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Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations

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To our wives
Doris and Kerstin

Contents

Chapter 1
Introduction

From the beginning, the theory of function spaces was more or less connected very closely with the investigation of partial differential equations. Here "function space" means a normed or quasi-normed space of functions or distributions defined on subsets of $\mathbb{R}^n$. The spaces C^m, $m = 0, 1, 2, \ldots$, of continuous and differentiable functions, the Hölder spaces C^s, $0 < s \neq$ integer, and the spaces L_p, $1 \leq p < \infty$, of p-integrable functions were useful tools for the study of differential equations. In the papers by S. L. Sobolev published between 1935 and 1938, new spaces were introduced which are nowadays called the classical Sobolev spaces W_p^m, $1 \leq p < \infty$, $m = 0, 1, 2, \ldots$. The calculus of distributions and embedding theorems were used successfully for the further development of the theory of linear partial differential equations and boundary value problems. These "classical" spaces were generalized above all in the fifties and sixties: the Zygmund spaces $\mathcal{C}^s$ with $s = 1, 2, 3, \ldots$, the Slobodeckij spaces W_p^s, $0 < s \neq$ integer, the Bessel-potential spaces H_p^s, $s \in \mathbb{R}$, and the classical Besov spaces $\Lambda_{p,q}^s$, $s > 0$, $1 < p < \infty$, $1 \leq q \leq \infty$. For instance, the Slobodeckij spaces W_p^s are necessary for exact description of traces of Sobolev spaces.

The theory of linear partial differential equations was extended by real and (classical) complex interpolation to the Besov and Bessel-potential spaces. We refer to H. Triebel [Tr 2]. Since the end of the sixties, many mathematicians considered all these spaces from the point of view of some general principles (interpolation theory, new methods of Fourier analysis, maximal function techniques). With the help of these powerful tools it was possible to study the above mentioned function spaces from a unified point of consideration: All these spaces are included in the two scales of function spaces of Besov type $B_{p,q}^s$, $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$ and of Triebel–Lizorkin type $F_{p,q}^s$, $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$. Generalizations of the classical theory of regular elliptic boundary value problems to these spaces may be found in H. Triebel [Tr 2] and J. Franke and T. Runst [FR 3]. Up to now, the theory of linear partial differential equations is one of the main applications of these two scales. On the other hand, in recent years it was shown that some parts of the theory of nonlinear partial differential equations are connected very closely with other mathematical aspects. For example, J. M. Bony [Bon 2] and Y. Meyer [Me] used successfully methods of Fourier analysis and the theory of so-called paradifferential operators to prove regularity results of solutions of nonlinear partial differential operators.

One of the main purposes of this book is to describe the solvability of semilinear elliptic boundary value problems in Besov–Triebel–Lizorkin spaces. More precisely, we study existence and multiplicity results of solutions of nonlinear partial differential equations of the type $Lu = Nu$, where L is a second order linear differential operator and N denotes special nonlinear Nemytskij operators generated by smooth functions. The study of such equations has become a very active field of research since the seventies. The paper by H. Triebel [Tr 7] in 1984 was the first one which considered the solvability of special classes of semilinear elliptic boundary problems within these two scales of function spaces. It was shown that one can derive existence results by application of special methods of the Fourier analysis, multiplication properties and mapping properties of nonlinear operators. As mentioned there, it is useful to deal with special problems not only in classical spaces but in more general function spaces. In this connection we refer also to H. Amann [Am 3]. One of the aims of this book is the description of some essential tools to study nonlinear elliptic value problems in the general theory of the two scales of Besov–Triebel–Lizorkin spaces. Of course, it is not the purpose of the book to give a treatment of the theory of nonlinear boundary value problems in function spaces in the widest sense. Here we are interested to demonstrate how one uses methods of Fourier analysis and the theory of function spaces to investigate nonlinear problems. We remark that the equations considered here are only prototypes for wider classes of semilinear elliptic boundary value problems. However, as we hope, the results of Chapters 4 and 5 will be helpful also for other fields in nonlinear analysis.

The book has six chapters. Every chapter has an introduction which explains what one will find there and ends with final remarks. The aim of Chapter 2 is twofold. For convenience, we list some known results concerning function spaces which are needed in the following parts of the book. Here we omit proofs if there are convenient references. We refer the reader to the monographs [Tr 2, Tr 6, Tr 9], J. Peetre [Pe 3] and M. Frazier, B. Jawerth and G. Weiss [FJW]. In Chapter 3, we study regular elliptic boundary value problems in spaces of Besov–Triebel–Lizorkin type. Here we follow essentially the recent paper by J. Franke and T. Runst [FR 3]. We show that the results given there are the most general if one deals with usual unweighted function spaces. In this sense, Chapter 3 can be considered as the continuation of the theory presented in H. Triebel [Tr 2, Tr 6]. It is the purpose of Chapter 4 to study the m-linear map

$$(\prod_{i=1}^{m} A^{s_i}_{p_i,q_i}) \longrightarrow A^{s}_{p,q}$$

induced by

$$(f_1, \ldots, f_m) \longrightarrow f_1 \cdot \ldots \cdot f_m$$

(pointwise multiplication), where $A^{s_i}_{p_i,q_i}$ denotes spaces of Besov–Triebel–Lizorkin type. The results about pointwise multiplication are of interest both in the theory of partial differential equations (variable coefficients) and for the study of nonlinear superposition operators. In contrast to the next chapters we prove sufficient and necessary conditions on the parameters $s, s_i, p, p_i, q, q_i, i = 1, \ldots, m$, excluding some limit cases. Chapter 5 deals with mapping properties of the Nemytskij operator T_G:

$$(f_1, \ldots, f_m) \longrightarrow G(f_1, \ldots, f_m),$$

generated by a given function $G\colon \mathbb{R}^m \longrightarrow \mathbb{R}^n$. It turns out that the mapping properties of those nonlinear operators depend strongly on the chosen domain of definition, in our case this means on the range of s, p and q. Here we are able to prove sharp results only for special classes of Nemytskij operators and special conditions on the parameters. On the other hand, there are a lot of open problems and this theory stands more or less at the beginning. Nevertheless, the purpose of Chapter 5 is to describe essential phenomena which show the difference between linear and nonlinear problems. This chapter summarizes a survey of recent results with respect to Besov–Triebel–Lizorkin spaces obtained partly by the Jena research group on function spaces. For this we use a uniform representation applying methods of Fourier analysis, especially the results of Chapter 4. Let us refer also to the monograph of J. Appell and P. P. Zabrejko [AZ], where such problems in classical Hölder–Zygmund and Sobolev spaces are investigated. In the last chapter, we consider existence and multiplicity results of solutions of semilinear elliptic boundary value problems. For this we apply the results of the preceding chapters and classical nonlinear methods extended to our case. We refer to the monographs K. Deimling [De], S. Fučík [Fu 7] and E. Zeidler [Ze 1, Ze 2]. Note that the spaces considered here are, in general, only quasi-normed. One of the aims is to show how one can use function spaces under consideration to investigate semilinear elliptic boundary value problems. Furthermore, we describe some difficulties which occur when we consider nonlinear problems in quasi-Banach spaces and how one can carry over some classical Banach space techniques. Of course, we are not able to give here a complete and detailed introduction to this part of nonlinear analysis. Therefore, the reader is expected to have a working knowledge of linear and nonlinear analysis as presented in classical textbooks. A familarity with the basic results of function spaces and methods in nonlinear analysis would be helpful.

A final remark concerning the range of parameters. In this text we tried, as much as possible, to cover spaces with $\min(p, q) < 1$ (the case of quasi-Banach spaces). Of course, part of motivation comes from the Fourier analytic tools we used and which allow us to deal with those spaces. A more natural motivation arises from nonlinear mappings itself. It is evident that

$$f \in L_p \longrightarrow f^m \in L_{\frac{p}{m}},$$

i.e. simple nonlinear maps lead naturally to a consideration of spaces with $p < 1$. It turns out that the description of those mappings $f \longrightarrow f^m$ can be made more complete and more satisfactory if $p < 1$ is allowed. A third motivation comes from the theory of function spaces itself. For example, so-called nonlinear approximation problems require necessarily Besov spaces with $p < 1$, see for example P. Oswald [Os 1] and R. de Vore, B. Jawerth and V. Popov [dVJP]. Since there is some connection to the approximation power of numerical algorithms for the solution of linear partial differential this is not very far from the topic of the book.

It is a great pleasure to thank all colleagues and friends for various hints and helpful discussions about the contents of the book. In particular, we are grateful to G. Bourdaud and J. Johnsen for critical reading of parts of the manuscript and for several hints for improvements of a first version, especially with respect to Chapters 4 and 5. Further, we thank our teacher H. Triebel for motivation to write this book and for his constant support during our work. Next, we express our gratitude to M. Malarski for his permanent assistance concerning the use of LaTeX and for producing various figures. Finally, we thank also the managing editor of this series, J. Appell. He supported our project from the very beginning and was responsible for smooth operation with the publishing house.

For better readability quotations of the literature are concentrated in "Notes and comments" at the end of each chapter, except for Chapter 2. There we give direct hints where the assertions are taken from.

Now a brief instruction on how to read the book. The text is devided in sections and the sections are then arranged into subsections. The book is organized by the decimal system. (n.k.l/m) refers to formula (m) in Subsection n.k.l. In a similar way theorems, propositions, etc. are quoted, whereas, for instance, Theorem n.k.l means the only theorem in Subsection n.k.l. All unimportant positive numbers will be denoted by c (with additional indices if there are several c's in the same formula).

Chapter 2
Function spaces of Besov–Triebel–Lizorkin type

As pointed out before, that will be not a book about function spaces. Our aim is to describe some possibilities how to apply spaces of Besov–Triebel–Lizorkin type in nonlinear partial differential equations. Therefore it will be convenient to include one chapter about properties of $F^s_{p,q}$ and $B^s_{p,q}$ which are of relevance in our context. Because of the generality we take a great care to illustrate the considerations by instructive examples. The general references about Besov–Triebel–Lizorkin spaces are the monographs H. Triebel [Tr 6, Tr 9], J. Peetre [Pe 3], S. M. Nikol'skij [Nik], and M. Frazier, B. Jawerth and G. Weiss [FJW] including the fundamental paper by M. Frazier and B. Jawerth [FJ]. Only in those cases, where a convenient reference was not available, we shall give proofs.

2.1 Definitions and fundamental properties

2.1.1 Definitions

In general, all functions, distributions, etc. are defined on the Euclidean n-space $\mathbb{R}^n$. If there is no danger of confusion we will not indicate this. By $\mathbb{N}$ we denote the set of all natural numbers, by $\mathbb{N}_0$ the same set including 0. $\mathbb{Z}^n$ means the set of all lattice-points in $\mathbb{R}^n$ having integer components. Let A be a (real or complex) linear vector space. $\| a \,|A\|$ is said to be a quasi-norm if $\| a \,|A\|$ satisfies the usual conditions of a norm with exception of the triangle inequality, which is replaced by

$$\| a_1 + a_2 \,|A\| \leq c \, (\| a_1 \,|A\| + \| a_2 \,|A\|)$$

(here c does not depend on $a_1,\ a_2 \in A$). A quasi-normed space is said to be a quasi-Banach space if it is complete (i.e. any fundamental sequence in A with respect to $\| \cdot \,|A\|$ converges).
By

$$\| a \|_1 \sim \| a \|_2 \,, \quad a \in M$$

we indicate the existence of two constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \, \| a \|_1 \leq \| a \|_2 \leq c_2 \, \| a \|_1$$

holds for all elements of M.
Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. The topology is generated by the semi-norms

$$p_N(\varphi) = \sup_{x \in \mathbb{R}^n} (1 + |x|)^N \sum_{|\alpha| \leq N} |D^\alpha \varphi(x)|, \quad N = 0, 1, 2, \dots ,$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Here

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdot \ldots \cdot \partial x_n^{\alpha_n}} \quad \text{and} \quad |\alpha| = \alpha_1 + \ldots + \alpha_n, \; \alpha_i \in \mathbb{N}_0, \quad i = 1, \dots, n \, .$$

Let $\mathcal{S}'(\mathbb{R}^n)$ denote the set of all tempered distributions, i.e. the topological dual of $\mathcal{S}(\mathbb{R}^n)$, equipped with the strong topology (if not otherwise stated). If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then

$$\mathcal{F}\varphi(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x) \, dx, \qquad \xi \in \mathbb{R}^n , \quad \varphi \in \mathcal{S}(\mathbb{R}^n) ,$$

($x\xi$ means the scalar product in $\mathbb{R}^n$) denotes the Fourier transform $\mathcal{F}\varphi$ of φ. The inverse transform $\mathcal{F}^{-1}\varphi$ of φ is given by

$$\mathcal{F}^{-1}\varphi(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} \varphi(x) \, dx, \qquad \xi \in \mathbb{R}^n , \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

One extends $\mathcal{F}$ and $\mathcal{F}^{-1}$ from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ in the usual way. Furthermore, let $\mathcal{F}'$ be the $(n-1)$-dimensional Fourier transform with respect to $x' = (x_1, \dots, x_{n-1})$, and let $\mathcal{F}_1$ be the one-dimensional Fourier transform with respect to x_n. The corresponding inverse transformations are $(\mathcal{F}')^{-1}$ and $\mathcal{F}_1^{-1}$.
Let Ω be an open subset of $\mathbb{R}^n$. The symbol $|\Omega|$ will be used for the n-dimensional Lebesgue measure of Ω. Further, let $D(\Omega)$ be the collection of all complex-valued, compactly supported and infinitely differentiable functions f in $\mathbb{R}^n$ with $\operatorname{supp} f \subset \Omega$. The topological dual is denoted by $D'(\Omega)$. Sometimes we also use $C_0^\infty(\Omega)$ in place of $D(\Omega)$. By $C_0^\infty(\Omega, \mathbb{R}^m)$ we mean all vector-valued functions $f = (f_1, \dots, f_m)$ such that $f_i \in C_0^\infty(\Omega)$, $i = 1, \dots, m$.
Finally, we shall make use of the following notations. Let $0 < p \leq \infty$ and $0 < q \leq \infty$. Then we put

$$\|f_k \,|L_p(\ell_q)\| = \left(\int_{\mathbb{R}^n} \left(\sum_{k=0}^{\infty} |f_k(x)|^q \right)^{p/q} dx \right)^{1/p}$$

and

$$\|f_k\,|\ell_p(L_q)\| = \left(\sum_{k=0}^{\infty}\left(\int_{\mathbb{R}^n} |f_k(x)|^p\,dx\right)^{q/p}\right)^{1/q}$$

(usual modification if $\max(p, q) = \infty$). If not otherwise stated the integration extends over all of $\mathbb{R}^n$. Sometimes we shall take the same abbreviations in case, where "$\sum_{k=0}^{\infty}$" is replaced by "$\sum_{k=-\infty}^{\infty}$". That will be clear from the context.
To introduce the spaces $F^s_{p,q}$and $B^s_{p,q}$we make use of the Fourier decomposition method. That will have two advantages. At first, from the very beginning it shows the great similarity of these two scales of distribution spaces, offering to us the possibility to treat these spaces $F^s_{p,q}$and $B^s_{p,q}$from a common point of view, and secondly, it will be applied in most of the proofs presented here. But, of course, the disadvantage of the following definition consists in its minor transparency and technical complexity. As we hope the contents of this chapter will be helpful to overcome this.
In general all functions, distributions etc. are defined on $\mathbb{R}^n$. So we omit $\mathbb{R}^n$ in notations. To introduce Besov–Triebel–Lizorkin spaces we need some special systems of functions contained in $\mathcal{S}$.

