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Algebra in the Stone-Čech Compactification

Theory and Applications

by

Neil Hindman
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Preface

The semigroup operation defined on a discrete semigroup $(S, \cdot)$ has a natural extension, also denoted by $\cdot$, to the Stone–Čech compactification βS of S. Under the extended operation, βS is a compact right topological semigroup with S contained in its topological center. That is, for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ is continuous and for each $s \in S$, the function $\lambda_s : \beta S \to \beta S$ is continuous, where $\rho_p(q) = q \cdot p$ and $\lambda_s(q) = s \cdot q$.

In Part I of this book, assuming only the mathematical background standardly provided in the first year of graduate school, we develop the basic background information about compact right topological semigroups, the Stone–Čech compactification of a discrete space, and the extension of the semigroup operation on S to βS. In Part II, we study in depth the algebra of the semigroup $(\beta S, \cdot)$ and in Part III present some of the powerful applications of the algebra of βS to the part of combinatorics known as *Ramsey Theory*. We conclude in Part IV with connections with Topological Dynamics, Ergodic Theory, and the general theory of semigroup compactifications.

The study of the semigroup $(\beta S, \cdot)$ has interested several mathematicians since it was first defined in the late 1950's. As a glance at the bibliography will show, a large number of research papers have been devoted to its properties.

There are several reasons for an interest in the algebra of βS.

It is intrinsically interesting as being a natural extension of S which plays a special role among semigroup compactifications of S. It is the largest possible compactification of this kind: If T is a compact right topological semigroup, φ is a continuous homomorphism from S to T, $\varphi[S]$ is dense in T, and $\lambda_{\varphi(s)}$ is continuous for each $s \in S$, then T is a quotient of βS.

We believe that $\beta\mathbb{N}$ is interesting and challenging for its own sake, as well as for its applications. Although it is a natural extension of the most familiar of all semigroups, it has an algebraic structure of extraordinary complexity, which is constantly surprising. For example, $\beta\mathbb{N}$ contains many copies of the free group on $2^{\mathfrak{c}}$ generators [152]. Algebraic questions about $\beta\mathbb{N}$ which sound deceptively simple have remained unsolved for many years. It is, for instance, not known whether $\beta\mathbb{N}$ contains any elements of finite order, other than idempotents. And the corresponding question about the existence of nontrivial finite groups was only very recently answered by E. Zelenuk. (His negative answer is presented in Chapter 7.)

The semigroup βS is also interesting because of its applications to combinatorial number theory and to topological dynamics.

Algebraic properties of βS have been a useful tool in Ramsey theory. Results in Ramsey Theory have a twin beauty. On the one hand they are representatives of pure mathematics at its purest: simple statements easy for almost anyone to understand (though not necessarily to prove). On the other hand, the area has been widely applied from its beginning. In fact a perusal of the titles of several of the original papers reveals that many of the classical results were obtained with applications in mind. (Hilbert's Theorem – Algebra; Schur's Theorem – Number Theory; Ramsey's Theorem – Logic; the Hales–Jewett Theorem – Game Theory).

The most striking example of an application of the algebraic structure of βS to Ramsey Theory is perhaps provided by the Finite Sums Theorem. This theorem says that whenever $\mathbb{N}$ is partitioned into finitely many classes (or in the terminology common within Ramsey Theory, is *finitely colored*), there is a sequence $\langle x_n\rangle_{n=1}^{\infty}$ with $\mathrm{FS}(\langle x_n\rangle_{n=1}^{\infty})$ contained in one class (or *monochrome*). (Here $\mathrm{FS}(\langle x_n\rangle_{n=1}^{\infty}) = \{\Sigma_{n\in F}\, x_n : F$ is a finite nonempty subset of $\mathbb{N}\}$.) This theorem had been an open problem for some decades, even though several mathematicians (including Hilbert) had worked on it. Although it was initially proved without using $\beta\mathbb{N}$, the first proof given was one of enormous complexity.

In 1975 F. Galvin and S. Glazer provided a brilliantly simple proof of the Finite Sums Theorem using the algebraic structure of $\beta\mathbb{N}$. Since this time numerous strong combinatorial results have been obtained using the algebraic structure of βS, where S is an arbitrary discrete semigroup. In the process, more detailed knowledge of the algebra of βS has been obtained.

Other famous combinatorial theorems, such as van der Waerden's Theorem or Rado's Theorem, have elegant proofs based on the algebraic properties of $\beta\mathbb{N}$. These proofs have in common with the Finite Sums Theorem the fact that they were initially established by combinatorial methods. A simple extension of the Finite Sums Theorem was first established using the algebra of $\beta\mathbb{N}$. This extension says that whenever $\mathbb{N}$ is finitely colored there exist sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ such that $\mathrm{FS}(\langle x_n\rangle_{n=1}^{\infty})\cup\mathrm{FP}(\langle y_n\rangle_{n=1}^{\infty})$ is monochrome, where $\mathrm{FP}(\langle y_n\rangle_{n=1}^{\infty}) = \{\Pi_{n\in F}\, y_n : F$ is a finite nonempty subset of $\mathbb{N}\}$. This combined additive and multiplicative result was first proved in 1975 using the algebraic structure of $\beta\mathbb{N}$ and it was not until 1993 that an elementary proof was found.

Other fundamental results have been established for which it seems unlikely that elementary proofs will be found. Among such results is a density version of the Finite Sums Theorem, which says roughly that the sequence $\langle x_n\rangle_{n=1}^{\infty}$ whose finite sums are monochrome can be chosen inductively in such a way that at each stage of the induction the set of choices for the next term has positive upper density. Another such result is the Central Set Theorem, which is a common generalization of many of the basic results of Ramsey Theory. Significant progress continues to be made in the combinatorial applications.

The semigroup βS also has applications in topological dynamics. A semigroup S of continuous functions acting on a compact Hausdorff space X has a closure in ${}^{X}X$ (the space of functions mapping X to itself with the product topology), which is a compact right topological semigroup. This semigroup, called the enveloping semigroup, was first studied by R. Ellis [86]. It is always a quotient of the Stone–Čech compactification

βS, as is every semigroup compactification of S, and is, in some important cases, equal to βS. In this framework, the algebraic properties of βS have implications for the dynamical behavior of the system.

The interaction with topological dynamics works both ways. Several notions which originated in topological dynamics, such as syndetic and piecewise syndetic sets, are important in describing the algebraic structure of βS. For example, a point p of βS is in the closure of the smallest ideal of βS if and only if for every neighborhood U of p, $U \cap S$ is piecewise syndetic.

This last statement can be made more concise when one notes the particular construction of βS that we use. That is, βS is the set of all ultrafilters on S, the principal ultrafilters being identified with the points of S. Under this construction, any point p of βS is precisely $\{U \cap S : U$ is a neighborhood of $p\}$. Thus p is in the closure of the smallest ideal of βS if and only if every member of p is piecewise syndetic.

In this book, we develop the algebraic theory of βS and present several of its combinatorial applications. We assume only that the reader has had graduate courses in algebra, analysis, and general topology as well as a familiarity with the basic facts about ordinal and cardinal numbers. In particular we develop the basic structure of compact right topological semigroups and provide an elementary construction of the Stone–Čech compactification of a discrete space.

With only three exceptions, this book is self contained for those with that minimal background. The three cases where we appeal to non elementary results not proved here are Theorem 6.36 (due to M. Rudin and S. Shelah) which asserts the existence of a collection of $2^{\mathfrak{c}}$ elements of $\beta\mathbb{N}$ no two of which are comparable in the Rudin–Keisler order, Theorem 12.37 (due to S. Shelah) which states that the existence of P-points in $\beta\mathbb{N}\backslash\mathbb{N}$ cannot be established in ZFC, and Theorem 20.13 (due to H. Furstenberg) which is an ergodic theoretic result that we use to derive Szemerédi's Theorem.

All of our applications involve Hausdorff spaces, so we will be assuming throughout, except in Chapter 7, that all hypothesized topological spaces are Hausdorff.

The first five chapters are meant to provide the basic preliminary material. The concepts and theorems given in the first three of these chapters are also available in other books. The remaining chapters of the book contain results which, for the most part, can only be found in research papers at present, as well as several previously unpublished results.

Notes on the historical development are given at the end of each chapter.

Let us make a few remarks about organization. Chapters are numbered consecutively throughout the book, regardless of which of the four parts of the book contains them. Lemmas, theorems, corollaries, examples, questions, comments, and remarks are numbered consecutively in one group within chapters (so that Lemma 2.4 will be found after Theorem 2.3, for example). There is no logical distinction between a theorem and a remark. The difference is that proofs are never included for remarks. Exercises come at the end of sections and are numbered consecutively within sections.

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April 1998

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Dona Strauss

Contents

Part I

Background Development

Notation

We write $\mathbb{N}$ for the set $\{1, 2, 3, \ldots\}$ of positive integers and $\omega = \{0, 1, 2, \ldots\}$ for the nonnegative integers. Also ω is the first infinite ordinal, and thus the first infinite cardinal. Each ordinal is the set of all smaller ordinals.

$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.

Given a function f and a set A contained in the domain of f, we write $f[A] = \{f(x) : x \in A\}$ and given any set B we write $f^{-1}[B] = \{x \in \text{Domain}(f) : f(x) \in B\}$.

Given a set A, $\mathcal{P}_f(A) = \{F : \emptyset \neq F \subseteq A \text{ and } F \text{ is finite}\}$.

Definitions of additional unfamiliar notation can be located by way of the index.

Chapter 1

Semigroups and Their Ideals

We assume that the reader has had an introductory modern algebra course. This assumption is not explicitly used in this chapter beyond the fact that we expect a certain amount of mathematical maturity.

1.1 Semigroups

Definition 1.1. A *semigroup* is a pair $(S, *)$ where S is a nonempty set and $*$ is a binary associative operation on S.

Formally a *binary operation* on S is a function $* : S \times S \to S$ and the operation is *associative* if and only if $*(*(x, y), z) = *\big(x, *(y, z)\big)$ for all x, y, and z in S. However, we customarily write $x * y$ instead of $*(x, y)$ so the associativity requirement becomes the more familiar $(x * y) * z = x * (y * z)$. The statement that $* : S \times S \to S$, i.e., that $x * y \in S$ whenever $x, y \in S$ is commonly referred to by saying that "S is *closed* under $*$".

Example 1.2. *Each of the following is a semigroup.*

(a) $(\mathbb{N}, +)$.

(b) $(\mathbb{N}, \cdot)$.

(c) $(\mathbb{R}, +)$.

(d) $(\mathbb{R}, \cdot)$.

(e) $(\mathbb{R}\backslash\{0\}, \cdot)$.

(f) $(\mathbb{R}^+, +)$.

(g) $(\mathbb{R}^+, \cdot)$.

(h) $(\mathbb{N}, \vee)$, *where* $x \vee y = \max\{x, y\}$.

(i) $(\mathbb{N}, \wedge)$, *where* $x \wedge y = \min\{x, y\}$. .

(j) $(\mathbb{R}, \wedge)$.

(k) $(S, *)$, *where* S *is any nonempty set and* $x * y = y$ *for all* $x, y \in S$.

(l) $(S, *)$, *where* S *is any nonempty set and* $x * y = x$ *for all* $x, y \in S$.

(m) $(S, *)$, *where* S *is any nonempty set and* $a \in S$ *and* $x * y = a$ *for all* $x, y \in S$.

(n) $({}^X X, \circ)$, *where* ${}^X X = \{f : f : X \to X\}$ *and* $\circ$ *represents the composition of functions.*

The semigroups of Example 1.2 (k) and (l) are called respectively *right zero* and *left zero* semigroups.

An important class of semigroups are the *free* semigroups. These require a more detailed explanation.

Definition 1.3. Let A be a nonempty set. The *free semigroup on the alphabet* A is the set $S = \{f : f$ is a function and range$(f) \subseteq A$ and there is some $n \in \mathbb{N}$ such that domain$(f) = \{0, 1, \ldots, n-1\}\}$. Given f and g in S, the operation (called *concatenation*) is defined as follows. Assume domain$(f) = \{0, 1, \ldots, n-1\}$ and domain$(g) = \{0, 1, \ldots, m-1\}$. Then domain$(f^\frown g) = \{0, 1, \ldots, m+n-1\}$ and given $i \in \{0, 1, \ldots, m+n-1\}$,

$$f^\frown g = \begin{cases} f(i) & \text{if } i < n \\ g(i-n) & \text{if } i \geq n. \end{cases}$$

The *free semigroup with identity on the alphabet* A is $S \cup \{\emptyset\}$ where S is the free semigroup on the alphabet A. Given $f \in S \cup \{\emptyset\}$ one defines $f^\frown \emptyset = \emptyset^\frown f = f$

One usually refers to the elements of a free semigroup as *words* and writes them by listing the values of the function in order. The *length* of a word is n where the domain of the word is $\{0, 1, \ldots, n-1\}$ (and the length of $\emptyset$ is 0). Thus if $A = \{2, 4\}$ and $f = \{(0, 4), (1, 2), (2, 2)\}$ (so that the length of f is 3 and $f(0) = 4$, $f(1) = 2$, and $f(2) = 2$), then one represents f as 422. Furthermore given the "words" 422 and 24424, one has $422^\frown 24424 = 42224424$.

We leave to the reader the routine verification of the fact that concatenation is associative, so that the free semigroup *is* a semigroup.

Definition 1.4. Let $(S, *)$ and $(T, \cdot)$ be semigroups.

