Zhi-Zhong Sun, Qifeng Zhang, and Guang-hua Gao
Finite Difference Methods for Nonlinear Evolution Equations

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Preface

The study of nonlinear phenomena is concerned in the field of natural science and even social science. Since many phenomena in nature are essentially nonlinear, nonlinear problems have aroused the interest and concern of engineers, physicists, mathematicians and many others. In the mathematical and physical sciences, nonlinearity is the phenomenon in which the change in output is not proportional to that of input. A large part of nonlinear phenomena can be described by nonlinear partial differential equations, among which two typical examples are the Navier–Stokes equation in fluid mechanics and Schrödinger equation in quantum mechanics. There are more than 118 nonlinear partial differential equations listed on Wikipedia.

The solution of the heat conduction equation with the Dirichlet boundary condition can be expressed as a linear combination of sinusoidal functions of different frequencies with time-dependent coefficients. The superposition principle makes it easy to solve linear problems. It is often possible to find several particular solutions for nonlinear problems, however, it is commonly very difficult to find general solutions from these particular solutions.

In the process of computerization of science, as a tool, a method and a new subject, science and engineering computation has begun its new development. Numerical solutions of differential equations have also been developed in an unprecedented way.

In this book, we study the difference methods to seek the numerical solutions by selecting 12 typical nonlinear partial differential equations. The 12 equations are respectively the Fisher equation, Burgers' equation, regularized long-wave equation, Korteweg-de Vries equation, Camassa–Holm equation, Schrödinger equation, Kuramoto–Tsuzuki equation, Zakharov equation, Ginzburg–Landau equation, Cahn–Hilliard equation, epitaxial growth model and phase field crystal model. Several effective difference schemes are established for each problem. The existence, uniqueness, conservation, boundedness and convergence of the solution of each difference scheme are proved.

The whole book is concise, hierarchical, gradually deepened in the level of difficulty, which is very suitable to be studied for primary scientific researchers. It is also ideal material for graduates to study and research.

The main part of the book originates from a translation of the monograph "Finite difference methods for nonlinear evolution equations" in Chinese (Science Press, 2018) written by Professor Zhi-Zhong Sun with the following modifications. Difference methods of the Fisher equation are added as a new Chapter 1; In Chapter 2, L^∞ error estimate of the solution to the initial-boundary value problem of the Burgers' equation and to the two-level nonlinear implicit difference scheme is added in Section 2.1 and Section 2.2, respectively; A new proposed compact difference scheme for the Burgers' equation is added in Section 2.5. In Chapter 4, the convergence and unique solvability analyses of two second-order schemes for the Korteweg–de Vries equation are supplemented in Section 4.4 and Section 4.5. In Chapter 12, the proof of Theorem 12.4 is updated. In addition,

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we have supplemented and collected no more than two numerical examples by taking a difference scheme as an example in the penultimate section of each chapter.

Zhi-Zhong Sun completed the main part of the book. Qifeng Zhang provided the translation of Chapters 2–9. He also supplemented and collected numerical examples in Chapters 2–12. Guang-hua Gao translated Chapters 1, 10–12 and supplemented numerical examples in Chapter 1. All of the authors have carefully checked and further polished the whole book.

Before the monograph was fully published, Qifeng and Guang-hua read many parts of the contents. After more than 10 years of study and research, both authors have benefited from the analytical methods and excellent skills. Good knowledge production should be shared with the entire world. This is one of the main motivations for translating and rewriting the book. The publication of this book was supported in part by the National Natural Science Foundation of China (Grant No. 11671081) and the Natural Science Foundation of Zhejiang Province (Grant No. LZ23A010007).

Most of the contents presented in this book originate from the work of the authors and collaborators. Here, we express our sincere thanks to all the collaborators. The authors are grateful to the editors of the press for their hard work. Due to the authors' limited ability, mistakes will be inevitable. We sincerely hope that experts and readers may provide valuable advice and suggestions.

November, 2022

Zhi-Zhong Sun
Qifeng Zhang
Guang-hua Gao

About the Authors

Zhi-Zhong Sun

Born in March 1963, received his Bachelor's degree from the Department of Mathematics, Nanjing University in 1984, Master's degree from the Department of Mathematics, Nanjing University in 1987, and PhD from the Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences in 1990. He is on the faculty at the School of Mathematics, Southeast University since 1990, and has been a full professor since April 1998 and a doctoral supervisor since July 2004.

Professor Sun is an academic leader of the Jiangsu Province "Qinglan Project" and the Executive Director of the Computational Mathematics Society in the Jiangsu Province. He majors in computational mathematics and scientific/engineering computing, and is interested in the theory of difference methods in the numerical solution of partial differential equations. Professor Sun teaches computational methods, numerical analysis, numerical solutions of partial differential equations and numerical methods of nonlinear evolution equations. He has trained 32 master students, 12 doctoral students and 2 post-doctoral students. He has chaired five National Natural Science Foundation projects in China and one Natural Science Foundation project of the Jiangsu Province. Professor Sun has published 6 monographs, 3 textbooks and 5 auxiliary textbooks, and more than 160 regular research papers. He is a highly cited scholar of Elsevier in 2020 and 2021. The course of numerical analysis for engineering graduates was awarded as the outstanding graduate course of the Innovation Project for Graduate Education in the Jiangsu Province. He won the first prize of the Jiangsu Higher Education Teaching Achievement Award (Rank 6), Jiangsu Excellent Postgraduate Textbook Award, Jiangsu Science and Technology Award (Rank 2) and the title of the National Excellent Coach in Mathematical Modeling.

Qifeng Zhang

Born in September 1987, received his PhD from the School of Mathematics and Statistics, Huazhong University of Science and Technology in 2014. He now is a faculty member in the Department of Mathematics, Zhejiang Sci-Tech University since 2014. He has been an associate professor since December 2017 and a master supervisor since June 2015. Professor Zhang engaged in postdoctoral research under the supervision of Professor Zhi-Zhong Sun during 2018–2021. During January 2020–January 2021, he visited Jan S. Hesthaven at the Ecole Polytechnique Federale de Lausanne.

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Professor Zhang majors in computational mathematics and scientific/engineering computing. He is now interested in the numerical solutions of partial differential equations. He teaches numerical analysis, numerical solutions of partial differential equations and linear algebra. As a project leader, Professor Zhang completed one National Natural Science Foundation project in China and chaired three Natural Science Foundation projects of the Zhejiang Province. He has coauthored one monograph and over 40 regular research papers.

Guang-hua Gao

Born in November 1985, obtained her PhD from the Department of Mathematics, Southeast University in 2012, and is a faculty member at the College of Science, Nanjing University of Posts and Telecommunications since 2012. She has been an associate professor since September 2016 and a master supervisor since March 2014. During March 2014–September 2014, she visited Professor Hai-Wei Sun at the University of Macau.

Professor Gao's research interests are in the numerical solutions of partial differential equations, especially fractional differential equations in recent years. More than 30 academic papers have been published as an author or coauthor and three monographs on numerical solutions of fractional differential equations have been published as a coauthor. As a project leader, she has completed the research work of two National Natural Science Foundation projects in China and two Natural Science Foundation projects of the Jiangsu Province. Until now, Professor Gao has supervised four graduate students.

Contents

1 Difference methods for the Fisher equation

1.1 Introduction

The Fisher equation belongs to the class of reaction-diffusion equations. In fact, it is one of the simplest semilinear reaction-diffusion equations, the one which has the inhomogeneous term $f(u) = \lambda u(1-u)$, which can exhibit traveling wave solutions that switch between equilibrium states given by $f(u) = 0$. Such an equation occurs, e. g., in ecology, physiology, combustion, crystallization, plasma physics and in general, phase transition problems. Fisher proposed this equation in 1937 to describe the spatial spread of an advantageous allele and explored its traveling wave solutions [12]. In the same year (1937) as Fisher, Kolmogorov, Petrovskii and Piskunov introduced a more general reaction-diffusion equation [18]. In this chapter, we consider the following initial and boundary value problem of a one-dimensional Fisher equation:

$$\begin{cases} u_t - u_{xx} = \lambda u(1-u), & 0 < x < L,\ 0 < t \leqslant T, \quad (1.1)\\ u(x,0) = \varphi(x), & 0 \leqslant x \leqslant L, \quad (1.2)\\ u(0,t) = \alpha(t), \quad u(L,t) = \beta(t), & 0 < t \leqslant T, \quad (1.3)\end{cases}$$

where λ is a positive constant, functions $\varphi(x)$, $\alpha(t)$, $\beta(t)$ are all given and $\varphi(0) = \alpha(0)$, $\varphi(L) = \beta(0)$. Suppose that the problem (1.1)–(1.3) has a smooth solution.

Before introducing the difference scheme, a priori estimate on the solution of the problem (1.1)–(1.3) is given.

Theorem 1.1. *Let $u(x,t)$ be the solution of the problem (1.1)–(1.3) with $\alpha(t) \equiv 0$, $\beta(t) \equiv 0$. Denote*

$$E(t) = \int_0^L u^2(x,t)\mathrm{d}x + 2\int_0^t \left[\int_0^L u_x^2(x,s)\mathrm{d}x + \lambda \int_0^L \left(u^3(x,s) - u^2(x,s)\right)\mathrm{d}x\right]\mathrm{d}s,$$

$$F(t) = \int_0^L u_x^2(x,t)\mathrm{d}x + \lambda \int_0^L \left[\frac{2}{3}u^3(x,t) - u^2(x,t)\right]\mathrm{d}x + 2\int_0^t \left[\int_0^L u_s^2(x,s)\mathrm{d}x\right]\mathrm{d}s.$$

Then

$$E(t) = E(0), \quad F(t) = F(0), \quad 0 < t \leqslant T.$$

Proof. (I) Multiplying both the right- and left-hand sides of (1.1) by $u(x,t)$ gives

$$u(x,t)u_t(x,t) - u(x,t)u_{xx}(x,t) + \lambda\left[u^3(x,t) - u^2(x,t)\right] = 0,$$

i. e.,

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$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[u^2(x,t)] - (u(x,t)u_x(x,t))_x + u_x^2(x,t) + \lambda[u^3(x,t) - u^2(x,t)] = 0.$$

Integrating both the right- and left-hand sides with respect to x on the interval $[0, L]$ and noticing (1.3) with $\alpha(t) = \beta(t) = 0$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u^2(x,t)\mathrm{d}x + \int_0^L u_x^2(x,t)\mathrm{d}x + \lambda\int_0^L [u^3(x,t) - u^2(x,t)]\mathrm{d}x = 0,$$

which can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^L u^2(x,t)\mathrm{d}x + 2\int_0^t\left[\int_0^L u_x^2(x,s)\mathrm{d}x + \lambda\int_0^L\Big(u^3(x,s) - u^2(x,s)\Big)\mathrm{d}x\right]\mathrm{d}s\right\} = 0.$$

Then $E(t) = E(0)$ is obtained.

(II) Multiplying both the right- and left-hand sides of (1.1) by $u_t(x,t)$ yields

$$u_t^2(x,t) - u_t(x,t)u_{xx}(x,t) - \lambda[u(x,t) - u^2(x,t)]u_t(x,t) = 0,$$

i. e.,

$$u_t^2(x,t) - (u_t(x,t)u_x(x,t))_x + \left(\frac{1}{2}u_x^2(x,t)\right)_t + \lambda\left[\frac{1}{3}u^3(x,t) - \frac{1}{2}u^2(x,t)\right]_t = 0.$$

Integrating both the right- and left-hand sides with respect to x on the interval $[0, L]$ and noticing (1.3) with $\alpha(t) = \beta(t) = 0$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u_x^2(x,t)\mathrm{d}x + \lambda\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\left[\frac{1}{3}u^3(x,t) - \frac{1}{2}u^2(x,t)\right]\mathrm{d}x + \int_0^L u_t^2(x,t)\mathrm{d}x = 0,$$

which can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^L u_x^2(x,t)\mathrm{d}x + \lambda\int_0^L\Big(\frac{2}{3}u^3(x,t) - u^2(x,t)\Big)\mathrm{d}x + 2\int_0^t\Big(\int_0^L u_s^2(x,s)\mathrm{d}x\Big)\mathrm{d}s\right] = 0,$$

i. e.,

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t} = 0, \quad 0 < t \leqslant T.$$

Thus, $F(t) = F(0)$ is followed. □

1.2 Notation and lemmas

In order to derive the difference scheme, we first divide the domain $[0, L] \times [0, T]$. Take two positive integers m, n. Divide $[0, L]$ into m equal subintervals, and $[0, T]$ into n subintervals. Denote $h = L/m$, $\tau = T/n$; $x_i = ih$, $0 \leqslant i \leqslant m$; $t_k = k\tau$, $0 \leqslant k \leqslant n$; $\Omega_h = \{x_i \mid 0 \leqslant i \leqslant m\}$, $\Omega_\tau = \{t_k \mid 0 \leqslant k \leqslant n\}$; $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$. We call all of the nodes $\{(x_i, t_k) \mid 0 \leqslant i \leqslant m\}$ on the line $t = t_k$ the k-th time-level nodes. In addition, denote $x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1})$, $t_{k+\frac{1}{2}} = \frac{1}{2}(t_k + t_{k+1})$, $r = \frac{\tau}{h^2}$.