Definition 1. Let Φ be the collection of all systems $\varphi = \{\varphi_j\}_{j=0}^{\infty} \subset \mathcal{S}$ such that

(i) there exist positive constants A, B, C and

$$\begin{cases} \text{supp } \varphi_0 \subset \{x \mid \ |x| \le A\}, \\ \text{supp } \varphi_j \subset \{x \mid \ B\,2^{j-1} \le |x| \le C\,2^{j+1}\} \quad \text{if } j = 1, 2, 3, \dots, \end{cases} \tag{1}$$

(ii) for every multi-index α there exists a positive number c_α and

$$\sup_x \sup_{j=0,1,\dots} 2^{j|\alpha|}\,|D^\alpha \varphi_j(x)| \le c_\alpha\,, \tag{2}$$

(iii)

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad \text{for every} \quad x \in \mathbb{R}^n. \tag{3}$$

An example

Let $\psi \in \mathcal{S}$ be a function with

$$0 \le \psi(x) \le 1, \quad \psi(x) = \begin{cases} 1 & \text{if} \quad |x| \le 1 \\ 0 & \text{if} \quad |x| \ge \frac{3}{2}\,. \end{cases} \tag{4}$$

We put

$$\left.\begin{array}{lcl} \varphi_0(x) & = & \psi(x) \\ \varphi_1(x) & = & \psi(\frac{x}{2}) - \psi(x) \\ \varphi_j(x) & = & \varphi_1(2^{-j+1}x), \quad j = 2, 3, \dots , \end{array}\right\} \tag{5}$$

This system belongs to Φ. Moreover, it holds

$$\sum_{j=0}^{M} \varphi_j(x) = \psi(2^{-M}x), \quad M = 0, 1, \dots , \tag{6}$$

$$\text{supp } \varphi_j \subset \{x |\, 2^{j-1} \le |x| \le 3 \cdot 2^{j-1}\}, \ j = 1, 2, \dots , \tag{7}$$

and if

$$\frac{3}{2}\, 2^{j-1} \le |x| \le 2^j \qquad \text{we have} \qquad \varphi_k(x) = \delta_{k,j}\,, \quad k = 0, 1, \dots , \tag{8}$$

($\delta_{k,j}$-Kronecker symbol). Having these smooth resolutions of unity we are able to introduce the Triebel–Lizorkin and Besov spaces.

Definition 2. Let $s \in \mathbb{R}$ and $0 < q \le \infty$. Let $\varphi = \{\varphi_j\}_{j=0}^{\infty} \in \Phi$.

(i) If $0 < p < \infty$, then

$$F^s_{p,q} = \left\{ f \in \mathcal{S}' : \quad \| f \,|F^s_{p,q}\, \|^{\varphi} = \|2^{sj} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f](\cdot) | L_p(\ell_q)\| < \infty \right\}. \tag{9}$$

(ii) If $0 < p \le \infty$, then

$$B^s_{p,q} = \left\{ f \in \mathcal{S}' : \quad \| f \,|B^s_{p,q}\, \|^{\varphi} = \|2^{sj} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f](\cdot) |\, \ell_q(L_p)\, \| < \infty \right\}. \tag{10}$$

Remark 1. Technical explanation. Since φ_j is smooth $\varphi_j(\xi) \cdot \mathcal{F} f(\xi)$ makes sense as a distribution in $\mathcal{S}'$. Because of the compactness of the support of φ_j the famous Paley–Wiener–Schwartz theorem tells us that $\mathcal{F}^{-1}[\varphi_j(\xi)\mathcal{F}(\xi)](\cdot)$ is an entire analytic function. So the quasi-norms in (9) and (10) make sense.

Remark 2. Interpretation. The aim of the Triebel–Lizorkin spaces $F^s_{p,q}$and the Besov spaces $B^s_{p,q}$is to measure smoothness. A rough interpretation gives that smoothness of f is measured via decay properties of Fourier transform $\mathcal{F} f$. Recall,

$$(-i)^{|\alpha|} \mathcal{F}(D^{\alpha} f)(\xi) = \xi^{\alpha} \mathcal{F} f(\xi)\,. \tag{11}$$

Hence, if

$$|(\mathcal{F} f)(\xi)| \le (1 + |\xi|)^{-(|\alpha|+m)}, \quad m > \frac{n}{2}\,, \tag{12}$$

we get

$$\|D^{\alpha} f \,|L_2\| = \|\mathcal{F}(D^{\alpha} f)(\cdot)|L_2\| = \|\xi^{\alpha} \mathcal{F} f(\xi)|L_2\| < \infty\,. \tag{13}$$

So, decay in the Fourier image means smoothness of the function itself and as it is seen from (13) also vice versa. In some sense the above definitions are a diversification of these simple arguments.

Remark 3. The quasi-norms in (9) and (10) are local in the following sense. Suppose

$$\inf\{\,|x-y|\,:\, x\in \operatorname{supp} f,\ y\in \operatorname{supp} g\,\}>0$$

then $f+g\in F^s_{p,q}$ if and only if $f\in F^s_{p,q}$ and $g\in F^s_{p,q}$ and $f+g\in B^s_{p,q}$ if and only if $f\in B^s_{p,q}$ and $g\in B^s_{p,q}$, cf. [Tr 8, 1.8.4].

Proposition 1. ([Tr 6, 2.3.3]) *$F^s_{p,q}$and $B^s_{p,q}$are quasi-Banach spaces (Banach spaces if* $\min(p,q)\geq 1$*). They are independent of the chosen system $\varphi\in\Phi$ (equivalent quasi-norms).*

Remark 4. We shall not distinguish between equivalent quasi-norms. So we shall write $\|\cdot|F^s_{p,q}\|$ and $\|\cdot|B^s_{p,q}\|$ instead of $\|\cdot|F^s_{p,q}\|^{\varphi}$ and $\|\cdot|B^s_{p,q}\|^{\varphi}$, respectively.

The spaces $F^s_{\infty,q}$

In the definition of the F-scale the case $p=\infty$ is missed. Unfortunately, to take simply the above definition makes no sense (the spaces $F^s_{\infty q}$, $q<\infty$, would depend on the chosen $\varphi\in\Phi$ then). Therefore we shall use the following modification. First, let

$$Q_{k,\ell}=\{x:\quad 2^{-k}\ell_i\leq x_i\leq 2^{-k}(\ell_i+1),\quad i=1,\dots,n\},\quad k\in\mathbb{Z},\ \ell\in\mathbb{Z}^n\,. \tag{14}$$

Definition 3. Let $s\in\mathbb{R}$ and $0<q\leq\infty$. Let $\varphi=\{\varphi_j\}_{j=0}^{\infty}\in\Phi$. We put

$$F^s_{\infty,q}=\Bigg\{f\in\mathcal{S}':\quad \|f\,|F^s_{\infty,q}\|= \tag{15}$$

$$\sup_{j=0,1,\dots}\ \sup_{\ell\in\mathbb{Z}^n}\left(2^{jn}\int_{Q_{j,\ell}}\left(\sum_{k=j}^{\infty}2^{ksq}\,|\mathcal{F}^{-1}[\varphi_k\mathcal{F}f](x)\,|^q\right)dx\right)^{1/q}<\infty\Bigg\}.$$

Proposition 2. ([FJ, Chapter 5]) (i) *The spaces $F^s_{\infty,q}$ are quasi-Banach spaces independent of $\varphi\in\Phi$ (equivalent norms).*
(ii) *Let $\varphi=\{\varphi_j\}_{j=0}^{\infty}\in\Phi$ and let $1\leq q\leq\infty$. Then*

$$F^s_{\infty,q}=\{f\in\mathcal{S}':\exists\,\{f_j\}_{j=0}^{\infty}\ \textit{with}\ f=\sum_{j=0}^{\infty}\mathcal{F}^{-1}[\varphi_j\mathcal{F}f_j]\ (\textit{convergence in}\ \mathcal{S}') \tag{16}$$

$$\textit{and} \quad \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \, |f_j(x)|^q \right)^{1/q} |L_\infty \right\| < \infty \Big\} \, . \tag{17}$$

An equivalent norm on $F^s_{\infty,q}$ is given by the infimum over all admissible representations (16) *of f in* (17).

Remark 5. The spaces $F^s_{\infty,q}$ will not play an essential role in our context. But they occur in some situations very naturally. At first, by considering the dual spaces of $F^s_{1,q}$, cf. Remark 2.1.5/2; secondly, they include the local *bmo* spaces, cf. 2.1.2; finally, they form important subspaces of the sets

$$M(F^s_{p,q}) = \{f \in \mathcal{S}' : \quad f \cdot g \in F^s_{p,q} \quad \text{for all } g \in F^s_{p,q}\},$$

cf. 4.7.2.

Remark 6. Beside the trivial equality $B^s_{p,p} = F^s_{p,p}$ we have always diversity. More exactly, it holds

(i) $F^{s_1}_{p_1,q_1} = F^{s_2}_{p_2,q_2}$ implies $s_1 = s_2$, $p_1 = p_2$ and $q_1 = q_2$,
(ii) $B^{s_1}_{p_1,q_1} = B^{s_2}_{p_2,q_2}$ implies $s_1 = s_2$, $p_1 = p_2$ and $q_1 = q_2$,
(iii) $F^{s_1}_{p_1,q_1} = B^{s_2}_{p_2,q_2}$ implies $s_1 = s_2$ and $p_1 = p_2 = q_1 = q_2$,

(cf. [Tr 6, 2.3.9]).

Remark 7. Other spaces. At the first glance one could think that the scales $F^s_{p,q}$and $B^s_{p,q}$are rich enough to cover all interesting tempered distributions. But this is not the case. If f is continuous and unbounded such that

$$\Big| \int_{|y|\le 1} f(x_j + y)\, dy \Big| \longrightarrow \infty \quad \text{for some sequence} \quad \{x_j\}_{j=0}^{\infty}, \quad |x_j| \to \infty \,,$$

then

$$f \not\in (F^s_{p,q} \cup B^s_{p,q}), \qquad s \in \mathbb{R}, \quad 0 < p \le \infty, \quad 0 < q \le \infty \, .$$

For instance, polynomials, which are not constant are not contained in these classes. To include those tempered distributions one has to investigate weighted counterparts of $F^s_{p,q}$ and $B^s_{p,q}$, respectively. We omit details and refer to [Tr 6, ST, Ya 2, Mar 3, Mar 6]. There is also a well-developed theory for anisotropic spaces, spaces of mixed dominated smoothness and its periodic counterparts, cf. [ST, Ya 2, Mar 3]. Function spaces related to more general differential operators than the Laplacian are investigated in H.-G. Leopold [Leo 1, Leo 2]. Let us refer also to the russian literature, where spaces of so-called generalized smoothness are extensively investigated, cf. e.g. [BIN, Gol 2, Gol 3, KL]. In any case, the starting point is a similar definition of these variants of $F^s_{p,q}$, $B^s_{p,q}$as given above.

For that reason there is some hope that some of the assertions presented here in the unweighted isotropic cases only, have appropriate general counterparts.

Remark 8. The general references for function spaces as above are J. Bergh and J. Löfström [BL], M. Frazier and B. Jawerth [FJ], M. Frazier, B. Jawerth and G. Weiss [FJW], J. Peetre [Pe 3], H.-J. Schmeisser and H. Triebel [ST], D. E. Edmunds and H. Triebel [ET 5] and H. Triebel [Tr 6, Tr 9]. For historical remarks we refer to H. Triebel [Tr 9].

Convention 1. Only additional restrictions of p, q and s will be given. That means if there are no restrictions for p, q or s given then the assertion holds for all admissible values in Definition 2. In particular, if not otherwise stated always $p < \infty$ in case of the F-scale is assumed.

Convention 2. If not otherwise stated then the quasi-norms $\| \cdot |F^s_{p,q}\|$ and $\| \cdot |B^s_{p,q}\|$ are always generated by using the system $\{\varphi_j\}_j \in \Phi$ constructed in (5).

2.1.2 Classical function spaces and their appearance in the scales $F^s_{p,q}$ and $B^s_{p,q}$

The aim of this subsection is twofold. On the one side we wish to show that many different types of, in some sense, "classical" function spaces can be identified with special cases of the above introduced scales $F^s_{p,q}$ and $B^s_{p,q}$, on the other side we hope to increase the transparency what type of spaces $F^s_{p,q}$ and $B^s_{p,q}$ are.
First we introduce the **Lebesgue spaces** L_p to be the set of all Lebesgue-measurable complex-valued functions on $\mathbb{R}^n$ such that

$$\left.\begin{aligned} \|f\,|L_p\| &= \Big(\int_{\mathbb{R}^n} |f(x)|^p\,dx\Big)^{\frac{1}{p}} < \infty, \quad 0 < p < \infty \\ \|f\,|L_\infty\| &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x)| \end{aligned}\right\} \tag{1}$$

(the measure is always Lebesgue measure) are finite. Corresponding to that $L_p^{\ell oc}$ means the set of functions satisfying

$$\int_B |f(x)|^p\,dx < \infty \tag{2}$$

for all compact sets B.
By C we denote the set of all complex-valued and uniformly conitinuous functions on $\mathbb{R}^n$ equipped with the sup-norm. Further, if $m = 1, 2, \ldots,$ we define

$$C^m = \{f \in C : D^\alpha f \in C \ \text{ for all } \ |\alpha| \le m\} \tag{3}$$

endowed with the norm

$$\|f\,|C^m\| = \sum_{|\alpha|\le m} \|D^\alpha f\,|L_\infty\|\,. \tag{4}$$

In (3) and (4) D^α means classical derivatives. After introduction of these basic spaces we come to the "constructive" spaces (cf. [Tr 6, 2.2.2]).

(i) **Hölder spaces C^s.** If s is a real number, then we put

$$s = [s] + \{s\} \quad \text{with } [s] \text{ integer and } \quad 0 \le \{s\} < 1. \tag{5}$$

If $s > 0$ is not an integer, then

$$C^s = \Big\{ f \in C^{[s]} : \tag{6}$$

$$\|f\,|C^s\| = \|f\,|C^{[s]}\| + \sum_{|\alpha|=[s]} \sup_{x\neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^{\{s\}}} < \infty \Big\}.$$

(ii) **Zygmund spaces $\mathcal{C}^s$.** If s is a real number, then we put

$$s = [s]^- + \{s\}^+ \quad \text{with } [s]^- \text{ integer and } \quad 0 < \{s\}^+ \le 1. \tag{7}$$

Furthermore, if $f(x)$ is an arbitrary function and $h \in \mathbb{R}^n$ we put

$$\Delta_h^m f(x) = \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^\ell f(x + (m-\ell)h\,), \quad m = 1, 2, \ldots . \tag{8}$$

If $s > 0$, then

$$\mathcal{C}^s = \Big\{ f \in C^{[s]^-} : \tag{9}$$

$$\|f\,|\mathcal{C}^s\| = \|f\,|C^{[s]^-}\| + \sum_{|\alpha|=[s]^-} \sup_{h\neq 0} |h\,|^{-\{s\}^+} \|\Delta_h^2 D^\alpha f\,|C\,\| < \infty \Big\}.$$

(iii) **Sobolev spaces W_p^m** . Let $1 \le p \le \infty$ and m=1,2, ..., then

$$W_p^m = \Big\{ f \in L_p : \quad \|f\,|W_p^m\| = \sum_{|\alpha|\le m} \|D^\alpha f\,|L_p\,\| < \infty \Big\}. \tag{10}$$

(iv) **Slobodeckij spaces W_p^s** . If $1 \le p < \infty$, $0 < s \neq$ integer, then

$$W_p^s = \Big\{ f \in W_p^{[s]} : \quad \|f\,|W_p^s\| = \|f\,|W_p^{[s]}\|$$

$$+ \sum_{|\alpha|=[s]} \left(\int\!\!\int \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{n+\{s\}p}}\, dx\, dy \right)^{\frac{1}{p}} < \infty \Big\}. \tag{11}$$

(v) **Besov (or Lipschitz) spaces $\Lambda^s_{p,q}$.** If $s > 0,\ 1 \le p < \infty$ and $1 \le q < \infty$, then

$$\Lambda^s_{p,q} = \Big\{ f \in W_p^{[s]^-} : \quad \| f \mid \Lambda^s_{p,q} \| = \| f \mid W_p^{[s]^-} \| + \sum_{|\alpha|=[s]^-} \left(\int |h|^{-\{s\}^+ q} \| \Delta_h^2 D^\alpha f \mid L_p \|^q \frac{dh}{|h|^n} \right)^{\frac{1}{q}} < \infty \Big\}. \tag{12}$$

If $s > 0,\ 1 \le p < \infty$, then

$$\Lambda^s_{p,\infty} = \Big\{ f \in W_p^{[s]^-} : \| f \mid \Lambda^s_{p,\infty} \| = \| f \mid W_p^{[s]^-} \| + \sum_{|\alpha|=[s]^-} \sup_{h \neq 0} | h |^{-\{s\}^+} \| \Delta_h^2 D^\alpha f \mid L_p \| < \infty \Big\}. \tag{13}$$

(vi) **Bessel-potential spaces (or Sobolev spaces of fractional order) H^s_p.** Let s be a real number and $1 < p < \infty$. Then

$$H^s_p = \Big\{ f \in \mathcal{S}' : \quad \| f \mid H^s_p \| = \| \mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F} f \mid L_p \| < \infty \Big\}. \tag{14}$$

(vii) **Local Hardy spaces h_p.** Let $0 < p < \infty$. For ψ as in (2.1.1/ 4) we put

$$h_p = \Big\{ f \in \mathcal{S}' : \| f \mid h_p \|^\psi = \| \sup_{0<t<1} | \mathcal{F}^{-1}[\psi(t\xi)\, \mathcal{F} f(\xi)](\cdot) | \quad |L_p\| < \infty \Big\}. \tag{15}$$

(viii) **Space of "local" bounded mean oscillation *bmo*.** If f is a locally Lebesgue-integrable function and if Q is a cube, then

$$f_Q = \frac{1}{|Q|} \int_Q f(x)\, dx \tag{16}$$

is the mean value of f with respect to Q. We put

$$bmo = \Big\{ f \in L_1^{\ell oc} : \quad \| f \mid bmo \| = \sup_{|Q| \le 1} \frac{1}{|Q|} \int\limits_Q | f(x) - f_Q |\, dx + \sup_{|Q|>1} \frac{1}{|Q|} \int\limits_Q | f(x) |\, dx < \infty \Big\}. \tag{17}$$

Remark 1. (Technical explanations). $D^\alpha f$ in (6) and (9) are classical derivatives. In all other cases it has to be interpreted in the sense of distributions. Further, $\sup_{|Q| \le 1}$ in (17) means that the supremum has to be taken over cubes Q with volume $|Q| \le 1$.