(a) A *homomorphism* from S to T is a function $\varphi : S \to T$ such that $\varphi(x * y) = \varphi(x) \cdot \varphi(y)$ for all $x, y \in S$.

(b) An *isomorphism* from S to T is a homomorphism from S to T which is both one-to-one and onto T.

(c) The semigroups S and T are *isomorphic* if and only if there exists an isomorphism from S to T. If S and T are isomorphic we write $S \approx T$.

(d) An *anti-homomorphism* from S to T is a function $\varphi : S \to T$ such that $\varphi(x * y) = \varphi(y) \cdot \varphi(x)$ for all $x, y \in S$.

(e) An *anti-isomorphism* from S to T is an anti-homomorphism from S to T which is both one-to-one and onto T.

(f) The semigroups S and T are *anti-isomorphic* if and only if there exists an anti-isomorphism from S to T.

Clearly, the composition of two homomorphism, if it exists, is also a homomorphism. The reader who is familiar with the concept of a category will recognize that there is a category of semigroups, in which the objects are semigroups and the morphisms are homomorphisms.

The free semigroup S on the alphabet A has the following property. Suppose that T is an arbitrary semigroup and that $g : A \to T$ is any mapping. Then there is a unique homomorphism $h : S \to T$ with the property that $h(a) = g(a)$ for every $a \in A$. (The proof of this assertion is Exercise 1.1.1.)

Definition 1.5. Let $(S, *)$ be a semigroup and let $a \in S$.

(a) The element a is a *left identity* for S if and only if $a * x = x$ for every $x \in S$.

(b) The element a is a *right identity* for S if and only if $x * a = x$ for every $x \in S$.

(c) The element a is a *two sided identity* (or simply an *identity*) for S if and only if a is both a left identity and a right identity.

Note that in a "free semigroup with identity" the element $\emptyset$ is a two sided identity (so the terminology is appropriate).

Note also that in a left zero semigroup, every element is a right identity and in a right zero semigroup, every element is a left identity. On the other hand we have the following simple fact.

Remark 1.6. *Let $(S, *)$ be a semigroup. If e is a left identity for S and f is a right identity for S, then $e = f$. In particular, a semigroup can have at most one two sided identity.*

Given a collection of semigroups $\langle (S_i, *_i) \rangle_{i \in I}$, the Cartesian product $\times_{i \in I} S_i$ is naturally a semigroup with the coordinatewise operations.

Definition 1.7. (a) Let $\langle (S_i, *_i) \rangle_{i \in I}$ be an indexed family of semigroups and let $S = \times_{i \in I} S_i$. With the operation $*$ defined by $(\vec{x} * \vec{y})_i = x_i *_i y_i$, the semigroup $(S, *)$ is called the *direct product* of the semigroups $(S_i, *_i)$.

(b) Let $\langle (S_i, *_i) \rangle_{i \in I}$ be an indexed family of semigroups where each S_i has a two sided identity e_i. Then the *direct sum* of the semigroups $(S_i, *_i)$ is $\bigoplus_{i \in I} S_i = \{\vec{x} \in \times_{i \in I} S_i : \{i \in I : x_i \neq e_i\} \text{ is finite}\}$.

We leave to the reader the easy verification that the direct product operation is associative as well as the verification that if $\vec{x}, \vec{y} \in \bigoplus_{i \in I} S_i$, then $\vec{x} * \vec{y} \in \bigoplus_{i \in I} S_i$.

Definition 1.8. Let $(S, *)$ be a semigroup and let $a, b, c \in S$.

(a) The element c is a *left a-inverse* for b if and only if $c * b = a$.

(b) The element c is a *right a-inverse* for b if and only if $b * c = a$.

(c) The element c is an *a-inverse* for b if and only if c is both a left a-inverse for b and a right a-inverse for b.

The terms *left a-inverse*, *right a-inverse*, and *a-inverse* are usually replaced by *left inverse*, *right inverse*, and *inverse* respectively. We introduce the more precise notions because one may have many left or right identities.

Definition 1.9. A *group* is a pair $(S, *)$ such that

(a) $(S, *)$ is a semigroup, and
(b) there is an element $e \in S$ such that
  (i) e is a left identity for S and
  (ii) for each $x \in S$ there exists $y \in S$ such that y is a left e-inverse for x.

Theorem 1.10. *Let $(S, *)$ be a semigroup. The following statements are equivalent.*

(a) *$(S, *)$ is a group.*
(b) *There is a two sided identity e for S with the property that for each $x \in S$ there is some $y \in S$ such that y is a (two sided) e-inverse for x.*
(c) *There is a left identity for S and given any left identity e for S and any $x \in S$ there is some $y \in S$ such that y is a left e-inverse for x.*
(d) *There is a right identity e for S such that for each $x \in S$ there is some $y \in S$ such that y is a right e-inverse for x.*
(e) *There is a right identity for S and given any right identity e for S and any $x \in S$ there is some $y \in S$ such that y is a right e-inverse for x.*

Proof. (a) implies (b). Pick e as guaranteed by Definition 1.9. We show first that any element has an e-inverse, so let $x \in S$ be given and let y be a left e-inverse for x. Let z be a left e-inverse for y. Then $x*y = e*(x*y) = (z*y)*(x*y) = z*\big(y*(x*y)\big) = z*\big((y*x)*y\big) = z*(e*y) = z*y = e$, so y is also a right e-inverse for x as required.

Now we show that e is a right identity for S, so let $x \in S$ be given. Pick an e-inverse y for x. Then $x*e = x*(y*x) = (x*y)*x = e*x = x$.

(b) implies (c). Pick e as guaranteed by (b). Given any left identity f for S we have by Remark 1.6 that $e = f$ so every element of S has a left f-identity.

That (c) implies (a) is trivial.

The implications (d) implies (b), (b) implies (e), and (e) implies (d) follow now by left-right switches, the details of which form Exercise 1.1.2. □

In a right zero semigroup S (Example 1.2 (k)) every element is a left identity and given any left identity e and any $x \in S$, e is a right e-inverse for x. This is essentially the only example of this phenomenon. That is, we shall see in Theorem 1.40 that any semigroup with a left identity e such that every element has a right e-inverse is the Cartesian product of a group with a right zero semigroup. In particular we see that if a semigroup has a unique left identity e and every element has a right e-inverse, then the semigroup is a group.

In the semigroup $(\mathbb{N}, \vee)$, 1 is the unique identity and the only element with an inverse.

When dealing with arbitrary semigroups it is customary to denote the operation by $\cdot$. Furthermore, given a semigroup $(S, \cdot)$ one customarily writes xy in lieu of $x \cdot y$. We shall now adopt these conventions. Accordingly, from this point on, when we write "Let S be a semigroup" we mean "Let $(S, \cdot)$ be a semigroup" and when we write "xy" we mean "$x \cdot y$".

Definition 1.11. Let S be a semigroup.

(a) S is *commutative* if and only if $xy = yx$ for all $x, y \in S$.

(b) The *center* of S is $\{x \in S : \text{for all } y \in S, xy = yx\}$.

(c) Given $x \in S$, $\lambda_x : S \to S$ is defined by $\lambda_x(y) = xy$.

(d) Given $x \in S$, $\rho_x : S \to S$ is defined by $\rho_x(y) = yx$.

(e) $L(S) = \{\lambda_x : x \in S\}$.

(f) $R(S) = \{\rho_x : x \in S\}$.

Remark 1.12. *Let S be a semigroup. Then $(L(S), \circ)$ and $(R(S), \circ)$ are semigroups.*

Since our semigroups are not necessarily commutative we need to specify what we mean by $\prod_{i=1}^n x_i$. There are 2 reasonable interpretations (and $n! - 2$ unreasonable ones). We choose it to mean the product in increasing order of indices because that is the order that naturally arises in our applications of right topological semigroups. More formally we have the following.

Definition 1.13. Let S be a semigroup. We define $\prod_{i=1}^n x_i$ for $\{x_1, x_2, \ldots, x_n\} \subseteq S$ inductively on $n \in \mathbb{N}$.

(a) $\prod_{i=1}^1 x_i = x_1$.

(b) Given $n \in \mathbb{N}$, $\prod_{i=1}^{n+1} x_i = (\prod_{i=1}^n x_i) \cdot x_{n+1}$.

Definition 1.14. Let S be a semigroup.

(a) An element $x \in S$ is *right cancelable* if and only if whenever $y, z \in S$ and $yx = zx$, one has $y = z$.

(b) An element $x \in S$ is *left cancelable* if and only if whenever $y, z \in S$ and $xy = xz$, one has $y = z$.

(c) S is *right cancellative* if and only if every $x \in S$ is right cancelable.

(d) S is *left cancellative* if and only if every $x \in S$ is left cancelable.

(e) S is *cancellative* if and only if S is both left cancellative and right cancellative.

Theorem 1.15. *Let S be a semigroup.*

(a) *The function $\lambda : S \to L(S)$ is a homomorphism onto $L(S)$.*

(b) *The function $\rho : S \to R(S)$ is an anti-homomorphism onto $R(S)$.*

(c) *If S is right cancellative, then S and $L(S)$ are isomorphic.*

(d) *If S is left cancellative, then S and $R(S)$ are anti-isomorphic.*

Proof. (a) Given x, y, and z in S one has $(\lambda_x \circ \lambda_y)(z) = \lambda_x(yz) = x(yz) = (xy)z = \lambda_{xy}(z)$ so $\lambda_x \circ \lambda_y = \lambda_{xy}$.

(c) This is part of Exercise 1.1.4. □

Right cancellation is a far stronger requirement than is needed to have $S \approx L(S)$. See Exercise 1.1.4.

Exercise 1.1.1. Let S be the free semigroup on the alphabet A and let T be an arbitrary semigroup. Assume that $g : A \to T$ is any mapping. Prove that there is a unique homomorphism $h : S \to T$ with the property that $h(a) = g(a)$ for every $a \in A$.

Exercise 1.1.2. Prove that statements (b), (d), and (e) of Theorem 1.10 are equivalent.

Exercise 1.1.3. Prove that, in the semigroup $({}^X X, \circ)$, the left cancelable elements are the injective functions and the right cancelable elements are the surjective functions.

Exercise 1.1.4. (a) Prove Theorem 1.15 (c).

(b) Give an example of a semigroup S which is not right cancellative such that $S \approx L(S)$.

Exercise 1.1.5. Let S be a right cancellative semigroup and let $a \in S$. Prove that if there is some $b \in S$ such that $ab = b$, then a is a *right* identity for S.

Exercise 1.1.6. Prove that "if S does not have an identity, one may be adjoined" (and in fact one may be adjoined even if S already has an identity). That is, Let S be a semigroup and let e be an element not in S. Define an operation $*$ on $S \cup \{e\}$ by $x * y = xy$ if $x, y \in S$ and $x * e = e * x = x$. Prove that $(S \cup \{e\}, *)$ is a semigroup with identity e. (Note that if S has an identity f, it is no longer the identity of $S \cup \{e\}$.)

Exercise 1.1.7. Suppose that S is a cancellative semigroup which does not have an identity. Prove that an identity can be adjoined to S so that the extended semigroup is also cancellative.

Exercise 1.1.8. Let S be a commutative cancellative semigroup. We define a relation $\equiv$ on $S \times S$ by stating that $(a, b) \equiv (c, d)$ if and only if $ad = bc$. Prove that this is an equivalence relation. Let $\overline{(a, b)}$ denote the equivalence class which contains the element $(a, b) \in S \times S$, and let G denote the set of all these equivalence classes. We define a binary relation $\cdot$ on G by stating that $\overline{(a, b)} \cdot \overline{(c, d)} = \overline{(ac, bd)}$. Prove that this is well defined, that $(G, \cdot)$ is a group and that it contains an isomorphic copy of S. (The group G is called the *group of quotients* of S. If $S = (\mathbb{N}, +)$, $G = (\mathbb{Z}, +)$; if $S = (\mathbb{N}, \cdot)$, $G = (\mathbb{Q}^+, \cdot)$, where $\mathbb{Q}^+ = \{x \in \mathbb{Q} : x > 0\}$.)

1.2 Idempotents and Subgroups

Our next subject is "idempotents". They will be very important to us throughout this book.

Definition 1.16. Let S be a semigroup.

(a) An element $x \in S$ is an *idempotent* if and only if $xx = x$.

(b) $E(S) = \{x \in S : x \text{ is an idempotent}\}$.

(c) T is a *subsemigroup* of S if and only if $T \subseteq S$ and T is a semigroup under the restriction of the operation of S.

(d) T is a *subgroup* of S if and only if $T \subseteq S$ and T is a group under the restriction of the operation of S.

(e) Let $e \in E(S)$. Then $H(e) = \bigcup\{G : G \text{ is a subgroup of } S \text{ and } e \in G\}$.

Lemma 1.17. *Let G be a group with identity e. Then $E(G) = \{e\}$.*

Proof. Assume $f \in E(G)$. Then $ff = f = fe$. Multiplying on the left by the inverse of f, one gets $f = e$. □

As a consequence of Lemma 1.17 the statement "$e \in G$" in the definition of $H(e)$ is synonymous with "e is the identity of G". Note that it is quite possible for $H(e)$ to equal $\{e\}$, but $H(e)$ is never empty.