Denote

$$\begin{aligned} \mathcal{U}_h &= \{u \mid u = (u_0, u_1, \ldots, u_m) \text{ is the grid function defined on } \Omega_h\}, \\ \mathring{\mathcal{U}}_h &= \{u \mid u \in \mathcal{U}_h, u_0 = u_m = 0\}. \end{aligned}$$

For any grid function $u \in \mathcal{U}_h$, introduce the following notation:

$$\delta_x u_{i+\frac{1}{2}} = \frac{1}{h}(u_{i+1} - u_i), \quad \delta_x^2 u_i = \frac{1}{h^2}(u_{i-1} - 2u_i + u_{i+1}), \quad \Delta_x u_i = \frac{1}{2h}(u_{i+1} - u_{i-1}).$$

It follows easily that

$$\delta_x^2 u_i = \frac{1}{h}(\delta_x u_{i+\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}), \quad \Delta_x u_i = \frac{1}{2}(\delta_x u_{i-\frac{1}{2}} + \delta_x u_{i+\frac{1}{2}}).$$

Suppose $u, v \in \mathcal{U}_h$. Introduce the inner products, norms and seminorms as

$$\begin{aligned} &(u, v) = h\left(\frac{1}{2}u_0 v_0 + \sum_{i=1}^{m-1} u_i v_i + \frac{1}{2}u_m v_m\right), \\ &\langle \delta_x u, \delta_x v\rangle = h\sum_{i=1}^{m}(\delta_x u_{i-\frac{1}{2}})(\delta_x v_{i-\frac{1}{2}}), \\ &\|u\|_\infty = \max_{0\leqslant i\leqslant m} |u_i|, \quad \|u\| = \sqrt{(u, u)}, \quad \|\delta_x u\|_\infty = \max_{1\leqslant i\leqslant m} |\delta_x u_{i-\frac{1}{2}}|, \\ &|u|_1 = \sqrt{\langle \delta_x u, \delta_x u\rangle}, \quad \|u\|_1 = \sqrt{\|u\|^2 + |u|_1^2}, \\ &|u|_2 = \sqrt{h\sum_{i=1}^{m-1}(\delta_x^2 u_i)^2}, \quad \|u\|_2 = \sqrt{\|u\|^2 + |u|_1^2 + |u|_2^2}. \end{aligned}$$

If $\mathcal{U}_h$ is a complex space, then the corresponding inner product is defined by

$$(u, v) = h\left(\frac{1}{2}u_0 \bar{v}_0 + \sum_{i=1}^{m-1} u_i \bar{v}_i + \frac{1}{2}u_m \bar{v}_m\right),$$

with $\bar{v}_i$ the conjugate of v_i.

Denote

$$\mathcal{S}_\tau = \{w \mid w = (w^0, w^1, \ldots, w^n) \text{ is the grid function defined on } \Omega_\tau\}.$$

For any $w \in \mathcal{S}_\tau$, introduce the following notation:

$$\begin{aligned}
w^{k+\frac{1}{2}} &= \frac{1}{2}(w^k + w^{k+1}), \quad & w^{\bar{k}} &= \frac{1}{2}(w^{k+1} + w^{k-1}),\\
D_t w^k &= \frac{1}{\tau}(w^{k+1} - w^k), \quad & D_{\bar{t}}\, w^k &= \frac{1}{\tau}(w^k - w^{k-1}),\\
\delta_t w^{k+\frac{1}{2}} &= \frac{1}{\tau}(w^{k+1} - w^k), \quad & \Delta_t w^k &= \frac{1}{2\tau}(w^{k+1} - w^{k-1}).
\end{aligned}$$

It is easy to know that

$$\Delta_t w^k = \frac{1}{2}(\delta_t w^{k-\frac{1}{2}} + \delta_t w^{k+\frac{1}{2}}).$$

Suppose $u = \{u_i^k \mid 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ is a grid function defined on $\Omega_{h\tau}$, then $v = \{u_i^k \mid 0 \leqslant i \leqslant m\}$ is a grid function defined on Ω_h, $w = \{u_i^k \mid 0 \leqslant k \leqslant n\}$ is a grid function defined on Ω_τ.

Lemma 1.1 ([25, 35]). (a) *Suppose $u, v \in \mathcal{U}_h$, then*

$$-h \sum_{i=1}^{m-1} (\delta_x^2 u_i) v_i = h \sum_{i=1}^{m} (\delta_x u_{i-\frac{1}{2}})(\delta_x v_{i-\frac{1}{2}}) + (\delta_x u_{\frac{1}{2}}) v_0 - (\delta_x u_{m-\frac{1}{2}}) v_m.$$

(b) *Suppose $u \in \mathring{\mathcal{U}}_h$, then*

$$\begin{aligned}
-h \sum_{i=1}^{m-1} (\delta_x^2 u_i) u_i &= |u|_1^2,\\
|u|_1^2 &\leqslant \|u\| \cdot |u|_2,\\
\|u\|_\infty &\leqslant \frac{\sqrt{L}}{2} |u|_1,\\
\|u\| &\leqslant \frac{L}{\sqrt{6}} |u|_1.
\end{aligned}$$

(c) *Suppose $u \in \mathring{\mathcal{U}}_h$, then*

$$\|u\|_\infty^2 \leqslant \|u\| \cdot |u|_1,$$

and for arbitrary $\varepsilon > 0$, it holds that

$$\|u\|_\infty \leqslant \varepsilon |u|_1 + \frac{1}{4\varepsilon}\|u\|, \quad \|u\|_\infty^2 \leqslant \varepsilon |u|_1^2 + \frac{1}{4\varepsilon}\|u\|^2.$$

(d) *Suppose* $u \in \mathcal{U}_h$, *then*

$$|u|_1^2 \leqslant \frac{4}{h^2}\|u\|^2.$$

(e) *Suppose* $u \in \mathcal{U}_h$, *then*

$$\|u\|_\infty^2 \leqslant 2\|u\| \cdot |u|_1 + \frac{1}{L}\|u\|^2,$$

and for arbitrary $\varepsilon > 0$, *it holds that*

$$\|u\|_\infty^2 \leqslant \varepsilon |u|_1^2 + \left(\frac{1}{\varepsilon} + \frac{1}{L}\right)\|u\|^2.$$

(f) *Suppose* $u \in \mathcal{U}_h$, *then for arbitrary* $\varepsilon > 0$, *it holds that*

$$\|\delta_x u\|_\infty^2 \leqslant \varepsilon |u|_2^2 + \left(\frac{1}{\varepsilon} + \frac{1}{L}\right)|u|_1^2.$$

Proof. We only prove (c) and (e).

(c) Noticing that $u_0 = 0$, when $1 \leqslant i \leqslant m-1$, we have

$$u_i^2 = \sum_{l=1}^{i}(u_l^2 - u_{l-1}^2) = \sum_{l=1}^{i}(u_l + u_{l-1})(u_l - u_{l-1}) = 2h\sum_{l=1}^{i} u_{l-\frac{1}{2}}\delta_x u_{l-\frac{1}{2}}.$$

Hence,

$$u_i^2 \leqslant 2h\sum_{l=1}^{i} |u_{l-\frac{1}{2}}| \cdot |\delta_x u_{l-\frac{1}{2}}|.$$

Similarly, noticing that $u_m = 0$, we have

$$u_i^2 \leqslant 2h\sum_{l=i+1}^{m} |u_{l-\frac{1}{2}}| \cdot |\delta_x u_{l-\frac{1}{2}}|.$$

Adding the above two inequalities together, we have

$$u_i^2 \leqslant h\sum_{l=1}^{m} |u_{l-\frac{1}{2}}| \cdot |\delta_x u_{l-\frac{1}{2}}| \leqslant \sqrt{h\sum_{l=1}^{m} |u_{l-\frac{1}{2}}|^2} \cdot \sqrt{h\sum_{l=1}^{m} |\delta_x u_{l-\frac{1}{2}}|^2} \leqslant \|u\| \cdot |u|_1.$$

It follows that

$$\|u\|_\infty^2 \leqslant \|u\| \cdot |u|_1.$$

For arbitrary $\varepsilon > 0$, then

$$\|u\|_\infty \leqslant \sqrt{\|u\| \cdot |u|_1} \leqslant \varepsilon |u|_1 + \frac{1}{4\varepsilon}\|u\|,$$
$$\|u\|_\infty^2 \leqslant \|u\| \cdot |u|_1 \leqslant \varepsilon |u|_1^2 + \frac{1}{4\varepsilon}\|u\|^2.$$

(e) When $i > j$,

$$\begin{aligned} u_i^2 &= u_j^2 + \sum_{l=j+1}^{i} (u_l^2 - u_{l-1}^2) \\ &= u_j^2 + 2h \sum_{l=j+1}^{i} u_{l-\frac{1}{2}} \delta_x u_{l-\frac{1}{2}} \\ &\leqslant u_j^2 + 2h \sum_{l=j+1}^{i} |u_{l-\frac{1}{2}}| \cdot |\delta_x u_{l-\frac{1}{2}}| \\ &\leqslant u_j^2 + 2h \sum_{l=1}^{m} |u_{l-\frac{1}{2}}| \cdot |\delta_x u_{l-\frac{1}{2}}| \\ &\leqslant u_j^2 + 2\|u\| \cdot |u|_1. \end{aligned} \tag{1.4}$$

It is easy to know that the above result holds also for $i \leqslant j$.

Denote

$$\omega_j = \begin{cases} 1, & 1 \leqslant j \leqslant m-1, \\ \frac{1}{2}, & j = 0, m. \end{cases}$$

Multiplying (1.4) by $h\omega_j$ on both the right- and left-hand sides and summing up for j from 0 to m, we have

$$h \sum_{j=0}^{m} \omega_j u_i^2 \leqslant h \sum_{j=0}^{m} \omega_j u_j^2 + 2h \sum_{j=0}^{m} \omega_j \|u\| \cdot |u|_1.$$

It easily follows that

$$L\|u\|_\infty^2 \leqslant \|u\|^2 + 2L\|u\| \cdot |u|_1,$$

namely,

$$\|u\|_\infty^2 \leqslant 2\|u\| \cdot |u|_1 + \frac{1}{L}\|u\|^2.$$

For arbitrary $\varepsilon > 0$, we have

$$\|u\|_\infty^2 \leqslant \varepsilon |u|_1^2 + \left(\frac{1}{\varepsilon} + \frac{1}{L}\right)\|u\|^2.$$

□

Similar results hold for the continuous functions.

Next, we will give several commonly used numerical differential formulas.

Lemma 1.2 ([35]). *Let c, h be given constants and $h > 0$.*

(a) *If $g(x) \in C^2[c-h, c+h]$, then*

$$g(c) = \frac{1}{2}[g(c-h) + g(c+h)] - \frac{h^2}{2} g''(\xi_0), \quad c-h < \xi_0 < c+h;$$

(b) *If $g(x) \in C^2[c, c+h]$, then*

$$g'(c) = \frac{1}{h}[g(c+h) - g(c)] - \frac{h}{2} g''(\xi_1), \quad c < \xi_1 < c+h;$$

(c) *If $g(x) \in C^2[c-h, c]$, then*

$$g'(c) = \frac{1}{h}[g(c) - g(c-h)] + \frac{h}{2} g''(\xi_2), \quad c-h < \xi_2 < c;$$

(d) *If $g(x) \in C^3[c-h, c+h]$, then*

$$g'(c) = \frac{1}{2h}[g(c+h) - g(c-h)] - \frac{h^2}{6} g'''(\xi_3), \quad c-h < \xi_3 < c+h;$$

(e) *If $g(x) \in C^4[c-h, c+h]$, then*

$$g''(c) = \frac{1}{h^2}[g(c+h) - 2g(c) + g(c-h)] - \frac{h^2}{12} g^{(4)}(\xi_4), \quad c-h < \xi_4 < c+h;$$

(f) *If $g(x) \in C^3[c, c+h]$, then*

$$g''(c) = \frac{2}{h}\left[\frac{g(c+h) - g(c)}{h} - g'(c)\right] - \frac{h}{3} g'''(\xi_5), \quad c < \xi_5 < c+h;$$

If $g(x) \in C^4[c, c+h]$, then

$$g''(c) = \frac{2}{h}\left[\frac{g(c+h) - g(c)}{h} - g'(c)\right] - \frac{h}{3} g'''(c) - \frac{h^2}{12} g^{(4)}(\xi_6), \quad c < \xi_6 < c+h;$$

(g) *If $g(x) \in C^3[c-h, c]$, then*

$$g''(c) = \frac{2}{h}\left[g'(c) - \frac{g(c) - g(c-h)}{h}\right] + \frac{h}{3} g'''(\xi_7), \quad c-h < \xi_7 < c;$$

If $g(x) \in C^4[c-h, c]$, then

$$g''(c) = \frac{2}{h}\left[g'(c) - \frac{g(c) - g(c-h)}{h}\right] + \frac{h}{3} g'''(c) - \frac{h^2}{12} g^{(4)}(\xi_8), \quad c-h < \xi_8 < c;$$

(h) *If* $g(x) \in C^6[c-h, c+h]$, *then*

$$\frac{1}{12}[g''(c-h) + 10g''(c) + g''(c+h)] = \frac{1}{h^2}[g(c+h) - 2g(c) + g(c-h)] + \frac{h^4}{240}g^{(6)}(\xi_9),$$
$$c - h < \xi_9 < c + h.$$

Now let us introduce some important Gronwall inequalities.