Proposition. ([Tr 6, 2.3.5]) *We have the identities (equivalent quasi-norms)*

(i)	$C^s = B^s_{\infty,\infty}$	*if*	$0 < s \neq$ *integer,*
(ii)	$\mathscr{C}^s = B^s_{\infty,\infty}$	*if*	$s > 0,$
(iii)	$L_p = F^0_{p,2}$	*if*	$1 < p < \infty,$
(iv)	$W^m_p = F^m_{p,2}$	*if*	$1 < p < \infty,\ m = 1, 2, \ldots,$
(v)	$W^s_p = F^s_{p,p} = (B^s_{p,p})$	*if*	$1 \leq p < \infty,\ 0 < s \neq$ *integer,*
(vi)	$\Lambda^s_{p,q} = B^s_{p,q}$	*if*	$1 \leq p < \infty,\ 1 \leq q \leq \infty$ *and* $s > 0,$
(vii)	$H^s_p = F^s_{p,2}$	*if*	$1 < p < \infty,$
(viii)	$h_p = F^0_{p,2}$	*if*	$0 < p < \infty,$
(ix)	$bmo = (F^0_{1,2})'$		*($()'$ means the topological dual).*

Remark 2. Not included in the scales $F^s_{p,q}$ and $B^s_{p,q}$ are L_1, L_∞, C, C^m, W^m_1, W^m_∞. For further informations see H. Triebel [Tr 6, 2.2.2] and [Tr 9, Chapter 1].

2.1.3 Basic properties

By $A_1 \subset A_2$ we mean the set-theoretical embedding. $A_1 \hookrightarrow A_2$ denotes continuous embedding, i.e. there exists a constant c such that

$$\| a \,|A_2\| \leq c \,\| a \,|A_1\| \qquad \text{for all} \quad a \in A_1 \,. \tag{1}$$

Finally, $A_1 \hookrightarrow\hookrightarrow A_2$ is reserved for compact embedding.

Proposition 1. ([Tr 6, 2.3.3]) (i) *We have*

$$\mathcal{S} \hookrightarrow F^s_{p,q} \hookrightarrow \mathcal{S}' \,. \tag{2}$$

Furthermore, $\mathcal{S}$ is dense in $F^s_{p,q}$ if $q < \infty$.
(ii) *We have*

$$\mathcal{S} \hookrightarrow B^s_{p,q} \hookrightarrow \mathcal{S}' \,. \tag{3}$$

Furthermore, $\mathcal{S}$ is dense in $B^s_{p,q}$ if $\max(p, q) < \infty$.
(iii) *In case $p = \infty$ we have*

$$\bigcap_{m \in \mathbb{N}} C^m \subset B^s_{\infty,q} \,. \tag{4}$$

Furthermore, $\bigcap_{m \in \mathbb{N}} C^m$ is dense in $B^s_{\infty,q}$ if $q < \infty$.

Since $\mathcal{S}$ is not dense in $F^s_{p,\infty}$, $B^s_{p,\infty}$ and $B^s_{\infty,q}$ the following definition is reasonable.

Definition 1. (i) $f^s_{p,q}$ denotes the closure of $\mathcal{S}$ in $F^s_{p,q}$ endowed with the same quasi-norm as $F^s_{p,q}$.

(ii) $b^s_{p,q}$ denotes the closure of $\mathcal{S}$ in $B^s_{p,q}$endowed with the same quasi-norm as $B^s_{p,q}$.

The Fatou property

Definition 2. Let A be a quasi-Banach space with $\mathcal{S} \hookrightarrow A \hookrightarrow \mathcal{S}'$. We say A has the Fatou property if there exists a constant c such that from

$$f_k \rightharpoonup f \quad \text{if} \quad k \to \infty \quad (\text{weak convergence in } \mathcal{S}')$$

and

$$\liminf_{k\to\infty} \|f_k\,|A\| \le D$$

it follows $f \in A$ and $\|f\,|A\| \le cD$ with c independent of f and $\{f_k\}_{k=0}^{\infty} \subset A$.

Proposition 2. ([Fr 3]) *$F^s_{p,q}$and $B^s_{p,q}$have the Fatou property.*

Remark 1. (a) There are simple examples to show that L_1, C, C^m, $b^s_{\infty,q}$ $b^s_{p,\infty}$ and $f^s_{p,\infty}$ do not satisfy the Fatou property.
(i) L_1: Take ψ from (2.1.1/4), then

$$2^{kn}\psi(2^k\,\cdot) \rightharpoonup \delta \quad \text{if} \quad k \to \infty \qquad (\delta\text{-Dirac distribution}).$$

(ii) C: Let φ be a continuous function with $\varphi(x) = 1$ for $x_n > 1$ and $\varphi(x) = 0$ for $x_n < 0$, where $x = (x_1, \ldots, x_n)$. Consider $f_k(\cdot) = \varphi(k\,\cdot)$ then $f_k \rightharpoonup \chi$, where χ denotes the characteristic function of the half-space. The same argument works in case C^m.
(iii) $b^s_{p,\infty}$, $f^s_{p,\infty}$, $b^s_{\infty,q}$: Consider

$$f_k(x) = \psi(2^{-k}x)\mathcal{F}^{-1}[\psi(2^{-k}\xi)\mathcal{F}f(\xi)](x).$$

We have $f_k \rightharpoonup f$, $k \to \infty$. Further it holds $f_k \in f^s_{p,q}$ $(b^s_{p,q})$ if $f \in F^s_{p,q}$ $(B^s_{p,q})$. Also the quasi-norms of f_k are uniformly bounded (cf. 2.1.5 and 4.7.1), but as pointed out after Proposition 1 f does not belong to $b^s_{p,q}$ $(f^s_{p,q})$ in general.
(b) L_∞ has the Fatou property, cf. [Fr 3].

Dilation in $F^s_{p,q}$ and $B^s_{p,q}$

We shall study the operation $f(\cdot) \longrightarrow f(\lambda\,\cdot)$, $\lambda > 0$. Also for later use we introduce the following abbreviations:

$$\sigma_p = n \cdot \max(0, \frac{1}{p} - 1) \tag{5}$$

and

$$\sigma_{p,q} = n \cdot \max(0, \frac{1}{p} - 1, \frac{1}{q} - 1)\,. \tag{6}$$

As we shall see in the next section $s > \sigma_p$ will guarantee

$$(F^s_{p,q} \cup B^s_{p,q}) \subset L_1^{\ell oc}\,. \tag{7}$$

Proposition 3. ([Tr 11], [ET 5, 2.3.1]) (i) *Let $s > \sigma_p$. Then there exists a constant c such that*

$$\| f(\lambda \cdot)\, |F^s_{p,q}\| \le c\, \lambda^{-\frac{n}{p}} \max(1,\lambda)^s \, \| f\, |F^s_{p,q}\| \tag{8}$$

holds for all $f \in F^s_{p,q}$. Here c is independent from $\lambda > 0$ and f.
(ii) *Let $s > \sigma_p$. Then there exists a constant such that*

$$\| f(\lambda \cdot)\, |B^s_{p,q}\| \le c\, \lambda^{-\frac{n}{p}} \max(1,\lambda)^s \, \| f\, |B^s_{p,q}\| \tag{9}$$

holds for all $f \in B^s_{p,q}$. Here c is independent from $\lambda > 0$ and f.

Remark 2. Both, (8) and (9) become false if $0 < p < 1$ and $s < n(\frac{1}{p} - 1)$ (cf. H. Triebel [Tr 11]). Sometimes limiting cases are of interest. Suppose $1 \le p \le \infty$ and $1 \le q \le \infty$. Then

$$\| f(\lambda \cdot) | B^0_{p,q}\| \le c\, \lambda^{-\frac{n}{p}} (1 + \log \lambda)^{\frac{1}{q}} \, \| f\, |B^0_{p,q}\| \tag{10}$$

if $\lambda \ge 1$, and

$$\| f(\lambda \cdot) | B^0_{p,q}\| \le c\, \lambda^{-\frac{n}{p}} (1 + |\log \lambda|)^{1-\frac{2}{q}} \, \| f\, |B^0_{p,q}\| \tag{11}$$

if $0 < \lambda < 1$, cf. G. Bourdaud [Bou 3, Bou 10].

The lemma of Ehrling and the triangle inequality

Proposition 4. *Let $s_0 < s < s_1$.*
(i) *For any $\varepsilon > 0$ there exists a constant c_ε such that*

$$\| f\, |F^s_{p,q}\| \le \varepsilon \, \| f\, |F^{s_1}_{p,q}\| + c_\varepsilon \, \| f\, |F^{s_0}_{p,q}\| \tag{12}$$

holds for all $f \in F^{s_1}_{p,q}$.

(ii) *For any* $\varepsilon > 0$ *there exists a constant* C_ε *such that*

$$\| f \, |B^s_{p,q} \| \le \varepsilon \| f \, |B^{s_1}_{p,q}\| + c_\varepsilon \| f \, |B^{s_0}_{p,q}\| \tag{13}$$

holds for all $f \in B^{s_1}_{p,q}$.

Proof. It holds

$$\| f \, |F^s_{p,q} \| \le c \left(\Big\| \Big(\sum_{j=0}^{M} 2^{jsq} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} f](\cdot)|^q \Big)^{\frac{1}{q}} |L_p \Big\| \right.$$

$$\left. + \Big\| \Big(\sum_{j=M+1}^{\infty} 2^{jsq} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} f](\cdot)|^q \Big)^{\frac{1}{q}} |L_p \Big\| \right)$$

$$\le c \left(2^{M(s-s_0)} \| f \, |F^{s_0}_{p,q}\| + 2^{(M+1)(s-s_1)} \| f \, |F^{s_1}_{p,q}\| \right)$$

for any $M \in \mathbb{N}$. The proof of (13) is the same. □

For each quasi-Banach space A there exists a number p, $0 < p \le 1$, and an equivalent quasi-norm such that

$$\| a_0 + a_1 \, |A\|^p \le \| a_0 \, |A\|^p + \| a_1 \, |A\|^p \tag{14}$$

holds for all a_0, $a_1 \in A$, cf. G. Köthe [Koe, 18.10].

Proposition 5. *Let* $d = \min(1, p, q)$.
(i) *It holds*

$$\| \sum_{j=0}^{\infty} f_j |F^s_{p,q} \|^d \le \sum_{j=0}^{\infty} \| f_j \, |F^s_{p,q} \|^d \tag{15}$$

for all $f_j \in F^s_{p,q}$, $j = 0, \ldots$.
(ii) *It holds*

$$\| \sum_{j=0}^{\infty} f_j |B^s_{p,q} \|^d \le \sum_{j=0}^{\infty} \| f_j \, |B^s_{p,q} \|^d \tag{16}$$

for all $f_j \in B^s_{p,q}$, $j = 0, \ldots$.

Proof. Both, (15) and (16), follow directly from the corresponding properties of L_p and ℓ_q. □

Invariance under diffeomorphic maps

Let

$$y = \eta(x), \quad \text{i.e. } y_j = \eta_j(x) \quad \text{if } j = 1, \dots, n, \tag{17}$$

be an infinitely differentiable one-to-one mapping from $\mathbb{R}^n$ onto $\mathbb{R}^n$. We call it a k-diffeomorphism if the components have classical derivatives up to the order k with $D^\alpha \eta_j \in C$, $0 < |\alpha| \le k$ and if $|\det \eta_*(x)| \le c > 0$ for some $c \in \mathbb{R}$ and all $x \in \mathbb{R}^n$. Here η_* stands for the Jacobian matrix. We call η a diffeomorphism if it is a diffeomorphism for any k.

Proposition 6. ([Tr 9, 4.3.1]) *Let η be a diffeomorphism. The linear mapping*

$$f(x) \longrightarrow f(\eta(x)) \tag{18}$$

yields an one-to-one mapping from $F^s_{p,q}$onto itself and from $B^s_{p,q}$onto itself, respectively.

2.1.4 Lifting property and related quasi-norms

Let

$$I_\sigma f(x) = \mathcal{F}^{-1}[(1 + |\xi|^2)^{\frac{\sigma}{2}} \mathcal{F} f(\xi)](x), \quad f \in \mathcal{S}', \ \sigma \in \mathbb{R}. \tag{1}$$

Proposition 1. ([Tr 6, 2.3.8]) (i) *I_σ maps $F^s_{p,q}$isomorphically onto $F^{s-\sigma}_{p,q}$.*
(ii) *I_σ maps $B^s_{p,q}$isomorphically onto $B^{s-\sigma}_{p,q}$.*

As usual, id denotes the identity operator and

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

denotes the Laplacian. We have

$$(-\Delta + \mathrm{id})^m f = \mathcal{F}^{-1}(1 + |\xi|^2)^m \mathcal{F} f, \quad m = 1, 2, \dots.$$

This shows that Proposition 1 describes the action of fractional powers of the operator $(-\Delta + \mathrm{id})$ on the function spaces under consideration here. The behaviour of

$$f \longrightarrow \frac{\partial^\alpha f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

will be described in the next proposition.