Theorem 1.18. *Let S be a semigroup and let $e \in E(S)$. Then $H(e)$ is the largest subgroup of S with e as identity.*

Proof. It suffices to show that $H(e)$ is a group since e is trivially an identity for $H(e)$ and $H(e)$ contains every group with e as identity. For this it in turn suffices to show that $H(e)$ is closed. So let $x, y \in H(e)$ and pick subgroups G_1 and G_2 of S with $e \in G_1 \cap G_2$ and $x \in G_1$ and $y \in G_2$. Let $G = \{\prod_{i=1}^{n} x_i : n \in \mathbb{N} \text{ and } \{x_1, x_2, \ldots, x_n\} \subseteq G_1 \cup G_2\}$. Then $xy \in G$ and $e \in G$ so it suffices to show that G is a group. For this the only requirement that is not immediate is the existence of inverses. So let $\prod_{i=1}^{n} x_i \in G$. For $i \in \{1, 2, \ldots, n\}$, pick y_i such that $x_{n+1-i} y_i = e$. Then $\prod_{i=1}^{n} y_i \in G$ and $(\prod_{i=1}^{n} x_i) \cdot (\prod_{i=1}^{n} y_i) = e$. □

The groups $H(e)$ are referred to as *maximal groups*. Indeed, given any group $G \subseteq S$, G has an identity e and $G \subseteq H(e)$.

Lemma 1.19. *Let S be a semigroup, let $e \in E(S)$, and let $x \in S$. Then the following statements are equivalent.*

(a) $x \in H(e)$.
(b) $xe = x$ *and there is some* $y \in S$ *such that* $ye = y$ *and* $xy = yx = e$.
(c) $ex = x$ *and there is some* $y \in S$ *such that* $ey = y$ *and* $xy = yx = e$.

Proof. We show the equivalence of (a) and (b); the equivalence of (a) and (c) then follows by a left-right switch. The fact that (a) implies (b) is immediate.

(b) implies (a). Let $G = \{x \in S : xe = x$ and there is some $y \in S$ such that $ye = y$ and $xy = yx = e\}$. It suffices to show that G is a group with identity e. To establish closure, let $x, z \in G$. Then $xze = xz$. Pick y and w in S such that $ye = y$, $we = w$, $xy = yx = e$, and $zw = wz = e$. Then $wye = wy$ and $xzwy = xey = xy = e = wz = wez = wyxz$.

Trivially, e is a right identity for G so it suffices to show that each element of G has a right e-inverse *in* G. Let $x \in G$ and pick $y \in S$ such that $ye = y$ and $yx = xy = e$. Note that indeed y does satisfy the requirements to be in G. □

Example 1.20. *Let X be any set. Then the idempotents in $({}^{X}X, \circ)$ are the functions $f \in {}^{X}X$ with the property that $f(x) = x$ for every $x \in f[X]$.*

We next define the concept of a free group on a given set of generators. The underlying idea is simple, but the rigorous definition may seem a little troublesome. The basic idea is that we want to construct all expressions of the form $a_1^{e_1} a_2^{e_2} \ldots a_k^{e_k}$, where each $a_i \in A$ and each exponent $e_i \in \mathbb{Z}$, and to combine them in the way that we are forced to by the group axioms.

Definition 1.21. Let S be the free semigroup with identity on the alphabet $A \times \{1, -1\}$ and let

$$G = \{g \in S : \text{ there do not exist } t, t+1 \in \text{domain}(g),\ a \in A \text{ and } i \in \{1, -1\} \text{ for which } g(t) = (a, i) \text{ and } g(t+1) = (a, -i)\}.$$

Given $f, g \in G \backslash \{\emptyset\}$ with

$$\text{domain}(f) = \{0, 1, \ldots, n-1\} \quad \text{and} \quad \text{domain}(g) = \{0, 1, \ldots, m-1\},$$

define $f \cdot g = f^\frown g$ unless there exist $a \in A$ and $i \in \{1, -1\}$ with $f(n-1) = (a, i)$ and $g(0) = (a, -i)$.

In the latter case, pick the largest $k \in \mathbb{N}$ such that for all $t \in \{1, 2, \ldots, k\}$, there exist $b \in A$ and $j \in \{1, -1\}$ such that $f(n-t) = (b, j)$ and $g(t-1) = (b, -j)$. If $k = m = n$, then $f \cdot g = \emptyset$. Otherwise, domain$(f \cdot g) = \{0, 1, \ldots, n+m-2k-1\}$ and for $t \in \{1, 2, \ldots, n+m-2k-1\}$,

$$(f \cdot g)(t) = \begin{cases} f(t) & \text{if } t < n-k \\ g(t+2k-n) & \text{if } t \geq n-k. \end{cases}$$

Then $(G, \cdot)$ is the *free group generated by A*.

It is not hard to prove that, with the operation defined above, G *is* a group.

We customarily write a in lieu of $(a, 1)$ and a^{-1} in lieu of $(a, -1)$. Then in keeping with the notation to be introduced in the next section (Section 1.3) we shall write the word $ab^{-1}b^{-1}b^{-1}a^{-1}a^{-1}bb$, for example, as $ab^{-3}a^{-2}b^2$. As an illustration, we have $(ab^{-3}a^{-2}b^2) \cdot (b^{-2}a^3b^{-4}) = ab^{-3}ab^{-4}$.

We observe that the free group G generated by A has a universal property given by the following lemma.

Lemma 1.22. *Let A be a set, let G be the free group generated by A, let H be an arbitrary group, and let $\phi : A \to H$ be any mapping. There is a unique homomorphism $\widehat{\phi} : G \to H$ for which $\widehat{\phi}(g) = \phi(g)$ for every $g \in A$.*

Proof. This is Exercise 1.2.1. □

We shall need the following result later.

Theorem 1.23. *Let A be a set, let G be the free group generated by A, and let $g \in G \backslash \{\emptyset\}$. There exist a finite group F and a homomorphism $\widehat{\phi} : G \to F$ such that $\widehat{\phi}(g)$ is not the identity of F.*

Proof. Let n be the length of g, let $X = \{0, 1, \ldots, n\}$, and let $F = \{f \in {}^{X}X : f$ is one-to-one and onto $X\}$. (Since X is finite, the "onto" requirement is redundant.) Then $(F, \circ)$ is a group whose identity is ι, the identity function from X to X. Given $a \in A$, let $D(a) = \{i \in \{0, 1, \ldots, n-1\} : g(i) = a^{-1}\}$ and let $E(a) = \{i \in \{1, 2, \ldots, n\} : g(i-1) = a\}$. Note that since $g \in G$, $D(a) \cap E(a) = \emptyset$. Define $\phi(a) : D(a) \cup E(a) \to X$ by

$$\phi(a)(i) = \begin{cases} i+1 & \text{if } i \in D(a) \\ i-1 & \text{if } i \in E(a), \end{cases}$$

and note that, because $g \in G$, $\phi(a)$ is one-to-one. Extend $\phi(a)$ in any way to a member of F. Let $\widehat{\phi} : G \to F$ be the homomorphism extending ϕ which was guaranteed by Lemma 1.22.

Suppose that $g = {a_0}^{i_0}{a_1}^{i_1} \ldots {a_{n-1}}^{i_{n-1}}$, where $a_r \in A$ and $i_r \in \{-1, 1\}$ for each $r \in \{0, 1, 2, \ldots, n-1\}$. We shall show that, for each $k \in \{1, 2, \ldots, n\}$, $\widehat{\phi}({a_{k-1}}^{i_{k-1}})(k) = k-1$.

To see this, first suppose that $i_{k-1} = 1$. Then $k \in E(a_{k-1})$ and so $\phi(a_{k-1})(k) = k-1$.

Now suppose that $i_{k-1} = -1$. Then $k-1 \in D(a_{k-1})$ and so $\phi(a_{k-1})(k-1) = k$. Thus $\widehat{\phi}({a_{k-1}}^{-1})(k) = \phi(a_{k-1})^{-1}(k) = k-1$.

It is now easy to see that $\widehat{\phi}(g)(n) = \widehat{\phi}({a_0}^{i_0})\widehat{\phi}({a_1}^{i_1}) \ldots \widehat{\phi}({a_{n-1}}^{i_{n-1}})(n) = 0$ and hence that $\widehat{\phi}(g)$ is not the identity map. □

Exercise 1.2.1. Prove Lemma 1.22.

1.3 Powers of a Single Element

Suppose that x is a given element in a semigroup S. For each $n \in \mathbb{N}$, we define an element x^n in S. We do this inductively, by stating that $x^1 = x$ and that $x^{n+1} = xx^n$ if x^n has already been defined. It is then straightforward to prove by induction that $x^m x^n = x^{m+n}$ for every $m, n \in \mathbb{N}$. Thus $\{x^n : n \in \mathbb{N}\}$ is a commutative subsemigroup of S. We shall say that x has *finite order* if this subsemigroup is finite; otherwise we shall say that x has *infinite order*.

If S has an identity e, we shall define x^0 for every $x \in S$ by stating that $x^0 = e$. If x has an inverse in S, we shall denote this inverse by x^{-1}, and we shall define x^{-n} for every $n \in \mathbb{N}$ by stating that $x^{-n} = (x^{-1})^n$. If x does have an inverse, it is easy to prove that $x^m x^n = x^{m+n}$ for every $m, n \in \mathbb{Z}$. Thus $\{x^n : n \in \mathbb{Z}\}$ forms a subgroup of S.

If additive notation is being used, x^n might be denoted by nx instead. The index law mentioned above would then be written as: $mx + nx = (m+n)x$.

Theorem 1.24. *Suppose that S is a semigroup and that $x \in S$ has infinite order. Then the subsemigroup $T = \{x^n : n \in \mathbb{N}\}$ of S is isomorphic to* $(\mathbb{N}, +)$.

Proof. The mapping $n \mapsto x^n$ from $(\mathbb{N}, +)$ onto T is a surjective homomorphism, and so it will be sufficient to show that it is one-to-one. Suppose then that $x^m = x^n$ for some $m, n \in \mathbb{N}$ satisfying $m < n$. Then x^{n-m} is an identity for x^m, and the same statement holds for $x^{q(n-m)}$, where q denotes any positive integer. Suppose that s is any integer satisfying $s > m$. We can write $s - m = q(n-m) + r$ where q and m are non-negative integers and $r < (n-m)$. So $x^s = x^{s-m}x^m = x^{q(n-m)+r}x^m = x^r x^m$. It follows that $\{x^s : s > m\}$ is finite and hence that T is finite, contradicting our assumption that x has infinite order. □

Theorem 1.25. *Any finite semigroup S contains an idempotent.*

Proof. This statement is obviously true if S contains only one element. We shall prove it by induction on the number of elements in S. We make the inductive assumption that the theorem is true for all semigroups with fewer elements than S. Choose any $x \in S$. There are positive integers m and n satisfying $x^m = x^n$ and $m < n$. Then $x^{n-m}x^m = x^m$. Consider the subsemigroup $\{y \in S : x^{n-m}y = y\}$ of S. If this is the whole of S it contains x^{n-m} and so x^{n-m} is idempotent. If it is smaller than S, it contains an idempotent, by our inductive assumption. □

Exercise 1.3.1. Prove that any finite cancellative semigroup is a group.

1.4 Ideals

The terminology "ideal" is borrowed from ring theory. Given subsets A and B of a semigroup S, by AB we of course mean $\{ab : a \in A \text{ and } b \in B\}$.

Definition 1.26. Let S be a semigroup.

(a) L is a *left ideal* of S if and only if $\emptyset \neq L \subseteq S$ and $SL \subseteq L$.

(b) R is a *right ideal* of S if and only if $\emptyset \neq R \subseteq S$ and $RS \subseteq R$.

(c) I is an *ideal* of S if and only if I is both a left ideal and a right ideal of S.

An ideal I of S satisfying $I \neq S$ is called a *proper* ideal of S.

Sometimes for emphasis an ideal is called a "two sided ideal". We often deal with semigroups in which the operation is denoted by $+$. In this case the terminology may seem awkward for someone who is accustomed to working with rings. That is, a left ideal L satisfies $S + L \subseteq L$ and a right ideal R satisfies $R + S \subseteq R$.

Of special importance for us is the notion of *minimal* left and right ideals. By this we mean simply left or right ideals which are minimal with respect to set inclusion.

Definition 1.27. Let S be a semigroup.

(a) L is a *minimal left ideal* of S if and only if L is a left ideal of S and whenever J is a left ideal of S and $J \subseteq L$ one has $J = L$.

(b) R is a *minimal right ideal* of S if and only if R is a right ideal of S and whenever J is a right ideal of S and $J \subseteq R$ one has $J = R$.

(c) S is *left simple* if and only if S is a minimal left ideal of S.
(d) S is *right simple* if and only if S is a minimal right ideal of S.
(e) S is *simple* if and only if the only ideal of S is S.

We do not define a minimal ideal. As a consequence of Lemma 1.29 below, we shall see that there is at most one minimal two sided ideal of a semigroup. Consequently we use the term "smallest" to refer to an ideal which does not properly contain another ideal.

Observe that S is left simple if and only if it has no proper left ideals. Similarly, S is right simple if and only if it has no proper right ideals. Whenever one has a theorem about left ideals, there is a corresponding theorem about right ideals. We shall not usually state both results.

Clearly any semigroup which is either right simple or left simple must be simple. The following simple example (pun intended) shows that the converse fails.

