activities of judgment, inference or linguistic practice. Turning to Frege's view about how we know the so-called "third realm" of entities that are neither physical nor mental, Burge raises the question: How could Frege believe that reason alone could supply knowledge of it? In the concluding part of his paper, Burge attempts to answer this question by explaining the role that the primitive laws of truth (or of logic) as well as our acknowledgement of them play in Frege's philosophy. It is suggested that in Frege's view, first, the justification for holding logical laws to be true rests on primitive laws of logic and, second, this dependence is manifest in two ways. (a) Any judgment by a particular person is necessarily subject to the primitive laws of logic conceived of as laws that prescribe how one ought to think (judge, infer) if one would attain truth. (b) Acknowledging certain basic laws of truth is a prerequisite for having reason and for engaging in rational thinking. Burge puts it in a nutshell: for Frege, reason and judgment are partly defined in terms of acknowledging the basic laws of truth. Questions of "access" to the third realm are said to be misconceived.

There are likewise three essays on Frege's philosophy of language. In his paper 'Fregean Theories of Truth and Meaning', Terence Parsons deals to a certain extent with a topic he had already explored in great detail in his almost classic paper 'Frege's Hierarchies of Indirect Senses and the Paradox of Analysis'[43]: the semantic analysis of indirect or oblique contexts along the lines of Frege's approach. The much broader objective is now to devise a semantic theory of natural language in terms of Frege's notions of referring and expressing, and to study how he thought natural language actually works as opposed to studying an ideal artificial language that works better. The theory designed by Parsons thus embodies Frege's view that in certain (nonextensional) contexts words refer to the senses which they express when they occur in

[43] *Midwest Studies in Philosophy* VI, University of Minnesota Press, Minneapolis 1981, 37–58.