Proposition 2. *Suppose* $\varphi = \{\varphi_j\}_{j=0}^{\infty} \in \Phi$ *and let* $m = 1, 2, \ldots$.
(i) *Then*

$$F^s_{p,q} = \left\{ f \in \mathcal{S}' : \sum_{|\alpha| \le m} \| D^\alpha f \,|F^{s-m}_{p,q}\| < \infty \right\}, \tag{2}$$

$$F^s_{p,q} = \left\{ f \in \mathcal{S}' : \| f \,|F^{s-m}_{p,q}\| + \sum_{j=1}^{n} \| \frac{\partial^m f}{\partial x_j^m} \,|F^{s-m}_{p,q}\| < \infty \right\}, \tag{3}$$

and

$$F^s_{p,q} = \left\{ f \in \mathcal{S}' : \| \mathcal{F}^{-1}[\varphi_0 \mathcal{F} f](\cdot) \,|L_p\| + \sum_{j=1}^{n} \| \frac{\partial^m f}{\partial x_j^m} \,|F^{s-m}_{p,q}\| < \infty \right\} \tag{4}$$

in the sense of equivalent quasi-norms.
(ii) *Then*

$$B^s_{p,q} = \left\{ f \in \mathcal{S}' : \sum_{|\alpha| \le m} \| D^\alpha f \,|B^{s-m}_{p,q}\| < \infty \right\}, \tag{5}$$

$$B^s_{p,q} = \left\{ f \in \mathcal{S}' : \| f \,|B^{s-m}_{p,q}\| + \sum_{j=1}^{n} \| \frac{\partial^m f}{\partial x_j^m} \,|B^{s-m}_{p,q}\| < \infty \right\}, \tag{6}$$

and

$$B^s_{p,q} = \left\{ f \in \mathcal{S}' : \| \mathcal{F}^{-1}[\varphi_0 \mathcal{F} f](\cdot) \,|L_p\| + \sum_{j=1}^{n} \| \frac{\partial^m f}{\partial x_j^m} \,|B^{s-m}_{p,q}\| < \infty \right\} \tag{7}$$

in the sense of equivalent quasi-norms.

Proof. A proof of (2), (3), (5), and (6) may be found in [Tr 6, 2.3.8]. Formulas (4) and (7) are not directly covered by this reference. However, an obvious modification of the proof given there yields these assertions, too. □

2.1.5 Dual spaces

From Proposition 2.1.3/1 we know that $(F^s_{p,q})'$, $(B^s_{p,q})'$ can be interpreted as subspaces of $\mathcal{S}'$, provided that $\max(p, q) < \infty$. More precisely, $g \in \mathcal{S}'$ belongs to the dual of $F^s_{p,q}$ $(B^s_{p,q})$, $\max(p, q) < \infty$ if and only if there exists a constant c such that

$$| g(\varphi) | \le c \,\| \varphi \,|F^s_{p,q} \| \tag{1}$$

and

$$| g(\varphi) | \le c \,\| \varphi \,|B^s_{p,q} \| \tag{2}$$

for all $\varphi \in \mathcal{S}$, respectively. If $1 \le r \le \infty$, then r' is determined in the usual way by $\frac{1}{r} + \frac{1}{r'} = 1$. If $0 < r < 1$ we put $r' = \infty$.

Proposition. ([Tr 6, 2.11], [Mar 2], [FJ])
(i) *Let* $1 \le p < \infty$ *and* $0 < q < \infty$. *Then*

$$(B^s_{p,q})' = B^{-s}_{p',q'} . \tag{3}$$

(ii) *Let* $1 \le p < \infty$ *and* $1 \le q < \infty$. *Then*

$$(F^s_{p,q})' = F^{-s}_{p',q'} . \tag{4}$$

(iii) *Let* $0 < p < 1$ *and* $0 < q < \infty$. *Then*

$$(B^s_{p,q})' = B^{-s+n(\frac{1}{p}-1)}_{\infty,q'} . \tag{5}$$

(iv) *Let* $0 < p < 1$ *and* $0 < q < \infty$. *Then*

$$(F^s_{p,q})' = B^{-s+n(\frac{1}{p}-1)}_{\infty,\infty} . \tag{6}$$

(v) *Let* $0 < q \le 1$. *Then*

$$(F^s_{1,q})' = B^{-s}_{\infty,\infty} . \tag{7}$$

Remark 1. Also limiting cases are of interest. Let $1 \le p \le \infty$ and $0 < q \le \infty$. Then

$$(b^s_{p,q})' = B^{-s}_{p',q'} \tag{8}$$

holds, cf. [Tr 6, 2.11]. If $0 < p < 1$ we have

$$(b^s_{p,\infty})' = B^{-s+\frac{n}{p}-n}_{\infty,1} , \tag{9}$$

cf. also [Tr 6, 2.11].

Remark 2. The counterpart in case of Triebel–Lizorkin spaces reads as follows. Let $1 \le p \le \infty$ and $1 \le q \le \infty$. Then

$$(f^s_{p,q})' = F^{-s}_{p'q'} \tag{10}$$

including

$$(f^s_{1,q})' = F^{-s}_{\infty,q'} \tag{11}$$

and hence

$$(h_1)' = bmo , \tag{12}$$

cf. J. Marschall [Mar 2]. If $0 < p < 1$ then the counterpart of (9) reads as

$$(f^s_{p,\infty})' = B^{-s+\frac{n}{p}-n}_{\infty,\infty}, \tag{13}$$

cf. J. Marschall [Mar 7].

2.1.6 Supplements: Fourier multipliers and maximal inequalities

Both, Fourier multipliers and maximal inequalities represent basic tools in the theory of function spaces, even if one applies the Fourier analytic approach (and we will do that very often). Therefore, but also for better reference we recall some basic facts, cf. H. Triebel [Tr 6, Tr 9].

The Hardy–Littlewood maximal function

If $f \in L_1^{\ell oc}$, then

$$Mf(x) = \sup \frac{1}{|B|} \int_B |f(y)|\, dy, \tag{1}$$

where the supremum is taken over all balls B centered at x. We have

$$(Mf)(x) \ge |f(x)| \quad \text{for almost every}\, x \in \mathbb{R}^n \tag{2}$$

(with respect to the Lebesgue measure).

Proposition 1. ([St 2, 2.1/2.2]) *Let $1 < p < \infty$ and $1 < q < \infty$. There exists a constant c such that*

$$\|M f_k \,|L_p(\ell_q)\| \le c\, \|f_k \,|L_p(\ell_q)\| \tag{3}$$

holds for all sequences $\{f_k\}_{k=0}^{\infty}$ of complex-valued locally Lebesgue-integrable functions.

Remark 1. In addition one knows

$$\|M f_k \,|L_p(\ell_\infty)\| \le c\, \|f_k \,|L_p(\ell_\infty)\|, \tag{4}$$

$$\|M f \,|L_p\| \le c\, \|f \,|L_p\|, \tag{5}$$

where $1 < p \le \infty$. These are obvious consequences of (1) and (3). Note, that (3) with $p = 1$ and/or $q = 1$ becomes false, cf. E. M. Stein [St 2, 2.5/A1].

The Peetre–Fefferman–Stein maximal function

For $\varphi = \{\varphi_j\}_{j=0}^\infty \in \Phi$, $a > 0$ and $f \in \mathcal{S}'$ we put

$$(\varphi_j^{*,a} f)(x) = \sup_{y\in\mathbb{R}^n} \frac{|\mathcal{F}^{-1}[\varphi_j \mathcal{F} f](x-y)|}{1+|2^j y|^a}\,, \quad x \in \mathbb{R}^n\,, \quad j = 0, 1, \ldots\,. \tag{6}$$

By $\operatorname{grad} f$ we denote the gradient of f.

Proposition 2. ([Tr 6, 2.3.6]) *Let $\varphi \in \Phi$. Let $0 < r < \infty$.*
(i) *There exist two constants c_1, c_2 such that*

$$\sup_{y\in\mathbb{R}^n} \frac{|\operatorname{grad} \mathcal{F}^{-1}[\varphi_0 \mathcal{F} f](x-y)|}{1+|y|^a} \le c_1 (\varphi_0^{*,a} f)(x) \le c_2 (M\,|\,\mathcal{F}^{-1}[\varphi_0 \mathcal{F} f](\cdot)|^r)^{\frac{1}{r}}(x) \tag{7}$$

holds for all $f \in \mathcal{S}'$ and all $x \in \mathbb{R}^n$.
(ii) *Let $a > \frac{n}{\min(p,q)}$. Then there exists a constant c such that*

$$\|\, 2^{js} \varphi_j^{*,a} f \;|L_p(\ell_q)\| \le c\, \|\, f \;|F^s_{p,q}\,\| \tag{8}$$

holds for all $f \in F^s_{p,q}$.
(iii) *Let $a > \frac{n}{p}$. Then there exists a constant c such that*

$$\|\, 2^{js} \varphi_j^{*,a} f \;|\ell_q(L_p)\| \le c\, \|\, f \;|B^s_{p,q}\,\| \tag{9}$$

holds for all $f \in B^s_{p,q}$.

Remark 2. From the definition of the maximal function $\varphi_j^{*,a} f(x)$ it follows that also the reverse inequalities to (8) and (9) are true. Hence, $F^s_{p,q}$ ($B^s_{p,q}$) is the set of all tempered distributions such that $\|2^{js}\varphi_j^{*,a} f\,|L_p(\ell_q)\| < \infty$ ($\|2^{js}\varphi_j^{*,a} f\,|\ell_q(L_p)\| < \infty$), cf. [Tr 9, 2.3.2].

Fourier multipliers for L_p and $L_p(\ell_q)$

Proposition 3. ([Tr 6, 1.5, 2.4.9]) *Let $\varphi \in \Phi$ and let $\{f_j\}_{j=0}^\infty \subset \mathcal{S}'$. Suppose*

$$\operatorname{supp}\ \mathcal{F} f_j \subset \{\xi : |\xi| \le D\ 2^j\}\,,\ j = 0, 1, \ldots\,, \tag{10}$$

for some $D \ge 1$.
(i) *Let $\kappa > \sigma_{p,q}$, cf.* (2.1.3/6). *Then there exists a constant c such that*

$$\|\, \mathcal{F}^{-1}[\varphi_j\, \mathcal{F} f_j](\cdot)\,|\, L_p(\ell_q)\| \le c\, D^\kappa\, \|\, f_j\, |L_p(\ell_q)\| \tag{11}$$

holds for all sequences $\{f_j\}_{j=0}^\infty$. Here c is independent of D and $\{f_j\}_{j=0}^\infty$.

(ii) *There exists a constant c such that*

$$\| \mathcal{F}^{-1}[\varphi_j \mathcal{F} f_j](\cdot) \, |L_p\| \le c \, D^{n \cdot \max(0, \frac{1}{p}-1)} \, \| f_j \, |L_p\| \tag{12}$$

holds for all sequences $\{f_j\}_{j=0}^{\infty}$. Here c is independent of D, j, and f_j.

If one drops condition (10), one has to restrict the investigations to values of $p \ge 1$.

Proposition 4. ([Ya 1]) (i) *Let $1 < p < \infty$ and $1 \le q \le \infty$. Then there exists a constant c such that*

$$\| \mathcal{F}^{-1}[\varphi(2^{-j} \xi) \mathcal{F} f_j(\xi)](\cdot) \, |L_p(\ell_q)\| \le c \, \| f_j \, |L_p(\ell_q)\| \tag{13}$$

holds for all $\varphi \in \mathcal{S}$ and all systems $\{f_j\}_{j=0}^{\infty} \subset \mathcal{S}'$.
(ii) *Let $1 \le p \le \infty$. Then*

$$\|\mathcal{F}^{-1}[\varphi(\xi) \mathcal{F} f(\xi)](\cdot) \, |L_p\| \le c \, \| \mathcal{F}^{-1}\varphi \, |L_1\| \, \| f \, |L_p\| \tag{14}$$

holds for all $\varphi \in \mathcal{S}$ and all $f \in \mathcal{S}'$.

Fourier multipliers for $F^s_{p,q}$ and $B^s_{p,q}$

We say φ satisfies the Michlin–Hörmander condition if

$$| D^\beta \varphi(\xi) \,| \le c_\beta \, (1 + |\xi\,|)^{-|\beta|} \tag{15}$$

holds for all multi-indices β and all $\xi \in \mathbb{R}^n$ and some $c_\beta < \infty$. We say that φ satisfies the Michlin–Hörmander condition for $N \in \mathbb{N}$ if (15) holds for all $|\beta\,| < N$.

Proposition 5. ([Tr 6, 2.3.7]) *Let φ satisfy the Michlin–Hörmander condition for sufficiently large $N = N(p, q)$.*
(i) *Then φ is a Fourier multiplier in $F^s_{p,q}$. That means, there exists a constant c_φ such that*

$$\| \mathcal{F}^{-1}[\varphi \, \mathcal{F} f](\cdot) \, |F^s_{p,q} \| \le c_\varphi \, \| f \, |F^s_{p,q} \| \tag{16}$$

holds for all $f \in F^s_{p,q}$.
(ii) *Then φ is a Fourier multiplier in $B^s_{p,q}$ and*

$$\| \mathcal{F}^{-1}[\varphi \, \mathcal{F} f](\cdot) \, |B^s_{p,q} \| \le c_\varphi \, \| f \, |B^s_{p,q} \| \tag{17}$$

holds for all $f \in B^s_{p,q}$.

More complicated are the following assertions, needed in Chapter 3. Let $n \ge 2$.

Proposition 6. ([FR 3]) *Let* $\varphi \in D(\mathbb{R}^{n-1})$ *and* $A > 0$.
(i) *There exists a positive constant* c_A *such that*

$$\| (\mathcal{F}')^{-1}[\varphi(\xi')\mathcal{F}' f](\cdot) \,|F^s_{p,q}\| \le c_A \|f \,|F^s_{p,q}\| \tag{18}$$

holds for all f *with*

$$\text{supp } \mathcal{F}f \subset \left\{ \xi \in \mathbb{R}^n : |\xi'| = (\sum_{i=1}^{n-1} |\xi_i|^2)^{\frac{1}{2}} < A \right\}.$$

(ii) *There exists a positive constant* c_A *such that*

$$\| \mathcal{F}^{-1}[\varphi(\xi')\mathcal{F} f](\cdot) \,|B^s_{p,q}\| \le c_A \|f \,|B^s_{p,q}\| \tag{19}$$

holds for all f *with*

$$\text{supp } \mathcal{F}f \subset \{ \xi \in \mathbb{R}^n : |\xi'| < A \}.$$

(iii) *If* $1 \le p < \infty$ *and* $1 \le q \le \infty$, *then* $\varphi(\xi')$ *is a Fourier multiplier in* $F^s_{p,q}$.
(iv) *If* $1 \le p \le \infty$ *and* $0 < q \le \infty$, *then* $\varphi(\xi')$ *is a Fourier multiplier in* $B^s_{p,q}$.

Proof. *Step* 1. We prove (i). The proof of (ii) will be almost the same. Let $x = (x', x_n) \in \mathbb{R}^n$ and recall that $\mathcal{F}'$, $\mathcal{F}'^{-1}$ mean the Fourier transform and its inverse with respect to $\mathbb{R}^{n-1}$ and applied to x' instead of x. For notational reasons we state the following identity.

$$\mathcal{F}^{-1}[\varphi_k \mathcal{F}\{\mathcal{F}'^{-1}[\varphi \mathcal{F}' f]\}](z) = c \int_{\mathbb{R}^{n-1}} (\mathcal{F}'^{-1}\varphi)(w') f_k(z - (w', 0)) dw'$$

with $f_k = \mathcal{F}^{-1}[\varphi_k \mathcal{F} f]$. Because of

$$\text{supp } \mathcal{F}'_{w'} f_k(z - (w', 0)) = \text{supp } \int_{\mathbb{R}} e^{iz_n x_n} \varphi_k(w', x_n)(\mathcal{F} f)(w', x_n) dx_n$$
$$\subset \{ w' \in \mathbb{R}^{n-1} : |w'| < A \}$$

(here z_n is playing the role of a parameter) we may apply the maximal inequality in H. Triebel [Tr 6, 1.6.3]. This yields

$$\| \int_{\mathbb{R}^{n-1}} (\mathcal{F}'^{-1}\varphi)(w') f_k(z - (w', 0)) dw' \,|L_p(\mathbb{R}^{n-1}, z')(\ell_q)\|$$
$$\le c_A \| f_k(z', z_n) |L_p(\mathbb{R}^{n-1}, z')(\ell_q)\|$$

with c_A independent of $f \in \mathcal{S}$ and z_n. Integrating with respect to z_n and using the above identity we end up with

$$\| (\mathcal{F}')^{-1}[\varphi(\xi')\mathcal{F}' f](\cdot) \,|F^s_{p,q}\| \le c_A \|f \,|F^s_{p,q}\|$$

for all $f \in \mathcal{S}$. The general result follows by a limit procedure. If $q < \infty$ this consists simply in a density argument. If $q = \infty$ we make use of

$$\liminf_{j} \|\mathcal{F}^{-1}[\psi(2^{-j}\xi)\,\mathcal{F}f(\xi)](\cdot)\,|F^s_{p,q}\| \sim \|f\,|F^s_{p,q}\|$$

cf. H. Triebel [Tr 9, 2.4.2]. Here ψ is the function defined in 2.1.1 /(4).
Note if

$$\operatorname{supp}\,\mathcal{F}f \subset \{\xi \in \mathbb{R}^n : |\xi'| < A\}$$

then

$$\operatorname{supp}\,\mathcal{F}[\psi(2^{-j}\xi)\mathcal{F}f(\xi)](\cdot) \subset \{\xi \in \mathbb{R}^n : |\xi'| < A\}.$$

Step 2. We prove (iii). Using the Minkowski inequality we obtain

$$\Big\| \int_{\mathbb{R}^{n-1}} (\mathcal{F}'^{-1}\varphi)(y')\,f(x'-y',x_n)\,dy'\,|F^s_{p,q}\Big\|$$

$$\le c \int_{\mathbb{R}^{n-1}} |\mathcal{F}'^{-1}\varphi(y')|\,\|f(\cdot-(y',0))|F^s_{p,q}\|\,dy' \le c\,\|f\,|F^s_{p,q}\|$$

whenever $F^s_{p,q}(\mathbb{R}^n)$ is a Banach space, cf. Proposition 2.1.1/1.
Step 3. We prove (iv). The assertion follows by real interpolation, cf. 2.5.1, taking into account Step 2. □

Proposition 7. ([FR 3]) *Let $\alpha = (\alpha', 0)$ be some multi-index. Let $A > 0$ be given. Then there exists some $c > 0$ such that*

$$\|D^{(\alpha',0)}f\,|F^s_{p,q}\| \le c\,\|f\,|F^s_{p,q}\| \tag{20}$$

and

$$\|D^{(\alpha',0)}f\,|B^s_{p,q}\| \le c\,\|f\,|B^s_{p,q}\| \tag{21}$$

holds for all f satisfying

$$\operatorname{supp}\,\mathcal{F}f \subset \{\xi : |\xi| \le A\}. \tag{22}$$

Here c is not depending on A and f.