Example 1.28. *Let $S = \{a, b, c, d\}$ where a, b, c, and d are any distinct objects and let S have the following multiplication table. Then S is simple but is neither left simple nor right simple.*

$\cdot$	a	b	c	d
a	a	b	a	b
b	a	b	a	b
c	c	d	c	d
d	c	d	c	d

One can laboriously verify that the table does define an associative operation. But 128 computations (of $(xy)z$ and $x(yz)$) are required, somewhat fewer if one is clever. It is usually much easier to establish associativity by representing the new semigroup as a subsemigroup of one with which we are already familiar. In this case, we can represent S as a semigroup of 3×3 matrices, by putting:

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

To verify the assertions of the example, note that $\{a, b\}$ and $\{c, d\}$ are right ideals of S and $\{a, c\}$ and $\{b, d\}$ are left ideals of S.

Lemma 1.29. *Let S be a semigroup.*

(a) *Let L_1 and L_2 be left ideals of S. Then $L_1 \cap L_2$ is a left ideal of S if and only if $L_1 \cap L_2 \neq \emptyset$.*

(b) *Let L be a left ideal of S and let R be a right ideal of S. Then $L \cap R \neq \emptyset$.*

Proof. Statement (a) is immediate. To see (b), let $x \in L$ and $y \in R$. Then $yx \in L$ because $x \in L$ and $yx \in R$ because $y \in R$. □

Lemma 1.30. *Let S be a semigroup.*

(a) *Let $x \in S$. Then xS is a right ideal, Sx is a left ideal and SxS is an ideal.*

(b) *Let $e \in E(S)$. Then e is a left identity for eS, a right identity for Se, and an identity for eSe.*

Proof. Statement (a) is immediate. For (b), let $e \in E(S)$. To see that e is a left identity for eS, let $x \in eS$ and pick $t \in S$ such that $x = et$. Then $ex = eet = et = x$. Likewise e is a right identity for Se. □

Theorem 1.31. *Let S be a semigroup.*

(a) *If S is left simple and $e \in E(S)$, then e is a right identity for S.*

(b) *If L is a left ideal of S and $s \in L$, then $Ss \subseteq L$.*

(c) *Let $\emptyset \neq L \subseteq S$. Then L is a minimal left ideal of S if and only if for each $s \in L$, $Ss = L$.*

Proof. (a) By Lemma 1.30 (a), Se is a left ideal of S, so $Se = S$ so Lemma 1.30 (b) applies.

(b) This follows immediately from the definition of left ideal.

(c) Necessity. By Lemma 1.30 (a) Ss is a left ideal and by (b) $Ss \subseteq L$ so, since L is minimal, $Ss = L$.

Sufficiency. Since $L = Ss$ for some $s \in L$, L is a left ideal. Let J be a left ideal of S with $J \subseteq L$ and pick $s \in J$. Then by (b), $Ss \subseteq J$ so $J \subseteq L = Ss \subseteq J$. □

We shall observe at the conclusion of the following definition that the objects defined there exist.

Definition 1.32. Let S be a semigroup.

(a) The smallest ideal of S which contains a given element $x \in S$ is called the *principal ideal generated by x*.

(b) The smallest left ideal of S which contains x is called the *principal left ideal of S generated by x*.

(c) The smallest right ideal of S which contains x is called the *principal right ideal generated by x*.

Theorem 1.33. *Let S be a semigroup and let $x \in S$.*

(a) *The principal ideal generated by x is $SxS \cup xS \cup Sx \cup \{x\}$.*

(b) *If S has an identity, then the principal ideal generated by x is SxS.*

(c) *The principal left ideal generated by x is $Sx \cup \{x\}$ and the principal right ideal generated by x is $xS \cup \{x\}$.*

Proof. This is Exercise 1.4.1. □

Exercise 1.4.1. Prove Theorem 1.33.

Exercise 1.4.2. Describe the ideals in each of the following semigroups. Also describe the minimal left ideals and the minimal right ideals in the cases in which these exist.

(i) $(\mathbb{N}, +)$.
(ii) $(\mathcal{P}(X), \cup)$, where X is any set.
(iii) $(\mathcal{P}(X), \cap)$, where X is any set.
(iv) $([0, 1], \cdot)$, where $\cdot$ denotes multiplication.
(v) The set of real-valued functions defined on a given set, with pointwise multiplication as the semigroup operation.
(vi) A left zero semigroup.
(vii) A right zero semigroup.

Exercise 1.4.3. Let X be any set. Describe the minimal left and right ideals in ${}^X X$.

Exercise 1.4.4. Let S be a commutative semigroup with an identity e. Prove that S has a proper ideal if and only if there is some $s \in S$ which has no e-inverse. In this case, prove that $\{s \in S : s \text{ has no } e\text{-inverse}\}$ is the unique maximal proper ideal of S.

1.5 Idempotents and Order

Intimately related to the notions of minimal left and minimal right ideals is the notion of minimal idempotents.

Definition 1.34. Let S be a semigroup and let $e, f \in E(S)$. Then
(a) $e \leq_L f$ if and only if $e = ef$,
(b) $e \leq_R f$ if and only if $e = fe$, and
(c) $e \leq f$ if and only if $e = ef = fe$.

In the semigroup of Example 1.28, one sees that $c \leq_L a$, $a \leq_L c$, $b \leq_L d$, $d \leq_L b$, $a \leq_R b$, $b \leq_R a$, $c \leq_R d$, and $d \leq_R c$, while the relation $\leq$ is simply equality on this semigroup.

Remark 1.35. *Let S be a semigroup. Then $\leq_L$, $\leq_R$, and $\leq$ are transitive and reflexive relations on $E(S)$. In addition, $\leq$ is antisymmetric.*

When we say that a point e is minimal with respect to a (not necessarily antisymmetric) relation $\preceq$ on a set B, we mean that if $f \in B$ and $f \preceq e$, then $e \preceq f$ (so if $\preceq$ *is* antisymmetric, the conclusion becomes $e = f$).

Theorem 1.36. *Let S be a semigroup and let $e \in E(S)$. The following statements are equivalent.*

(a) *The element e is minimal with respect to $\leq$.*
(b) *The element e is minimal with respect to $\leq_R$.*
(c) *The element e is minimal with respect to $\leq_L$.*

Proof. (b) implies (a). Assume that e is minimal with respect to $\leq_R$ and let $f \leq e$. Then $f = ef$ so $f \leq_R e$ so $e \leq_R f$. Then $e = fe = f$.

We show that (a) implies (b). (Then the equivalence of (a) and (c) follows by a left-right switch.) Assume that e is minimal with respect to $\leq$ and let $f \leq_R e$. Let $g = fe$. Then $gg = fefe = ffe = fe = g$ so $g \in E(S)$. Also, $g = fe = efe$ so $eg = eefe = efe = g = efee = ge$. Thus $g \leq e$ so $g = e$ by the minimality of e. That is, $e = fe$ so $e \leq_R f$ as required. □

As a consequence of Theorem 1.36, we are justified in making the following definition.

Definition 1.37. Let S be a semigroup. Then e is a *minimal idempotent* if and only if $e \in E(S)$ and e is minimal with respect to any (hence all) of the orders $\leq$, $\leq_R$, or $\leq_L$.

We see that the notions of "minimal idempotent" and "minimal left ideal" and "minimal right ideal" are intimately related. We remind the reader that there is a corresponding "right" version of the following theorem.

Theorem 1.38. *Let S be a semigroup and let $e \in E(S)$.*

(a) *If e is a member of some minimal left ideal (equivalently if Se is a minimal left ideal), then e is a minimal idempotent.*

(b) *If S is simple and e is minimal, then Se is a minimal left ideal.*

(c) *If every left ideal of S contains an idempotent and e is minimal, then Se is a minimal left ideal.*

(d) *If S is simple or every left ideal of S has an idempotent then the following statements are equivalent.*

(i) *e is minimal.*
(ii) *e is a member of some minimal left ideal of S.*
(iii) *Se is a minimal left ideal of S.*

Proof. (a) Let L be a minimal left ideal with $e \in L$. (The existence of a set L with this property is equivalent to Se being minimal, by Theorem 1.31 (c).) Then $L = Se$. Let $f \in E(S)$ with $f \leq e$. Then $f = fe$ so $f \in L$ so (by Theorem 1.31(c)) $L = Sf$ so $e \in Sf$ so by Lemma 1.30(b), $e = ef$ so $e = ef = f$.

(b) Let L be a left ideal with $L \subseteq Se$. We show that $Se \subseteq L$ (and hence $Se = L$). Pick some $s \in L$. Then $s \in Se$ so by Lemma 1.30(b), $s = se$. Also, since S is simple $SsS = S$, so pick u and v in S with $e = vsu$. Let $r = eue$ and $t = ev$. Then $tsr = evseue = evsue = eee = e$ and $er = eeue = eue = r$. Let $f = rts$. Then $ff = rtsrts = r(tsr)ts = rets = reevs = revs = rts = f$, so $f \in E(S)$. Also, $fe = rtse = rts = f$ and $ef = erts = rts = f$ so $f \leq e$ so $f = e$. Thus $Se = Sf = Srts \subseteq Ss \subseteq L$.

(c) Let L be a left ideal with $L \subseteq Se$. We show that $e \in L$ (so that $Se \subseteq L$ and hence $Se = L$). Pick an idempotent $t \in L$, and let $f = et$. Then $f \in L$. Since $t \in Se$, $t = te$. Thus $f = et = ete$. Therefore $ff = etet = ett = et = f$ so $f \in E(S)$. Also $ef = eete = ete = f$ and $fe = etee = ete = f$ so $f \leq e$ so $f = e$ and hence $e \in L$.

(d) This follows from (a), (b), and (c). □

We now obtain several characterizations of a group.

Theorem 1.39. *Let S be a semigroup. The following statements are equivalent.*

(a) *S is cancellative and simple and $E(S) \neq \emptyset$.*
(b) *S is both left simple and right simple.*
(c) *For all a and b in S, the equations $ax = b$ and $ya = b$ have solutions x, y in S.*
(d) *S is a group.*

Proof. (a) implies (b). Pick an idempotent e in S. We show first that e is a (two sided) identity for S. Let $x \in S$. Then $ex = eex$ so by left cancellation $x = ex$. Similarly, $x = xe$. To see that S is left simple, let L be a left ideal of S. Then LS is an ideal of S so $LS = S$, so pick $t \in L$ and $s \in S$ such that $e = ts$. Then $sts = se = s = es$ so cancelling on the right one has $st = e$. Thus $e \in L$ so $S = SL \subseteq L$. Consequently S is left simple, and similarly S is right simple.

(b) implies (c). Let $a, b \in S$. Then $aS = S$ so there is some $x \in S$ such that $ax = b$. Similarly, since $Sa = S$, there is some $y \in S$ such that $ya = b$.

(c) implies (d). Pick $a \in S$ and pick $e \in S$ such that $ea = a$. We show that e is a left identity for S. Let $b \in S$. We show that $eb = b$. Pick some $y \in S$ such that $ay = b$. Then $eb = eay = ay = b$.

Now given any $x \in S$ there is some $y \in S$ such that $yx = e$ so every element of S has a left e-inverse.

(d) implies (a). Trivially S is cancellative and $E(S) \neq \emptyset$. To see that S is simple, let I be an ideal of S and pick $x \in I$. Let y be the inverse of x. Then $xy \in I$ so $I = S$. □

As promised earlier, we now see that any semigroup with a left identity e such that every element has a right e-inverse must be (isomorphic to) the Cartesian product of a group with a right zero semigroup.

Theorem 1.40. *Let S be a semigroup and let e be a left identity for S such that for each $x \in S$ there is some $y \in S$ with $xy = e$. Let $Y = E(S)$ and let $G = Se$. Then Y is a right zero semigroup and G is a group and $S = GY \approx G \times Y$.*

Proof. We show first that:

($*$) For all $x \in Y$ and for all $y \in S$, $xy = y$.

To establish ($*$), let $x \in Y$ and $y \in S$ be given. Pick $z \in S$ such that $xz = e$. Then $xe = xxz = xz = e$. Therefore $xy = x(ey) = (xe)y = ey = y$, as required.

From ($*$) it follows that for all $x, y \in Y$, $xy = y$, and $Y \neq \emptyset$ because $e \in Y$, so to see that Y is a right zero semigroup, it suffices to show that it is a semigroup, that is that Y is closed. But this also follows from ($*$) since, given $x, y \in Y$ one has $xy = y \in Y$.

Now we establish that $G = Se$ is a group. By Lemma 1.30(b), e is a right identity for G. Now every element in S has a right e-inverse *in* S. So every element of G has a right e-inverse *in* S. By Theorem 1.10 we need only to show that every element of G has a right e-inverse *in* G. To this end let $x \in G$ be given and pick $y \in S$ such that $xy = e$. Then $ye \in G$ and $xye = ee = e$ so ye is as required. Since we also have $GG = SeSe \subseteq SSSe \subseteq Se = G$, it follows that G is a group.

Now define $\varphi : G \times Y \to S$ by $\varphi(g, y) = gy$. To see that φ is a homomorphism, let $(g_1, y_1), (g_2, y_2) \in G \times Y$. Then

$$\begin{aligned}\varphi(g_1, y_1)\varphi(g_2, y_2) &= g_1y_1g_2y_2 \\ &= g_1g_2y_2 && \text{(by } (*)\text{)} \\ &= g_1g_2y_1y_2 && \text{(by } (*)\text{)} \\ &= \varphi(g_1g_2, y_1y_2).\end{aligned}$$

To see that φ is surjective, let $s \in S$ be given. Then $se \in Se = G$, and so there exists $x \in Se$ such that $x(se) = (se)x = e$. We claim that $xs \in Y = E(S)$. Indeed,

$$\begin{aligned}xsxs &= xsexs && (\text{since } x \in G,\ ex = x) \\ &= xes \\ &= xs && (\text{since } x \in G,\ xe = x).\end{aligned}$$

Thus $(se, xs) \in G \times Y$ and $\varphi(se, xs) = sexs = es = s$. Since φ is onto S, we have established that $S = GY$.