Theorem 1.2. (a) *Suppose* $\{F^k\}_{k=0}^{\infty}$ *is a nonnegative sequence; c and g are two nonnegative constants satisfying*

$$F^{k+1} \leqslant (1 + c\tau)F^k + \tau g, \quad k = 0, 1, 2, \ldots,$$

then

$$F^k \leqslant \mathrm{e}^{ck\tau}\left(F^0 + \frac{g}{c}\right), \quad k = 0, 1, 2, \ldots.$$

(b) *Suppose* $\{F^k\}_{k=0}^{\infty}$ *and* $\{g^k\}_{k=0}^{\infty}$ *are two nonnegative sequences; c is a nonnegative constant satisfying*

$$F^{k+1} \leqslant (1 + c\tau)F^k + \tau g^k, \quad k = 0, 1, 2, \ldots,$$

then

$$F^k \leqslant \mathrm{e}^{ck\tau}\left(F^0 + \tau\sum_{l=0}^{k-1} g^l\right), \quad k = 0, 1, 2, \ldots.$$

(c) *Suppose* $\{F^k\}_{k=0}^{\infty}$ *is a nonnegative sequence; c and g are two nonnegative constants satisfying*

$$F^k \leqslant c\tau\sum_{l=0}^{k-1} F^l + g, \quad k = 0, 1, 2, \ldots,$$

then

$$F^k \leqslant \mathrm{e}^{ck\tau}g, \quad k = 0, 1, 2, \ldots.$$

(d) *Suppose* $\{F^k\}_{k=0}^{\infty}$ *is a nonnegative sequence and* $\{g^k\}_{k=0}^{\infty}$ *is nonnegative monotonically increasing (allowed not strictly monotonic) sequence satisfying*

$$F^k \leqslant c\tau\sum_{l=0}^{k-1} F^l + g^k, \quad k = 0, 1, 2, \ldots,$$

then

$$F^k \leqslant \mathrm{e}^{ck\tau}g^k, \quad k = 0, 1, 2, \ldots.$$

Proof. (a)

$$\begin{aligned}
F^{k+1} &\leqslant (1+c\tau)F^k + \tau g \\
&\leqslant (1+c\tau)\big[(1+c\tau)F^{k-1} + \tau g\big] + \tau g \\
&= (1+c\tau)^2 F^{k-1} + \big[(1+c\tau)+1\big]\tau g \\
&\leqslant (1+c\tau)^2\big[(1+c\tau)F^{k-2} + \tau g\big] + \big[(1+c\tau)+1\big]\tau g \\
&= (1+c\tau)^3 F^{k-2} + \big[(1+c\tau)^2 + (1+c\tau)+1\big]\tau g \\
&\leqslant \cdots \\
&\leqslant (1+c\tau)^k F^1 + \big[(1+c\tau)^{k-1} + (1+c\tau)^{k-2} + \cdots + 1\big]\tau g \\
&\leqslant (1+c\tau)^k\big[(1+c\tau)F^0 + \tau g\big] + \big[(1+c\tau)^{k-1} + (1+c\tau)^{k-2} + \cdots + 1\big]\tau g \\
&= (1+c\tau)^{k+1} F^0 + \big[(1+c\tau)^k + (1+c\tau)^{k-1} + \cdots + 1\big]\tau g \\
&= (1+c\tau)^{k+1} F^0 + \frac{(1+c\tau)^{k+1}-1}{c\tau}\cdot \tau g \\
&\leqslant \mathrm{e}^{c(k+1)\tau}\Big(F^0 + \frac{g}{c}\Big), \quad k = 0,1,\ldots.
\end{aligned}$$

(b)

$$\begin{aligned}
F^{k+1} &\leqslant (1+c\tau)F^k + \tau g^k \\
&\leqslant (1+c\tau)\big[(1+c\tau)F^{k-1} + \tau g^{k-1}\big] + \tau g^k \\
&= (1+c\tau)^2 F^{k-1} + (1+c\tau)\tau g^{k-1} + \tau g^k \\
&\leqslant (1+c\tau)^2\big[(1+c\tau)F^{k-2} + \tau g^{k-2}\big] + (1+c\tau)\tau g^{k-1} + \tau g^k \\
&= (1+c\tau)^3 F^{k-2} + (1+c\tau)^2\tau g^{k-2} + (1+c\tau)\tau g^{k-1} + \tau g^k \\
&\leqslant (1+c\tau)^3\big[(1+c\tau)F^{k-3} + \tau g^{k-3}\big] + (1+c\tau)^2\tau g^{k-2} + (1+c\tau)\tau g^{k-1} + \tau g^k \\
&= (1+c\tau)^4 F^{k-3} + (1+c\tau)^3\tau g^{k-3} + (1+c\tau)^2\tau g^{k-2} + (1+c\tau)\tau g^{k-1} + \tau g^k \\
&\leqslant \cdots \\
&\leqslant (1+c\tau)^{k+1} F^0 + \tau\sum_{l=0}^{k}(1+c\tau)^{k-l} g^l \\
&\leqslant (1+c\tau)^{k+1}\Big(F^0 + \tau\sum_{l=0}^{k} g^l\Big) \leqslant \mathrm{e}^{c(k+1)\tau}\Big(F^0 + \tau\sum_{l=0}^{k} g^l\Big), \quad k = 0,1,2,\ldots.
\end{aligned}$$

(c) It is easy to know that

$$F^0 \leqslant g.$$

Let

$$G^k = c\tau \sum_{l=0}^{k-1} F^l + g, \quad k = 0, 1, 2, \ldots.$$

Then

$$\begin{aligned}
&G^0 = g, \\
&F^k \leqslant G^k, \quad k = 0, 1, 2, \ldots, \\
&G^k = G^{k-1} + c\tau F^{k-1} \leqslant G^{k-1} + c\tau G^{k-1} = (1 + c\tau) G^{k-1}, \quad k = 1, 2, 3, \ldots,
\end{aligned}$$

by recursion, we have

$$G^k \leqslant (1 + c\tau)^k G^0 \leqslant \mathrm{e}^{ck\tau} g, \quad k = 0, 1, 2, \ldots,$$

so that

$$F^k \leqslant G^k \leqslant \mathrm{e}^{ck\tau} g, \quad k = 0, 1, 2, \ldots.$$

(d) It is easy to know that

$$F^0 \leqslant g^0.$$

Let

$$G^k = c\tau \sum_{l=0}^{k-1} F^l + g^k, \quad k = 0, 1, 2, \ldots,$$

then

$$\begin{aligned}
G^0 &= g^0, \\
F^k &\leqslant G^k, \quad k = 0, 1, 2, \ldots, \\
G^k &= c\tau \sum_{l=0}^{k-2} F^l + g^{k-1} + c\tau F^{k-1} + (g^k - g^{k-1}) \\
&= G^{k-1} + c\tau F^{k-1} + (g^k - g^{k-1}) \\
&\leqslant (1 + c\tau) G^{k-1} + (g^k - g^{k-1}), \quad k = 1, 2, \ldots.
\end{aligned}$$

Applying the result of (b), we have

$$F^k \leqslant G^k \leqslant \mathrm{e}^{ck\tau} \left[G^0 + \sum_{l=1}^{k} (g^l - g^{l-1}) \right] = \mathrm{e}^{ck\tau} g^k, \quad k = 0, 1, 2, \ldots.$$

□

1.3 Forward Euler difference scheme

1.3.1 Derivation of the difference scheme

Define the grid function $U = \{U_i^k \mid 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ on $\Omega_{h\tau}$, where

$$U_i^k = u(x_i, t_k), \quad 0 \leqslant i \leqslant m,\ 0 \leqslant k \leqslant n.$$

Denote

$$c_0 = \max_{\substack{0 \leqslant x \leqslant L \\ 0 \leqslant t \leqslant T}} |u(x,t)|.$$

Considering equation (1.1) at the point (x_i, t_k), we have

$$u_t(x_i, t_k) - u_{xx}(x_i, t_k) = \lambda u(x_i, t_k)[1 - u(x_i, t_k)], \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1. \tag{1.5}$$

With the help of Lemma 1.2, we have

$$u_t(x_i, t_k) = \frac{1}{\tau}(U_i^{k+1} - U_i^k) + O(\tau) = D_t U_i^k + O(\tau), \tag{1.6}$$

$$u_{xx}(x_i, t_k) = \frac{1}{h^2}(U_{i+1}^k - 2U_i^k + U_{i-1}^k) + O(h^2) = \delta_x^2 U_i^k + O(h^2), \tag{1.7}$$

$$u(x_i, t_k) = u(x_i, t_{k+1}) + O(\tau) = U_i^{k+1} + O(\tau). \tag{1.8}$$

Substituting (1.6)–(1.8) into (1.5) arrives at

$$D_t U_i^k - \delta_x^2 U_i^k = \lambda(U_i^k - U_i^k U_i^{k+1}) + (R_1)_i^k, \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1, \tag{1.9}$$

where there is a constant c_1 such that

$$|(R_1)_i^k| \leqslant c_1(\tau + h^2), \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1. \tag{1.10}$$

Noticing the initial-boundary value conditions (1.2)–(1.3), we have

$$\begin{cases} U_i^0 = \varphi(x_i), & 0 \leqslant i \leqslant m, \qquad (1.11) \\ U_0^k = \alpha(t_k), \quad U_m^k = \beta(t_k), & 1 \leqslant k \leqslant n. \qquad (1.12) \end{cases}$$

Neglecting the small term $(R_1)_i^k$ in (1.9) and replacing the exact solution U_i^k by its numerical one u_i^k, the following forward Euler difference scheme is obtained as

$$\begin{cases} D_t u_i^k - \delta_x^2 u_i^k = \lambda(u_i^k - u_i^k u_i^{k+1}), & 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1, \qquad (1.13) \\ u_i^0 = \varphi(x_i), & 0 \leqslant i \leqslant m, \qquad (1.14) \\ u_0^k = \alpha(t_k), \quad u_m^k = \beta(t_k), & 1 \leqslant k \leqslant n. \qquad (1.15) \end{cases}$$

It is easy to get the following conclusion.