Proof. Both, (20) and (21) are immediate consequences of the preceding proposition with $\varphi(\xi') = (\xi')^\alpha\,\psi(A^{-1}\xi')$. □

Some counterexamples

First we show that there is no extension of the assertions (iii) and (iv) of the above proposition to values $p < 1$.

Lemma 1. ([FR 3]) *Let* $\omega \in D(\mathbb{R})$ *with* supp $\omega \subset (\frac{13}{8}, \frac{15}{8})$ *and let* $\varphi \in D(\mathbb{R}^{n-1})$ *with*

$$\varphi(\xi') = 1 \quad \textit{if} \quad |\xi| \le 1 \quad \textit{and} \quad \varphi(\xi') = 0 \quad \textit{if} \quad |\xi| \ge 2 .$$

It holds

$$\| \mathcal{F}^{-1} [\varphi(2^{-k} \xi')\omega(2^{-k} |\xi_n|)](\cdot) \, |B^s_{p,q} \| \sim c_1 \, 2^{k(s+n-\frac{n}{p})} \tag{23}$$

and

$$\| \mathcal{F}'^{-1}[\varphi(\xi') \, \mathcal{F}'(\mathcal{F}^{-1}(\varphi(2^{-k}\eta')\omega(2^{-k}|\eta_n|)))] \, (\cdot) \, |B^s_{p,q} \| \sim c_2 \, 2^{k(s+1-\frac{1}{p})} \tag{24}$$

where $c_1, c_2 > 0$ *are not depending on* $k \in \mathbb{N}$.

Proof. *Step* 1. We prove (23). Investigating the support of $\varphi(2^{-k}\xi') \, \omega(2^{-k}|\xi|)$ we find

$$\| \mathcal{F}^{-1}[\varphi(2^{-k}\xi')\omega(2^{-k}|\xi_n|)] \, (\cdot) \, |B^s_{p,q} \|$$

$$= \left(\sum_{j=-1}^{1} \| \, 2^{(k+j)s} \mathcal{F}^{-1} \, [\varphi_1(2^{-(k+j)}\xi)\varphi(2^{-k}\xi')\omega(2^{-k}|\xi_n|)] \, (\cdot) |L_p\|^q \right)^{\frac{1}{q}}$$

$$= \left(\sum_{j=-1}^{1} 2^{(k+j)sq} \, 2^{knq} \, 2^{-k\frac{n}{p}q} \, \|\mathcal{F}^{-1} \, [\varphi_1(2^{-j}\xi)\varphi(\xi')\omega(|\xi_n|)](\cdot) \, |L_p\|^q \right)^{\frac{1}{q}} .$$

This proves (23).
Step 2. Proof of (24). Similar as in proof of Proposition 6 we get

$$\mathcal{F}^{-1} \left\{ \varphi_j \, \mathcal{F} \left(\mathcal{F}'^{-1} \left[\varphi(\xi')\mathcal{F}(\mathcal{F}^{-1}(\varphi(2^{-k}\eta_n)\omega(2^{-k}|\eta_n|)) \right] \right) \right\} (z)$$

$$= c \, \mathcal{F}'^{-1} \, [\varphi_j(\xi) \, \varphi(\xi')\omega(2^{-k}|\xi_n|)](z) ,$$

where we used $\varphi_j(\xi) \cdot \varphi(2^{-k}\xi') = \varphi_j(\xi)$, $k = 1, 2, \ldots$. Taking into account the various conditions on the supports of φ, φ_1, ω we obtain

$$\varphi_j(\xi) \, \varphi(\xi') \, \omega(2^{-k}|\xi_n|) = \delta_{j,k} \, \varphi_k(\xi) \, \varphi(\xi') \, \omega(2^{-k}|\xi_n|)$$

making k large enough. Hence, we have derived

$$\| \mathcal{F}'^{-1}[\, \varphi(\xi') \, \mathcal{F}'(\mathcal{F}^{-1}\varphi(2^{-k}\xi') \, \omega(2^{-k}|\xi_n|))](\cdot) \, |B^s_{p,q} \, (\mathbb{R}^n)\|$$

$$= c \, 2^{k(s+1-1/p)} \, \| \mathcal{F}^{-1}[\varphi(\xi') \, \omega(|\xi_n|)](\cdot) \, |L_p(\mathbb{R}^n)\|$$

for k sufficiently large. The proof is complete. □

Remark 2. As a direct consequence of the lemma we have that Proposition 6(iv) can not be true if $p < 1$. Real interpolation (cf. Proposition 2.5.1) yields Proposition 6(iii) can not be extended to $p < 1$.

Finally, we deal with $0 < q < 1$ in Proposition 6(iii). That is more interesting as the preceding case but also more delicate. For the sake of simplicity we suppose $n = 2$.

Lemma 2. ([FR 3]) *Let $\varphi,\ \varrho \in \mathcal{S}(\mathbb{R})$ be two functions such that*

$$\operatorname{supp} \mathcal{F}_1\, \varrho \subset \left\{ \xi \ : \ |\xi| < \frac{1}{2} \right\}, \tag{25}$$

$$\mathcal{F}_1\, \varrho(\xi) = 1 \quad \textit{if} \quad |\xi| < \frac{1}{4}, \tag{26}$$

$$\operatorname{supp} \varphi \subset \left\{ \xi \ : \ |\xi| < \frac{1}{2} \right\}, \tag{27}$$

and

$$\mathcal{F}_1^{-1}\, \varphi(\xi) > 1 \quad \textit{if} \quad |\xi| < 2. \tag{28}$$

Define

$$f_k(x_1, x_2) = \sum_{j=k+1}^{2^k} \varrho(2^k x_1 - j)\, \varrho(x_2)\, e^{i\lambda_j x_2} = \sum_{j=k+1}^{2^k} f_{k,j}(x_1, x_2), \tag{29}$$

$\lambda_j = 3 \cdot 2^{j-1} + \frac{3}{4}$. *Then there exist two positive constants A, B, such that*

$$\| f_k \,|F^0_{p,q}(\mathbb{R}^2)\| \leq A < \infty \tag{30}$$

and

$$\| \mathcal{F}'^{-1}[\varphi(\xi_1)\, \mathcal{F}' f_k](\cdot) \,|F^0_{p,q}(\mathbb{R}^2)\| \geq B\, 2^{k(\frac{1}{q}-1)} \tag{31}$$

for all sufficiently large k.

Proof. *Step* 1. We prove (30). The Fourier transform of f_k reads as

$$\mathcal{F} f_k(\xi_1, \xi_2) = \sum_{j=k+1}^{2^k} 2^{-k}\, \mathcal{F}'\varrho(2^{-k}\xi_1)\, \mathcal{F}'\varrho(\xi_2 + \lambda_j)\, e^{-ij2^{-k}\xi_1}. \tag{32}$$

Employing (25), (27), (29) and making use of Proposition 3 we obtain

$$\| f_k \,|F^0_{p,q}(\mathbb{R}^2)\| \leq c \, \Bigg\| \left(\sum_{j=k+1}^{2^k} |\mathcal{F}^{-1}[\varphi_j\, \mathcal{F} f_{k,j}](\cdot)\,|^q \right)^{1/q} |L_p(\mathbb{R}^2)\Bigg\|$$

$$\leq c \, \Big\| \Big(\sum_{j=k+1}^{2^k} |\, f_{k,j}\,|^q \Big)^{1/q} \, \Big| L_p(\mathbb{R}^2) \Big\| \, .$$

Thanks to $\varrho \in \mathcal{S}(\mathbb{R})$ there exists for any $M \in \mathbb{N}$ some constant c_M such that

$$|\, \varrho(t)\,| \leq c_M \, (1 + |t|)^{-M} \, .$$

Consequently,

$$\| f_k \, |F^0_{p,q}(\mathbb{R}^2)\| \leq c \, \| \varrho \, |L_p(\mathbb{R})\| \, \Big\| \Big(\sum_{j=0}^{2^k} |\, \varrho(2^k t - j)\,|^q \Big)^{1/q} \, \Big| L_p(\mathbb{R}) \Big\| \qquad (33)$$

$$\leq c \, 2^{-k\frac{1}{p}} \left(\sum_{t=-\infty}^{\infty} \Big(\sum_{j=0}^{2^k} (1 + |t - j|)^{-Mq} \Big)^{p/q} \right)^{1/p} .$$

Because of

$$\Big(\sum_{j=0}^{2^k} (1 + |t - j|)^{-Mq} \Big)^{1/q} \leq c \begin{cases} 1 & 0 \leq t \leq 2^k \\ (\min(|t|, |t - 2^k|))^{-M+1/q} & t \not\in [0, 2^k] \end{cases}$$

the estimate (30) follows from (33).
Step 2. We prove (31). A simple calculation yields

$$\mathcal{F}'^{-1}[\varphi(\xi_1)\mathcal{F}' f_k](z_1, z_2) = \sum_{j=k+1}^{2^k} 2^{-k} \, \mathcal{F}'^{-1}[\varphi(\xi_1) e^{-i\, j 2^{-k} \xi_1}](z_1) \, \varrho(z_2) \, e^{i \lambda_j z_2}$$

using (32) and (4) provided that k is sufficiently large. This implies

$$\mathcal{F}^{-1}[\varphi_\ell \mathcal{F} \, (\mathcal{F}'^{-1}[\varphi(\xi_1)\mathcal{F}' f_k])(z) \qquad (34)$$

$$= \sum_{j=k+1}^{2^k} 2^{-k} \, \mathcal{F}^{-1}[\varphi_\ell(\xi_1, \xi_2) \, \varphi(\xi_1) \, e^{-i\, j 2^{-k} \xi_1} \, \mathcal{F}\varrho(\xi_2 + \lambda_j)](z).$$

Again we make use of (25), (27) and (8), which shows that

$$\varphi_\ell(\xi_1, \xi_2) \, \varphi(\xi_1) \, e^{-i\, j 2^{-k} \xi_1} \, \mathcal{F}\varrho(\xi_2 + \lambda_j) = \delta_{\ell,j} \, \varphi(\xi_1) \, e^{-i\, j 2^{-k} \xi_1} \, \varrho(\xi_2 + \lambda_j). \qquad (35)$$

Both, (34) and (35) lead to

$$\| \, \mathcal{F}'^{-1} \, [\varphi(\xi_1)\mathcal{F}' f_k](\cdot) \, |F^0_{p,q}(\mathbb{R}^2)\|$$

$$= \Big\| \Big(\sum_{j=k+1}^{2^k} |2^{-k} \, \mathcal{F}^{-1}[\varphi(\xi_1) e^{-i\, j 2^{-k} \xi_1} \, \varrho(\xi_2 + \lambda_j)](\cdot) \, |^q \Big)^{1/q} \, \Big| L_p(\mathbb{R}^2) \Big\|$$

$$= \| \varrho \,|L_p(\mathbb{R})\| \; \| \left(\sum_{j=k+1}^{2^k} 2^{-kq} \, |\mathcal{F}'^{-1} \varphi(z_1 - j2^{-k})|^q \right)^{1/q} |L_p(\mathbb{R})\|.$$

Consider only integration over the unit interval and applying (28) we find

$$\| \left(\sum_{j=k+1}^{2^k} 2^{-kq} \, |\mathcal{F}'^{-1} \varphi(z_1 - j2^{-k})|^q \right)^{1/q} |L_p(\mathbb{R})\|$$

$$\geq \left(\sum_{j=k+1}^{2^k} 2^{-kq} \right)^{1/q} = \left(2^k - (k+1) \right)^{1/q} 2^{-k}.$$

Making k large enough (31) follows. The proof is complete. □

Remark 3. The importance of the above Lemma 2 is twofold. On the one hand it shows that we can not extend Proposition 6(iii) to values $q < 1$ (note that the set of Fourier multipliers of $F^s_{p,q}$ does not depend on s). That is in contrast to the case of Besov spaces. There q and s do not influence the set of Fourier multipliers. So, and that is the second consequence, Lemma 2 gives us a hint that an additional q-dependence of several assertions with respect to Triebel–Lizorkin spaces (in comparison with the Besov spaces) may occur. In some sense $F^s_{p,q}$, $q < 1$, behaves like spaces $F^s_{p,q}$, $p < 1$.

2.2 Embeddings

Various embedding relations between $F^s_{p,q}$ and $B^s_{p,q}$ will play a major role in what follows, especially in Chapters 4 and 5.

2.2.1 Elementary embeddings

We collect consequences of the monotonicity of the ℓq-quasi-norms and of the convergence of geometric series.

Proposition. ([Tr 6, 2.3.2]) *Let $\varepsilon > 0$ and suppose $q_0 < q_1$. Then it holds*

$$F^s_{p,q_0} \hookrightarrow F^s_{p,q_1}, \tag{1}$$

$$F^{s+\varepsilon}_{p,\infty} \hookrightarrow F^s_{p,q}, \tag{2}$$

$$B^s_{p,q_0} \hookrightarrow B^s_{p,q_1}, \tag{3}$$

and

$$B^{s+\varepsilon}_{p,\infty} \hookrightarrow B^s_{p,q} \,. \tag{4}$$

Remark. In view of (1)–(4) one can interprete q as a fein-index. The main role is played by s and p.

2.2.2 Embeddings with constant smoothness

Theorem. ([SiTr]) (i) *We have the equivalence*

$$B^s_{p,u} \hookrightarrow F^s_{p,q} \hookrightarrow B^s_{p,v} \tag{1}$$

if and only if

$$0 < u \le \min(p,q) \quad \textit{and} \quad \max(p,q) \le v \le \infty \,. \tag{2}$$

(ii) *It holds*

$$B^0_{1,u} \hookrightarrow L_1 \hookrightarrow B^0_{1,v} \tag{3}$$

if and only if

$$0 < u \le 1 \quad \textit{and} \quad v = \infty \,. \tag{4}$$

(iii) *It holds*

$$F^0_{1,u} \hookrightarrow L_1 \tag{5}$$

if and only if

$$0 < u \le 2 \,. \tag{6}$$

Furthermore, we have

$$L_1 \not\subset F^0_{1,\infty} \,. \tag{7}$$

(iv) *It holds*

$$B^0_{\infty,u} \hookrightarrow L_\infty \hookrightarrow B^0_{\infty,v} \tag{8}$$

if and only if

$$0 < u \le 1 \quad \textit{and} \quad v = \infty \,. \tag{9}$$

Remark 1. In (8) we can replace the space L_∞ by C and the equivalence remains true.