Finally to see that φ is one-to-one, let $(g, y) \in G \times Y$ and let $s = \varphi(g, y)$ We show that $g = se$ and $y = xs$ where x is the (unique) inverse of se in Se. Now $s = gy$ so

$$\begin{aligned}se &= gye \\ &= ge && (\text{by } (*)\ ye = e) \\ &= g && (\text{since } g \in Se).\end{aligned}$$

Also

$$\begin{aligned}xs &= xgy \\ &= xgyey && (ye \in Y \text{ so by } (*)\ yey = y) \\ &= xsey \\ &= ey \\ &= y.\end{aligned}$$

□

We know that the existence of a left identity e for a semigroup S such that every element of S has a right e-inverse does not suffice to make S a group. A right zero semigroup is the standard example. Theorem 1.40 tells us that is essentially the only example.

Corollary 1.41. *Let S be a semigroup and assume that S has a unique left identity e and that every element of S has a right e-inverse. Then S is a group.*

Proof. This is Exercise 1.5.1. □

Exercise 1.5.1. Prove Corollary 1.41. (Hint: Consider $|Y|$ in Theorem 1.40.)

1.6 Minimal Left Ideals

We shall see in this section that many significant consequences follow from the existence of minimal left (or right) ideals, especially those with idempotents. This is important for us, because, as we shall see in Corollary 2.6, any compact right topological semigroup has minimal left ideals with idempotents.

We begin by establishing an easy consequence of Theorem 1.40.

Theorem 1.42. *Let S be a semigroup and assume that there is a minimal left ideal L of S which has an idempotent e. Then $L = XG \approx X \times G$ where X is the (left zero) semigroup of idempotents of L, and $G = eL = eSe$ is a group. All maximal groups in L are isomorphic to G.*

Proof. Given $x \in L$, Lx is a left ideal of S and $Lx \subseteq L$ so $Lx = L$ and hence there is some $y \in L$ such that $yx = e$. By Lemma 1.30(b), e is a right identity for $Le = L$. Therefore the right-left switch of Theorem 1.40 applies (with L replacing S). It is a routine exercise to show that the maximal groups of $X \times G$ are the sets of the form $\{x\} \times G$. □

Lemma 1.43. *Let S be a semigroup, let L be a left ideal of S, and let T be a left ideal of L.*

(a) *For all $t \in T$, Lt is a left ideal of S and $Lt \subseteq T$.*

(b) *If L is a minimal left ideal of S, then $T = L$. (So minimal left ideals are left simple.)*

(c) *If T is a minimal left ideal of L, then T is a left ideal of S.*

Proof. (a) $S(Lt) = (SL)t \subseteq Lt$ and $Lt \subseteq LT \subseteq T$.

(b) Pick any $t \in T$. By (a), Lt is a left ideal of S and $Lt \subseteq T \subseteq L$ so $Lt = L$ so $T = L$.

(c) Pick any $t \in T$. By (a), Lt is a left ideal of S, so Lt is a left ideal of L. Since $Lt \subseteq T$, $Lt = T$. Therefore, $ST = S(Lt) = (SL)t \subseteq Lt = T$. □

As a consequence of Lemma 1.43, if L is a left ideal of S and T is a left ideal of L and either L is minimal in S or T is minimal in L, then T is a left ideal of S. Of course, the right-left switch of this statement also holds. That is, if R is a right ideal of S and T is a right ideal of R and either R is minimal in S or T is minimal in R, then T is a right ideal of S. We see now that without *some* assumptions, T need not be a right ideal of S.

Example 1.44. *Let $X = \{0, 1, 2\}$ and let $S = {}^{X}X$. Let $R = \big\{f \in X : \mathrm{Range}(f) \subseteq \{0, 1\}\big\}$ and let $T = \{\overline{0}, a\}$ where $\overline{0}$ is the constant function and $a : \begin{array}{l} 0 \to 0 \\ 1 \to 0 \\ 2 \to 1 \end{array}$. Then R is a right ideal of S and T is a right ideal of R, but T is not a right ideal of S.*

Lemma 1.45. *Let S be a semigroup, let I be an ideal of S and let L be a minimal left ideal of S. Then $L \subseteq I$.*

Proof. This is Exercise 1.6.1. □

We now see that all minimal left ideals of a semigroup are intimately connected with each other.

Theorem 1.46. *Let S be a semigroup, let L be a minimal left ideal of S, and let $T \subseteq S$. Then T is a minimal left ideal of S if and only if there is some $a \in S$ such that $T = La$.*

Proof. Necessity. Pick $a \in T$. Then $SLa \subseteq La$ and $La \subseteq ST \subseteq T$ so La is a left ideal of S contained in T so $La = T$.

Sufficiency. Since $SLa \subseteq La$, La is a left ideal of S. Assume that B is a left ideal of S and $B \subseteq La$. Let $A = \{s \in L : sa \in B\}$. Then $A \subseteq L$ and $A \neq \emptyset$. We claim that A is a left ideal of S, so let $s \in A$ and let $t \in S$. Then $sa \in B$ so $tsa \in B$ and, since $s \in L$, $ts \in L$, so $ts \in A$ as required. Thus $A = L$ so $La \subseteq B$ so $La = B$. □

Corollary 1.47. *Let S be a semigroup. If S has a minimal left ideal, then every left ideal of S contains a minimal left ideal.*

Proof. Let L be a minimal left ideal of S and let J be a left ideal of S. Pick $a \in J$. Then by Theorem 1.46, La is a minimal left ideal which is contained in J. □

Theorem 1.48. *Let S be a semigroup and let $e \in E(S)$. Statements* (a) *through* (f) *are equivalent and imply statement* (g). *If either S is simple or every left ideal of S has an idempotent, then all statements are equivalent.*

(a) *Se is a minimal left ideal.*

(b) *Se is left simple.*

(c) *eSe is a group.*

(d) *$eSe = H(e)$.*

(e) *eS is a minimal right ideal.*

(f) *eS is right simple.*

(g) *e is a minimal idempotent.*

Proof. By Theorem 1.38(a), we have that (a) implies (g) and by Theorem 1.38(d), if either S is simple or every left ideal of S has an idempotent, then (g) implies (a).

We show that $\begin{array}{ccc} \text{(a)} & \Rightarrow & \text{(b)} \\ \Uparrow & & \Downarrow \\ \text{(d)} & \Leftarrow & \text{(c)} \end{array}$ from which $\begin{array}{ccc} \text{(e)} & \Rightarrow & \text{(f)} \\ \Uparrow & & \Downarrow \\ \text{(d)} & \Leftarrow & \text{(c)} \end{array}$ follows by left-right duality and the fact that (c) and (d) are two sided statements.

That (a) implies (b) follows from Lemma 1.43(b).

(b) implies (c). Trivially eSe is closed. By Lemma 1.30 e is a two sided identity for eSe. Also let $x = ese \in eSe$ be given. One has $x \in Se$ so Sx is a left ideal of Se and consequently $Sx = Se$, since Se is left simple. Thus $e \in Sx$, so pick $y \in S$ such that $e = yx$. Then $eye \in eSe$ and $eyex = eyx = ee = e$ so x has a left e-inverse in eSe.

(c) implies (d). Since eSe is a group and $e \in eSe$, one has $eSe \subseteq H(e)$. On the other hand, by Theorem 1.18, e is the identity of $H(e)$ so given $x \in H(e)$, one has that $x = exe \in eSe$, so $H(e) \subseteq eSe$

(d) implies (a). Let L be a left ideal of S with $L \subseteq Se$ and pick $t \in L$. Then $t \in Se$ so $et \in eSe$. Pick $x \in eSe$ such that $x(et) = e$. Then $xt = (xe)t = x(et) = e$ so $e \in L$ so $Se \subseteq SL \subseteq L$. □

We note that in the semigroup $(\mathbb{N}, \cdot)$, 1 is the only idempotent, and is consequently minimal, while $\mathbb{N}1$ is not a minimal left ideal. Thus Theorem 1.48(g) does not in general imply the other statements of Theorem 1.48.

We recall that in a ring there may be many minimal two sided ideals. This is because a "minimal ideal" in a ring is an ideal minimal among all ideals not equal to $\{0\}$, and one may have ideals I_1 and I_2 with $I_1 \cap I_2 = \{0\}$. By contrast, we see that a semigroup can have at most one minimal two-sided ideal.

Lemma 1.49. *Let S be a semigroup and let K be an ideal of S. If K is minimal in $\{J : J$ is an ideal of $S\}$ and I is an ideal of S, then $K \subseteq I$.*

Proof. By Lemma 1.29(b), $K \cap I \neq \emptyset$ so $K \cap I$ is an ideal contained in K so $K \cap I = K$. □

The terminology "minimal ideal" is widely used in the literature. Since, by Lemma 1.49, there can be at most one minimal ideal in a semigroup, we prefer the terminology "smallest ideal".

Definition 1.50. Let S be a semigroup. If S has a smallest ideal, then $K(S)$ is that smallest ideal.

We see that a simple condition guarantees the existence of $K(S)$.

Theorem 1.51. *Let S be a semigroup. If S has a minimal left ideal, then $K(S)$ exists and $K(S) = \bigcup\{L : L$ is a minimal left ideal of $S\}$.*

Proof. Let $I = \bigcup\{L : L$ is a minimal left ideal of $S\}$. By Lemma 1.45, if J is any ideal of S, then $I \subseteq J$, so it suffices to show that I is an ideal of S. We have that $I \neq \emptyset$ by assumption, so let $x \in I$ and let $s \in S$. Pick a minimal left ideal L of S such that $x \in L$. Then $sx \in L \subseteq I$. Also, by Theorem 1.46, Ls is a minimal left ideal of S so $Ls \subseteq I$ while $xs \in Ls$. □

Observe, however, that many common semigroups do not have a smallest ideal. This is true for example of both $(\mathbb{N}, +)$ and $(\mathbb{N}, \cdot)$.

Lemma 1.52. *Let S be a semigroup.*

(a) *Let L be a left ideal of S. Then L is minimal if and only if $Lx = L$ for every $x \in L$.*

(b) *Let I be an ideal of S. Then I is the smallest ideal if and only if $IxI = I$ for every $x \in I$.*

Proof. (a) If L is minimal and $x \in L$, then Lx is a left ideal of S and $Lx \subseteq L$ so $Lx = L$. Now assume $Lx = L$ for every $x \in L$ and let J be a left ideal of S with $J \subseteq L$. Pick $x \in J$. Then $L = Lx \subseteq LJ \subseteq J \subseteq L$.

(b) This is Exercise 1.6.2. □

Theorem 1.53. *Let S be a semigroup. If L is a minimal left ideal of S and R is a minimal right ideal of S, then $K(S) = LR$.*

Proof. Clearly LR is an ideal of S. We use Lemma 1.52 to show that $K(S) = LR$. So, let $x \in LR$. Then $LRxL$ is a left ideal of S which is contained in L so $LRxL = L$ and hence $LRxLR = LR$. □

Theorem 1.54. *Let S be a semigroup and assume that $K(S)$ exists and $e \in E(S)$. The following statements are equivalent and are implied by any of the equivalent statements* (a) *through* (f) *of Theorem* 1.48.

(h) $e \in K(S)$.

(i) $K(S) = SeS$.

Proof. By Theorem 1.51, it follows that Theorem 1.48(a) implies (h).

(h) implies (i). Since SeS is an ideal, we have $K(S) \subseteq SeS$. Since $e \in K(S)$, we have $SeS \subseteq K(S)$.

(i) implies (h). We have $e = eee \in SeS = K(S)$. □

Two natural questions are raised by Theorems 1.51 and 1.54. First, if $K(S)$ exists, is it the union of all minimal left ideals or at least is it either the union of all minimal left ideals or be the union of all minimal right ideals? Second, given that $K(S)$ exists and e is an idempotent in $K(S)$, must Se be a minimal left ideal, or at least must e be a minimal idempotent? The following example, known as the *bicyclic* semigroup, answers the weaker versions of both of these questions in the negative. Recall that $\omega = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$.

Example 1.55. *Let $S = \omega \times \omega$ and define an operation $\cdot$ on S by*

$$(m, n) \cdot (r, s) = \begin{cases} (m, s + n - r) & \text{if } n \geq r \\ (m + r - n, s) & \text{if } n < r. \end{cases}$$

Then S is a simple semigroup (so $K(S) = S$), S has no minimal left ideals and no minimal right ideals, $E(S) = \{(n, n) : n \in \omega\}$, and for each $n \in \omega$, $(n + 1, n + 1) \leq (n, n)$.