Theorem 1.3 ([29]). *Let $\{u_i^k \mid 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ be the solution of the difference scheme (1.13)–(1.15). If $0 \leqslant \varphi(x) \leqslant 1$, $0 \leqslant \alpha(t) \leqslant 1$, $0 \leqslant \beta(t) \leqslant 1$ and $r \leqslant \frac{1}{2}$, then it holds that*

$$0 \leqslant u_i^k \leqslant 1, \quad 0 \leqslant i \leqslant m, \ 0 \leqslant k \leqslant n.$$

Proof. Reformulate (1.13) as

$$(1 + \lambda\tau u_i^k)u_i^{k+1} = (1 - 2r)u_i^k + r(u_{i-1}^k + u_{i+1}^k) + \lambda\tau u_i^k, \quad 1 \leqslant i \leqslant m-1, \ 0 \leqslant k \leqslant n-1,$$

or

$$u_i^{k+1} = \frac{1}{1 + \lambda\tau u_i^k}\left[(1 - 2r)u_i^k + r(u_{i-1}^k + u_{i+1}^k) + \lambda\tau u_i^k\right], \quad 1 \leqslant i \leqslant m-1, \ 0 \leqslant k \leqslant n-1.$$

If $0 \leqslant u_i^k \leqslant 1$, $0 \leqslant i \leqslant m$ and $r \leqslant \frac{1}{2}$, then we have

$$u_i^{k+1} \geqslant 0, \quad 1 \leqslant i \leqslant m-1$$

and

$$u_i^{k+1} \leqslant \frac{1}{1 + \lambda\tau u_i^k}\left[(1 - 2r) \times 1 + r \times (1 + 1) + \lambda\tau u_i^k\right] = 1, \quad 1 \leqslant i \leqslant m-1.$$

By induction, the conclusion is true. □

1.3.2 Solvability and convergence of the difference scheme

Theorem 1.4. *Let $\{U_i^k \mid 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ and $\{u_i^k \mid 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ be solutions of the problem (1.1)–(1.3) and the difference scheme (1.13)–(1.15), respectively. Denote*

$$e_i^k = U_i^k - u_i^k, \quad 0 \leqslant i \leqslant m, \ 0 \leqslant k \leqslant n,$$
$$c_2 = \frac{c_1}{2\lambda(c_0 + 1)} \mathrm{e}^{3\lambda(c_0+1)T}.$$

Then, when $r \leqslant \frac{1}{2}$, $c_2(\tau + h^2) \leqslant 1$ and $\lambda(c_0 + 1)\tau \leqslant \frac{1}{3}$, it holds that
(I) *the solution of the difference scheme (1.13)–(1.15) exists;*
(II)

$$\|e^k\|_\infty \leqslant c_2(\tau + h^2), \quad 0 \leqslant k \leqslant n. \tag{1.16}$$

Proof. Subtracting (1.13)–(1.15) from (1.9), (1.11) and (1.12), respectively, the system of error equations can be produced as

$$\begin{cases} D_t e_i^k - \delta_x^2 e_i^k = \lambda e_i^k - \lambda(U_i^k U_i^{k+1} - u_i^k u_i^{k+1}) + (R_1)_i^k, & 1 \leqslant i \leqslant m-1, 0 \leqslant k \leqslant n-1, \quad (1.17) \\ e_i^0 = 0, & 0 \leqslant i \leqslant m, \quad (1.18) \\ e_0^k = 0, \quad e_m^k = 0, & 1 \leqslant k \leqslant n. \quad (1.19) \end{cases}$$

Rewrite (1.17) as

$$e_i^{k+1} = (1-2r)e_i^k + r(e_{i-1}^k + e_{i+1}^k) + \lambda\tau e_i^k - \lambda\tau(u_i^k e_i^{k+1} + e_i^k U_i^{k+1}) + \tau(R_1)_i^k,$$
$$1 \leqslant i \leqslant m-1, \ 0 \leqslant k \leqslant n-1. \tag{1.20}$$

When $r \leqslant \frac{1}{2}$, taking the absolute value on both the right- and left-hand sides of (1.20) and using the triangle inequality, with the help of (1.10), we have

$$\begin{aligned} |e_i^{k+1}| &\leqslant (1-2r)\|e^k\|_\infty + r(\|e^k\|_\infty + \|e^k\|_\infty) + \lambda\tau\|e^k\|_\infty \\ &\quad + \lambda\tau(\|u^k\|_\infty\|e^{k+1}\|_\infty + \|e^k\|_\infty\|U^{k+1}\|_\infty) + c_1\tau(\tau + h^2) \\ &= (1+\lambda\tau)\|e^k\|_\infty + \lambda\tau(\|u^k\|_\infty\|e^{k+1}\|_\infty + c_0\|e^k\|_\infty) + c_1\tau(\tau + h^2) \\ &= [1 + \lambda(c_0+1)\tau]\|e^k\|_\infty + \lambda\tau\|u^k\|_\infty\|e^{k+1}\|_\infty + c_1\tau(\tau + h^2), \\ &1 \leqslant i \leqslant m-1. \end{aligned}$$

It follows by noticing (1.19) that

$$\|e^{k+1}\|_\infty \leqslant [1 + \lambda(c_0+1)\tau]\|e^k\|_\infty + \lambda\tau\|u^k\|_\infty\|e^{k+1}\|_\infty + c_1\tau(\tau + h^2), \quad 0 \leqslant k \leqslant n-1. \tag{1.21}$$

In view of (1.18),

$$\|e^0\|_\infty = 0,$$

which implies the truth of (1.16) for $k = 0$.

Now assume that (1.16) is true for $0 \leqslant k \leqslant l$, i. e.,

$$\|e^k\|_\infty \leqslant c_2(\tau + h^2), \quad 0 \leqslant k \leqslant l.$$

Then noticing $e_i^k = U_i^k - u_i^k$, when $c_2(\tau + h^2) \leqslant 1$, it follows that

$$\|u^k\|_\infty \leqslant \|U^k\|_\infty + \|e^k\|_\infty \leqslant c_0 + 1, \quad 0 \leqslant k \leqslant l.$$

Considering (1.13) with $k = l$ and noticing $1 + \lambda\tau u_i^l \geqslant 1 - \lambda\tau(c_0+1) \geqslant \frac{2}{3}$, we obtain

$$u_i^{l+1} = \frac{1}{1+\lambda\tau u_i^l}[(1-2r)u_i^l + r(u_{i-1}^l + u_{i+1}^l) + \lambda\tau u_i^l], \quad 1 \leqslant i \leqslant m-1,$$

which means that u^{l+1} can be solved explicitly and uniquely. In addition, by (1.21), we have

$$\|e^{k+1}\|_\infty \leqslant [1+\lambda(c_0+1)\tau]\|e^k\|_\infty + \lambda\tau(c_0+1)\|e^{k+1}\|_\infty + c_1\tau(\tau+h^2), \quad 0 \leqslant k \leqslant l,$$

i. e.,

$$[1-\lambda\tau(c_0+1)]\|e^{k+1}\|_\infty \leqslant [1+\lambda(c_0+1)\tau]\|e^k\|_\infty + c_1\tau(\tau+h^2), \quad 0 \leqslant k \leqslant l.$$

When $\lambda\tau(c_0+1) \leqslant \frac{1}{3}$, we have

$$\|e^{k+1}\|_\infty \leqslant [1+3\lambda(c_0+1)\tau]\|e^k\|_\infty + \frac{3}{2}c_1\tau(\tau+h^2), \quad 0 \leqslant k \leqslant l.$$

The application of the Gronwall inequality (Theorem 1.2(a)) yields

$$\|e^{l+1}\|_\infty \leqslant \mathrm{e}^{3\lambda(c_0+1)T} \cdot \frac{c_1}{2\lambda(c_0+1)}(\tau+h^2) = c_2(\tau+h^2),$$

from which (1.16) also holds for $k = l+1$. By induction, (1.16) is true for all k $(0 \leqslant k \leqslant n)$. □

1.4 Backward Euler difference scheme

The forward Euler scheme requires the step size ratio $r \leqslant \frac{1}{2}$, which implies that the temporal step size must be much smaller than the spatial one. Next, an unconditionally stable difference scheme will be introduced.

1.4.1 Derivation of the difference scheme

Considering equation (1.1) at the node point (x_i, t_{k+1}), we have

$$u_t(x_i,t_{k+1}) - u_{xx}(x_i,t_{k+1}) = \lambda u(x_i,t_{k+1})[1-u(x_i,t_{k+1})], \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1. \tag{1.22}$$

With the help of Lemma 1.2, we have

$$u_t(x_i,t_{k+1}) = \frac{1}{\tau}(U_i^{k+1} - U_i^k) + O(\tau) = D_{\bar{t}}\, U_i^{k+1} + O(\tau), \tag{1.23}$$

$$u_{xx}(x_i,t_{k+1}) = \frac{1}{h^2}(U_{i+1}^{k+1} - 2U_i^{k+1} + U_{i-1}^{k+1}) + O(h^2) = \delta_x^2 U_i^{k+1} + O(h^2), \tag{1.24}$$

$$u(x_i,t_{k+1}) = u(x_i,t_k) + O(\tau) = U_i^k + O(\tau). \tag{1.25}$$

Substituting (1.23)–(1.25) into (1.22) arrives at

$$D_{\bar{t}}\, U_i^{k+1} - \delta_x^2 U_i^{k+1} = \lambda\big(U_i^k - U_i^k U_i^{k+1}\big) + (R_2)_i^{k+1}, \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1, \tag{1.26}$$

where there is a constant c_3 such that

$$\big|(R_2)_i^{k+1}\big| \leqslant c_3(\tau + h^2), \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1. \tag{1.27}$$

Noticing the initial-boundary value conditions (1.2)–(1.3), we have

$$\begin{cases} U_i^0 = \varphi(x_i), & 0 \leqslant i \leqslant m, \quad (1.28)\\ U_0^k = \alpha(t_k), \quad U_m^k = \beta(t_k), & 1 \leqslant k \leqslant n. \quad (1.29)\end{cases}$$

Neglecting the small term $(R_2)_i^{k+1}$ in (1.26) and replacing the exact solution U_i^k by its numerical one u_i^k, the backward Euler difference scheme reads

$$\begin{cases} D_{\bar{t}}\, u_i^{k+1} - \delta_x^2 u_i^{k+1} = \lambda(u_i^k - u_i^k u_i^{k+1}), & 1 \leqslant i \leqslant m-1, 0 \leqslant k \leqslant n-1, \quad (1.30)\\ u_i^0 = \varphi(x_i), & 0 \leqslant i \leqslant m, \quad (1.31)\\ u_0^k = \alpha(t_k), \quad u_m^k = \beta(t_k), & 1 \leqslant k \leqslant n. \quad (1.32)\end{cases}$$

Note that (1.30)–(1.32) is a two-level linearized difference scheme.

It is easy to get the following conclusion.

Theorem 1.5 ([29]). *Let* $\{u_i^k \mid 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ *be the solution of the difference scheme (*1.30*)–(*1.32*). If* $0 \leqslant \varphi(x) \leqslant 1$, $0 \leqslant \alpha(t) \leqslant 1$ *and* $0 \leqslant \beta(t) \leqslant 1$, *then it holds that*

$$0 \leqslant u_i^k \leqslant 1, \quad 0 \leqslant i \leqslant m,\ 0 \leqslant k \leqslant n.$$

Proof. Rewrite (1.30) as

$$(1 + 2r + \lambda\tau u_i^k)u_i^{k+1} = r(u_{i-1}^{k+1} + u_{i+1}^{k+1}) + u_i^k + \lambda\tau u_i^k, \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1. \tag{1.33}$$

Suppose $0 \leqslant u_i^k \leqslant 1$, $0 \leqslant i \leqslant m$ and notice $0 \leqslant \alpha(t_{k+1}) \leqslant 1$, $0 \leqslant \beta(t_{k+1}) \leqslant 1$. Denote

$$\min_{0\leqslant i\leqslant m} u_i^{k+1} = u_{i_*}^{k+1}, \quad \max_{0\leqslant i\leqslant m} u_i^{k+1} = u_{i^*}^{k+1}.$$

If $i_* \neq 0, m$, letting $i = i_*$ in (1.33), we have

$$(1 + 2r + \lambda\tau u_{i_*}^k)u_{i_*}^{k+1} = r(u_{i_*-1}^{k+1} + u_{i_*+1}^{k+1}) + u_{i_*}^k + \lambda\tau u_{i_*}^k \geqslant 2r u_{i_*}^{k+1} + u_{i_*}^k + \lambda\tau u_{i_*}^k,$$

i. e.,

$$(1 + \lambda\tau u_{i_*}^k)u_{i_*}^{k+1} \geqslant (1 + \lambda\tau)u_{i_*}^k,$$

which implies

$$u_{i_*}^{k+1} \geqslant 0.$$

If $i^* \neq 0, m$, letting $i = i^*$ in (1.33), we have

$$(1 + 2r + \lambda\tau u_{i^*}^k)u_{i^*}^{k+1} = r(u_{i^*-1}^{k+1} + u_{i^*+1}^{k+1}) + u_{i^*}^k + \lambda\tau u_{i^*}^k \leqslant 2ru_{i^*}^{k+1} + u_{i^*}^k + \lambda\tau u_{i^*}^k,$$

i. e.,

$$(1 + \lambda\tau u_{i^*}^k)u_{i^*}^{k+1} \leqslant u_{i^*}^k + \lambda\tau u_{i^*}^k \leqslant 1 + \lambda\tau u_{i^*}^k,$$

which implies

$$u_{i^*}^{k+1} \leqslant 1.$$

The result $0 \leqslant u_i^{k+1} \leqslant 1$ is followed.