Remark 2. By (1) and (2) we know $F^0_{1,\infty} \hookrightarrow B^0_{1,\infty}$. The assertion (7) shows that (3) can not improved by replacing $B^0_{1,\infty}$ by $F^0_{1,\infty}$.

2.2.3 Embeddings with constant differential dimension

Recall that $s - \frac{n}{p}$ is called the differential dimension both of $F^s_{p,q}$ and $B^s_{p,q}$. It is a characteristic number which plays a crucial role in the theory of these spaces, see for instance Proposition 2.1.3/3.

Theorem. ([Jaw 1, SiTr]) (i) *Let* $0 < p_0 < p < p_1$ *and suppose*

$$s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1}. \tag{1}$$

Then

$$B^{s_0}_{p_0,u} \hookrightarrow F^s_{p,q} \hookrightarrow B^{s_1}_{p_1,v} \tag{2}$$

if and only if

$$0 < u \le p \le v \le \infty. \tag{3}$$

(ii) *Let* $p < p_1$ *and suppose*

$$s - \frac{n}{p} = s_1 - \frac{n}{p_1}. \tag{4}$$

Then

$$F^s_{p,\infty} \hookrightarrow F^{s_1}_{p_1,q}. \tag{5}$$

(iii) *Let* $0 < p < 1$. *Then*

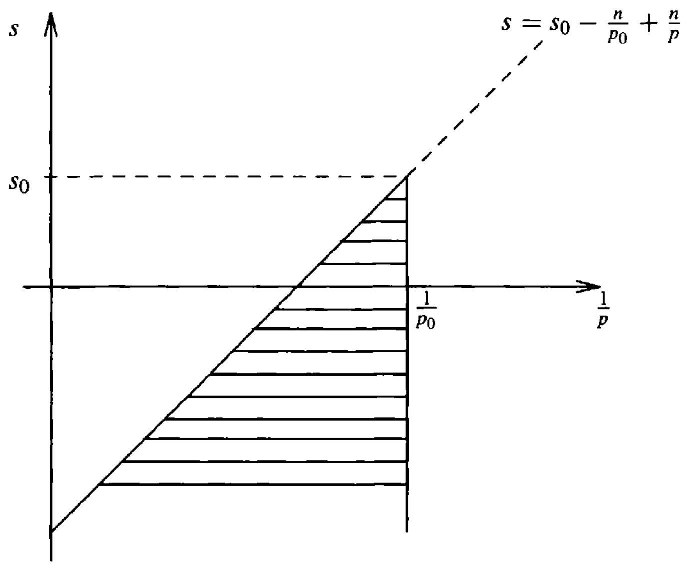
$$B^{n(\frac{1}{p}-1)}_{p,q} \hookrightarrow L_1 \tag{6}$$

if and only if

$$0 < q \le 1. \tag{7}$$

Remark 1. Of course, by the monotonicity of the $F^s_{p,q}$-spaces one may replace $F^s_{p,\infty}$ by $F^s_{p,r}$ with r arbitrary in (5).

Remark 2. Consider $F^{s_0}_{p_0,q_0}$. From Proposition 2.2.1 and the above theorem, in particular (5), we get $F^{s_0}_{p_0,q_0} \hookrightarrow F^s_{p,q}$ if $s_0 - \frac{n}{p_0} \ge s - \frac{n}{p}$ and if $p_0 \le p$, cf. the figure. Similarily in case of the Besov spaces $B^{s_0}_{p_0,q_0}$ (with exception of the line $s = s_0 - \frac{n}{p_0} + \frac{n}{p}$).



Remark 3. To complete the picture we state the following embeddings containing $F^s_{\infty,q}$. We have

$$B^{s+n/p}_{p,\infty} \hookrightarrow F^s_{\infty,q}, \tag{8}$$

$0 < p < \infty$ and

$$B^s_{\infty,q} \hookrightarrow F^s_{\infty,q} \hookrightarrow F^s_{\infty,\infty} = B^s_{\infty,\infty}, \quad 0 < q < \infty, \tag{9}$$

cf. [Mar 2, Mar 7].

Remark 4. Suppose (1). Then the embedding $B^{s_0}_{p_0,u} \hookrightarrow B^{s_1}_{p_1,u}$ becomes a consequence of the so-called Nikol'skij inequality, cf. [Tr 6, 1.3.2]: let $f \in L_{p_1}$ be a function such that

$$\operatorname{supp} \mathcal{F} f \subset \{\xi : |\xi| \le A\}$$

for some $A > 0$. Then there exists some constant c, independent of f and A such that

$$\|f\,|L_{p_1}\| \le c\, A^{n(\frac{1}{p_0}-\frac{1}{p_1})}\, \|f\,|L_{p_0}\| \tag{10}$$

holds.

2.2.4 Embeddings in L_∞, $L_1^{\ell oc}$ and L_p

Theorem 1. ([SiTr]) (i) *The following three assertions are equivalent :*

(a) $$F^s_{p,q} \hookrightarrow L_\infty, \tag{1}$$

(b) $$F^s_{p,q} \hookrightarrow C\,, \tag{2}$$

(c) $$\textit{either} \quad s > \frac{n}{p} \quad \textit{or} \quad s = \frac{n}{p} \quad \textit{and} \quad 0 < p \le 1\,. \tag{3}$$

(ii) *The following three assertions are equivalent :*

(a) $$B^s_{p,q} \hookrightarrow L_\infty\,, \tag{4}$$

(b) $$B^s_{p,q} \hookrightarrow C\,, \tag{5}$$

(c) $$\textit{either} \quad s > \frac{n}{p} \quad \textit{or} \quad s = \frac{n}{p} \quad \textit{and} \quad 0 < q \le 1\,. \tag{6}$$

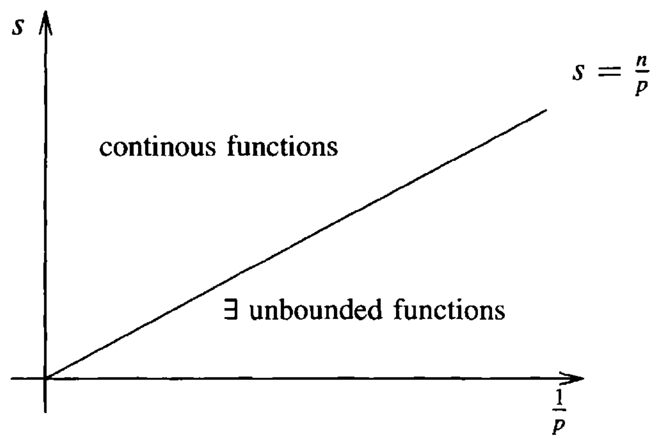


Fig. 1

Remark 1. In addition to (4) note that

$$B^0_{\infty,q} \hookrightarrow bmo \hookrightarrow B^0_{\infty,\infty} \tag{7}$$

$0 < q \le 2$ and

$$W^m_1 \hookrightarrow L_\infty \tag{8}$$

if $\quad m \ge n\,.$

In the next theorem we shall interprete $L_1^{\ell oc}$ as the set of regular distributions.

Theorem 2. ([SiTr]) (i) *The following two assertions are equivalent :*

(a) $$F^s_{p,q} \subset L_1^{\ell oc}\,, \tag{9}$$

(b) $$\textit{either} \quad 0 < p < 1, \quad s \ge \sigma_p\,, \quad 0 < q \le \infty\,, \tag{10}$$

$$\textit{or} \quad 1 \le p < \infty, \quad s > 0, \quad 0 < q \le \infty\,, \tag{11}$$

$$\textit{or} \quad 1 \le p < \infty, \quad s = 0, \quad 0 < q \le 2\,. \tag{12}$$

(ii) *The following two assertions are equivalent :*

(a) $$B^s_{p,q} \subset L_1^{\ell oc}\,, \tag{13}$$

(b) $$\textit{either} \quad 0 < p \le \infty, \quad s > \sigma_p\,, \quad 0 < q \le \infty\,, \tag{14}$$

$$\textit{or} \quad 0 < p \le 1, \quad s = n\left(\frac{1}{p} - 1\right), \quad 0 < q \le 1\,, \tag{15}$$

$$1 < p \le \infty, \quad s = 0, \quad 0 < q \le \min(p,2)\,. \tag{16}$$

Remark 2. For short, if $s > \sigma_p$ then $(F^s_{p,q} \cup B^s_{p,q}) \subset L_1^{\ell oc}$ holds, whereas in case $s < \sigma_p$ singular distributions are contained in $F^s_{p,q}$and $B^s_{p,q}$, cf. Fig. 2.

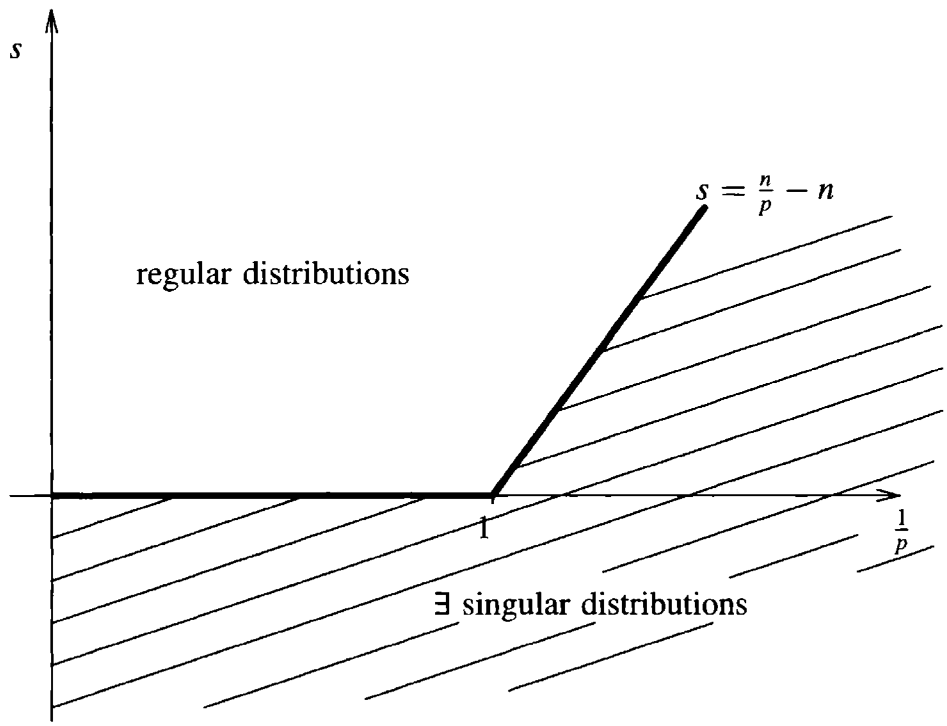


Fig. 2

Remark 3. Let δ be the Dirac distribution. Then

$$\delta = \sum_{j=0}^{\infty} \mathcal{F}^{-1}[\varphi_j \mathcal{F}\delta](\cdot) = (\frac{1}{2\pi})^{n/2} \sum_{j=0}^{\infty} \mathcal{F}^{-1}\varphi_j(\cdot)$$

with $\varphi = \{\varphi_j\}_j \in \Phi$. Choosing φ to be the system from (2.1.1/5) we get

$$\| \mathcal{F}^{-1}[\varphi_j \mathcal{F}\delta](\cdot) \, |L_p\| = c \, \| 2^{jn} \mathcal{F}^{-1}\varphi_1(2^j \cdot) \, |L_p\| = c \, 2^{jn(1-\frac{1}{p})} \| \mathcal{F}^{-1}\varphi_1 \, |L_p\|\,, \tag{17}$$

$j = 1, \ldots$. Directly from (17) it follows

$$\delta \in B^s_{p,q} \quad \text{if and only if} \quad \text{either} \quad s < \frac{n}{p} - n \quad \text{or} \quad s = \frac{n}{p} - n \quad \text{and} \quad q = \infty\,,$$

which illustrates the restrictions on s appearing in the above theorem if $p \leq 1$. To prove (15) only a small modification of these arguments is needed, cf. [SiTr] for details.

Remark 4. Let $1 < p < q \leq \infty$ and $s = 0$. The construction of a singular distribution g belonging to $B^0_{p,q}$ is less obvious than in the previous case. We start this construction with a smooth, non-trivial function f, supported around the origin, $\int f(x)\,dx = 0$ and $|f(x)| \leq 1$. Let $\sigma > 1$. Define $\kappa_0 = 0$ and $\kappa_j = \sum_{\ell=1}^{j} \ell^{-1} (\log(\ell+1))^{-\sigma}$. Since $\sigma > 1$ there exists a limit κ of this sequence $\{\kappa_j\}_j$. By R_j, $j = 1, 2, \dots$, we denote the cube

$$\{x = (x_1, \dots, x_n) : \kappa_{j-1} < x_1 \leq \kappa_j,\ 0 < x_\ell < 1,\ \ell = 2, \dots, n\}.$$

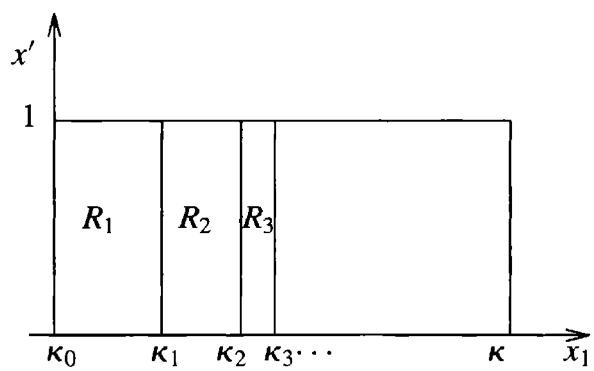


Fig. 3

Next we subdivide R_j in

$$N_j = 2^{j(n-1)} \left[2^j j^{-1} (\log(j+1))^{-\sigma} \right]$$

([ ] integer part) cubes of side-length 2^{-j} and denote the centre by $x^{j,r}$. Then the announced singular distribution $g \in B^0_{p,q}$ is given by

$$g = \sum_{j=1}^{\infty} \sum_{r=1}^{N_j} (\log(j+1))^{\sigma} f(2^{j+1}x - x^{j,r}) . \tag{18}$$

We omit the details and refer to [SiTr].

For the readers convenience and better reference we shall formulate some further consequences of the above embedding relations and of Subsection 2.1.2. Sometimes we shall use the generic notation $A^s_{p,q}$ in place of $F^s_{p,q}$and $B^s_{p,q}$.