One may verify directly that the operation in Example 1.55 is associative. It is probably easier, however, to observe that S is isomorphic to a subsemigroup of ${}^{\mathbb{N}}\mathbb{N}$. Specifically define $f, g \in {}^{\mathbb{N}}\mathbb{N}$ by $f(t) = t + 1$ and $g(t) = \begin{cases} t - 1 & \text{if } t > 1 \\ 1 & \text{if } t = 1 \end{cases}$. Then given $n, r \in \omega$ one has $g^n \circ f^r = \begin{cases} g^{n-r} & \text{if } n \geq r \\ f^{r-n} & \text{if } n < r \end{cases}$. Consequently, one has

$$(f^m \circ g^n) \circ (f^r \circ g^s) = \begin{cases} f^m \circ g^{s+n-r} & \text{if } n \geq r \\ f^{m+r-n} \circ g^s & \text{if } n < r. \end{cases}$$

To see that the semigroup in Example 1.55 is simple, note that given any (m, n), $(k, r) \in S$, $(k, m) \cdot (m, n) \cdot (n, r) = (k, r)$. To see that S has no minimal left ideals, let L be a left ideal of S and pick $(m, n) \in L$. Then $\{(k, r) \in S : r > n\}$ is a left ideal of S which is properly contained in L. Similarly, if R is a right ideal of S and $(m, n) \in R$ then $\{(k, r) \in S : k > m\}$ is a right ideal of S which is properly contained in R.

It is routine to verify the assertions about the idempotents in Example 1.55.

Exercise 1.6.1. Prove Lemma 1.45.

Exercise 1.6.2. Prove Lemma 1.52(b).

Exercise 1.6.3. Let $S = \{f \in {}^{\mathbb{N}}\mathbb{N} : f$ is one-to-one and $\mathbb{N} \setminus f[\mathbb{N}]$ is infinite$\}$. Prove that $(S, \circ)$ is left simple (so S is a minimal left ideal of S) and S has no idempotents.

Exercise 1.6.4. Suppose that a minimal left ideal L of a semigroup is commutative. Prove that L is a group.

Exercise 1.6.5. Let S be a semigroup and assume that there is a minimal left ideal of S. Prove that, if $K(S)$ is commutative, then it is a group.

1.7 Minimal Left Ideals with Idempotents

We present here several results that have as hypothesis "Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent". These are important to us because, as we shall see in Corollary 2.6, this hypothesis holds in any compact right topological semigroup. (See Exercise 1.6.3 to show that the reference to the existence of an idempotent cannot be deleted from this hypothesis.)

Theorem 1.56. *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Then every minimal left ideal has an idempotent.*

Proof. Let L be a minimal left ideal with an idempotent e and let J be a minimal left ideal. By Theorem 1.46, there is some $x \in S$ such that $J = Lx$. By Theorem 1.42, $eL = eSe$ is a group, so let $y = eye$ be the inverse of exe in this group. Then $yx \in Lx = J$ and $yxyx = (ye)x(ey)x = y(exe)yx = eyx = yx$. □

We shall get left and right conclusions from this one sided hypothesis. We see now that in fact the right version follows from the left.

Lemma 1.57. *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Then there is a minimal right ideal of S which has an idempotent.*

Proof. Pick a minimal left ideal L of S and an idempotent $e \in L$. By Theorem 1.31(c) Se is a minimal left ideal of S so by Theorem 1.48 eS is a minimal right ideal of S and e is an idempotent in eS. □

Theorem 1.58. *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Let $T \subseteq S$.*

(a) *T is a minimal left ideal of S if and only if there is some $e \in E(K(S))$ such that $T = Se$.*

(b) *T is a minimal right ideal of S if and only if there is some $e \in E(K(S))$ such that $T = eS$.*

Proof. Pick a minimal left ideal L of S and an idempotent $f \in L$.

(a) Necessity. Since Sf is a left ideal contained in L, $Sf = L$. Thus by Theorem 1.48, fSf is a group. Pick any $a \in T$. Then $faf \in fSf$ so pick $x \in fSf$ such that $x(faf) = f$. Then

$$\begin{aligned} xaxa &= (xf)a(fx)a \\ &= (xfaf)xa \\ &= fxa \\ &= xa. \end{aligned}$$

Consequently, xa is an idempotent. Also $xa \in T$ while $T \subseteq K(S)$ by Theorem 1.51 so $xa \in E(K(S))$. Finally, Sxa is a left ideal contained in T, so $T = Sxa$.

Sufficiency. Since $e \in K(S)$, pick by Theorem 1.51 a minimal left ideal I of S with $e \in I$. Then $Se = I$ by Theorem 1.31(c).

(b) As a consequence of Lemma 1.57 this follows by a left-right switch. □

Theorem 1.59. *Let S be a semigroup, assume that there is a minimal left ideal of S which has an idempotent, and let $e \in E(S)$. The following statements are equivalent.*

(a) *Se is a minimal left ideal.*
(b) *Se is left simple.*
(c) *eSe is a group.*
(d) *$eSe = H(e)$.*
(e) *eS is a minimal right ideal.*
(f) *eS is right simple.*
(g) *e is a minimal idempotent.*
(h) *$e \in K(S)$.*
(i) *$K(S) = SeS$.*

Proof. By Corollary 1.47 and Theorem 1.56 every left ideal of S contains an idempotent so by Theorem 1.48 statements (a) through (g) are equivalent. By Theorems 1.51 and 1.54 we need only show that (h) implies (a). But this follows from Theorem 1.58. □

Theorem 1.60. *Let S be a semigroup, assume that there is a minimal left ideal of S which has an idempotent, and let e be an idempotent in S. There is a minimal idempotent f of S such that $f \leq e$.*

Proof. Se is a left ideal which thus contains a minimal left ideal L with an idempotent g by Corollary 1.47 and Theorem 1.56. Now $g \in Se$ so $ge = g$ by Lemma 1.30. Let $f = eg$. Then $ff = egeg = egg = eg = f$ so f is an idempotent. Also $f \in L$ so $L = Sf$ so by Theorem 1.59 f is a minimal idempotent. Finally $ef = eeg = eg = f$ and $fe = ege = eg = f$ so $f \leq e$. □

Theorem 1.61. *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Given any minimal left ideal L of S and any minimal right ideal R of S, there is an idempotent $e \in R \cap L$ such that $R \cap L = RL = eSe$ and eSe is a group.*

Proof. Let R and L be given. Pick by Theorem 1.58 an idempotent $f \in K(S)$ such that $L = Sf$. By Theorem 1.48, fSf is a group. Pick $a \in R$ and let x be the inverse of faf in fSf. Then $x \in Sf = L$ so $ax \in R \cap L$. By Theorem 1.51 $ax \in K(S)$. Also

$$\begin{aligned} axax &= a(xf)a(fx) \\ &= a(xfaf)x \\ &= afx \\ &= ax. \end{aligned}$$

Let $e = ax$. Then $eSe \subseteq Sx \subseteq L$ and $eSe \subseteq aS \subseteq R$ so $eSe \subseteq R \cap L$. To see that $R \cap L \subseteq eSe$, let $b \in R \cap L$. By Theorem 1.31 $L = Se$ and $R = eS$ so by Lemma 1.30, $b = eb = be$. Thus $b = eb = ebe \in eSe$.

Now $RL = eSSe \subseteq eSe \subseteq RL$, so $RL = eSe$.

As we have observed, $e \in K(S)$, so by Theorem 1.59 eSe is a group. □

Lemma 1.62. *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Then all minimal left ideals of S are isomorphic.*

Proof. Let L be a minimal left ideal of S with an idempotent e. Then $L = Se$ so by Theorem 1.59 eSe is a group.

We claim first that given any $s \in K(S)$ and any $t \in S$, $s(ese)^{-1} = st(este)^{-1}$, where the inverses are taken in eSe. Indeed, using the fact that $(ese)^{-1}e = e(ese)^{-1} = (ese)^{-1}$,

$$s(ese)^{-1}s(ese)^{-1} = s(ese)^{-1}ese(ese)^{-1} = s(ese)^{-1}e = s(ese)^{-1}$$

and similarly $st(este)^{-1}$ is an idempotent. By Lemma 1.57 and Theorem 1.51, $K(S) = \bigcup\{R : R \text{ is a minimal right ideal of } S\}$. Pick a minimal right ideal R of S such that $s \in R$. Then $s(ese)^{-1}$ and $st(este)^{-1}$ are both idempotents in $R \cap L$, which is a group by Theorem 1.61. Thus $s(ese)^{-1} = st(este)^{-1}$ as claimed.

Now let L' be any other minimal left ideal of S. By Theorem 1.59, eS is a minimal right ideal of S so by Theorem 1.61 $L' \cap eS$ is a group so pick an idempotent $d \in L' \cap eS$. Notice that $L' = Sd$ and $dS = eS$. In particular, by Lemma 1.30(b), $de = e$, $ed = d$, and for any $s \in L'$, $sd = s$.

Define $\psi : Sd \to Se$ by $\psi(s) = s(ese)^{-1}dse$, where the inverse is in the group eSe. We claim first that ψ is a homomorphism. To this end, let $s, t \in Sd$. Then

$$\begin{aligned}
\psi(s)\psi(t) &= s(ese)^{-1}dset(ete)^{-1}dte \\
&= s(ese)^{-1}dsete(ete)^{-1}dte && (\, e(ete)^{-1} = (ete)^{-1}\,) \\
&= s(ese)^{-1}dsedte \\
&= s(ese)^{-1}dste && (\, ed = d \text{ and } sd = s\,) \\
&= st(este)^{-1}dste && (\, s(ese)^{-1} = st(este)^{-1}\,) \\
&= \psi(st).
\end{aligned}$$

Now define $\gamma : Se \to Sd$ by $\gamma(t) = t(dtd)^{-1}etd$ where the inverse is in dSd, which is a group by Theorem 1.59. We claim that γ is the inverse of ψ (and hence ψ takes Sd one-to-one onto Se). To this end, let $s \in L'$. Then $ds \in L'$ so Sds is a left ideal contained in L' and thus $L' = Sds$. So pick $x \in S$ such that $s = xds$. Then

$$\begin{aligned}
\gamma\big(\psi(s)\big) &= \psi(s)(d\psi(s)d)^{-1}e\psi(s)d \\
&= s(ese)^{-1}dse(ds(ese)^{-1}dsed)^{-1}es(ese)^{-1}dsed \\
&= xds(ese)^{-1}dsed(ds(ese)^{-1}dsed)^{-1}ese(ese)^{-1}dsed \\
&= xdedsed \\
&= xddsd \\
&= xds \\
&= s.
\end{aligned}$$

Similarly, if $t \in L$, then $\psi\big(\gamma(t)\big) = t$. □

We now analyze in some detail the structure of a particular semigroup. Our motive is that this allows us to analyze the structure of the smallest ideal of any semigroup that has a minimal left ideal with an idempotent.

Theorem 1.63. *Let X be a left zero semigroup, let Y be a right zero semigroup, and let G be a group. Let e be the identity of G, fix $u \in X$ and $v \in Y$ and let $[\,,\,] : Y \times X \to G$ be a function such that $[y, u] = [v, x] = e$ for all $y \in Y$ and all $x \in X$. Let $S = X \times G \times Y$ and define an operation $\cdot$ on S by $(x, g, y) \cdot (x', g', y') = (x, g[y, x']g', y')$. Then S is a simple semigroup (so that $K(S) = S = X \times G \times Y$) and each of the following statements holds.*

(a) *For every $(x, y) \in X \times Y$, $(x, [y, x]^{-1}, y)$ is an idempotent (where the inverse is taken in G) and all idempotents are of this form. In particular, the idempotents in $X \times G \times \{v\}$ are of the form (x, e, v) and the idempotents in $\{u\} \times G \times Y$ are of the form (u, e, y).*

(b) *For every $y \in Y$, $X \times G \times \{y\}$ is a minimal left ideal of S and all minimal left ideals of S are of this form.*

(c) *For every $x \in X$, $\{x\} \times G \times Y$ is a minimal right ideal of S and all minimal right ideals of S are of this form.*

(d) *For every* $(x, y) \in X \times Y$, $\{x\} \times G \times \{y\}$ *is a maximal group in* S *and all maximal groups in* S *are of this form.*

(e) *The minimal left ideal* $X \times G \times \{v\}$ *is the direct product of* X, G, *and* $\{v\}$ *and the minimal right ideal* $\{u\} \times G \times Y$ *is the direct product of* $\{u\}$, G, *and* Y.

(f) *All maximal groups in* S *are isomorphic to* G.

(g) *All minimal left ideals of* S *are isomorphic to* $X \times G$ *and all minimal right ideals of* S *are isomorphic to* $G \times Y$.

Proof. The associativity of $\cdot$ is immediate. To see that S is simple, let (x, g, y), $(x', g', y') \in S$. By Lemma 1.52(b), it suffices to show that $(x', g', y') \in S(x, g, y)S$. To see this, let $h = g'g^{-1}[y, x]^{-1}g^{-1}[y, x]^{-1}$. Then $(x', g', y') = (x', h, y)\cdot(x, g, y)\cdot(x, g, y')$.

(a) That $(x, [y, x]^{-1}, y)$ is an idempotent is immediate. Given an idempotent (x, g, y), one has that $g[y, x]g = g$ so $g = [y, x]^{-1}$.

(b) Let $y \in Y$. Trivially $X \times G \times \{y\}$ is a left ideal of S. To see that it is minimal, let $(x, g), (x', g') \in X \times G$. It suffices by Lemma 1.52(a) to note that $(x', g', y) = (x', g'g^{-1}[y, x]^{-1}, y)\cdot(x, g, y)$. Given any minimal left ideal L of S, pick $(x, g, y) \in L$. Then $L \cap (X \times G \times \{y\}) \neq \emptyset$ so $L = X \times G \times \{y\}$.

Statement (c) is the right-left switch of statement (b).

(d) By the equivalence of (h) and (d) in Theorem 1.59 we have that the maximal groups in S are precisely the sets of the form fSf where f is an idempotent of $K(S) = S$. That is, by (a), where $f = (x, [y, x]^{-1}, y)$. Since

$$(x, [y, x]^{-1}, y)S(x, [y, x]^{-1}, y) = \{x\} \times G \times \{y\},$$

we are done.