By induction, the proof is completed. □

1.4.2 Existence and convergence of the difference solution

Theorem 1.6. *Let $\{U_i^k \mid 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ and $\{u_i^k \mid 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ be solutions of the problem (1.1)–(1.3) and the difference scheme (1.30)–(1.32), respectively. Denote*

$$e_i^k = U_i^k - u_i^k, \quad 0 \leqslant i \leqslant m,\ 0 \leqslant k \leqslant n,$$
$$c_4 = \frac{c_3}{2\lambda(c_0+1)} \mathrm{e}^{3\lambda(c_0+1)T}.$$

Then, when $c_4(\tau + h^2) \leqslant 1$ and $\lambda(c_0 + 2)\tau \leqslant \frac{1}{3}$, it holds that

(I) *the difference scheme (1.30)–(1.32) is uniquely solvable;*

(II)

$$\|e^k\|_\infty \leqslant c_4(\tau + h^2), \quad 0 \leqslant k \leqslant n. \tag{1.34}$$

Proof. Subtracting (1.30)–(1.32) from (1.26), (1.28)–(1.29), respectively, the system of error equations is produced as

$$\begin{cases} D_{\bar{t}}\, e_i^{k+1} - \delta_x^2 e_i^{k+1} = \lambda e_i^k - \lambda(U_i^k U_i^{k+1} - u_i^k u_i^{k+1}) + (R_2)_i^{k+1}, \\ \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1, & (1.35) \\ e_i^0 = 0, \qquad\qquad 0 \leqslant i \leqslant m, & (1.36) \\ e_0^k = 0, \quad e_m^k = 0, \qquad 1 \leqslant k \leqslant n. & (1.37) \end{cases}$$

Rewrite (1.35) as

$$(1 + 2r)e_i^{k+1} = r(e_{i-1}^{k+1} + e_{i+1}^{k+1}) + (1 + \lambda\tau)e_i^k - \lambda\tau(u_i^k e_i^{k+1} + e_i^k U_i^{k+1}) + \tau(R_2)_i^{k+1},$$
$$1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1.$$

Taking the absolute value on both the right- and left-hand sides of the equality above and using the triangle inequality, with the help of (1.27), we have

$$
\begin{aligned}
&(1+2r)|e_i^{k+1}| \\
&\leqslant 2r\|e^{k+1}\|_\infty + (1+\lambda\tau)\|e^k\|_\infty + \lambda\tau(\|u^k\|_\infty\|e^{k+1}\|_\infty + \|e^k\|_\infty\|U^{k+1}\|_\infty) + c_3\tau(\tau+h^2), \\
&1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1.
\end{aligned}
$$

It follows by noticing (1.37) that

$$
\begin{aligned}
&(1+2r)\|e^{k+1}\|_\infty \\
&\leqslant 2r\|e^{k+1}\|_\infty + (1+\lambda\tau)\|e^k\|_\infty + \lambda\tau(\|u^k\|_\infty\|e^{k+1}\|_\infty + c_0\|e^k\|_\infty) + c_3\tau(\tau+h^2), \\
&0 \leqslant k \leqslant n-1,
\end{aligned}
$$

i. e.,

$$
\|e^{k+1}\|_\infty \leqslant [1+\lambda(c_0+1)\tau]\|e^k\|_\infty + \lambda\tau\|u^k\|_\infty\|e^{k+1}\|_\infty + c_3\tau(\tau+h^2), \quad 0 \leqslant k \leqslant n-1. \tag{1.38}
$$

From (1.36), we easily have

$$
\|e^0\|_\infty = 0, \tag{1.39}
$$

which means that (1.34) holds for $k = 0$. Now suppose that the values of $u^0, u^1, \dots, u^l$ have been obtained from (1.30)–(1.32) and the inequality (1.34) is true for $0 \leqslant k \leqslant l$. Then when $c_4(\tau+h^2) \leqslant 1$, it follows:

$$
\|e^k\|_\infty \leqslant c_4(\tau+h^2) \leqslant 1, \quad 0 \leqslant k \leqslant l
$$

and

$$
\|u^k\|_\infty \leqslant \|U^k\|_\infty + \|e^k\|_\infty \leqslant c_0 + 1, \quad 0 \leqslant k \leqslant l. \tag{1.40}
$$

(I) Proof for the unique solvability.

The system of linear equations in u^{l+1} can be obtained from (1.30) and (1.32) as

$$
\begin{cases}
D_{\bar{t}}\, u_i^{l+1} - \delta_x^2 u_i^{l+1} = \lambda(u_i^l - u_i^l u_i^{l+1}), & 1 \leqslant i \leqslant m-1, \\
u_0^{l+1} = \alpha(t_{l+1}), \quad u_m^{l+1} = \beta(t_{l+1}).
\end{cases}
$$

Consider its homogeneous one:

$$
\begin{cases}
\frac{1}{\tau} u_i^{l+1} - \delta_x^2 u_i^{l+1} = -\lambda u_i^l u_i^{l+1}, \quad 1 \leqslant i \leqslant m-1, & (1.41) \\
u_0^{l+1} = 0, \quad u_m^{l+1} = 0. & (1.42)
\end{cases}
$$

Rewrite (1.41) as

$$(1+2r)u_i^{l+1} = r(u_{i-1}^{l+1} + u_{i+1}^{l+1}) - \lambda\tau u_i^l u_i^{l+1}, \quad 1 \leqslant i \leqslant m-1.$$

Suppose $|u_{i^*}^{l+1}| = \|u^{l+1}\|_\infty$, $1 \leqslant i^* \leqslant m-1$. Letting $i = i^*$ in the equality above and taking the absolute value on both the right- and left-hand sides, with the help of the triangle inequality, we get

$$(1+2r)\|u^{l+1}\|_\infty \leqslant 2r\|u^{l+1}\|_\infty + \lambda\tau\|u^l\|_\infty\|u^{l+1}\|_\infty.$$

By (1.40), it further follows:

$$\|u^{l+1}\|_\infty \leqslant \lambda(c_0+1)\tau\|u^{l+1}\|_\infty.$$

When $\lambda(c_0+1)\tau < 1$, it implies $\|u^{l+1}\|_\infty = 0$. Thus, (1.30) and (1.32) are uniquely solvable in u^{l+1}.

(II) Proof for (1.34).

From (1.38) and (1.40), we have

$$\|e^{k+1}\|_\infty \leqslant [1+\lambda(c_0+1)\tau]\|e^k\|_\infty + \lambda(c_0+1)\tau\|e^{k+1}\|_\infty + c_3\tau(\tau+h^2), \quad 0 \leqslant k \leqslant l,$$

i. e.,

$$[1-\lambda(c_0+1)\tau]\|e^{k+1}\|_\infty \leqslant [1+\lambda(c_0+1)\tau]\|e^k\|_\infty + c_3\tau(\tau+h^2), \quad 0 \leqslant k \leqslant l.$$

When $\lambda(c_0+1)\tau \leqslant \frac{1}{3}$, it follows:

$$\|e^{k+1}\|_\infty \leqslant [1+3\lambda(c_0+1)\tau]\|e^k\|_\infty + \frac{3}{2}c_3\tau(\tau+h^2), \quad 0 \leqslant k \leqslant l.$$

Noticing (1.39), the application of the Gronwall inequality (Theorem 1.2(a)) yields

$$\|e^{l+1}\|_\infty \leqslant \frac{c_3}{2\lambda(c_0+1)}\mathrm{e}^{3\lambda(c_0+1)T}(\tau+h^2) = c_4(\tau+h^2),$$

which says that (1.34) is also true for $k = l+1$.

By induction, (1.34) is true for all k $(0 \leqslant k \leqslant n)$. □

1.5 Crank–Nicolson difference scheme

This section is devoted to the derivation of an unconditionally convergent difference scheme with the accuracy $O(\tau^2 + h^2)$.

1.5.1 Derivation of the difference scheme

Considering equation (1.1) at the point $(x_i, t_{k+\frac{1}{2}})$, we have

$$u_t(x_i, t_{k+\frac{1}{2}}) - u_{xx}(x_i, t_{k+\frac{1}{2}}) = \lambda[u(x_i, t_{k+\frac{1}{2}}) - u^2(x_i, t_{k+\frac{1}{2}})], \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1. \tag{1.43}$$

By Lemma 1.2, we have

$$u_t(x_i, t_{k+\frac{1}{2}}) = \delta_t U_i^{k+\frac{1}{2}} + O(\tau^2), \tag{1.44}$$

$$\begin{aligned} u_{xx}(x_i, t_{k+\frac{1}{2}}) &= \frac{1}{2}[u_{xx}(x_i, t_{k+1}) + u_{xx}(x_i, t_k)] + O(\tau^2) \\ &= \frac{1}{2}(\delta_x^2 U_i^{k+1} + \delta_x^2 U_i^k) + O(h^2) + O(\tau^2) \\ &= \delta_x^2 U_i^{k+\frac{1}{2}} + O(\tau^2 + h^2), \end{aligned} \tag{1.45}$$

$$u(x_i, t_{k+\frac{1}{2}}) = U_i^{k+\frac{1}{2}} + O(\tau^2), \tag{1.46}$$

$$\begin{aligned} u^2(x_i, t_{k+\frac{1}{2}}) &= \left[U_i^k + \frac{\tau}{2}u_t(x_i, t_{k+\frac{1}{2}}) + O(\tau^2)\right]\left[U_i^{k+1} - \frac{\tau}{2}u_t(x_i, t_{k+\frac{1}{2}}) + O(\tau^2)\right] \\ &= U_i^k U_i^{k+1} + O(\tau^2). \end{aligned} \tag{1.47}$$

Inserting (1.44)–(1.47) into (1.43) arrives at

$$\delta_t U_i^{k+\frac{1}{2}} - \delta_x^2 U_i^{k+\frac{1}{2}} = \lambda(U_i^{k+\frac{1}{2}} - U_i^k U_i^{k+1}) + (R_3)_i^{k+\frac{1}{2}}, \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1, \tag{1.48}$$

where there is a constant c_5 such that

$$|(R_3)_i^{k+\frac{1}{2}}| \leqslant c_5(\tau^2 + h^2), \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1. \tag{1.49}$$

Noticing the initial-boundary value conditions (1.2)–(1.3), we have

$$\begin{cases} U_i^0 = \varphi(x_i), & 0 \leqslant i \leqslant m, \quad (1.50) \\ U_0^k = \alpha(t_k), \quad U_m^k = \beta(t_k), & 1 \leqslant k \leqslant n. \quad (1.51) \end{cases}$$

Neglecting the small term $(R_3)_i^{k+\frac{1}{2}}$ in (1.48) and replacing the exact solution U_i^k by its numerical one u_i^k, the Crank–Nicolson difference scheme is derived as

$$\begin{cases} \delta_t u_i^{k+\frac{1}{2}} - \delta_x^2 u_i^{k+\frac{1}{2}} = \lambda(u_i^{k+\frac{1}{2}} - u_i^k u_i^{k+1}), & 1 \leqslant i \leqslant m-1, 0 \leqslant k \leqslant n-1, \quad (1.52) \\ u_i^0 = \varphi(x_i), & 0 \leqslant i \leqslant m, \quad (1.53) \\ u_0^k = \alpha(t_k), \quad u_m^k = \beta(t_k), & 1 \leqslant k \leqslant n. \quad (1.54) \end{cases}$$

The difference scheme (1.52)–(1.54) is a two-level linearized difference scheme.