Corollary 1. ([SiTr]) (i) *Let $p < \infty$. Then the following two assertions are equivalent:*

$$\text{(a)} \qquad A^s_{p,q} \subset L_1^{\ell oc}, \tag{19}$$

$$\text{(b)} \qquad A^s_{p,q} \hookrightarrow L_{\overline{p}}, \quad \overline{p} = \max(p, 1). \tag{20}$$

(ii) *The following two assertions are equivalent:*

$$\text{(a)} \qquad B^s_{\infty,q} \subset L_1^{\ell oc}, \tag{21}$$

$$\text{(b)} \qquad B^s_{\infty,q} \hookrightarrow bmo. \tag{22}$$

Corollary 2. ([SiTr]) *Let $p < p_1 < \infty$ and suppose $p_1 \geq 1$.*
(i) *We have the equivalence of the following two assertions:*

$$\text{(a)} \qquad F^s_{p,q} \hookrightarrow L_{p_1}, \tag{23}$$

$$\text{(b)} \qquad s \geq n\left(\frac{1}{p} - \frac{1}{p_1}\right). \tag{24}$$

(ii) *We have the equivalence of the following two assertions:*

$$\text{(a)} \quad B^s_{p,q} \hookrightarrow L_{p_1}, \tag{25}$$

$$\text{(b)} \quad \textit{either} \quad s > n\left(\frac{1}{p} - \frac{1}{p_1}\right) \quad \textit{or} \quad s = n\left(\frac{1}{p} - \frac{1}{p_1}\right) \quad \textit{and} \quad q \leq p_1. \tag{26}$$

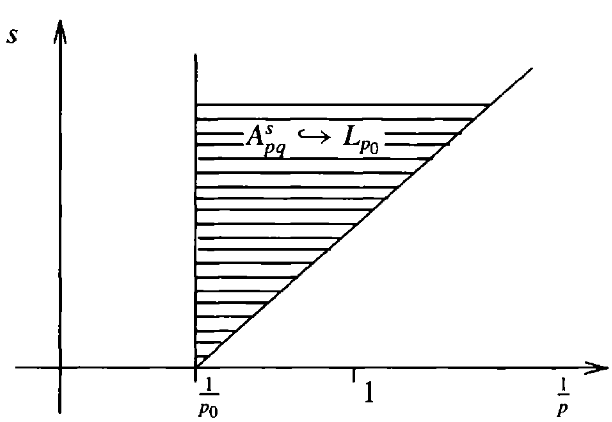


Fig. 4

It remains to clarify what happens in case $p_1 < 1$. In general, the topologies in $\mathcal{S}'$ and L_p are incompatible. To see this consider the sequence

$$f_k = \frac{1}{|Q_k|}\chi_{Q_k}, \; k = 1, 2, \ldots,$$

where Q_k is the cube

$$Q_k = \{x = (x_1, \ldots, x_n) : |x_i| \le \frac{1}{k}, \ i = 1, \ldots, n\}$$

and χ_{Q_k} denotes the corresponding characteristic function. For $k \to \infty$ we have $f_k \rightharpoonup \delta$ in $\mathcal{S}'$. On the contrary, $\| f_k \, | L_p \| = |Q_k|^{\frac{1}{p}-1} \to 0$ as $k \to \infty$, and hence $f_k \to 0$ in L_p. Corollary 2 and Theorem 2.2.3 offer us the possibility to interprete $F^s_{p,q}$ and $B^s_{p,q}$ as subspaces of L_1 provided that $p < 1$ and $s > \frac{n}{p} - n$. Making use of these arguments the following result is known.

Lemma. ([Tr 6, 2.5.3]) *Let* $0 < p < 1$ *and* $s > \frac{n}{p} - n$.
(i) *There exists a constant* c *such that*

$$\| f \, | L_p \| \le c \, \| f \, | F^s_{p,q} \| \tag{27}$$

holds for all $f \in F^s_{p,q}$.
(ii) *There exists a constant* c *such that*

$$\| f \, | L_p \| \le c \, \| f \, | B^s_{p,q} \| \tag{28}$$

holds for all $f \in B^s_{p,q}$.

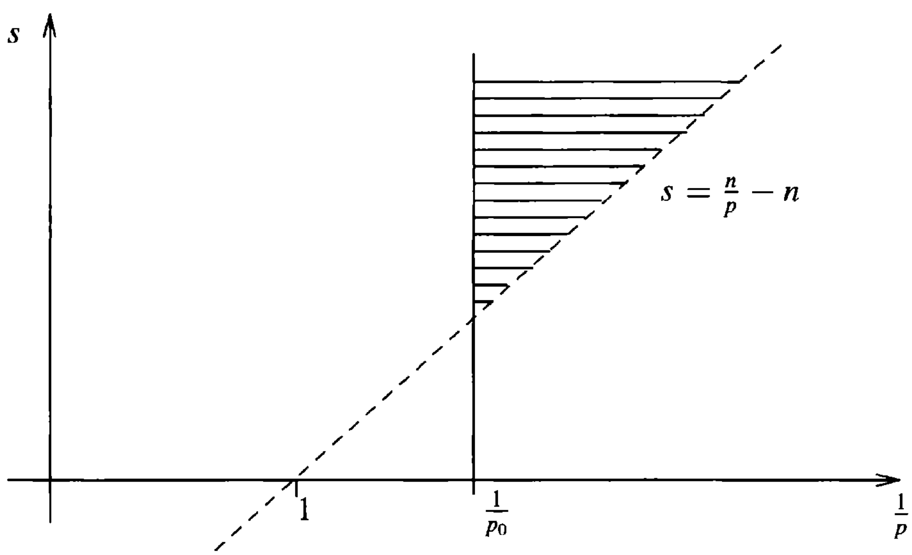


Fig. 5

2.2.5 Embeddings for spaces of bounded functions

Spaces of type $F^s_{p,q} \cap L_\infty$ and $B^s_{p,q} \cap L_\infty$ will play an important role in what follows. We shall need some refinement of the embeddings using this additional information. Recall $L_\infty \hookrightarrow B^0_{\infty,\infty}$, cf. (2.2.2/8).

Theorem. ([Ru 3]) *Let $0 < \Theta < 1$ and suppose $s > 0$.*
(i) *There exists a constant c such that*

$$\| f \,|F^{\Theta s}_{\frac{p}{\Theta},r}\| \le c \, \| f \,|F^s_{p,q}\|^{\Theta} \, \| f \,|B^0_{\infty,\infty}\|^{1-\Theta} \tag{1}$$

holds for all $f \in F^s_{p,q} \cap B^0_{\infty,\infty}$.
(ii) *There exists a constant c such that*

$$\| f \,|B^{\Theta s}_{\frac{p}{\Theta},\frac{q}{\Theta}}\| \le c \, \| f \,|B^s_{p,q}\|^{\Theta} \, \| f \,|B^0_{\infty,\infty}\|^{1-\Theta} \tag{2}$$

holds for all $f \in B^s_{p,q} \cap B^0_{\infty,\infty}$.

Proof. Step 1. (Proof of (i)). Recall

$$\int_{\mathbb{R}^n} |g(x)|^p \, dx = p \int_0^\infty t^{p-1} \, \big| \{x : \, |g(x)| > t \} \big| \, dt \,,$$

where $\big|\{x : \ldots\}\big|$ denotes the Lebesgue measure of the set $\{x : \ldots\}$. Writing

$$\| f \,|F^{\Theta s}_{\frac{p}{\Theta},r}\|^{\Theta/p} = \frac{p}{\Theta} \int_0^\infty t^{\frac{p}{\Theta}-1} \left| \left\{ x : \, \Big(\sum_{k=0}^\infty |2^{k\Theta s} \mathcal{F}^{-1}[\varphi_k \mathcal{F} f](x)|^r \Big)^{1/r} > t \right\} \right| dt \tag{3}$$

the idea of the proof consists in to split the sum over k in dependence of t. For simplicity we assume $\| f \,|B^0_{\infty,\infty}\| = 1$ (the general result follows then by a homogeneity argument). By assumption we have

$$\Big(\sum_{k=0}^K 2^{k\Theta s r} \, |\mathcal{F}^{-1}[\varphi_k \mathcal{F} f](x)|^r \Big)^{1/r} \tag{4}$$

$$\le c \, 2^{K\Theta s} \Big(\sup_k \sup_x |\mathcal{F}^{-1}[\varphi_k \mathcal{F} f](x)| \Big) \le c \, 2^{K\Theta s} \,,$$

where c is independent of $K \ge 0$ and f. We choose K to be the largest natural number such that

$$c \, 2^{K\Theta s} \le \frac{t}{2} \tag{5}$$

(c is the constant from (4)). Now we split the integral in (3) into two parts, one over $(0, 2c)$ and one over $(2c, \infty)$, where c is again determined by (4). Suppose $t > 2c$. It follows from $\Theta s < s$

$$\left| \left\{ x : \, \Big(\sum_{k=0}^\infty |2^{k\Theta s} \mathcal{F}^{-1}[\varphi_k \mathcal{F} f](x)|^r \Big)^{1/r} > t \right\} \right|$$

$$\leq \left| \left\{ x : \left(\sum_{k=K+1}^{\infty} | 2^{k\Theta s} \mathcal{F}^{-1}[\varphi_k \mathcal{F} f](x) |^r \right)^{1/r} > \frac{t}{2} \right\} \right|$$

$$\leq \left| \left\{ x : \sup_{k=0,1,\ldots} 2^{ks} | \mathcal{F}^{-1}[\varphi_k \mathcal{F} f](x) | > \frac{t}{2} 2^{Ks(1-\Theta)} \right\} \right| . \tag{6}$$

Using (5) we derive $\frac{t}{2} 2^{Ks(1-\Theta)} \geq c' t^{1/\Theta}$, where c' is independent of t, K and f. That leads to

$$\int_{2c}^{\infty} t^{\frac{p}{\Theta}-1} | \{x : \ldots\} | \, dt$$

$$\leq C \int_{c'(2c)^{1/\Theta}}^{\infty} v^{p-1} | \{x : \sup_{k=0,1,\ldots} 2^{ks} | \mathcal{F}^{-1}[\varphi_k \mathcal{F} f](x) | > v \} | \, dv$$

$$\leq C \| f \, | F^s_{p,\infty} \|^p . \tag{7}$$

It remains to estimate $\int_0^{2c}$. Because of $\frac{p}{\Theta} - p > 0$ we have

$$\int_0^{2c} t^{\frac{p}{\Theta}-1} | \{x : \ldots\} | \, dt$$

$$\leq C \int_0^{2c} t^{p-1} | \{x : \sup_{k=0,1,\ldots} 2^{ks} | \mathcal{F}^{-1}[\varphi_k \mathcal{F} f](x) | > t \} | \, dt$$

$$\leq C \| f \, | F^s_{p,\infty} \|^p . \tag{8}$$

From (7), (8) and a homogeneity argument the desired estimate (1) follows.
Step 2. Formula (2) is obtained from

$$\| f \, | B^{\Theta s}_{\frac{p}{\Theta}, \frac{q}{\Theta}} \| = \left(\sum_{k=0}^{\infty} 2^{ksq} \left(\int | \mathcal{F}^{-1}[\varphi_k \mathcal{F} f](x) |^{p/\Theta} \, dx \right)^{q/p} \right)^{\Theta/q}$$

$$\leq \left(\sum_{k=0}^{\infty} 2^{ksq} \left(\int | \mathcal{F}^{-1}[\varphi_k \mathcal{F} f](x) |^p \, \| \mathcal{F}^{-1}[\varphi_k \mathcal{F} f](\cdot) \, | L_\infty \|^{\frac{p}{\Theta}-p} \, dx \right)^{q/p} \right)^{\Theta/q}$$

$$\leq \| f \, | B^s_{p,q} \|^\Theta \, \| f \, | B^0_{\infty,\infty} \|^{1-\Theta} .$$

The proof is complete. □

Remark 1. Let $s > 0$ and $0 < \Theta < 1$. Weaker versions of the above theorem are

$$F^s_{p,q} \cap bmo \hookrightarrow F^{\Theta s}_{\frac{p}{\Theta}, r} , \tag{9}$$

$$B^s_{p,q} \cap bmo \hookrightarrow B^{\Theta s}_{\frac{p}{\Theta}, \frac{q}{\Theta}} , \tag{10}$$

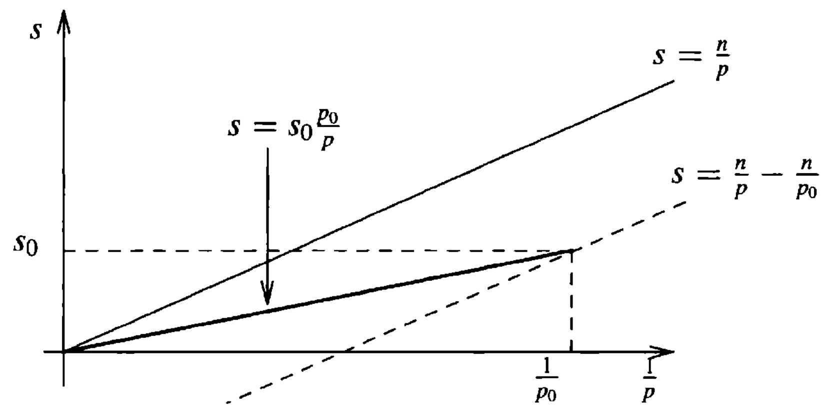
and

$$F^s_{p,q} \cap L_\infty \hookrightarrow F^{\Theta s}_{\frac{p}{\Theta},r}\,, \tag{11}$$

$$B^s_{p,q} \cap L_\infty \hookrightarrow B^{\Theta s}_{\frac{p}{\Theta},\frac{q}{\Theta}}\,, \tag{12}$$

cf. Remark 2.2.4/1 and Theorem 2.2.2/(iv).

Remark 2. Of course, only under the additional assumption $s < n/p$ the inequalities (1), (2), (9)–(12) are improvements of Theorem 2.2.3, cf. the figure.



Remark 3. Recall $\varphi_k^{*,a} f$ denotes the Peetre–Fefferman–Stein maximal function, cf. (2.1.6/6). The same proof as above yields

$$\| 2^{k\Theta s}\, \varphi_k^{*,a} f \;|L_{\frac{p}{\Theta}}(\ell_r)\| \le c \,\| 2^{ks}\, \varphi_k^{*,a} f \;|L_p(\ell_\infty)\|^{\Theta}\; \| \varphi_k^{*,a} f \;|L_\infty(\ell_\infty)\|^{1-\Theta} \tag{13}$$

if $s > 0$, $0 < \Theta < 1$, and $a > 0$. The only point, where we used special properties of $\mathcal{F}^{-1}[\varphi_k \mathcal{F} f](x)$ was inequality (4), which is obviously fulfiled with $\varphi_k^{*,a} f$ instead of $\mathcal{F}^{-1}[\varphi_k \mathcal{F} f](x)$, whenever $a > 0$.

Remark 4. We repeat the arguments used in Remark 3, now with the Hardy-Littlewood maximal function $M\, |\mathcal{F}^{-1}[\varphi_k \mathcal{F} f](\cdot)|(x)$ instead of $\mathcal{F}^{-1}[\varphi_k \mathcal{F} f](x)$. Formula (4) has a direct counterpart, cf. Remark 2.1.6/1. Consequently, we have

$$\begin{aligned} &\| 2^{k\Theta s} \Big(M\, |\mathcal{F}^{-1}[\varphi_k \mathcal{F} f]\,|^{\alpha}\Big)^{1/\alpha} \;|L_{\frac{p}{\Theta}}(\ell_r)\| \\ &\qquad \le c \,\| 2^{ks} \Big(M\, |\mathcal{F}^{-1}[\varphi_k \mathcal{F} f\;|^{\alpha}\Big)^{1/\alpha} \;|L_p(\ell_\infty)\|^{\Theta}\; \| \\ &\qquad\qquad \Big(M\, |\mathcal{F}^{-1}[\varphi_k \mathcal{F} f\;|^{\alpha}\Big)^{1/\alpha} \;|L_\infty(\ell_\infty)\|^{1-\Theta} \end{aligned} \tag{14}$$

if $s > 0$, $0 < \Theta < 1$ and $\alpha > 0$.

2.3 Some equivalent characterizations of $F^s_{p,q}$ and $B^s_{p,q}$

2.3.1 Characterizations by differences and some representatives of $F^s_{p,q}$ and $B^s_{p,q}$

Recall that Δ_h^m, σ_p and $\sigma_{p,q}$ have been defined in (2.1.2/8), (2.1.3/5) and (2.1.3/6), respectively.

Theorem. ([Tr 9, 3.5.3]) *Let M be a natural number.*
(i) *Suppose*

$$\sigma_{p,q} < s < M . \tag{1}$$

Then

$$F^s_{p,q} = \Big\{ f \in L_{\max(1,p)} : \tag{2}$$

$$\| f \,|L_p\| + \Big\| \Big(\int_0^1 t^{-sq} \big(\frac{1}{t^n} \int_{|h|<t} |\Delta_h^M f(x)|\, dh \big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} |L_p \Big\| < \infty \Big\}$$

in the sense of equivalent quasi-norms.
(ii) *Suppose*

$$\sigma_p < s < M . \tag{3}$$

Then

$$B^s_{p,q} = \Big\{ f \in L_{\max(1,p)} : \ \| f \,|L_p\| + \Big(\int |h|^{-sq} \|\Delta_h^M f \,|\, |L_p\|^q \frac{dh}{|h|^n} \Big)^{\frac{1}{q}} < \infty \Big\} \tag{4}$$

in the sense of equivalent quasi-norms. In (4) *the term* $(\int \ldots dh)$ *may be replaced by* $(\int_{\{h:|h|<\varepsilon\}} \ldots)$ *for any* $\varepsilon > 0$.