(e) We show that $X \times G \times \{v\}$ is a direct product, the other statement being similar. Let $(x, g), (x', g') \in X \times G$. Then

$$\begin{aligned}(x, g, v) \cdot (x', g', v) &= (x, g[v, x']g', v)\\ &= (x, geg', v)\\ &= (xx', gg', vv).\end{aligned}$$

(f) Trivially $\{u\} \times G \times \{v\}$ is isomorphic to G. Now, let $(x, y) \in X \times Y$. Then $\{x\} \times G \times \{v\}$ and $\{u\} \times G \times \{v\}$ are maximal groups in the minimal left ideal $X \times G \times \{v\}$, hence are isomorphic by Theorem 1.42. Also $\{x\} \times G \times \{v\}$ and $\{x\} \times G \times \{y\}$ are maximal groups in the minimal right ideal $\{x\} \times G \times Y$, hence are isomorphic by the left-right switch of Theorem 1.42.

(g) By Lemma 1.62 all minimal left ideals of S are isomorphic and by (e) $X \times G \times \{v\}$ is isomorphic to $X \times G$. The other conclusion follows similarly. □

Note that in Theorem 1.63, the set S is the cartesian product of X, G, and Y, but is not the direct product unless $[y, x] = e$ for every $(y, x) \in Y \times X$.

Observe that as a consequence of Theorem 1.63(g) we have that for any $y \in Y$, $X \times G \times \{y\} \approx X \times G$. However, there is no transparent isomorphism unless $[y, x] = e$ for all $x \in X$, such as when $y = v$.

Theorem 1.63 spells out in detail the structure of $X \times G \times Y$. We see now that this is in fact the structure of the smallest ideal of any semigroup which has a minimal left ideal with an idempotent.

Theorem 1.64 (The Structure Theorem). *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Let R be a minimal right ideal of S, let L be a minimal left ideal of S, let $X = E(L)$, let $Y = E(R)$, and let $G = RL$. Define an operation $\cdot$ on $X \times G \times Y$ by $(x, g, y) \cdot (x', g', y') = (x, gyx'g', y')$. Then $X \times G \times Y$ satisfies the conclusions of Theorem 1.63 (where $[y, x] = yx$) and $K(S) \approx X \times G \times Y$. In particular:*

(a) *The minimal right ideals of S partition $K(S)$ and the minimal left ideals of S partition $K(S)$.*

(b) *The maximal groups in $K(S)$ partition $K(S)$.*

(c) *All minimal right ideals of S are isomorphic and all minimal left ideals of S are isomorphic.*

(d) *All maximal groups in $K(S)$ are isomorphic.*

Proof. After noting that, by Lemma 1.43 and Theorem 1.51, the minimal left ideals of S and of $K(S)$ are identical (and the minimal right ideals of S and of $K(S)$ are identical), the "in particular" conclusions follow immediately from Theorem 1.63. So it suffices to show that $X \times G \times Y$ satisfies the hypotheses of Theorem 1.63 with $[y, x] = yx$ and that $K(S) \approx X \times G \times Y$.

We know by Lemma 1.57 that S has a minimal right ideal with an idempotent (so R exists) and hence by Theorem 1.56 R has an idempotent. We know by Theorem 1.61 that RL is a group and we know by Theorem 1.42 that X is a left zero semigroup and Y is a right zero semigroup. Let e be the identity of $RL = R \cap L$ and let $u = v = e$. Given $y \in Y$ one has, since Y is a right zero semigroup, that $[y, u] = yu = u = e$. Similarly, given $x \in X$, $[v, x] = e$. Consequently the hypotheses of Theorem 1.63 are satisfied.

Define $\varphi : X \times G \times Y \to S$ by $\varphi(x, g, y) = xgy$. We claim that φ is an isomorphism onto $K(S)$. From the definition of the operation in $X \times G \times Y$ we see immediately that φ is a homomorphism. By Theorem 1.42 we have that $L = XG$ and $R = GY$. By Theorem 1.53, $K(S) = LR = XGGY = XGY = \varphi[X \times G \times Y]$. Thus it suffices to produce an inverse for φ.

For each $t \in K(S)$, let $\gamma(t)$ be the inverse of ete in $eSe = G$. Then $t\gamma(t) = t\gamma(t)e \in Se = L$ and

$$\begin{aligned} t\gamma(t)t\gamma(t) &= t\gamma(t)ete\gamma(t) \\ &= te\gamma(t) \\ &= t\gamma(t), \end{aligned}$$

so $t\gamma(t) \in X$. Similarly, $\gamma(t)t \in Y$.

Define $\tau : K(S) \to X \times G \times Y$ by $\tau(t) = (t\gamma(t), ete, \gamma(t)t)$. We claim that $\tau = \varphi^{-1}$. So let $(x, g, y) \in X \times G \times Y$. Then

$$\tau\big(\varphi(x, g, y)\big) = (xgy\gamma(xgy), exgye, \gamma(xgy)xgy).$$

Now

$$\begin{aligned} xgy\gamma(xgy) &= xxgy\gamma(xgy) && (x = xx) \\ &= xexgye\gamma(xgy) && \big(x = xe \text{ and } \gamma(xgy) = e\gamma(xgy)\big) \\ &= xe \\ &= x. \end{aligned}$$

Similarly $\gamma(xgy)xgy = y$. Since also $exgye = ege = g$, we have that $\tau = \varphi^{-1}$ as required. □

The following theorem enables us to identify the smallest ideal of many semigroups that arise in topological applications.

Theorem 1.65. *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Let T be a subsemigroup of S and assume also that T has a minimal left ideal with an idempotent. If $K(S) \cap T \neq \emptyset$, then $K(T) = K(S) \cap T$.*

Proof. By Theorem 1.51, $K(T)$ exists so, since $K(S) \cap T$ is an ideal of T, $K(T) \subseteq K(S) \cap T$. For the reverse inclusion, let $x \in K(S) \cap T$ be given. Then Tx is a left ideal of T so by Corollary 1.47 and Theorem 1.56 Tx contains a minimal left ideal Te of T for some idempotent $e \in T$. Now $x \in K(S)$ so by Theorem 1.51 pick a minimal left ideal L of S with $x \in L$. Then $L = Sx$ and $e \in Tx \subseteq Sx$ so $L = Se$ so $x \in Se$ so by Lemma 1.30, $x = xe \in Te \subseteq K(T)$. □

We know from the Structure Theorem (Theorem 1.64) that maximal groups in the smallest ideal are isomorphic. It will be convenient for us later to know an explicit isomorphism between them.

Theorem 1.66. *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Let $e, f \in E\big(K(S)\big)$. If g is the inverse of efe in eSe, then the function $\varphi : eSe \to fSf$ defined by $\varphi(x) = fxgf$ is an isomorphism.*

Proof. To see that φ is a homomorphism, let $x, y \in eSe$. Then

$$\begin{aligned} \varphi(x)\varphi(y) &= fxgffygf \\ &= fxgfygf \\ &= fxgefeygf && (ge = g,\ ey = y) \\ &= fxeygf \\ &= fxygf \\ &= \varphi(xy). \end{aligned}$$

To see that φ is one-to-one, let x be in the kernel of φ. Then

$$\begin{aligned} fxgf &= f \\ efxgfe &= efe \\ efexgefe &= efe && (ex = x,\ ge = g) \\ efexe &= efe \\ efex &= efee \\ x &= e && \text{(left cancellation in } eSe). \end{aligned}$$

To see that φ is onto fSf, let $y \in fSf$ and let h and k be the inverses of fgf and fef respectively in fSf. Then $ekyhe \in eSe$ and

$$\begin{aligned}\varphi(ekyhe) &= fekyhegf \\ &= fefkyhgf \qquad (fk = k,\ eg = g) \\ &= fyhgf \qquad (fefk = f) \\ &= fyhfgf \qquad (h = hf) \\ &= fyf \qquad (hfgf = f) \\ &= y.\end{aligned}$$

□

We conclude the chapter with a theorem characterizing arbitrary elements of $K(S)$.

Theorem 1.67. *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Let $s \in S$. The following statements are equivalent.*

(a) $s \in K(S)$.
(b) *For all $t \in S$, $s \in Sts$.*
(c) *For all $t \in S$, $s \in stS$.*
(d) *For all $t \in S$, $s \in stS \cap Sts$.*

Proof. (a) implies (d). Pick by Theorem 1.51 and Lemma 1.57 a minimal left ideal L of S and a minimal right ideal R of S with $s \in L \cap R$. Let $t \in S$. Then $ts \in L$ so Sts is a left ideal contained in L so $Sts = L$. Similarly $stS = R$.

The facts that (d) implies (c) and (d) implies (b) are trivial.

(b) implies (a). Pick $t \in K(S)$. Then $s \in Sts \subseteq K(S)$.

Similarly (c) implies (a). □

Exercise 1.7.1. Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Prove that if $K(S) \neq S$ and $x \in K(S)$, then x is neither left nor right cancelable in S. (Hint: If x is a member of the minimal left ideal L, then $L = Sx = Lx$.)

Exercise 1.7.2. Identify $K(S)$ for those semigroups S in Exercises 1.4.2 and 1.4.3 for which the smallest ideal exists.

Exercise 1.7.3. Let S and T be semigroups and let $h : S \to T$ be a surjective homomorphism. If S has a smallest ideal, show that T does as well and that $K(T) = h[K(S)]$.

Notes

Much of the material in this chapter is based on the treatment in [39, Section II.1]. The presentation of the Structure Theorem was suggested to us by J. Pym and is based on his treatment in [202]. The Structure Theorem (Theorem 1.64) is due to A. Suschkewitsch [231] in the case of finite semigroups and to D. Rees [210] in the general case.

Chapter 2

Right Topological (and Semitopological and Topological) Semigroups

In this (and subsequent) chapters, we assume that the reader has mastered an introductory course in general topology. In particular, we expect familiarity with the notions of continuous functions, nets, and compactness.

2.1 Topological Hierarchy

Definition 2.1. (a) A *right topological semigroup* is a triple $(S, \cdot, \mathcal{T})$ where $(S, \cdot)$ is a semigroup, $(S, \mathcal{T})$ is a topological space, and for all $x \in S$, $\rho_x : S \to S$ is continuous.

(b) A *left topological semigroup* is a triple $(S, \cdot, \mathcal{T})$ where $(S, \cdot)$ is a semigroup, $(S, \mathcal{T})$ is a topological space, and for all $x \in S$, $\lambda_x : S \to S$ is continuous.

(c) A *semitopological semigroup* is a right topological semigroup which is also a left topological semigroup.

(d) A *topological semigroup* is a triple $(S, \cdot, \mathcal{T})$ where $(S, \cdot)$ is a semigroup, $(S, \mathcal{T})$ is a topological space, and $\cdot : S \times S \to S$ is continuous.

(e) A *topological group* is a triple $(S, \cdot, \mathcal{T})$ such that $(S, \cdot)$ is a group, $(S, \mathcal{T})$ is a topological space, $\cdot : S \times S \to S$ is continuous, and $\mathrm{In} : S \to S$ is continuous (where $\mathrm{In}(x)$ is the inverse of x in S).

We did not include any separation axioms in the definitions given above. However, all of our applications involve Hausdorff spaces. So we shall be assuming throughout, except in Chapter 7, that all hypothesized topological spaces are Hausdorff.

In a right topological semigroup we say that the operation "$\cdot$" is "right continuous". We should note that many authors use the term "left topological" for what we call "right topological" and vice versa. One may reasonably ask why someone would refer to an operation for which multiplication on the right is continuous as "left continuous". The people who do so ask why we refer to an operation which is continuous in the left variable as "right continuous".

We shall customarily not mention either the operation or the topology and say something like "let S be a right topological semigroup".

Note that trivially each topological group is a topological semigroup, each topological semigroup is a semitopological semigroup and each semitopological semigroup is both a left and right topological semigroup.

Of course any semigroup which is not a group provides an example of a topological semigroup which is not a topological group simply by providing it with the discrete topology. It is the content of Exercise 2.1.1 to show that there is a topological semigroup which is a group but is not a topological group.

It is a celebrated theorem of R. Ellis [84], that if S is a locally compact *semitopological* semigroup which is a group then S is a topological group. That is, if S is locally compact and a group, then separate continuity implies joint continuity and continuity of the inverse. We shall prove this theorem in the last section of this chapter. For an example of a semitopological semigroup which is a group but is not a topological semigroup see Exercise 9.2.7.

It is the content of Exercise 2.1.2 that there is a semitopological semigroup which is not a topological semigroup.

Recall that given any topological space $(X, \mathcal{T})$, the *product topology* on ${}^X X$ is the topology with subbasis $\{\pi_x^{-1}[U] : x \in X \text{ and } U \in \mathcal{T}\}$, where for $f \in {}^X X$ and $x \in X$, $\pi_x(f) = f(x)$. Whenever we refer to a "basic" or "subbasic" open set in ${}^X X$, we mean sets defined in terms of this subbasis. The product topology is also often referred to as the *topology of pointwise convergence* . The reason for this terminology is that a net $\langle f_\iota \rangle_{\iota \in I}$ converges to f in ${}^X X$ if and only if $\langle f_\iota(x) \rangle_{\iota \in I}$ converges to $f(x)$ for every $x \in X$.