1.5.2 Existence and convergence of the difference solution

Theorem 1.7. *Let $\{U_i^k \mid 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ and $\{u_i^k \mid 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ be solutions of the problem (1.1)–(1.3) and the difference scheme (1.52)–(1.54), respectively. Denote*

$$e_i^k = U_i^k - u_i^k, \quad 0 \leqslant i \leqslant m,\ 0 \leqslant k \leqslant n,$$
$$c_6 = \frac{c_5}{\lambda(c_0+1)}\sqrt{\frac{3}{L}}\mathrm{e}^{\frac{L^2}{2}\lambda^2(c_0+1)^2T}.$$

Then when $\frac{\sqrt{L}}{2}c_6(\tau^2+h^2) \leqslant 1$, $L^2\lambda^2[1+2(c_0+1)^2]\tau \leqslant 2$ and $\lambda(\frac{3}{2}+c_0)\tau < 1$, it holds that
(I) *the difference scheme (1.52)–(1.54) is uniquely solvable;*
(II)

$$|e^k|_1 \leqslant c_6(\tau^2+h^2), \quad 0 \leqslant k \leqslant n. \tag{1.55}$$

Proof. Subtracting (1.52)–(1.54) from (1.48), (1.50)–(1.51), respectively, we get the system of error equations as follows:

$$\begin{cases} \delta_t e_i^{k+\frac{1}{2}} - \delta_x^2 e_i^{k+\frac{1}{2}} = \lambda e_i^{k+\frac{1}{2}} - \lambda(U_i^k U_i^{k+1} - u_i^k u_i^{k+1}) + (R_3)_i^{k+\frac{1}{2}}, \\ \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1, & (1.56) \\ e_i^0 = 0, \qquad 0 \leqslant i \leqslant m, & (1.57) \\ e_0^k = 0, \quad e_m^k = 0, \qquad 1 \leqslant k \leqslant n. & (1.58) \end{cases}$$

Taking the inner product of (1.56) with $\delta_t e^{k+\frac{1}{2}}$ on both the right- and left-hand sides, and using the summation by parts, we have

$$\begin{aligned} &\|\delta_t e^{k+\frac{1}{2}}\|^2 + \frac{1}{2\tau}(|e^{k+1}|_1^2 - |e^k|_1^2) \\ &= \lambda h \sum_{i=1}^{m-1} [e_i^{k+\frac{1}{2}} - (u_i^k e_i^{k+1} + e_i^k U_i^{k+1})]\delta_t e_i^{k+\frac{1}{2}} + h\sum_{i=1}^{m-1}(R_3)_i^{k+\frac{1}{2}}\delta_t e_i^{k+\frac{1}{2}} \\ &\leqslant \lambda(\|e^{k+\frac{1}{2}}\| \cdot \|\delta_t e^{k+\frac{1}{2}}\| + \|u^k\|_\infty \|e^{k+1}\| \cdot \|\delta_t e^{k+\frac{1}{2}}\| \\ &\quad + \|U^{k+1}\|_\infty \|e^k\| \cdot \|\delta_t e^{k+\frac{1}{2}}\|) + \|(R_3)^{k+\frac{1}{2}}\| \cdot \|\delta_t e^{k+\frac{1}{2}}\| \\ &\leqslant \left(\frac{1}{4}\|\delta_t e^{k+\frac{1}{2}}\|^2 + \lambda^2\|e^{k+\frac{1}{2}}\|^2\right) + \left(\frac{1}{4}\|\delta_t e^{k+\frac{1}{2}}\|^2 + \lambda^2\|u^k\|_\infty^2\|e^{k+1}\|^2\right) \\ &\quad + \left(\frac{1}{4}\|\delta_t e^{k+\frac{1}{2}}\|^2 + \lambda^2\|U^{k+1}\|_\infty^2\|e^k\|^2\right) + \left(\frac{1}{4}\|\delta_t e^{k+\frac{1}{2}}\|^2 + \|(R_3)^{k+\frac{1}{2}}\|^2\right), \\ &\quad 0 \leqslant k \leqslant n-1, \end{aligned}$$

which follows by noticing (1.49) that

$$\frac{1}{2\tau}(|e^{k+1}|_1^2 - |e^k|_1^2) \leqslant \lambda^2\|e^{k+\frac{1}{2}}\|^2 + \lambda^2\|u^k\|_\infty^2\|e^{k+1}\|^2 + \lambda^2\|U^{k+1}\|_\infty^2\|e^k\|^2 + \|(R_3)^{k+\frac{1}{2}}\|^2$$

$$\begin{aligned}&\leqslant \frac{1}{2}\lambda^2(\|e^k\|^2+\|e^{k+1}\|^2)+\lambda^2\|u^k\|_\infty^2\|e^{k+1}\|^2\\&\quad+\lambda^2c_0^2\|e^k\|^2+Lc_5^2(\tau^2+h^2)^2,\quad 0\leqslant k\leqslant n-1.\end{aligned}\tag{1.59}$$

In view of (1.57), we know $|e^0|_1=0$, which means that (1.55) holds for $k=0$. Now assume that (1.55) is true for $0\leqslant k\leqslant l$, i. e.,

$$|e^k|_1\leqslant c_6(\tau^2+h^2),\quad 0\leqslant k\leqslant l.$$

By Lemma 1.1, when $\frac{\sqrt{L}}{2}c_6(\tau^2+h^2)\leqslant 1$, we have

$$\begin{aligned}&\|e^k\|_\infty\leqslant\frac{\sqrt{L}}{2}|e^k|_1\leqslant\frac{\sqrt{L}}{2}c_6(\tau^2+h^2)\leqslant 1,\quad 0\leqslant k\leqslant l,\\&\|u^k\|_\infty\leqslant\|U^k\|_\infty+\|e^k\|_\infty\leqslant c_0+1,\quad 0\leqslant k\leqslant l.\end{aligned}$$

(I) Proof for the unique solvability.

From (1.52) and (1.54), the system of linear equations in u^{l+1} can be obtained as

$$\begin{cases}\delta_t u_i^{l+\frac{1}{2}}-\delta_x^2u_i^{l+\frac{1}{2}}=\lambda(u_i^{l+\frac{1}{2}}-u_i^lu_i^{l+1}),\quad 1\leqslant i\leqslant m-1,\\ u_0^{l+1}=\alpha(t_{l+1}),\quad u_m^{l+1}=\beta(t_{l+1}).\end{cases}$$

Consider its homogeneous one:

$$\begin{cases}\frac{1}{\tau}u_i^{l+1}-\frac{1}{2}\delta_x^2u_i^{l+1}=\lambda(\frac{1}{2}u_i^{l+1}-u_i^lu_i^{l+1}),\quad 1\leqslant i\leqslant m-1, & (1.60)\\ u_0^{l+1}=0,\quad u_m^{l+1}=0. & (1.61)\end{cases}$$

Taking the inner product of (1.60) on both the right- and left-hand sides with u^{l+1} gives

$$\frac{1}{\tau}\|u^{l+1}\|^2+\frac{1}{2}|u^{l+1}|_1^2\leqslant\lambda\Big(\frac{1}{2}\|u^{l+1}\|^2+\|u^l\|_\infty\|u^{l+1}\|^2\Big),$$

which further implies

$$\frac{1}{\tau}\|u^{l+1}\|^2\leqslant\lambda\Big(\frac{1}{2}+\|u^l\|_\infty\Big)\|u^{l+1}\|^2\leqslant\lambda\Big[\frac{1}{2}+(c_0+1)\Big]\|u^{l+1}\|^2.$$

Thus, when $\tau<\frac{1}{\lambda(3/2+c_0)}$, the equality $\|u^{l+1}\|=0$ is followed. Therefore, the value of u^{l+1} is uniquely determined by (1.52) and (1.54).

(II) Proof for (1.55).

By (1.59) and Lemma 1.1, we have

$$\begin{aligned}
&\frac{1}{2\tau}\left(|e^{k+1}|_1^2-|e^k|_1^2\right)\\
\leqslant&\frac{1}{2}\lambda^2\left(\|e^{k+1}\|^2+\|e^k\|^2\right)+\lambda^2(c_0+1)^2\|e^{k+1}\|^2+\lambda^2c_0^2\|e^k\|^2+Lc_5^2(\tau^2+h^2)^2\\
\leqslant&\frac{1}{12}\lambda^2L^2\left(|e^{k+1}|_1^2+|e^k|_1^2\right)+\frac{1}{6}\lambda^2(c_0+1)^2L^2|e^{k+1}|_1^2+\frac{1}{6}\lambda^2c_0^2L^2|e^k|_1^2\\
&+Lc_5^2(\tau^2+h^2)^2,\quad 0\leqslant k\leqslant l,
\end{aligned}$$

i. e.,

$$\begin{aligned}
&\left[1-\frac{1}{6}\lambda^2L^2(1+2(c_0+1)^2)\tau\right]|e^{k+1}|_1^2\\
\leqslant&\left[1+\frac{1}{6}\lambda^2L^2(1+2c_0^2)\tau\right]|e^k|_1^2+2Lc_5^2\tau(\tau^2+h^2)^2,\quad 0\leqslant k\leqslant l.
\end{aligned}$$

When $\frac{1}{6}\lambda^2L^2[1+2(c_0+1)^2]\tau\leqslant\frac{1}{3}$, it follows:

$$|e^{k+1}|_1^2\leqslant[1+\lambda^2L^2(c_0+1)^2\tau]|e^k|_1^2+3Lc_5^2\tau(\tau^2+h^2)^2,\quad 0\leqslant k\leqslant l.$$

The application of the Gronwall inequality (Theorem 1.2(a)) yields

$$|e^{l+1}|_1^2\leqslant \mathrm{e}^{\lambda^2L^2(c_0+1)^2l\tau}\left[|e^0|_1^2+\frac{3c_5^2}{L\lambda^2(c_0+1)^2}(\tau^2+h^2)^2\right]\leqslant\frac{3c_5^2}{L\lambda^2(c_0+1)^2}\mathrm{e}^{\lambda^2L^2(c_0+1)^2T}(\tau^2+h^2)^2.$$

Taking the square root on both the right- and left-hand sides of the inequality above produces

$$|e^{l+1}|_1\leqslant c_6(\tau^2+h^2).$$

By induction, the theorem is proved. □

1.6 Fourth-order compact difference scheme

In this section, an unconditionally convergent compact difference scheme with the accuracy $O(\tau^2+h^4)$ will be developed.

1.6.1 Derivation of the difference scheme

For $w=\{w_i\mid 0\leqslant i\leqslant m\}\in\mathcal{U}_h$, define an averaging operator by

$$\mathcal{A}w_i=\begin{cases}\frac{1}{12}(w_{i-1}+10w_i+w_{i+1}), & 1\leqslant i\leqslant m-1,\\ w_i, & i=0,\ m.\end{cases}$$

Considering equation (1.1) at the point $(x_i, t_{k+\frac{1}{2}})$, we have

$$u_t(x_i, t_{k+\frac{1}{2}}) - u_{xx}(x_i, t_{k+\frac{1}{2}}) = \lambda[u(x_i, t_{k+\frac{1}{2}}) - u^2(x_i, t_{k+\frac{1}{2}})], \quad 0 \leqslant i \leqslant m,\ 0 \leqslant k \leqslant n-1.$$

By Lemma 1.2, we have

$$\delta_t U_i^{k+\frac{1}{2}} - \frac{1}{2}[u_{xx}(x_i, t_{k+1}) + u_{xx}(x_i, t_k)] = \lambda(U_i^{k+\frac{1}{2}} - U_i^k U_i^{k+1}) + O(\tau^2),$$
$$0 \leqslant i \leqslant m,\ 0 \leqslant k \leqslant n-1. \tag{1.62}$$

Performing the operator $\mathcal{A}$ on both the right- and left-hand sides of (1.62) and noticing (Lemma 1.2(h))

$$\mathcal{A}u_{xx}(x_i, t_k) = \delta_x^2 U_i^k + O(h^4),$$

we have

$$\mathcal{A}\delta_t U_i^{k+\frac{1}{2}} - \delta_x^2 U_i^{k+\frac{1}{2}} = \lambda\mathcal{A}(U_i^{k+\frac{1}{2}} - U_i^k U_i^{k+1}) + (R_4)_i^{k+\frac{1}{2}}, \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1, \tag{1.63}$$

where there is a constant c_7 such that

$$\left|(R_4)_i^{k+\frac{1}{2}}\right| \leqslant c_7(\tau^2 + h^4), \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1. \tag{1.64}$$

Noticing the initial-boundary value conditions (1.2)–(1.3), we have

$$\begin{cases} U_i^0 = \varphi(x_i), & 0 \leqslant i \leqslant m, & (1.65)\\ U_0^k = \alpha(t_k), \quad U_m^k = \beta(t_k), & 1 \leqslant k \leqslant n. & (1.66)\end{cases}$$

Neglecting the small term $(R_4)_i^{k+\frac{1}{2}}$ in (1.63) and replacing the exact solution U_i^k by its numerical one u_i^k, a compact difference scheme is derived in the form of

$$\begin{cases} \mathcal{A}\delta_t u_i^{k+\frac{1}{2}} - \delta_x^2 u_i^{k+\frac{1}{2}} = \lambda\mathcal{A}(u_i^{k+\frac{1}{2}} - u_i^k u_i^{k+1}), & 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1, & (1.67)\\ u_i^0 = \varphi(x_i), & 0 \leqslant i \leqslant m, & (1.68)\\ u_0^k = \alpha(t_k), \quad u_m^k = \beta(t_k), & 1 \leqslant k \leqslant n. & (1.69)\end{cases}$$

The difference scheme (1.67)–(1.69) is also a two-level linearized difference scheme.