Remark 1. Sometimes we will find it convenient to replace the ball-means $(\frac{1}{t^n} \int_{|h|\le t} \ldots)$ by means over cubes in (2).

Further, we collect some properties of differences we shall need in Chapter 3. Therefore, let $k = 1, \ldots, n$ and $h > 0$. Then we put

$$\Delta_{k,h} f(x) = \frac{f(x_1, \ldots, x_k + h, \ldots, x_n) - f(x)}{h} . \tag{5}$$

Proposition. ([FR 3]) *Let $0 < h < 1$ and $k \in \{1, \ldots, n\}$.*
(i) *There exists a constant c such that*

$$\|\Delta_{k,h} f \,|F^{s-1}_{p,q}\| \le c \|f \,|F^s_{p,q}\| \tag{6}$$

and

$$\|\Delta_{k,h} f - \frac{\partial f}{\partial x_k} |F^{s-2}_{p,q}\| \leq c\, h\, \|f\, |F^s_{p,q}\| \tag{7}$$

holds for all $f \in F^s_{p,q}$ *and all* h.

(ii) *There exists a constant c such that*

$$\|\Delta_{k,h} f\, |B^{s-1}_{p,q}\| \leq c \|f\, |B^s_{p,q}\| \tag{8}$$

and

$$\|\Delta_{k,h} f - \frac{\partial f}{\partial x_k} |B^{s-2}_{p,q}\| \leq c\, h\, \|f\, |B^s_{p,q}\| \tag{9}$$

holds for all $f \in B^s_{p,q}$ *and all* h.

Proof. Step 1. We prove (6), the proof of (8) may be derived similarly. Recall $\varphi^{*,a}_j f$ has been defined in (2.1.6/6). Observe

$$\Delta_{k,h} f(x) = \sum_{j=0}^{\ell} (\Delta_{k,h} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f])(x) + \sum_{j=\ell+1}^{\infty} (\Delta_{k,h} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f])(x)\,.$$

Let $2^{-(\ell+1)} < h \leq 2^{-\ell}$. Then the mean-value theorem, the maximal inequality (2.1.6/7) together with a homogeneity argument imply

$$|\,(\Delta_{k,h} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f])(x)\,| = \left| \frac{\partial \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]}{\partial x_k}(x_1, \ldots, x_k + \lambda h, \ldots, x_n) \right|$$

$$\leq c\,(1 + |2^j \lambda h|^a) \sup_{y \in \mathbb{R}^n} \frac{\left| \frac{\partial \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]}{\partial x_k}(x-y) \right|}{1 + |2^j y|^a} \leq c\, 2^j\, (\varphi^{*,a}_j f)(x)$$

provided that $\ell \geq j$ and $\lambda \in (0,1)$ is some number depending on x, k, h and j. If $a > n/\min(p,q)$ then we can continue with (2.1.6/8) which leads to

$$\|\, \{2^{j(s-1)}\, \Delta_{k,h} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]\}_{j=0}^{\ell}\, |L_p(\ell_q)\| \leq c \|f\, |F^s_{p,q}\|\,. \tag{10}$$

We estimate the second summand. It holds

$$\begin{aligned} &\|\, \{2^{j(s-1)}\ \Delta_{k,h} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]\, \}_{j=\ell+1}^{\infty}\, |L_p(\ell_q)\| \\ &\quad \leq c\, h^{-1}\, \|\, \{2^{j(s-1)}\, \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]\, \}_{j=\ell+1}^{\infty}\, |L_p(\ell_q)\| \\ &\quad \leq c\, 2^{\ell+1}\, \|\, \{2^{j(s-1)}\, \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]\, \}_{j=\ell+1}^{\infty}\, |L_p(\ell_q)\| \leq c \|f\, |F^s_{p,q}\|\,. \end{aligned} \tag{11}$$

Now (10) and (11) prove (6).

Step 2. Again we are concentrated on (7), the proof of (9) can be done complete similarly. As above, if $j \le \ell$ then

$$\begin{aligned} &\left| (\Delta_{k,h}\ \mathcal{F}^{-1}[\varphi_j \mathcal{F} f])(x) - \frac{\partial \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]}{\partial x_k}(x) \right| \\ &\quad = \frac{h}{2} \left| \frac{\partial^2 \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]}{\partial x_k^2}(x_1, \ldots, x_k + \lambda h, \ldots, x_n) \right| \le c\, 2^{2j}\, \varphi_j^{*,a} f(x)\,. \end{aligned} \tag{12}$$

This estimate can be complemented as in (10). Based on Step 1 and the lifting property, cf. Proposition 2.1.4, we derive

$$\begin{aligned} &\| \left\{ 2^{j(s-2)} \left(\Delta_{k,h} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]\right) - \frac{\partial \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]}{\partial x_k} \right\}_{j=\ell+1}^{\infty} | L_p(\ell_q) \| \\ &\le c \| \left\{ 2^{j(s-2)} \left(\Delta_{k,h} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]\right) \right\}_{j=\ell+1}^{\infty} | L_p(\ell_q) \| \\ &\qquad\qquad + \| \left\{ 2^{j(s-2)} \frac{\partial \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]}{\partial x_k} \right\}_{j=\ell+1}^{\infty} | L_p(\ell_q) \| \\ &\le c \| \left\{ 2^{j(s-1)} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]) \right\}_{j=\ell+1}^{\infty} | L_p(\ell_q) \| \\ &\qquad\qquad + \| \left\{ 2^{j(s-1)} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f] \right\}_{j=\ell+1}^{\infty} | L_p(\ell_q) \| \\ &\le c\, 2^{-\ell}\, \| 2^{js} \mathcal{F}^{-1}[\varphi_j \mathcal{F} f]\, | L_p(\ell_q) \|\,, \end{aligned}$$

which completes the proof of (7). □

Representatives of $F^s_{p,q}$ and $B^s_{p,q}$

To make things more transparent, it is often helpful to consider some model functions. To this end we investigate a series of examples. Most of the calculations are based on Theorem 2.3.1.

Example 1. Functions with a local singularity. We put

$$f_{\alpha,\delta}(x) = \varrho(x)\, |x|^{\alpha}\, (-\log |x|)^{-\delta} \tag{13}$$

where $\alpha^2 + \delta^2 > 0$, $\delta \ge 0$ and ϱ is a smooth cut-off function with supp $\varrho \subset \{x : |x| \le \vartheta\}$, $\vartheta > 0$ sufficiently small. That means, we are interested in a local singularity at the origin.

Lemma 1. *Let $s > \sigma_p$ and suppose $\alpha \neq 0$.*
(i) *Let $\delta > 0$. Then*

$$f_{\alpha,\delta} \in B^s_{p,q} \quad \text{if and only if either} \quad s < \frac{n}{p} + \alpha \quad \text{or} \quad s = \frac{n}{p} + \alpha \ \text{ and } \ q\delta > 1\,. \tag{14}$$

(ii) *We have*

$$f_{\alpha,0} \in B^s_{p,q} \quad \text{if and only if either} \quad s < \frac{n}{p} + \alpha \quad \text{or} \quad s = \frac{n}{p} + \alpha \ \text{ and } \ q = \infty\,. \tag{15}$$

(iii) *Let $\delta > 0$. Then*

$$f_{\alpha,\delta} \in F^s_{p,q} \quad \text{if and only if either} \quad s < \frac{n}{p} + \alpha \quad \text{or} \quad s = \frac{n}{p} + \alpha \ \text{ and } \ \delta p > 1\,. \tag{16}$$

(iv) *We have*

$$f_{\alpha,0} \in F^s_{p,q} \quad \text{if and only if} \quad s < \frac{n}{p} + \alpha\,. \tag{17}$$

Proof. Step 1. Proof of sufficiency in (i) and (ii). To begin with let $\delta > 0$. Let M be a natural number large enough. Let $M\,|h| < \frac{1}{4}$. It holds

$$\int\limits_{|x| \le 2M|h|} |\Delta_h^M f_{\alpha,\delta}(x)|^p\, dx \le c \int\limits_0^{3M|h|} r^{\alpha p}\, r^{n-1} (-\log r)^{-\delta p}\, dr$$
$$\le c\, |h|^{\alpha p + n} (-\log |h|)^{-\delta p}\,, \tag{18}$$

where c is independent of h.
Using

$$|\Delta_h^M f_{\alpha,\delta}(x)| \le c\, |h|^M \max_{|\gamma| = M} \sup_{|x-y| \le M|h|} |D^\gamma f_{\alpha,\delta}(y)| \tag{19}$$

if $0 \notin \{y : |x - y| \le M\,|h|\}$
and

$$|D^\gamma f_{\alpha,\delta}(x)| \le c\, |x|^{\alpha - M} (-\log |x|)^{-\delta} \quad , |\gamma| = M \ge 1\,, \tag{20}$$

we obtain

$$\int\limits_{|x| > 2M|h|} |\Delta_h^M f_{\alpha,\delta}(x)|^p\, dx \tag{21}$$
$$\le c \sum_{j=1}^{j_0} \int\limits_{2jM|h|}^{2(j+1)M|h|} |h|^{Mp} (jM\,|h|)^{(\alpha - M)p}\, |\log(jM\,|h|)\,|^{-\delta p}\, r^{n-1}\, dr$$

$$\leq c\,|h|^{\alpha p+n}\sum_{j=1}^{j_0} j^{n-1+(\alpha-M)p}|\log(jM|h|)|^{-\delta p}\,,$$

where j_0 is defined to be the smallest integer such that

$$2(j_0+1)M\,|h|>\vartheta. \tag{22}$$

Since $|h|\leq\varepsilon$ (cf. Theorem 2.3.1(ii)) we may assume $2(j_0+1)M\,|h|<\frac{1}{2}$ for sufficiently small $\varepsilon>0$. Note that (22) guarantees

$$\sum_{j=1}^{j_0} j^{n-1+(\alpha-M)p}\,|\log(jM|h|)\,|^{-\delta p}\leq c\,|\,\log|h|\,|^{-\delta p} \tag{23}$$

for sufficiently large M. Again c is independent of h.
Hence, (21) and (23) leed to

$$\int\limits_{|x|>2M|h|}|\Delta_h^M f_{\alpha,\delta}(x)|^p\,dx\leq c\,|h|^{\alpha p+n}(-\log|h|)^{-\delta p} \tag{24}$$

Consequently, (18) and (24) prove

$$\int\limits_{|h|<\varepsilon}|h|^{-(\frac{n}{p}+\alpha)q}\|\Delta_h^M f_{\alpha,\delta}|L_p\|^q\,\frac{dh}{|h|^n}\leq c\int\limits_0^\varepsilon r^{-1}\,|\log r|^{-\delta q}\,dr<\infty \tag{25}$$

if $\delta q>1$. This proves sufficiency in (i). By similar arguments one obtains also sufficiency in part (ii).
Step 2. Necessity in part(i) and (ii). Again we deal with $\delta>0$. Let $s=\frac{n}{p}+\alpha$ and $q\delta=1$. From $f_{\alpha,\delta}\in B^s_{p,q}$ we derive the existence of some $r>0$ such that

$$f_{\alpha,\delta}\in B_{r,q}^{s-n(\frac{1}{p}-\frac{1}{r})}\,,\qquad \max(1,p)<r<\infty\,,\quad 0<s-n(\frac{1}{p}-\frac{1}{r})<1\,,$$

using Theorem 2.2.3. The quasi-norm in $B_{r,q}^{s-(\frac{n}{p}-\frac{n}{r})}$ is bounded from below by

$$\Big(\int\limits_{|h|\leq\varepsilon}|h|^{-(s-\frac{n}{p}+\frac{n}{r})q}\|\Delta_h^1 f_{\alpha,\delta}|L_r\|^q\,\frac{dh}{|h|^n}\Big)^{\frac{1}{q}}. \tag{26}$$

Applying the inequality

$$\big|\,|x|^\alpha(-\log|x|)^{-\delta}-|x+h|^\alpha(-\log|x+h|)^{-\delta}\big|\geq c\,|h|^\alpha(-\log|h|)^{-\delta}\,, \tag{27}$$

where $x = (x_1, \ldots, x_n)$, $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$, $x_i \geq 0$, $h_i \geq 0$, $i = 1, \ldots, n$ and $|x| \leq \frac{|h|}{M} < h_0$ for some $c > 0$ and some $M > 0$ we obtain

$$\int\limits_{|h|<\varepsilon} |h|^{-(s-\frac{n}{p}+\frac{n}{r})q} \|\Delta_h^1 f_{\alpha,\delta}|L_r\|^q \frac{dh}{|h|^n} \geq \int\limits_0^{r_0} (-\log t)^{-\delta q} \frac{dt}{t} = \infty .$$

In view of (26) this yields $f_{\alpha,\delta} \notin B^s_{p,q}$. A modification of the same argument yields (ii).

Step 3. Proof of sufficiency in (iii) and (iv). This can be done by using (i), (ii) and embedding theorems only. Suppose $\sigma_p < s = \frac{n}{p} + \alpha$ and $\delta p > 1$. In particular, (i) yields

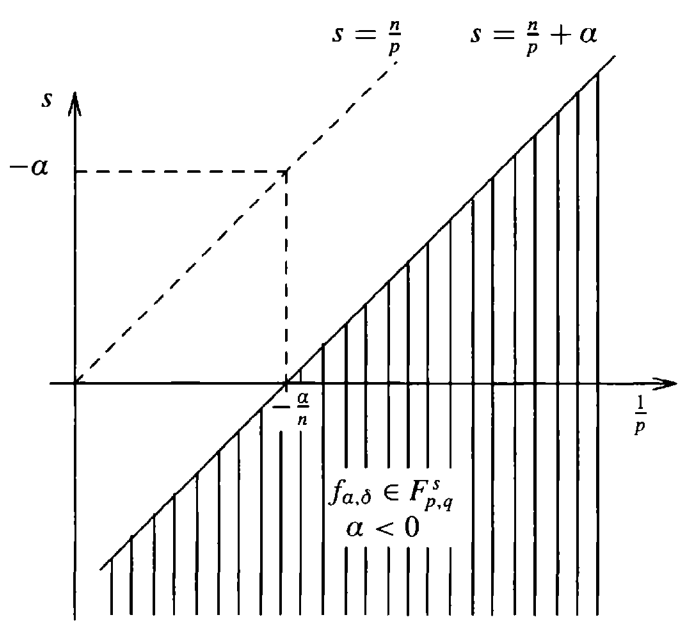
$$f_{\alpha,\delta} \in B^{\frac{n}{p_0}+\alpha}_{p_0,p} , \qquad p_0 < p .$$

By Theorem 2.2.3 it follows $f_{\alpha,\delta} \in B^{\frac{n}{p}+\alpha}_{p,q}$ which proves sufficiency in (iii). Sufficiency in (iv) follows using the same procedure.

Step 4. Necessity in (iii) and (iv). Again we use (i) in connection with embeddings. Let $\sigma_p < s < \frac{n}{p} + \alpha$ and $\delta p = 1$. Theorem 2.2.3 yields

$$F^{\frac{n}{p}+\alpha}_{p,q} \hookrightarrow B^{\frac{n}{p_1}+\alpha}_{p_1,p} , \qquad p_1 > p .$$

Therefore, (i) tells us, that $\delta p = 1$ implies $f_{\alpha,\delta} \notin B^{\frac{n}{p_1}+\alpha}_{p_1,p}$ and consequently $f_{\alpha,\delta} \notin F^{\frac{n}{p_1}+\alpha}_{p_1,p}$. The proof in case $\delta = 0$ is the same. $\square$



Remark 2. The restriction $s > \sigma_p$ is natural, except may be the limit case $s = \sigma_p$. In case $s - \sigma_p = \frac{n}{p} + \alpha$, $p \leq 1$ it follows $-\alpha = n$ and consequently $f_{\alpha,0} \notin \mathcal{S}'$.