Theorem 2.2. *Let $(X, \mathcal{T})$ be any topological space and let $\mathcal{V}$ be the product topology on ${}^X X$.*

(a) *$({}^X X, \circ, \mathcal{V})$ is a right topological semigroup.*

(b) *For each $f \in {}^X X$, λ_f is continuous if and only if f is continuous.*

Proof. Let $f \in {}^X X$. Suppose that the net $\langle g_\iota \rangle_{\iota \in I}$ converges to g in $({}^X X, \mathcal{V})$. Then, for any $x \in X$, $\langle g_\iota(f(x)) \rangle_{\iota \in I}$ converges to $g(f(x))$ in X. Thus $\langle g_\iota \circ f \rangle_{\iota \in I}$ converges to $g \circ f$ in $({}^X X, \mathcal{V})$, and so ρ_f is continuous. This establishes (a).

Now λ_f is continuous if and only if $\langle f(g_\iota(x)) \rangle_{\iota \in I}$ converges to $f(g(x))$ for every net $\langle g_\iota \rangle_{\iota \in I}$ converging to g in $({}^X X, \mathcal{V})$ and every $x \in X$. This is obviously the case if f is continuous. Conversely, suppose that λ_f is continuous. Let $\langle x_\iota \rangle_{\iota \in I}$ be a net converging to x in X. We define $g_\iota = \overline{x_\iota}$, the function in ${}^X X$ which is constantly equal to x_ι and $g = \overline{x}$. Then $\langle g_\iota \rangle_{\iota \in I}$ converges to g in ${}^X X$ and so $\langle f \circ g_\iota \rangle_{\iota \in I}$ converges to $f \circ g$. This means that $\langle f(x_\iota) \rangle_{\iota \in I}$ converges to $f(x)$. Thus f is continuous, and we have established (b). □

Corollary 2.3. *Let X be a topological space. The following statements are equivalent.*

(a) *${}^X X$ is a topological semigroup.*

(b) *${}^X X$ is a semitopological semigroup.*

(c) *For all $f \in {}^X X$, f is continuous.*

(d) *X is discrete.*

Proof. Exercise 2.1.3 □

If X is any nondiscrete space, it follows from Theorem 2.2 and Corollary 2.3 that ${}^X X$ is a right topological semigroup which is not left topological. Of course, reversing the order of operation yields a left topological semigroup which is not right topological.

Definition 2.4. Let S be a right topological semigroup. The *topological center* of S is the set $\Lambda(S) = \{x \in S : \lambda_x \text{ is continuous}\}$.

Thus a right topological semigroup S is a semitopological semigroup if and only if $\Lambda(S) = S$. Note that trivially the algebraic center of a right topological semigroup is contained in its topological center.

Exercise 2.1.1. Let $\mathcal{T}$ be the topology on $\mathbb{R}$ with basis $\mathcal{B} = \{(a, b] : a, b \in \mathbb{R}$ and $a < b\}$. Prove that $(\mathbb{R}, +, \mathcal{T})$ is a topological semigroup but not a topological group.

Exercise 2.1.2. Let $S = \mathbb{R} \cup \{\infty\}$, let S have the topology of the one point compactification of $\mathbb{R}$ (with its usual topology), and define an operation $*$ on S by

$$x * y = \begin{cases} x + y & \text{if } x, y \in \mathbb{R} \\ \infty & \text{if } x = \infty \text{ or } y = \infty. \end{cases}$$

(a) Prove that $(S, *)$ is a semitopological semigroup.
(b) Show that $* : S \times S \to S$ is not continuous at (∞, ∞).

Exercise 2.1.3. Prove Corollary 2.3.

2.2 Compact Right Topological Semigroups

We shall be concerned throughout this book with certain compact right topological semigroups. Of fundamental importance is the following theorem.

Theorem 2.5. *Let S be a compact right topological semigroup. Then $E(S) \neq \emptyset$.*

Proof. Let $\mathcal{A} = \{T \subseteq S : T \neq \emptyset, T \text{ is compact, and } T \cdot T \subseteq T\}$. That is, $\mathcal{A}$ is the set of compact subsemigroups of S. We show that $\mathcal{A}$ has a minimal member using Zorn's Lemma. Since $S \in \mathcal{A}$, $\mathcal{A} \neq \emptyset$. Let $\mathcal{C}$ be a chain in $\mathcal{A}$. Then $\mathcal{C}$ is a collection of closed subsets of the compact space S with the finite intersection property, so $\bigcap \mathcal{C} \neq \emptyset$ and $\bigcap \mathcal{C}$ is trivially compact and a semigroup. Thus $\bigcap \mathcal{C} \in \mathcal{A}$, so we may pick a minimal member A of $\mathcal{A}$.

Pick $x \in A$. We shall show that $xx = x$. (It will follow that $A = \{x\}$, but we do not need this.) We start by showing that $Ax = A$. Let $B = Ax$. Then $B \neq \emptyset$ and since $B = \rho_x[A]$, B is the continuous image of a compact space, hence compact. Also

$BB = AxAx \subseteq AAAx \subseteq Ax = B$. Thus $B \in \mathcal{A}$. Since $B = Ax \subseteq AA \subseteq A$ and A is minimal, $B = A$.

Let $C = \{y \in A : yx = x\}$. Since $x \in A = Ax$, we have $C \neq \emptyset$. Also, $C = A \cap \rho_x^{-1}[\{x\}]$, so C is closed and hence compact. Given $y, z \in C$ one has $yz \in AA \subseteq A$ and $yzx = yx = x$ so $yz \in C$. Thus $C \in \mathcal{A}$. Since $C \subseteq A$ and A is minimal, we have $C = A$ so $x \in C$ and so $xx = x$ as required. □

In Section 1.7 there were several results which had as part of their hypotheses "Let S be a semigroup and assume there is a minimal left ideal of S which has an idempotent." Because of the following corollary, we are able to incorporate all of these results.

Corollary 2.6. *Let S be a compact right topological semigroup. Then every left ideal of S contains a minimal left ideal. Minimal left ideals are closed, and each minimal left ideal has an idempotent.*

Proof. If L is any left ideal L of S and $x \in L$, then Sx is a compact left ideal contained in L. (It is compact because $Sx = \rho_x[S]$.) Consequently any minimal left ideal is closed and by Theorem 2.5 any minimal left ideal contains an idempotent. Thus we need only show that any left ideal of S contains a minimal left ideal. So let L be a left ideal of S and let $\mathcal{A} = \{T : T \text{ is a closed left ideal of } S \text{ and } T \subseteq L\}$. Applying Zorn's Lemma to $\mathcal{A}$, one gets a left ideal M minimal among all closed left ideals contained in L. But since every left ideal contains a closed left ideal, M is a minimal left ideal. □

We now deduce some consequences of Corollary 2.6. Note that these consequences apply in particular to any finite semigroup S, since S is a compact topological semigroup when provided with the discrete topology.

Theorem 2.7. *Let S be a compact right topological semigroup.*

(a) *Every right ideal of S contains a minimal right ideal which has an idempotent.*

(b) *Let $T \subseteq S$. Then T is a minimal left ideal of S if and only if there is some $e \in E\big(K(S)\big)$ such that $T = Se$.*

(c) *Let $T \subseteq S$. Then T is a minimal right ideal of S if and only if there is some $e \in E\big(K(S)\big)$ such that $T = eS$.*

(d) *Given any minimal left ideal L of S and any minimal right ideal R of S, there is an idempotent $e \in R \cap L$ such that $R \cap L = eSe$ and eSe is a group.*

Proof. (a) Corollary 2.6, Lemma 1.57, Corollary 1.47, and Theorem 1.56.

(b) and (c). Corollary 2.6 and Theorem 1.58.

(d) Corollary 2.6 and Theorem 1.61. □

Theorem 2.8. *Let S be a compact right topological semigroup. Then S has a smallest (two sided) ideal $K(S)$ which is the union of all minimal left ideals of S and also the union of all minimal right ideals of S. Each of $\{Se : e \in E\big(K(S)\big)\}$, $\{eS : e \in E\big(K(S)\big)\}$, and $\{eSe : e \in E\big(K(S)\big)\}$ are partitions of $K(S)$.*

Proof. Corollary 2.6 and Theorems 1.58, 1.61, and 1.64. □

Theorem 2.9. *Let S be a compact right topological semigroup and let $e \in E(S)$. The following statements are equivalent.*

(a) *Se is a minimal left ideal.*
(b) *Se is left simple.*
(c) *eSe is a group.*
(d) *$eSe = H(e)$.*
(e) *eS is a minimal right ideal.*
(f) *eS is right simple.*
(g) *e is a minimal idempotent.*
(h) *$e \in K(S)$.*
(i) *$K(S) = SeS$.*

Proof. Corollary 2.6 and Theorem 1.59. □

Theorem 2.10. *Let S be a compact right topological semigroup. Let $s \in S$. The following statements are equivalent.*

(a) *$s \in K(S)$.*
(b) *For all $t \in S$, $s \in Sts$.*
(c) *For all $t \in S$, $s \in stS$.*
(d) *For all $t \in S$, $s \in stS \cap Sts$.*

Proof. Corollary 2.6 and Theorem 1.67. □

The last few results have had purely algebraic conclusions. We now obtain a result with both topological and algebraic conclusions. Suppose that we have two topological spaces which are also semigroups. We say that they are *topologically and algebraically isomorphic* if there is a function from one of them onto the other which is both an isomorphism and a homeomorphism.

Theorem 2.11. *Let S be a compact right topological semigroup.*

(a) *All maximal subgroups of $K(S)$ are (algebraically) isomorphic.*

(b) *Maximal subgroups of $K(S)$ which lie in the same minimal right ideal are topologically and algebraically isomorphic.*

(c) *All minimal left ideals of S are homeomorphic. In fact, if L and L' are minimal left ideals of S and $z \in L'$, then $\rho_{z|L}$ is a homeomorphism from L onto L'.*

Proof. (a) Corollary 2.6 and Theorem 1.66.

(b) Let R be a minimal right ideal of S and let $e, f \in E(R)$. Then eS and fS are right ideals contained in R and so $R = eS = fS$. Then by Lemma 1.30, $ef = f$ and $fe = e$. Let g be the inverse of efe in the group eSe and define $\varphi : eSe \to fSf$ by $\varphi(x) = fxgf$. Then by Theorem 1.66, φ is an isomorphism from eSe onto fSf. To see that φ is continuous, we show that φ is the restriction of ρ_{gf} to eSe. To this end, let $x \in eSe$. Then

$$\begin{aligned} \varphi(x) &= fxgf \\ &= fexgf && (x = ex) \\ &= exgf && (fe = e) \\ &= xgf && (ex = x). \end{aligned}$$

Now let h and k be the inverses in fSf of fgf and fef respectively. We showed in the proof of Theorem 1.66 that if $y \in fSf$, then $\varphi^{-1}(y) = ekyhe$. Thus

$$\begin{aligned} \varphi^{-1}(y) &= ekyhe \\ &= fefkyhe && (fk = k \text{ and } fe = e) \\ &= fyhe && (fefk = f) \\ &= yhe && (fy = y). \end{aligned}$$

So φ^{-1} is the restriction of ρ_{he} to fSf and hence is continuous.

(c) Let L and L' be minimal left ideals of S and let $z \in L'$. By Theorem 2.7(b), pick $e \in E\big(K(S)\big)$ such that $L = Se$. Then $\rho_{z|L}$ is a continuous function from Se to $Sz = L'$ and $\rho_z[Se] = L'$ because Sez is a left ideal of S which is contained in L'. To see that ρ_z is one-to-one on Se, let g be the inverse of eze in eSe. We show that for $x \in Se$, $\rho_g\big(\rho_z(x)\big) = x$, so let $x \in Se$ be given.

$$\begin{aligned} xzg &= xezeg && (x = xe \text{ and } g = eg) \\ &= xe \\ &= x. \end{aligned}$$

Since $\rho_{z|L}$ is one-to-one and continuous and L is compact, $\rho_{z|L}$ is continuous. $\square$

Recall that given any idempotents e, f in a semigroup S, $e \leq_R f$ if and only if $fe = e$.

Theorem 2.12. *Let S be a compact right topological semigroup and let $e \in E(S)$. There is a $\leq_R$-maximal idempotent f in S with $e \leq_R f$.*

Proof. Let $A = \{x \in E(S) : e \leq_R x\}$. Then $A \neq \emptyset$ because $e \in A$. Let C be a $\leq_R$-chain in A. Then $\{c\ell\{r \in C : x \leq_R r\} : x \in C\}$ is a collection of closed subsets of S with the finite intersection property, so $\bigcap_{x\in C} c\ell\{r \in C : x \leq_R r\} \neq \emptyset$. Since S is Hausdorff, $\bigcap_{x\in C} c\ell\{r \in C : x \leq_R r\} \subseteq \{t \in S : \text{for all } x \in C, tx = x\}$. Consequently, $\{t \in S : \text{for all } x \in C, tx = x\}$ is a compact subsemigroup of S and hence by Theorem 2.5 there is an idempotent y such that for all $x \in C$, $yx = x$. This y is an upper bound for C, so A has a maximal member. $\square$

Given $e, f \in E\big(K(S)\big)$ and an assignment to find an isomorphism from eSe onto fSf, most of us would try first the function $\tau : eSe \to fSf$ defined by $\tau(y) = fyf$. In fact, if $eS = fS$, this works (Exercise 2.2.1). We see now that this natural function need not be a homomorphism if $eS \neq fS$ and $Se \neq Sf$