1.6.2 Existence and convergence of difference solution

Theorem 1.8. *Let $\{U_i^k \mid 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ and $\{u_i^k \mid 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ be solutions of the problem (1.1)–(1.3) and the difference scheme (1.67)–(1.69), respectively. Denote*

$$e_i^k = U_i^k - u_i^k, \quad 0 \leqslant i \leqslant m, \ 0 \leqslant k \leqslant n,$$
$$c_8 = \frac{c_7}{\lambda}\sqrt{\frac{6}{L[1+2(c_0+1)^2]}}\, \mathrm{e}^{\frac{3}{8}L^2\lambda^2[1+2(c_0+1)^2]T}.$$

Then when $\frac{\sqrt{L}}{2}c_8(\tau^2+h^4) \leqslant 1$, $\frac{L^2}{4}\lambda^2[1+2(c_0+1)^2]\tau \leqslant \frac{1}{3}$ *and* $\frac{3}{2}(\frac{3}{2}+c_0)\lambda\tau < 1$, *it holds that*
(I) *the difference scheme* (1.67)–(1.69) *is uniquely solvable;*
(II)

$$|e^k|_1 \leqslant c_8(\tau^2+h^4), \quad 0 \leqslant k \leqslant n. \tag{1.70}$$

Proof. Subtracting (1.67)–(1.69) from (1.63), (1.65)–(1.66), respectively, the system of error equations is obtained as

$$\begin{cases} \mathcal{A}\delta_t e_i^{k+\frac{1}{2}} - \delta_x^2 e_i^{k+\frac{1}{2}} = \lambda\mathcal{A}(e_i^{k+\frac{1}{2}} - u_i^k e_i^{k+1} - U_i^{k+1}e_i^k) + (R_4)_i^{k+\frac{1}{2}}, \\ \quad 1 \leqslant i \leqslant m-1, \ 0 \leqslant k \leqslant n-1, & (1.71) \\ e_i^0 = 0, \qquad\qquad\qquad\qquad 0 \leqslant i \leqslant m, & (1.72) \\ e_0^k = 0, \quad e_m^k = 0, \qquad\qquad 1 \leqslant k \leqslant n. & (1.73) \end{cases}$$

Taking the inner product on both the right- and left-hand sides of (1.71) with $\delta_t e^{k+\frac{1}{2}}$, we have

$$\begin{aligned} &(\mathcal{A}\delta_t e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) - (\delta_x^2 e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) \\ &= \lambda[(\mathcal{A}e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) - (\mathcal{A}(u^k e^{k+1}), \delta_t e^{k+\frac{1}{2}}) - (\mathcal{A}(U^{k+1}e^k), \delta_t e^{k+\frac{1}{2}})] \\ &\quad + ((R_4)^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}), \quad 0 \leqslant k \leqslant n-1. \end{aligned} \tag{1.74}$$

Now each term in (1.74) will be analyzed:

$$\begin{aligned} (\mathcal{A}\delta_t e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) &= \left(\left(I + \frac{h^2}{12}\delta_x^2\right)\delta_t e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}\right) \\ &= \|\delta_t e^{k+\frac{1}{2}}\|^2 - \frac{h^2}{12}\|\delta_x\delta_t e^{k+\frac{1}{2}}\|^2 \\ &\geqslant \|\delta_t e^{k+\frac{1}{2}}\|^2 - \frac{h^2}{12}\cdot\frac{4}{h^2}\|\delta_t e^{k+\frac{1}{2}}\|^2 \\ &= \frac{2}{3}\|\delta_t e^{k+\frac{1}{2}}\|^2, \end{aligned} \tag{1.75}$$

$$-(\delta_x^2 e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) = \frac{1}{2\tau}(|e^{k+1}|_1^2 - |e^k|_1^2), \tag{1.76}$$

$$\begin{aligned} \lambda(\mathcal{A}e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) &\leqslant \frac{1}{6}\|\delta_t e^{k+\frac{1}{2}}\|^2 + \frac{3}{2}\lambda^2\|\mathcal{A}e^{k+\frac{1}{2}}\|^2 \\ &\leqslant \frac{1}{6}\|\delta_t e^{k+\frac{1}{2}}\|^2 + \frac{3}{2}\lambda^2\|e^{k+\frac{1}{2}}\|^2, \end{aligned} \tag{1.77}$$

$$\lambda(\mathcal{A}(u^k e^{k+1}), \delta_t e^{k+\frac{1}{2}}) \leqslant \frac{1}{6}\|\delta_t e^{k+\frac{1}{2}}\|^2 + \frac{3}{2}\lambda^2\|\mathcal{A}(u^k e^{k+1})\|^2$$

$$\leqslant \frac{1}{6}\|\delta_t e^{k+\frac{1}{2}}\|^2 + \frac{3}{2}\lambda^2\|u^k\|_\infty^2\|e^{k+1}\|^2, \tag{1.78}$$

$$\lambda(\mathcal{A}(U^{k+1} e^{k}), \delta_t e^{k+\frac{1}{2}}) \leqslant \frac{1}{6}\|\delta_t e^{k+\frac{1}{2}}\|^2 + \frac{3}{2}\lambda^2\|\mathcal{A}(U^{k+1} e^{k})\|^2$$

$$\leqslant \frac{1}{6}\|\delta_t e^{k+\frac{1}{2}}\|^2 + \frac{3}{2}\lambda^2\|U^{k+1}\|_\infty^2\|e^{k}\|^2, \tag{1.79}$$

$$((R_4)^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) \leqslant \frac{1}{6}\|\delta_t e^{k+\frac{1}{2}}\|^2 + \frac{3}{2}\|(R_4)^{k+\frac{1}{2}}\|^2. \tag{1.80}$$

Inserting (1.75)–(1.80) into (1.74) and noticing (1.64) lead to

$$\begin{aligned}
&\frac{1}{2\tau}(|e^{k+1}|_1^2 - |e^k|_1^2)\\
&\leqslant \frac{3}{2}\lambda^2\|e^{k+\frac{1}{2}}\|^2 + \frac{3}{2}\lambda^2\|u^k\|_\infty^2\|e^{k+1}\|^2 + \frac{3}{2}\lambda^2\|U^{k+1}\|_\infty^2\|e^k\|^2 + \frac{3}{2}\|(R_4)^{k+\frac{1}{2}}\|^2\\
&\leqslant \frac{3}{4}\lambda^2(\|e^{k+1}\|^2 + \|e^k\|^2) + \frac{3}{2}\lambda^2\|u^k\|_\infty^2\|e^{k+1}\|^2\\
&\quad + \frac{3}{2}\lambda^2 c_0^2\|e^k\|^2 + \frac{3}{2}Lc_7^2(\tau^2 + h^4)^2, \quad 0 \leqslant k \leqslant n-1.
\end{aligned} \tag{1.81}$$

In view of (1.72), we know $|e^0|_1 = 0$, which means that (1.70) holds for $k = 0$. Now assume that (1.70) is true for $0 \leqslant k \leqslant l$, i. e.,

$$|e^k|_1 \leqslant c_8(\tau^2 + h^4), \quad 0 \leqslant k \leqslant l.$$

By Lemma 1.1, when $\frac{\sqrt{L}}{2}c_8(\tau^2 + h^4) \leqslant 1$, we have

$$\|e^k\|_\infty \leqslant \frac{\sqrt{L}}{2}|e^k|_1 \leqslant \frac{\sqrt{L}}{2}c_8(\tau^2 + h^4) \leqslant 1, \quad 0 \leqslant k \leqslant l,$$

$$\|u^k\|_\infty \leqslant \|U^k\|_\infty + \|e^k\|_\infty \leqslant c_0 + 1, \quad 0 \leqslant k \leqslant l.$$

(I) Proof for the unique solvability.

From (1.67) and (1.69), the system of linear equations in u^{l+1} can be obtained as

$$\begin{cases} \mathcal{A}\delta_t u_i^{l+\frac{1}{2}} - \delta_x^2 u_i^{l+\frac{1}{2}} = \lambda\mathcal{A}(u_i^{l+\frac{1}{2}} - u_i^l u_i^{l+1}), \quad 1 \leqslant i \leqslant m-1,\\ u_0^{l+1} = \alpha(t_{l+1}), \quad u_m^{l+1} = \beta(t_{l+1}). \end{cases}$$

Consider its homogeneous one:

$$\begin{cases} \frac{1}{\tau}\mathcal{A}u_i^{l+1} - \frac{1}{2}\delta_x^2 u_i^{l+1} = \lambda\mathcal{A}(\frac{1}{2}u_i^{l+1} - u_i^l u_i^{l+1}), \quad 1 \leqslant i \leqslant m-1, & (1.82)\\ u_0^{l+1} = 0, \quad u_m^{l+1} = 0. & (1.83) \end{cases}$$

Taking the inner product on both the right- and left-hand sides of (1.82) with u^{l+1} gives

$$
\begin{aligned}
&\frac{1}{\tau}(\mathcal{A}u^{l+1}, u^{l+1}) + \frac{1}{2}|u^{l+1}|_1^2 \\
&= \lambda\Big(\mathcal{A}\Big(\frac{1}{2}u^{l+1} - u^l u^{l+1}\Big), u^{l+1}\Big) \\
&\leqslant \lambda\Big\|\mathcal{A}\Big(\frac{1}{2}u^{l+1} - u^l u^{l+1}\Big)\Big\| \cdot \|u^{l+1}\| \\
&\leqslant \lambda\Big\|\Big(\frac{1}{2} + u^l\Big)u^{l+1}\Big\| \cdot \|u^{l+1}\| \\
&\leqslant \lambda\Big(\frac{1}{2} + \|u^l\|_\infty\Big)\|u^{l+1}\|^2 \\
&\leqslant \lambda\Big(\frac{1}{2} + c_0 + 1\Big)\|u^{l+1}\|^2,
\end{aligned}
$$

which further implies

$$
\frac{1}{\tau} \cdot \frac{2}{3}\|u^{l+1}\|^2 \leqslant \lambda\Big(\frac{3}{2} + c_0\Big)\|u^{l+1}\|^2.
$$

Thus, when $\frac{3}{2}(\frac{3}{2} + c_0)\lambda\tau < 1$, the equality $\|u^{l+1}\| = 0$ is followed. Therefore, the value of u^{l+1} is uniquely determined by (1.67) and (1.69).

(II) Proof for (1.70).

From (1.81), we have

$$
\begin{aligned}
&\frac{1}{2\tau}(|e^{k+1}|_1^2 - |e^k|_1^2) \\
&\leqslant \frac{3}{4}\lambda^2(\|e^{k+1}\|^2 + \|e^k\|^2) + \frac{3}{2}\lambda^2(c_0+1)^2\|e^{k+1}\|^2 + \frac{3}{2}\lambda^2 c_0^2\|e^k\|^2 + \frac{3}{2}Lc_7^2(\tau^2 + h^4)^2 \\
&\leqslant \frac{3}{4}\lambda^2[1 + 2(c_0+1)^2](\|e^{k+1}\|^2 + \|e^k\|^2) + \frac{3}{2}Lc_7^2(\tau^2 + h^4)^2 \\
&\leqslant \frac{3}{4}\lambda^2[1 + 2(c_0+1)^2]\frac{L^2}{6}(|e^{k+1}|_1^2 + |e^k|_1^2) + \frac{3}{2}Lc_7^2(\tau^2 + h^4)^2, \quad 0 \leqslant k \leqslant l,
\end{aligned}
$$

i. e.,

$$
\begin{aligned}
&\Big[1 - \frac{L^2}{4}\lambda^2(1 + 2(c_0+1)^2)\tau\Big]|e^{k+1}|_1^2 \\
&\leqslant \Big[1 + \frac{L^2}{4}\lambda^2(1 + 2(c_0+1)^2)\tau\Big]|e^k|_1^2 + 3Lc_7^2\tau(\tau^2 + h^4)^2, \quad 0 \leqslant k \leqslant l.
\end{aligned}
$$

When $\frac{L^2}{4}\lambda^2[1 + 2(c_0+1)^2]\tau \leqslant \frac{1}{3}$, it follows:

$$
|e^{k+1}|_1^2 \leqslant \Big[1 + \frac{3}{4}L^2\lambda^2(1 + 2(c_0+1)^2)\tau\Big]|e^k|_1^2 + \frac{9}{2}Lc_7^2\tau(\tau^2 + h^4)^2, \quad 0 \leqslant k \leqslant l.
$$

The application of the Gronwall inequality (Theorem 1.2(a)) leads to

$$\begin{aligned}|e^{l+1}|_1^2 &\leqslant e^{\frac{3}{4}L^2\lambda^2[1+2(c_0+1)^2]l\tau}\left\{|e^0|_1^2 + \frac{\frac{9}{2}Lc_7^2}{\frac{3}{4}L^2\lambda^2[1+2(c_0+1)^2]}(\tau^2+h^4)^2\right\}\\ &\leqslant \frac{6c_7^2}{L\lambda^2[1+2(c_0+1)^2]}e^{\frac{3}{4}L^2\lambda^2[1+2(c_0+1)^2]T}(\tau^2+h^4)^2.\end{aligned}$$

Taking the square root on both the right- and left-hand sides of the inequality above produces

$$|e^{l+1}|_1 \leqslant c_8(\tau^2+h^4),$$

which says that (1.70) also holds for $k = l+1$.

By induction, (1.70) is true for all k $(0 \leqslant k \leqslant n)$. □

1.7 Three-level linearized difference scheme

This part will focus on an unconditionally convergent and conservative three-level linearized difference scheme for solving (1.1)–(1.3) with the convergence order $O(\tau^2+h^2)$.

1.7.1 Derivation of the difference scheme

Considering equation (1.1) at the point $(x_i, t_{\frac{1}{2}})$, we have

$$u_t(x_i,t_{\frac{1}{2}}) - u_{xx}(x_i,t_{\frac{1}{2}}) = \lambda[u(x_i,t_{\frac{1}{2}}) - u^2(x_i,t_{\frac{1}{2}})], \quad 1 \leqslant i \leqslant m-1.$$

By Lemma 1.2, we have

$$\delta_t U_i^{\frac{1}{2}} - \delta_x^2 U_i^{\frac{1}{2}} = \lambda(U_i^{\frac{1}{2}} - U_i^0 U_i^1) + (R_5)_i^0, \quad 1 \leqslant i \leqslant m-1, \tag{1.84}$$

where there is a constant c_9 such that

$$|(R_5)_i^0| \leqslant c_9(\tau^2+h^2), \quad 1 \leqslant i \leqslant m-1. \tag{1.85}$$

Considering equation (1.1) at the node point (x_i, t_k), we have

$$u_t(x_i,t_k) - u_{xx}(x_i,t_k) = \lambda[u(x_i,t_k) - u^2(x_i,t_k)], \quad 1 \leqslant i \leqslant m-1,\ 1 \leqslant k \leqslant n-1.$$

By Lemma 1.2, we have

$$\begin{aligned}&\Delta_t U_i^k - \delta_x^2 U_i^{\bar{k}} = \lambda\left[U_i^{\bar{k}} - \frac{1}{3}(U_i^{k-1} + U_i^k + U_i^{k+1})U_i^k\right] + (R_5)_i^k,\\ &1 \leqslant i \leqslant m-1,\ 1 \leqslant k \leqslant n-1,\end{aligned} \tag{1.86}$$

where there is a constant c_{10} such that

$$|(R_5)_i^k| \leqslant c_{10}(\tau^2 + h^2), \quad 1 \leqslant i \leqslant m-1,\ 1 \leqslant k \leqslant n-1. \tag{1.87}$$

Noticing the initial-boundary value conditions (1.2)–(1.3), we have

$$\begin{cases} U_i^0 = \varphi(x_i), & 0 \leqslant i \leqslant m, & (1.88)\\ U_0^k = \alpha(t_k), \quad U_m^k = \beta(t_k), & 1 \leqslant k \leqslant n. & (1.89)\end{cases}$$

Neglecting the small terms in (1.84) and (1.86), and replacing the exact solution U_i^k by its numerical one u_i^k, the following difference scheme can be derived in the form:

$$\begin{cases} \delta_t u_i^{\frac{1}{2}} - \delta_x^2 u_i^{\frac{1}{2}} = \lambda(u_i^{\frac{1}{2}} - u_i^0 u_i^1), & 1 \leqslant i \leqslant m-1, & (1.90)\\ \Delta_t u_i^k - \delta_x^2 u_i^{\bar{k}} = \lambda[u_i^{\bar{k}} - \frac{1}{3}u_i^k(u_i^{k-1} + u_i^k + u_i^{k+1})], & 1 \leqslant i \leqslant m-1,\ 1 \leqslant k \leqslant n-1, & (1.91)\\ u_i^0 = \varphi(x_i), & 0 \leqslant i \leqslant m, & (1.92)\\ u_0^k = \alpha(t_k), \quad u_m^k = \beta(t_k), & 1 \leqslant k \leqslant n. & (1.93)\end{cases}$$

The next result illustrates the conservative property of this difference scheme.

Theorem 1.9. *Suppose* $\{u_i^k \mid 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ *is the solution of the difference scheme (*1.90*)–(*1.93*) with* $\alpha(t) \equiv 0$*,* $\beta(t) \equiv 0$*. Denote*

$$\begin{aligned} E^k = {} & \frac{1}{2}(\|u^{k+1}\|^2 + \|u^k\|^2) + 2\tau\left(\frac{1}{2}|u^{\frac{1}{2}}|_1^2 + \sum_{l=1}^{k}|u^{\bar{l}}|_1^2\right) + 2\lambda\tau\left\{\frac{1}{2}[(u^0u^1, u^{\frac{1}{2}}) - \|u^{\frac{1}{2}}\|^2]\right.\\ & \left. + \sum_{l=1}^{k}\left[\left(\frac{1}{3}(u^{l-1} + u^l + u^{l+1})u^l, u^{\bar{l}}\right) - \|u^{\bar{l}}\|^2\right]\right\}, \quad 0 \leqslant k \leqslant n-1,\\ F^k = {} & \frac{1}{2}(|u^{k+1}|_1^2 + |u^k|_1^2) + \lambda\left\{\frac{1}{3}[(u^k, (u^{k+1})^2) + ((u^k)^2, u^{k+1})]\right.\\ & \left. - \frac{1}{2}(\|u^{k+1}\|^2 + \|u^k\|^2)\right\} + 2\tau\left(\frac{1}{2}\|\delta_t u^{\frac{1}{2}}\|^2 + \sum_{l=1}^{k}\|\Delta_t u^l\|^2\right), \quad 0 \leqslant k \leqslant n-1. \end{aligned}$$

Then we have

$$E^k = \|u^0\|^2, \quad 0 \leqslant k \leqslant n-1, \tag{1.94}$$

$$F^k = \hat{F}^0, \quad 0 \leqslant k \leqslant n-1, \tag{1.95}$$

where

$$\hat{F}^0 = |u^0|_1^2 + \lambda\left[\frac{4}{3}((u^0)^2, u^1) - \frac{2}{3}(u^0, (u^1)^2) - \|u^0\|^2\right].$$

Proof. (I) Taking the inner product of (1.90) on both the right- and left-hand sides with $u^{\frac{1}{2}}$ gives

$$(\delta_t u^{\frac{1}{2}}, u^{\frac{1}{2}}) - (\delta_x^2 u^{\frac{1}{2}}, u^{\frac{1}{2}}) = \lambda[\|u^{\frac{1}{2}}\|^2 - (u^0 u^1, u^{\frac{1}{2}})].$$

Noticing

$$(\delta_t u^{\frac{1}{2}}, u^{\frac{1}{2}}) = \frac{1}{2\tau}(\|u^1\|^2 - \|u^0\|^2) \quad \text{and} \quad -(\delta_x^2 u^{\frac{1}{2}}, u^{\frac{1}{2}}) = |u^{\frac{1}{2}}|_1^2,$$

we have

$$\frac{1}{2\tau}(\|u^1\|^2 - \|u^0\|^2) + |u^{\frac{1}{2}}|_1^2 + \lambda[(u^0 u^1, u^{\frac{1}{2}}) - \|u^{\frac{1}{2}}\|^2] = 0,$$

which can be rewritten as

$$\frac{1}{2}(\|u^1\|^2 + \|u^0\|^2) + \tau|u^{\frac{1}{2}}|_1^2 + \lambda\tau[(u^0 u^1, u^{\frac{1}{2}}) - \|u^{\frac{1}{2}}\|^2] = \|u^0\|^2,$$

i. e.,

$$E^0 = \|u^0\|^2. \tag{1.96}$$

Taking the inner product of (1.91) on both the right- and left-hand sides with $u^{\bar{k}}$ yields

$$(\Delta_t u^k, u^{\bar{k}}) - (\delta_x^2 u^{\bar{k}}, u^{\bar{k}}) = \lambda\left[\|u^{\bar{k}}\|^2 - \left(\frac{1}{3}(u^{k-1} + u^k + u^{k+1})u^k, u^{\bar{k}}\right)\right],$$
$$1 \leqslant k \leqslant n-1.$$

Noticing

$$(\Delta_t u^k, u^{\bar{k}}) = \frac{1}{4\tau}(\|u^{k+1}\|^2 - \|u^{k-1}\|^2) \quad \text{and} \quad -(\delta_x^2 u^{\bar{k}}, u^{\bar{k}}) = |u^{\bar{k}}|_1^2,$$

we have

$$\begin{aligned}&\frac{1}{2\tau}\left(\frac{\|u^{k+1}\|^2 + \|u^k\|^2}{2} - \frac{\|u^k\|^2 + \|u^{k-1}\|^2}{2}\right) + |u^{\bar{k}}|_1^2 \\ &\quad + \lambda\left[\left(\frac{1}{3}(u^{k-1} + u^k + u^{k+1})u^k, u^{\bar{k}}\right) - \|u^{\bar{k}}\|^2\right] = 0, \quad 1 \leqslant k \leqslant n-1.\end{aligned}$$

Replacing k by l in the equality above and summing over l from 1 to k will arrive at

$$\begin{aligned}&\frac{1}{2}(\|u^{k+1}\|^2 + \|u^k\|^2) + 2\tau\sum_{l=1}^{k}|u^{\bar{l}}|_1^2 + 2\lambda\tau\sum_{l=1}^{k}\left[\left(\frac{1}{3}(u^{l-1} + u^l + u^{l+1})u^l, u^{\bar{l}}\right) - \|u^{\bar{l}}\|^2\right] \\ &\quad = \frac{1}{2}(\|u^1\|^2 + \|u^0\|^2), \quad 1 \leqslant k \leqslant n-1.\end{aligned}$$

Adding $\tau|u^{\frac{1}{2}}|_1^2+\lambda\tau[(u^0u^1,u^{\frac{1}{2}})-\|u^{\frac{1}{2}}\|^2]$ on both the right- and left-hand sides of the equality above yields

$$E^k = E^0, \quad 1 \leqslant k \leqslant n-1. \tag{1.97}$$

Then the equality (1.94) is followed from (1.96) and (1.97).

(II) Taking the inner product of (1.90) on both the right- and left-hand sides with $\delta_t u^{\frac{1}{2}}$ gives

$$\|\delta_t u^{\frac{1}{2}}\|^2 - (\delta_x^2 u^{\frac{1}{2}}, \delta_t u^{\frac{1}{2}}) = \lambda[(u^{\frac{1}{2}}, \delta_t u^{\frac{1}{2}}) - (u^0u^1, \delta_t u^{\frac{1}{2}})].$$

Noticing

$$\begin{aligned} -(\delta_x^2 u^{\frac{1}{2}}, \delta_t u^{\frac{1}{2}}) &= \frac{1}{2\tau}(|u^1|_1^2 - |u^0|_1^2), \\ (u^{\frac{1}{2}}, \delta_t u^{\frac{1}{2}}) &= \frac{1}{2\tau}(\|u^1\|^2 - \|u^0\|^2), \\ (u^0u^1, \delta_t u^{\frac{1}{2}}) &= \frac{1}{\tau}[(u^0,(u^1)^2) - ((u^0)^2, u^1)], \end{aligned}$$

we have

$$\|\delta_t u^{\frac{1}{2}}\|^2 + \frac{1}{2\tau}(|u^1|_1^2 - |u^0|_1^2) + \lambda\left\{\frac{1}{\tau}[(u^0,(u^1)^2) - ((u^0)^2,u^1)] - \frac{1}{2\tau}(\|u^1\|^2 - \|u^0\|^2)\right\} = 0,$$

which can be rewritten as

$$F^0 = |u^0|_1^2 + \lambda\left[\frac{4}{3}((u^0)^2, u^1) - \frac{2}{3}(u^0,(u^1)^2) - \|u^0\|^2\right] \equiv \hat{F}^0. \tag{1.98}$$

Taking the inner product of (1.91) with $\Delta_t u^k$ on both the right- and left-hand sides yields

$$\|\Delta_t u^k\|^2 - (\delta_x^2 u^{\bar{k}}, \Delta_t u^k) = \lambda\left[(u^{\bar{k}}, \Delta_t u^k) - \frac{1}{3}((u^{k-1}+u^k+u^{k+1})u^k, \Delta_t u^k)\right],$$
$$1 \leqslant k \leqslant n-1.$$

Noticing

$$-(\delta_x^2 u^{\bar{k}}, \Delta_t u^k) = \frac{1}{4\tau}(|u^{k+1}|_1^2 - |u^{k-1}|_1^2), \quad (u^{\bar{k}}, \Delta_t u^k) = \frac{1}{4\tau}(\|u^{k+1}\|^2 - \|u^{k-1}\|^2)$$

and

$$\begin{aligned} &\frac{1}{3}((u^{k-1}+u^k+u^{k+1})u^k, \Delta_t u^k) \\ &= \frac{1}{6\tau}[((u^{k+1}+u^{k-1})u^k, u^{k+1}-u^{k-1}) + ((u^k)^2, u^{k+1}-u^{k-1})] \end{aligned}$$